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The rational homology of the outer automorphism group of F_7

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Abstract. We compute the homology groups $H_*(\mathrm{Out}(F_7);\mathbb{Q})$ of the outer automorphism group of the free group of rank 7.

We produce in this manner the first rational homology classes of $\mathrm{Out}(F_n)$ that are neither constant ($* = 0$) nor Morita classes ($* = 2n-4$).

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1. Introduction

The homology groups $H_k(\mathrm{Out}(F_n);\mathbb{Q})$ are intriguing objects. On the one hand, they are known to "stably vanish", i.e., for all $n \in \mathbb{N}$ we have $H_k(\mathrm{Out}(F_n);\mathbb{Q}) = 0$ as soon as k is large enough [3]. Hatcher and Vogtmann prove that the natural maps

$$H_k \mathrm{Out}(F_n) \to H_k \mathrm{Aut}(F_n) \quad \text{and} \quad H_k \mathrm{Aut}(F_n) \to H_k \mathrm{Aut}(F_{n+1})$$

are isomorphisms for $n \geq 2k+2$ respectively $n \geq 2k+4$, see [4,5]. On the other hand, $H_k(\mathrm{Out}(F_n);\mathbb{Q}) = 0$ for $k > 2n-3$, since $\mathrm{Out}(F_n)$ acts geometrically on a contractible space (the "spine of outer space", see [2]) of dimension $2n-3$. Combining these results, the only $k \geq 1$ for which $H_k(\mathrm{Out}(F_n);\mathbb{Q})$ could possibly be nonzero are in the range $\frac{n}{2} - 2 < k \leq 2n-3$. Morita conjectures in [9, page 390] that $H_{2n-3}(\mathrm{Out}(F_n);\mathbb{Q})$ always vanishes; this

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would improve the upper bound to $k = 2n - 4$, and $H_{2n-4}(\mathrm{Out}(F_n); \mathbb{Q})$ is also conjectured to be nontrivial.

We shall see that the first conjecture does not hold. Indeed, the first few values of $H_k(\mathrm{Out}(F_n); \mathbb{Q})$ may be computed by a combination of human and computer work, and yield

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
2	1	0										
3	1	0	0	0								
4	1	0	0	0	1	0						
5	1	0	0	0	0	0	0	0				
6	1	0	0	0	0	0	0	0	1	0		
7	1	0	0	0	0	0	0	0	1	0	0	1

The values for $n \le 6$ were computed by Ohashi in [12]. They reveal that, for $n \le 6$, only the constant class ($k = 0$) and the Morita classes $k = 2n-4$ yield nontrivial homology. The values for $n = 7$ are the object of this Note, and reveal that the picture changes radically:

Theorem. *The nontrivial homology groups $H_k(\mathrm{Out}(F_7); \mathbb{Q})$ occur for $k \in \{0, 8, 11\}$ and are all 1-dimensional.*

Previously, only the rational Euler characteristic

$$\chi_{\mathbb{Q}}(\mathrm{Out}(F_7)) = \sum (-1)^k \dim H_k(\mathrm{Out}(F_7); \mathbb{Q})$$

was known, and shown to be 1 by Morita, Sakasai and Suzuki [10]. These authors computed in fact the rational Euler characteristics up to $n = 11$ in that paper and the sequel [11].

2. Methods

We make fundamental use of a construction of Kontsevich [6], explained in [1]. We follow the simplified description from [12].

Let F_n denote the free group of rank n. This parameter n is fixed once and for all, and will in fact be omitted from the notation as often as possible. An *admissible graph of rank n* is a graph G that is 2-connected (G remains connected even after an arbitrary edge is removed), without loops, with fundamental group isomorphic to F_n, and without vertices of valency ≤ 2. Its *degree* is

$$\deg(G) := \sum_{v \in V(G)} (\deg(v) - 3).$$

In particular, G has $2n - 2 - \deg(G)$ vertices and $3n - 3 - \deg(G)$ edges, and is trivalent if and only if $\deg(G) = 0$. If Φ is a collection of edges in a graph G, we denote by G/Φ the graph quotient, obtained by contracting all edges in Φ to points.

A *forested graph* is a pair (G, Φ) with Φ an oriented forest in G, namely an ordered collection of edges that do not form any cycle. We note that the symmetric group $\mathrm{Sym}(k)$ acts on the set of forested graphs whose forest contains k edges, by permuting the forest's edges.

