# Minimal growth harmonic functions on lamplighter groups

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ABSTRACT. We study the minimal possible growth of harmonic functions on lamplighters. We find that $(\mathbb{Z}/2) \wr \mathbb{Z}$ has no sublinear harmonic functions, $(\mathbb{Z}/2) \wr \mathbb{Z}^2$ has no sublogarithmic harmonic functions, and neither has the repeated wreath product $(\cdots (\mathbb{Z}/2 \wr \mathbb{Z}^2) \wr \mathbb{Z}^2) \wr \cdots \wr \mathbb{Z}^2$. These results have implications on attempts to quantify the Derriennic–Kaimanovich–Vershik theorem.

## CONTENTS

## 1. Introduction

The celebrated Derriennic–Kaimanovich–Vershik theorem states that for any finitely generated group $G$ and any set of generators $S$, the Cayley graph of $G$ with respect to $S$ has bounded nonconstant harmonic functions if and only if the entropy of the position of a random walk on the same Cayley graph at time $n$ grows linearly with $n$ [7, 12]. This result was a landmark in the understanding of the *Poisson boundary* of a group, i.e., the space of bounded harmonic functions.

The "if" and the "only if" directions of the theorem are quite different in nature. The first direction states that once the entropy is sublinear the graph is Liouville, i.e., does not admit a nonconstant bounded harmonic function (this direction was proved earlier [2]). This direction may be quantified, e.g., one may show that there are no harmonic functions growing slower than $\sqrt{n/H(X_n)}$ where $H(X_n)$ is the entropy of the random walk. This is a known fact [8, 3] but for completeness we give the proof in the appendix.

In this paper we wish to study the question "how tight is the bound $\sqrt{n/H(X_n)}$?" As a simple example let us take the lamplighter group

$$(\mathbb{Z}/2) \wr \mathbb{Z}$$

(precise definitions will be given later, §1.3). Our first result is the following:

**Theorem 1.** *The lamplighter group $(\mathbb{Z}/2) \wr \mathbb{Z}$ with the standard generators does not support any nonconstant harmonic function $h$ with $h(x) = o(|x|)$ where $|\cdot|$ is the word metric.*

It is well-known and easy to see that the entropy is $\sqrt{n}$ and hence the bound $\sqrt{n/H(X_n)}$ gives only that harmonic functions growing slower than $n^{1/4}$ are constant. Thus on the lamplighter group the bound $\sqrt{n/H(X_n)}$ is not tight. As Theorem 1 is quite simple but still instructive, let us sketch its proof.

**Proof sketch.** Let us use the generators "move or switch," i.e., if we write any element of $(\mathbb{Z}/2) \wr \mathbb{Z}$ as a couple $(\omega, n)$ with $\omega : \mathbb{Z} \to \mathbb{Z}/2$ and $n \in \mathbb{Z}$ then the generators are $\{(\mathbf{1}_0, 0), (\vec{0}, 1), (\vec{0}, -1)\}$. Examine two elements $g_1, g_2 \in (\mathbb{Z}/2) \wr \mathbb{Z}$ which differ only in the configuration at 0, i.e., if $g_i = (\omega_i, n_i)$, then $n_1 = n_2$ and $\omega_1(k) = \omega_2(k)$ for all $k \neq 0$.

Let $X_n^i$ be two lazy random walks (with laziness probability $\frac{1}{4}$) starting from $g_i$, and couple them as follows:

- Changes to the $\mathbb{Z}$ component are done identically so that the $\mathbb{Z}$ components of $X_n^1$ and $X_n^2$ are always identical.
- Changes to the configuration are also done identically except when the walkers "are at 0" (i.e., their $\mathbb{Z}$ component is 0) and their configurations are still different. In this case, if one walker switches (i.e., goes in the $(\mathbf{1}_0, 0)$ direction) then the other walker stays lazily at its place, and vice versa.

• In all other cases, the moves are done together.

It is now clear that each time both walkers are at 0 they have a probability $\frac{1}{2}$ to "glue," i.e., to have $X_n^1 = X_n^2$, and when this happens this is preserved forever. For any $r \in \mathbb{N}$, define $T_r$ to be the first time the walkers are at $\pm r$. The fact that $h$ is harmonic means that $h(X_n^i)$ are martingales, and because $h$ is bounded for all time up to $T_r$ we may use the Optional Stopping Theorem to claim that

$$h(g_i) = \mathbb{E}(h(X_{T_r}^i)).$$

Let $E$ be the gluing time. Then we can write

$$
\begin{aligned}
h(g_1) - h(g_2) &= \mathbb{E}(h(X_{T_r}^1) - h(X_{T_r}^2)) \\
&= \mathbb{E}\left((h(X_{T_r}^1) - h(X_{T_r}^2))\mathbf{1}\{E < T_r\}\right) \\
&\quad + \mathbb{E}\left((h(X_{T_r}^1) - h(X_{T_r}^2))\mathbf{1}\{E \geq T_r\}\right).
\end{aligned}
$$

The first term is simply 0, as if the walkers glued before $T_r$ then $X_{T_r}^1 = X_{T_r}^2$. The second term is bounded by

$$\mathbb{P}(E \geq T) \cdot 2 \max\{h(g) : g \text{ can be the value of } X_{T_r}\}.$$