For $k \in \mathbb{N}$, let C_k denote the $\mathbb{Q}$-vector space spanned by isomorphism classes of forested graphs of rank n with a forest of size k, subject to the relation

$$(G, \pi\Phi) = (-1)^{\pi}(G, \Phi) \text{ for all } \pi \in \mathrm{Sym}(k).$$

Note, in particular, that if $(G, \Phi) \sim (G, \pi\Phi)$ for an odd permutation π then $(G, \Phi) = 0$ in C_k. These spaces (C_*) form a chain complex for the differential $\partial = \partial_C - \partial_R$, defined respectively on $(G, \Phi) = (G, \{e_1, \dots, e_p\})$ by

$$\partial_C(G, \Phi) = \sum_{i=1}^{p} (-1)^i (G/e_i, \Phi \setminus \{e_i\}),$$

$$\partial_R(G, \Phi) = \sum_{i=1}^{p} (-1)^i (G, \Phi \setminus \{e_i\}),$$

and the homology of (C_*, ∂) is $H_*(\mathrm{Out}(F_n); \mathbb{Q})$.

The spaces C_k may be filtered by degree: let F_pC_k denote the subspace spanned by forested graphs (G, Φ) with $\deg(G/\Phi) \leq p$. The differentials satisfy respectively

$$\partial_C(F_pC_k) \subseteq F_pC_{k-1}, \qquad \partial_R(F_pC_k) \subseteq F_{p-1}C_{k-1}.$$

A spectral sequence argument gives

$$H_p(\mathrm{Out}(F_n); \mathbb{Q}) = E^2_{p,0} = \frac{\ker(\partial_C | F_pC_p) \cap \ker(\partial_R | F_pC_p)}{\partial_R(\ker(\partial_C | F_{p+1}C_{p+1}))}. \tag{1}$$

Note that if $(G, \Phi) \in F_pC_p$ then G is trivalent. We compute explicitly bases for the vector spaces F_pC_p, and matrices for the differentials ∂_C, ∂_R, to prove the theorem.

3. Implementation

We follow for $n = 7$ the procedure sketched in [12]. Using the software program `nauty` [8], we enumerate all trivalent graphs of rank n and vertex valencies ≥ 3. The libraries in `nauty` produce a canonical ordering of a graph, and compute generators for its automorphism group. We then weed out the non-2-connected ones.

For given $p \in \mathbb{N}$, we then enumerate all p-element oriented forests in these graphs, and weed out those that admit an odd symmetry. The remaining ones are stored as a basis for F_pC_p. Let a_p denote the dimension of F_pC_p.

For (G, Φ) a basis vector in F_pC_p, the forested graphs that appear as summands in $\partial_C(G, \Phi)$ and $\partial_R(G, \Phi)$ are numbered and stored in a hash table as they occur, and the matrices ∂_C and ∂_R are computed as sparse matrices with a_p columns.

The nullspace $\ker(\partial_C | F_pC_p)$ is then computed: let b_p denote its dimension; then the nullspace is stored as a sparse $(a_p \times b_p)$-matrix N_p. The computation is greatly aided by the fact that ∂_C is a block matrix, whose row and column blocks are spanned by $\{(G,\Phi) : G/\Phi = G_0\}$ for all choices of the fully contracted graph G_0. The matrices N_p are computed using the linear algebra library `linbox` [7], which provides exact linear algebra over $\mathbb{Q}$ and finite fields.

Finally, the rank c_p of $\partial_R \circ N_p$ is computed, again using `linbox`. By (1), we have

$$\dim H_p(\mathrm{Out}(F_n); \mathbb{Q}) = b_p - c_p - c_{p+1}.$$

For memory reasons (the computational requirements reached 200GB of RAM at its peak), some of these ranks were computed modulo a large prime (65521 and 65519 were used in two independent runs).

Computing modulo a prime can only reduce the rank; so that the values c_p we obtained are underestimates of the actual ranks of $\partial_R \circ N_p$. However, we also know *a priori* that $b_p - c_p - c_{p+1} \geq 0$ since it is the dimension of a vector space; and none of the c_p we computed can be increased without at the same time causing a homology dimension to become negative, so our reduction modulo a prime is legal.

For information, the parameters a_p, b_p, c_p for $n = 7$ are as follows:

p	0	1	2	3	4	5	6	7	8	9	10	11
a_p	365	3712	23227	$\approx 105k$	$\approx 348k$	$\approx 854k$	$\approx 1.6m$	$\approx 2.3m$	$\approx 2.6m$	$\approx 2.1m$	$\approx 1.2m$	$\approx 376k$
b_p	365	1784	5642	14766	28739	39033	38113	28588	16741	6931	1682	179
c_p	0	364	1420	4222	10544	18195	20838	17275	11313	5427	1504	178

The largest single matrix operations that had to be performed were computing the nullspace of a 2038511×536647 matrix (16 CPU hours) and the rank modulo 65519 of a (less sparse) 1355531×16741 matrix (10 CPU hours).