The probability is $\leq C/r$ from known properties of random walk on $\mathbb{Z}$. On the other hand, for $r > \max\{|\operatorname{supp}\omega_i|, |n_i|\}$ the only $g$ that can be values of $X_{T_r}$ have distance $\leq 5r$ from the identity of $(\mathbb{Z}/2) \wr \mathbb{Z}$ and by the sublinearity of $h$ we get $h(g) = o(r)$. We get that

$$h(g_1) - h(g_2) = 0 + \frac{C}{r}o(r) \xrightarrow{r \to \infty} 0$$

and we get that $h((\omega, n))$ does not depend on the value of $\omega(0)$. Translating we get that it does not depend on the value of any lamp, i.e., on any $\omega(i)$. This means that it is a function of $n$ only, which is harmonic, implying that it is a harmonic function on $\mathbb{Z}$. But a harmonic function on $\mathbb{Z}$ (with the generators $\pm 1$) is linear, which can be proved by a simple induction. Thus, $h$ is constant. □

The result is sharp since for the lamplighter there is an obvious linear growth harmonic function: the $\mathbb{Z}$ component. We remark also that, in general, every finitely-generated group supports a nonconstant linear growth harmonic function. See, e.g., [13, 20]. It is also instructive at this point to compare the lamplighter to nilpotent groups. Similarly to the lamplighter, nilpotent groups do not support any nonconstant sublinear growth harmonic functions (see, e.g., Remark 22, below). However nilpotent groups have much lower entropy: $\log n$ vs. $\sqrt{n}$ for the lamplighter.

Our main result concerns wreath products with $\mathbb{Z}^2$, or more generally any recurrent group.

**Theorem 2.** *Let $L$ be a finitely generated group and $\mu$ a symmetric measure over a finite set of generators such that $L$ supports no $\mu$-harmonic subloga-rithmic nonconstant function. Let $G$ be a recurrent group with respect to a*

*measure $\nu$. Let $\nu \wr \mu$ be the "move or switch" (each with probability $\frac{1}{2}$) measure on $L \wr G$. Then $L \wr G$ does not support any $\nu \wr \mu$-harmonic sublogarithmic nonconstant function.*

In particular, this means that repeated wreath products with $\mathbb{Z}^2$, i.e.

$$\underbrace{(\cdots (\mathbb{Z}/2 \wr \mathbb{Z}^2) \wr \mathbb{Z}^2) \wr \cdots \wr \mathbb{Z}^2}_{k \text{ times}}$$

all do not support any sublogarithmic nonconstant harmonic functions (with respect to the natural set of generators). As we will see below (Proposition 16), this group has entropy $n/\log^{(k)} n$. Heuristically this says that the Derriennic–Kaimanovich–Vershik theorem cannot be quantified in the logarithmic scale, as these examples have entropy extremely close to linear, but no corresponding harmonic functions (which should be extremely close to constant). We remark that constructing nonconstant harmonic functions growing logarithmically (which shows that Theorem 2 is sharp) is easy, and we do it in §2.3.

One may consider a similar statement for wreath products with $\mathbb{Z}$. Even for the second simplest case, $\mathbb{Z} \wr \mathbb{Z}$, getting the sharp order of growth (which is $|x|^{2/3}(\log \log |x|)^{1/3}$) is significantly harder. We plan to tackle it in a future paper. Our methods can be used for some of the analysis, but these methods require information regarding the speed of the random walk on the lamp group, and thus the analysis is more delicate.

It is not known whether the Liouville property depends on the choice of generators and this is a major open problem. Similarly, we do not know whether claims such as "$G$ does not support a nonconstant sublinear harmonic function" are group properties. As this is not the focus of the paper, we will always work with the most convenient system of generators. Theorem 1 can be strengthened to hold for any symmetric finitely-supported generating measure, and Theorem 2 may be strengthened so that the conclusion on iterated wreath products would hold for any set of generators, but we will not do it here.

**1.1. Notation.** For a graph $G$, we write $x \sim y$ to denote two adjacent vertices in $G$. The graph metric will be denoted by $\operatorname{dist}(\cdot, \cdot)$. If $G$ is a group, $1_G$ denotes the unit element in $G$. For two random variables $X$ and $Y$ the notation $X \sim Y$ will mean that $X$ and $Y$ have the same distribution — we hope no confusion will arise from the two different uses of $\sim$.

All our groups will be finitely generated, and we will not repeat this fact. Suppose therefore that $G = \langle S \rangle$ is generated by a finite set $S$ such that $S = S^{-1}$, (i.e., $S$ is symmetric). In this case it is natural to consider the Cayley graph of $G$ with respect to $S$, and the graph distance in this graph as the metric on $G$ (this is also known as the word metric on $G$ with respect to $S$). For every $g \in G$ we denote $|g| = \operatorname{dist}(1_G, g)$ (both dist and $|\cdot|$ depend on the group $G$ and on the generating set $S$ but we will suppress it

in the notation). Let $\mu$ be a symmetric probability measure on $S$; that is, $\mu(s^{-1}) = \mu(s)$ for all $s \in S$. Then $\mu$ induces a Markov chain on $G$, namely the process which transitions from $x$ to $xs$ with probability $\mu(s)$. We call this process the *(right) random walk* on $G$, or, if we want to stress the role of $\mu$, the $\mu$-random walk. The law of the random walk on $G$ started from $x \in G$ is denoted by $\mathbb{P}_x$ (again, dependence on $G$ and $S$ is suppressed in the notation). When we refer to the walk started at $1_G$, we omit the reference to the starting point, i.e., $\mathbb{P} = \mathbb{P}_{1_G}$.

We use $C$ and $c$ to denote positive constants, depending only on the group and the set of generators in question, and, if there is a specific harmonic function studied, on it too. The value of $C$ and $c$ may change from formula to formula or even within the same formula, and we will also often omit sentences like "there exists a constant $C$ such that" from beginnings of lemmas. The letter $C$ will be used for constants which are large enough, and $c$ for constants which are small enough. We will denote $f \approx g$ if $cg \leq f \leq Cg$.

We denote the indicator function of a set $E$ by either $\mathbf{1}_E$ or $\mathbf{1}\{E\}$.

**1.2. Harmonic growth definitions.** For a group $G$ and a finitely-supported measure $\mu$ on $G$, a function $h : G \to \mathbb{R}$ is called $\mu$-*harmonic* if for every $x \in G$, $h(x) = \mathbb{E}_x[h(X_1)]$, where $(X_n)_{n \geq 0}$ is a $\mu$-random walk on $G$. In other words, $h(X_n)$ is a martingale. If $\mu$ is clear from the context we will just call such functions *harmonic*.

The harmonic growth of a graph $G$ is the smallest rate of growth of a nonconstant harmonic function on $G$. (In this paper we only work with Cayley graphs, so we will consider growth around $1_G$.) For a monotone nondecreasing function $f : \mathbb{N} \to [0, \infty)$, we say that $G$ has *harmonic growth at least $f$* (this is denoted by $\mathrm{har}(G) \succeq f$ — note that we do not claim $\mathrm{har}(G)$ is some well-defined function, this is just a shorthand notation), if every harmonic $h : G \to \mathbb{R}$, with $|h(g)| = o(f(|g|))$ is constant. The graph $G$ is said to have harmonic growth at most $f$ if there exists a harmonic function $h : G \to \mathbb{R}$ such that $|h(g)| < Cf(|g|)$ for some constant $C > 0$ (this is denoted by $\mathrm{har}(G) \preceq f$). If the harmonic growth of $G$ is at least $f$ and at most $f$ then we say that $G$ has harmonic growth $f$, and denote this $\mathrm{har}(G) \approx f$. Note that the harmonic growth of a graph is an asymptotic notion. In particular, it depends only on the behaviour of $f$ at infinity. Let us mention a few properties of the harmonic growth:

(1) The harmonic growth of a Cayley graph is always at most linear since every such graph possesses a linearly growing harmonic function [13, 20].

(2) The harmonic growth of the group $\mathbb{Z}^d$ is linear (the function

$$h(x_1, \ldots, x_d) = x_1$$

is harmonic, and there are no nonconstant sublinear harmonic functions).

**1.3. Lamplighters.** We now define the groups that are of interest to us, as well as their natural set of generators. These are called *wreath products* or *generalised lamplighters*, with the lamplighter group being the simplest example, $(\mathbb{Z}/2) \wr \mathbb{Z}$.

Let $L, G$ be groups. The *wreath product* $L \wr G$ is the semi-direct product $L^G \rtimes G$, where $L^G$ is the group of all functions from $G$ to $L$ which are $1_L$ for all but finitely many elements of $G$ (such a function is said to have finite support) and where $G$ acts on $L^G$ by translations. We will denote elements of $L \wr G$ by $(\omega, g)$ with $\omega \in L^G$ and $g \in G$ so the product is

$$(\omega, g)(\xi, k) = (\omega(\cdot)\xi(g^{-1}\cdot), gk).$$

For an element $(\omega, g) \in L\wr G$, and $k \in G$, we call $g$ the *lamplighter (position)*, and $\omega(k)$ is the *(status of the) lamp at $k$*. The group $G$ is sometimes called the *base* group and the group $L$ the group of *lamps*.

For $\ell \in L$, define the delta function $\delta_\ell \in L^G$ by

$$\delta_\ell(g) = \begin{cases} \ell & \text{if } g = 1_G \\ 1_L & \text{otherwise.} \end{cases}$$

Let also $\mathbf{1}$ denote the function which is constant $1_L$. Let $S$ be a generating set of $L$ and $U$ a generating set of $G$. Consider the set

$$\Gamma = \{(\delta_s, 1) \,:\, s \in S\} \cup \{(\mathbf{1}, u) \,:\, u \in U\}.$$

It is not difficult to see that $\Gamma$ generates $L \wr G$. Right multiplication by $(\mathbf{1}, u)$ corresponds to moving the lamplighter in $G$ while right-multiplying by $(\delta_s, 1)$ corresponds to changing the status of the current lamp by right-multiplying it by $s$. Given symmetric probability measures, $\mu$ supported on $S$ and $\nu$ supported on $U$, we can define the *move or switch* measure, which is a symmetric probability measure $\mu \wr \nu$ supported on $\Gamma$, by

$$(\mu \wr \nu)(\mathbf{1}, u) := \frac{1}{2} \cdot \nu(u) \qquad \text{and} \qquad (\mu \wr \nu)(\delta_s, 1) := \frac{1}{2} \cdot \mu(s).$$

That is, under the measure $\mu\wr\nu$, the walk on $L\wr G$ has the following behaviour: with probability $1/2$ the lamplighter moves in $G$ according to the distribution given by $\nu$; with the remaining probability $1/2$ the lamplighter does not move but rather changes the status of the current lamp according to the distribution given by $\mu$.