The source files used for the computations are available as supplemental material. Compilation requires `g++` version 4.7 or later, a functional `linbox` library, available from the site `http://www.linalg.org`, as well as the `nauty` program suite, available from the site

`http://pallini.di.uniroma1.it`.

It may also be directly downloaded and installed by typing

'`make nauty25r9`'

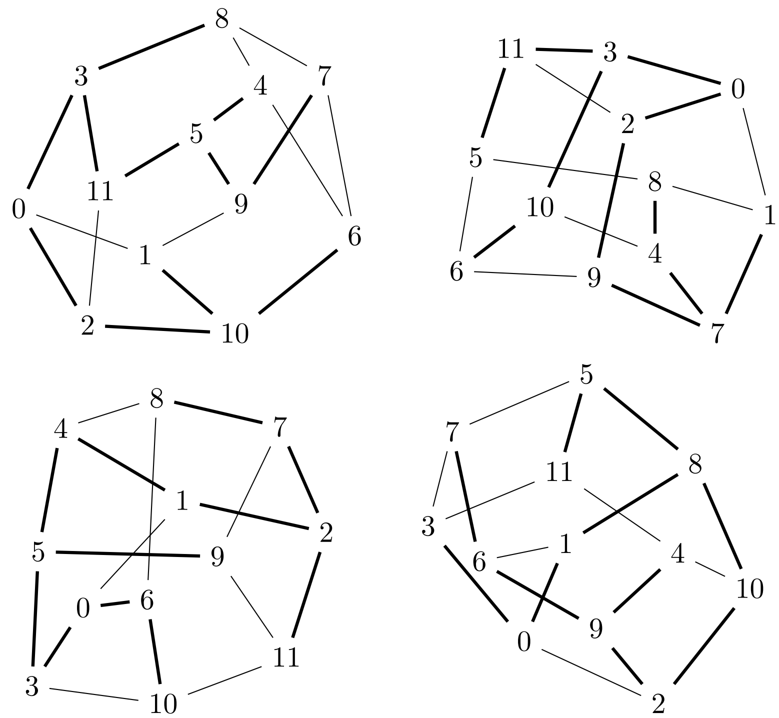
in the directory in which the supplemental material was downloaded. Beware that the calculations required for $n = 7$ are prohibitive for most desktop computers.

Conclusion

Computing the dimensions of the homology groups is only the first step in understanding them; much more interesting would be to know visually, or graph-theoretically, where these nontrivial classes come from.

It seems almost hopeless to describe, via computer experiments, the nontrivial class in degree 8, unless it is somehow related to the nontrivial class in $H_8(\mathrm{Out}(F_6);\mathbb{Q})$. It may be possible, however, to arrive at a reasonable understanding of the nontrivial class in degree 11.

This class may be interpreted as a linear combination w of trivalent graphs on 12 vertices, each marked with an oriented spanning forest. There are 376365 such forested graphs that do not admit an odd symmetry. The class $w \in \mathbb{Q}^{376365}$ is a $\mathbb{Z}$-linear combination of 70398 different forested graphs, with coefficients in $\{\pm 1, \dots, \pm 16\}$. For illustration, eleven graphs occur with coefficient ± 13; four of them have indices $25273, 53069, 53239, 53610$ respectively, and are, with the spanning tree in bold,



The coefficients of w, and corresponding graphs, are distributed as ancillary material in the file `w_cycle`, in format

‘`coefficient [edge1 edge2 ...]`’,

where each edge is ‘`x-y`’ or ‘`x+y`’ to indicate whether the edge is absent or present in the forest. Edges always satisfy $\mathtt{x} < \mathtt{y}$, and the forest is oriented so that its edges are lexicographically ordered. Edges are numbered from 0. There are no loops nor multiple edges.

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Links to ancillary material

The source and data files are stored at `http://arxiv.org/src/1512.03075v2/anc`; here are some direct links to them, embedded in the PDF document.

The cycle in degree 11: `w_cycle`.

The source files of the program that computed the b_p and c_p:

- `Makefile`
- `homology.h`
- `homology_boundary.C`
- `homology_graphs.C`
- `homology_print.C`
- `murmur3/README.md`
- `murmur3/example.c`
- `murmur3/makefile`
- `murmur3/murmur3.c`
- `murmur3/murmur3.h`
- `murmur3/test.c`.

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