If the base group $G$ is transient, then $L \wr G$ admits bounded nonconstant harmonic functions (i.e., is not Liouville). For instance, one may consider the function $h(\omega)$ to be the probability that the status of the lamp at $1_G$ differs eventually from $1_L$. As a consequence, $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^3$ is an example of an amenable non-Liouville group [12, §6.2]. See also [9] for a proof that these groups nevertheless do not support nonconstant harmonic functions of bounded energy.

## 2. Proof of Theorem 2

Recall the statement of Theorem 2: if $G$ is recurrent and if $L$ does not support any nonconstant sublogarithmic functions, then neither does $L \wr G$. Before starting the proof proper let us remark that the difficulty lies in the case that $L$ is infinite. If $L$ is finite then the theorem may be proved quite similarly to the proof of Theorem 1 (see page 834). Let us recall quickly the argument:

**Sketch of the finite $L$ case.** Let $x_1, x_2 \in L \wr G$ differ only in the configuration at $1_G$. Examine two lazy random walkers starting from the $x_i$ and coupled to walk together except when they are both at $1_G$, where they have positive probability to glue for all time. We define $E$ to be the gluing time and $T_r$ to be the first time that the walker reaches distance $r$ from $1_G$. Known estimate for return probabilities on recurrent groups (which, by Gromov's theorem are finite extensions of $\mathbb{Z}$ or $\mathbb{Z}^2$) show that $\mathbb{P}(E \geq T_r) \leq C/\log r$. The sublogarithmicity of $h$ shows that the contribution of this event decays as $r \to \infty$ and the coupling shows that $h$ does not depend on the lamp at $1_G$. Translating we get that $h$ does not depend on the state of the lamps at all, and hence may be considered as a harmonic function on $G$. But any sublinear harmonic functions on a virtually nilpotent group is constant (Remark 22, below). $\square$

Where things change for $L$ infinite is that one can no longer claim that the probability that $E$ occurred before the $k^{\text{th}}$ return to $1_G$ increases to 1 exponentially fast in $k$. Even in the simplest case that the lamp group is $\mathbb{Z}$, this probability decays only like $1/\sqrt{k}$ and had we translated the proof literally we would only get that $\mathbb{Z} \wr \mathbb{Z}^2$ has no sub-$\sqrt[3]{\log}$ nonconstant harmonic functions.

To solve this problem we replace our $x_1, x_2$ with infinitely many $x$, which differ only at the lamp at $1_G$. This gives a function $\psi : L \to \mathbb{R}$ with $\psi(\ell) = h(x_\ell)$, where $x_\ell \in L \wr G$ is $x$ with the status of the lamp at $1_G$ set to $\ell$. Now $\psi$ is sublogarithmic on $L$ but is not necessarily harmonic on it. However, the harmonicity and sublogarithmic growth of $h$ on $L \wr G$ allows to use the strong Markov property and represent $\psi(\ell)$ as the value of $h$ at the $k^{\text{th}}$ return of a random walker to $1_G$. This means that $\psi$ may be written as $Q^k f_k$ where $f_k$ is the value of $h$ had the lamp at $1_H$ never moved (and $Q$ is the transition kernel of lazy random walk on $L$). The sublogarithmic growth of $h$ allows to show that $f_k(\ell) \leq Ck^3 \log|\ell|$ (the polynomial growth in $k$ is the important fact here), see Proposition 11. We will show (Proposition 5) that such estimates imply that $\psi$ is constant. The laziness of the walk plays an important role in this step.

The approach is significantly complicated by the fact that we do not know a priori that the value of $h$ at the $k^{\text{th}}$ return to $1_G$ is integrable. This complicates the definition of $f_k$ and some parts of the argument. The details are provided in the next sections.

## 2.1. Preliminaries. We begin with some preliminary results.

**Lemma 3.** *Fix $p \in (0,1)$ and $n \in \mathbb{N}$. Let $b(k) = \binom{n}{k}p^k(1-p)^{n-k}$ and $b(k) = 0$ for $k \in \mathbb{Z} \setminus \{0, \ldots, n\}$. Then for the difference operator $\partial$ defined by $\partial\psi(k) = \psi(k) - \psi(k-1)$ we have that, for any $k$,*

$$(1) \qquad\qquad |\partial^m b(k)| \le \left(\frac{m}{p(1-p)n}\right)^{m/2}.$$

**Proof.** From the binomial formula,

$$\left(1 - p + pe^{it}\right)^n = \sum_k b(k)e^{itk}$$

which leads to

$$b(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(1 - p + pe^{it}\right)^n \cdot e^{-itk} dt.$$

Applying $\partial$ is the same as multiplying by $1 - e^{it}$ in the Fourier domain hence

$$\partial^m b(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(1 - p + pe^{-it}\right)^n \cdot (1 - e^{it})^m \cdot e^{-itk} \, dt.$$

We estimate the integral by the maximum of the absolute value of the integrand. The expression for the maximum would be shorter if we use the quantity $u = 2p(1-p)(1 - \cos(t)) \in [0,1]$. We get

$$|\partial^m b(k)| \le \sup_{u \in [0,1]} (1 - u)^{n/2} \cdot u^{m/2} \cdot p^{-m/2}(1-p)^{-m/2},$$

which is maximised at $u = \frac{m}{m+n}$. Hence,

$$|\partial^m b(k)| \le \left(\frac{m}{p(1-p)n}\right)^{m/2}. \qquad\qquad \square$$

**Lemma 4.** *Let $G$ be a group and let $P$ be the transition matrix of some random walk on $G$. Let $\psi : G \to \mathbb{R}$ be a function with sub-linear growth. Then if $(I - P)\psi$ is constant, then this constant must be zero (and then $\psi$ is harmonic).*

**Remark.** In this lemma the random walk need not be symmetric.

**Proof.** Let $(X_t)$ be the random walk on $G$ with transitions given by $P$ and let $K$ be the constant from the statement of the lemma, i.e., $(P - I)\psi \equiv K$. Then $M_t := \psi(X_t) - tK$ is a martingale, and hence for all $t$,

$$\psi(x) = M_0 = \mathbb{E}_x[M_t] = \mathbb{E}_x[\psi(X_t)] - tK.$$

But since $\psi$ has sub-linear growth,

$$K = \frac{1}{t}\mathbb{E}_x[\psi(X_t)] - \frac{1}{t}\psi(x) \longrightarrow 0 \text{ as } t \text{ tends to } \infty.$$

So $K = 0$ and $\psi$ is harmonic with respect to $P$. $\qquad\qquad \square$

**Proposition 5.** *Let $G$ be a group and let $P$ be the transition matrix of some random walk on $G$, let $\alpha \in (0,1)$ and let $Q = \alpha I + (1-\alpha)P$ be the transition matrix of the corresponding lazy random walk. Let $\psi : G \to \mathbb{R}$ be a function with sub-linear growth.*

*Suppose that for infinitely many $k$ there exists functions $f_k : G \to \mathbb{R}$ with $f_k(g) \le Ck^C |g|^C$ such that $\psi = Q^k f_k$. Then, there exists $m$ such that $(I - P)^m \psi \equiv 0$.*

*Moreover, if $\psi$ grows slower than the harmonic growth of $G$ (with respect to $P$), then $\psi$ is constant.*

**Proof.** Observe that

$$Q^k = (\alpha I + (1-\alpha)P)^k = \sum_{j=0}^{k} \binom{k}{j} \alpha^{k-j}(1-\alpha)^j P^j = \sum_{j \in \mathbb{N}} b(j) P^j,$$

for $b(j)$ as in Lemma 3, with $p_{\text{lemma } 3} = 1 - \alpha$ and $n_{\text{lemma } 3} = k$. Thus, we may write $\psi = Q^k f_k$ as

$$(2) \qquad \psi = \sum_{j \in \mathbb{N}} b(j) P^j f_k.$$

Thus for every $k$ for which (2) holds we may write

$$(3) \qquad |(I - P)^m \psi(g)|$$

$$\overset{(2)}{=} \sum_{j \in \mathbb{N}} \partial^m b(j) P^j f_k \le \sum_{j \in \mathbb{N}} |\partial^m b(j)| \cdot |P^j f(g)|$$

$$\overset{(*)}{\le} (k+1) \cdot \left( \frac{m}{\alpha(1-\alpha)k} \right)^{m/2} \cdot \sup \{ |f(h)| \ : \ \text{dist}_G(g,h) \le k \}$$

$$\le C \left( \frac{m}{\alpha(1-\alpha)} \right)^{m/2} \cdot (|g| + k)^C \cdot k^{1-m/2}.$$

The inequality marked by $(*)$ has three parts. First, we use the fact that the sum has only $k + 1$ nonzero terms to bound it by $k + 1$ times the maximal term. Second, we estimate the term $\partial^m b(j)$ using Lemma 3. Third, for the term $P^j f_k$ we note that because the generator $P$ is supported on the generators of the Cayley graph $P^j f(g)$ contains only terms with distance $\le j \le k$ from $g$, and the coefficients sum to 1, so $P^j f_k$ can be bounded by the maximum of $f_k$ in the given ball.

Provided that $m > 2(C+1)$, the last term in (3) converges to 0 as $k \to \infty$. This implies the first part of the claim.

Let us now assume that $\psi$ grows slower than har$(G)$. Then, $(I - P)^{m-1}\psi$ also grows slower than har$(G)$, as it is a finite combination of translates of $\psi$. Since $(I - P)^{m-1}\psi$ is harmonic (via the first part of the claim), this implies that it is constant. However, because every group has harmonic growth at most linear, we have that $(I - P)^{m-2}\psi$ is a sub-linear function with constant Laplacian. By Lemma 4, we get that $(I - P)^{m-2}\psi$ is harmonic. Iterating this reasoning, we obtain that $\psi$ is harmonic and thus constant. $\qquad \square$

**Lemma 6.** *Let $(X_t)_t = (\omega_t, g_t)_t$ be a random walk on $L \wr G$ with step measure $\mu \wr \nu$ and define stopping times*

$$(4) \qquad U_k := \inf \left\{ t \geq 0 \ : \ \sum_{j=0}^{t} \mathbf{1}\{g_t = 1_G\} \geq k \right\},$$

$$T_r := \inf \left\{ t \geq 0 \ : \ \mathrm{dist}(g_t, 1_G) > r \right\}.$$

*Then, the random variable $\omega_{U_k}(1_G)$ is independent of $\{\omega_{U_k}(g)\}_{g \neq 1_G}$ and of the event $\{U_k < T_r\}$. Furthermore, its law is the law of a lazy $\mu$-random walk on $L$ with laziness probability $1/2$, at time $k$.*

The proof of this statement is elementary and will be omitted.

We finish this section with a few standard facts on recurrent groups. Most readers would want to skip to §2.2.

**Lemma 7.** *Let $G$ be a recurrent group, let $g \in G$ and let $r > |g|$. Let $E$ be the event that random walk on $G$ starting from $g$ reaches distance $r$ from $1_G$ before reaching $1_G$ itself. Then $\mathbb{P}(E) \leq C \log |g| / \log r$.*

**Proof.** Any recurrent group contains as a subgroup of finite index one of $0$, $\mathbb{Z}$ or $\mathbb{Z}^2$, see, e.g., [22, Theorem 3.24]. The proof uses deep results by Varopoulos, Gromov, Bass, and Guivarc'h, see [22] for details and references. The theorem of Gromov has had new proofs recently, see [13, 20, 18]. A reader unfamiliar with this literature may simply read Theorem 2 replacing the sentence "Let $G$ be a recurrent group" with "Let $G$ be a finite extension of $0$, $\mathbb{Z}$ or $\mathbb{Z}^2$" with no loss in understanding.

Let us start with the case that $G = \mathbb{Z}^2$. In this case it is known that there is a function $a : \mathbb{Z}^2 \to \mathbb{R}$ harmonic everywhere except at $(0,0)$ and satisfying $a(x) = c \log |x| + O(1)$ (see, e.g., [14, §4.4]). Let $b$ be the harmonic extension of the values of $a$ on the boundary of the ball of radius $r$ to its interior (so $b(x) = c \log r + O(1)$). We see that $h = 1 - (b - a)/(b(0,0) - a(0,0))$ is harmonic on the ball except at $(0,0)$, is $0$ at $(0,0)$, $1$ on the boundary and $(\log |g| + O(1))/ \log r$ at $g$. By the strong Markov property, $h(g)$ is exactly the probability sought, and the claim is proved in this case.

The case that the group is a finite extension of $\mathbb{Z}^2$ (i.e., that it contains it as a subgroup of finite index) may be done similarly: the function $a$ on $G$ can be defined, as in [14], by

$$a(h) = \sum_{n=0}^{\infty} (p_n(1_G) - p_n(h)),$$

where $p_n$ is the heat kernel on $G$. Since $G$ satisfies a Berry-Essen estimate (see, e.g., [1]), $a$ would still satisfy $a(x) = c \log(|x|) + O(1)$. If $G$ is a finite extension of $\mathbb{Z}$ a similar argument holds except this time $a(g) = O(|g|)$ and we get $\mathbb{P}(E) \leq C/r$. If $G$ is finite then $\mathbb{P}(E) = 0$ for $r$ sufficiently large. $\qquad \square$

**Lemma 8.** *Let $G$ be a recurrent group and let $T_r$ be the exit time from a ball of radius $r$. Then*

$$\mathbb{P}(T_r > M) \leq 2 \exp(-cr^{-2}M).$$

**Proof.** Random walk on any group satisfies the weak Poincaré inequality, see [19, Lemma 4.1.1] for the statement of the inequality and the proof. Since, as in the proof of the previous lemma, it is a finite extension of $\{0\}$, $\mathbb{Z}$ or $\mathbb{Z}^2$, it also satisfies volume doubling, i.e., $|B(2r)| \leq C|B(r)|$. This means, by Delmotte's theorem [6] that $p_t(x, y) \leq C/|B(\sqrt{t})|$. Summing this inequality we see that after time $C_1 r^2$ for some $C_1$ sufficiently large the probability to stay in a ball of radius $2r$ is $\leq \frac{1}{2}$. This means that if in time $t$ you are at some $g \in B(1_G, r)$ then by time $t + C_1 r^2$ you have probability $\geq \frac{1}{2}$ to exit $B(g, 2r) \supset B(1_G, r)$. In the language of $T_r$ this means that

$$\mathbb{P}(T_r > t + C_1 r^2 \mid T_r > t) \leq \frac{1}{2}.$$

The lemma follows readily. $\qquad\square$

**Lemma 9.** *Recall the definition of $U_k$ and $T_r$ from (4). For every $k \geq 1$, every $M \geq 0$ and every starting point $x = (\omega, g) \in L \wr G$, we have that $\log(T_r + M)$ is integrable and*

$$(5) \qquad \mathbb{E}_x[\log(T_r + M)\mathbf{1}_{\{T_r \leq U_k\}}] < Ck \frac{\log(r + M)\log(|g|)}{\log r}.$$

**Proof.** The integrability clause is an immediate corollary of Lemma 8 so we move to prove (5). Write

$$(6) \qquad \mathbb{E}[\cdot] = \mathbb{E}[\cdot\mathbf{1}_{\{T_r < r^3\}}] + \sum_{i=0}^{\infty} \mathbb{E}[\cdot\mathbf{1}_{\{T_r \in [r^3 2^i, r^3 2^{i+1})\}}].$$

In the first term, the integrand $\log(T_r + M)\mathbf{1}_{\{T_r < r^3\}}$ is bounded by

$$\log(M + r^3) \leq 3\log(M + r)$$

and the probability of the event $\{T_r \leq U_k\}$ is at most $Ck\log(|g|)/\log r$ by Lemma 7. For the second term in (6), we drop the condition $T_r \leq U_k$ and write

$$\mathbb{E}\left[\log(E(r) + M) \cdot \mathbf{1}_{\{T_r \leq U_k\}} \cdot \mathbf{1}_{\{T_r \in [r^3 2^i, r^3 2^{i+1})\}}\right]$$
$$\leq \log(r^3 2^{i+1} + M) \cdot \mathbb{P}[T_r \geq r^3 2^i] \leq \log(r^3 2^{i+1} + M)\exp(-cr2^i)$$

which may be readily summed over $i$ and the sum is bounded by

$$C\log(r + M)/\log r. \qquad\square$$

**2.2. The main step.** We now reach the heart of the proof of Theorem 2. Throughout this section, we fix a group $L$ with $\mathrm{har}(L) \succeq \log$, and a recurrent group $G$. We fix one $x \in L \wr G$ for the rest of the proof.

For $\ell \in L$, let $\phi_\ell : L \wr G \to L \wr G$ be the function that changes the status of the lamp at $1_G$ to $\ell$, leaving all other lamps unchanged. In a formula

$$\phi_\ell(\sigma, g) = (\tau, g) \text{ with } \tau(k) = \begin{cases} \sigma(k) & \text{if } k \neq 1_G \\ \ell & \text{otherwise.} \end{cases}$$

We note immediately that $\phi$ does not change distances by much:

(7) $$|\phi_\ell(g)| \leq |g| + |\ell|,$$

which holds for our "move or switch" generators.

**Definition 10.** Let $h : L \wr G \to \mathbb{R}$ be a harmonic function of sub-logarithmic growth and let $x \in L \wr G$. For $k \geq 1$ and $\ell \in L$ define

(8) $$f_k(\ell) = \lim_{r \to \infty} \mathbb{E}_x[h(\phi_\ell(X_{\min\{U_k, T_r\}}))].$$

where $U_k$ and $T_r$ are defined by (4). Note that $f_k$ depends also on $h$ and $x$, but these will be suppressed in the notation.

Here and below $\mathbb{E}$ refers to expectation with respect to the random walk on the group $L \wr G$ with the "move or switch" generators.

It is not clear a-priori that $f_k$ is a well-defined function as we have not shown that $h(\phi_\ell(X_{\min\{U_k, T_r\}}))$ is integrable, nor that the limit exists. We will show this in Proposition 11, and, more importantly, give a useful estimate on $f_k$.

**Proposition 11.** *For every $k \geq 1$, $f_k$ is well-defined and satisfies*

$$|f_k(\ell)| \leq C \log(|x| + |\ell|) \cdot k^3,$$

*for some constant $C > 0$ (which may depend on $h$).*

**Proof.** Denote

$$M(r) = \sup \left\{ \frac{|h(y)|}{\log(|y|)} : y \in L \wr G \text{ such that } |y| \geq r \right\}$$

and note that $M$ is decreasing in $r$ and $M(r) \to 0$ as $r \to \infty$. A simple initial reduction is the following:

**Claim 12.**

$$\lim_{r \to \infty} \mathbb{E}_x[h(\phi_\ell(X_{T_r})) \cdot \mathbf{1}\{T_r < U_k\}] = 0.$$

**Proof.** We first note that

$$r \leq |\phi_\ell(X_{T_r})| \overset{(7)}{\leq} |X_{T_r}| + |\ell| \leq T_r + |x| + |\ell|$$

and hence

$$|h(\phi_\ell(X_{T_r}))| \leq M(|\phi_\ell(X_{T_r})|) \log(|\phi_\ell(X_{T_r})|)$$
$$\leq M(r) \log(T_r + |x| + |\ell|).$$

Lemma 8 gives that $\log(T_r + |x| + |\ell|)$ is integrable and consequently so is $h(\phi_\ell(X_{T_r}))$. Taking expectation (and assuming $r > |x|$) we get

$$\mathbb{E}_x[h(\phi_\ell(X_{T_r})) \cdot \mathbf{1}\{T_r < U_k\}]$$
$$\leq M(r) \, \mathbb{E}_x[\log(T_r + |x| + |\ell|) \cdot \mathbf{1}\{T_r < U_k\}]$$
$$\overset{(5)}{\leq} M(r) \cdot Ck \frac{\log(r + |x| + |\ell|) \log(|x|)}{\log r} \to 0 \quad \text{as } r \to \infty$$

as needed. $\square$

A similar calculation shows that the variable $h(\phi_\ell(X_{\min\{U_k, T_r\}}))$ integrated over in the definition of $f_k$ (8) is indeed integrable. All similar quantities (i.e., that involve only the walk in the ball of radius $r$ in $G$) are proved to be integrable using the same argument and we will not return to this point.

At some point during the proof, it will be convenient to assume that the walk is not degenerate, i.e., does not spend all its time at $1_G$ (in particular the degenerate case can happen only if the $G$-component of the starting point $x$ is $1_G$). Let us remove this event now. Define therefore

$$(9) \qquad \mathscr{A} = \{\text{the walk stays at } 1_G \text{ until } U_k\},$$
$$\mathscr{B} = \{U_k < T_r\} \setminus \mathscr{A},$$
$$f(r) = \mathbb{E}_x[h(\phi_\ell(X_{U_k})) \cdot \mathbf{1}_{\mathscr{B}}].$$

**Claim 13.** *The proposition will be proved once we show that $f(r)$ converges as $r \to \infty$, and that $\lim f(r) \leq Ck^3 \log(|x| + |\ell|)$.*

**Proof.** Clearly $\mathbb{P}(\mathscr{A}) \leq \exp(-c(k-1))$. Further, when $\mathscr{A}$ happens then $U_k = k - 1$ so $|X_{U_k}| \leq |x| + k$. Writing

$$\mathbb{E}_x[h(\phi_\ell(X_{\min\{U_k, T_r\}}))]$$
$$= f(r) + \mathbb{E}_x[h(\phi_\ell(X_{U_k})) \cdot \mathbf{1}_{\mathscr{A}}] + \mathbb{E}_x[h(\phi_\ell(X_{T_r})) \cdot \mathbf{1}\{T_r < U_k\}]$$

(clearly $T_r$ can never be equal to $U_k$) we get that the second term is independent of $r$ and bounded by $C \exp(-c(k-1)) \log(|x| + |\ell| + k)$, while the third term goes to zero as $r \to \infty$ by Claim 12. $\square$

Define now $g_t$ to be the position of the lighter at time $t$ (or the $G$-component of $X_t$ if you want) and define

$$\Lambda_j = \max\{|g_t| : t \in [U_j, U_{j+1}]\},$$

where the $U_j$ are still defined by (4). We call $\Lambda_j$ the *height* of the $j^{\text{th}}$ excursion. We need to single out the excursion with the largest height (denote it by $i$ — if there are ties take the last longest walk). Define therefore the following two random elements of $L \wr G$:

$$V = X_{U_i}^{-1} X_{U_{i+1}} \qquad W = X_{U_i} X_{U_{i+1}}^{-1} X_{U_k}.$$

In words, $V$ is the excursion of largest height and $W$ are all the rest. We note the following

**Claim 14.** *Under $\mathscr{B}$ the variables $X_{U_k}$ and $WV$ have the same distribution.*

**Proof.** Since the $G$ component of all $X_{U_i}$ is $1_G$ then we need only consider the lamps. However, whether we take the steps of the random walk in the original order $(X_{U_k})$ or with the largest excursion taken out and performed in the end $(WV)$, each lamp is visited exactly the same number of times. So conditioning on the steps in the $G$ direction, each lamp does a simple random walk on $L$ of equal length (we use here that the event $\mathscr{B}$ depends only on the $G$ component). This shows that $X_{U_k}$ and $WV$ have the same distribution after conditioning on the walk in the $G$ direction. Integrating gives the claim. $\qquad\square$

In particular,
$$f(r) = \mathbb{E}[h(\phi_\ell(WV)) \cdot \mathbf{1}_{\mathscr{B}}].$$
Condition on $W$ and examine $V$. It is the value of simple random walk on $L \wr G$, conditioned to have larger height that all other excursions, at the time when it first returns to $1_G$. Examine the time $\tau$ when the walker "knows" this excursion is the longest (this could be either the time when it reaches the same height as the highest excursion in $W$, or when it surpasses it, depending on how one resolves ties, but in all cases it is a stopping time). We also modify $\tau$ in the degenerate case that all excursions in $W$ stay in $1_G$ and require from $\tau$ to be at least $U_i + 1$ even if the walker knows it was the largest already at time $U_i$, so that the walker also knows it did not spend all time in $1_G$. After $\tau$, the walk is a simple random walk, unconditioned. Write $V = V_1 V_2$ with
$$V_1 = X_{U_i}^{-1} X_\tau \qquad V_2 = X_\tau^{-1} X_{U_{i+1}}$$
and condition also on $V_1$. Write
$$f(r) = \mathbb{E}_x[\mathbb{E}[h(\phi_\ell(WV_1V_2)) \cdot \mathbf{1}_{\mathscr{B}} \,|\, i, W, V_1]].$$
We notice two facts. First, the condition $\neg\mathscr{A}$ (recall that $\mathscr{A}$ is our degenerate event, see (9)) affects only $W$ and $V_1$, and can be taken from the inner expectation to the outer. Second, we can write $\phi_\ell(WV_1V_2) = \phi_\ell(W)V_1V_2$ because the value of the lamp at $1_G$ is changed only in excursions of height 0 (here is where we use $\neg\mathscr{A}$, to say that $V$ is not of length 0). Denote $y = \phi_\ell(W)V_1$. We get

(10)      $$f(r) = \mathbb{E}_x\left[\mathbb{E}\left[h(yV_2) \cdot \mathbf{1}\{T_r > U_k\} \,|\, i, W, V_1\right]\mathbf{1}_{\neg\mathscr{A}}\right].$$

We now apply the strong Markov property at the stopping time $\tau$. The event $T_r > U_k$ for the "external" random walk becomes $T_r > U_1$ for the random walk after $\tau$, and $V_2$ becomes $X_{U_1}$. Hence
$$\mathbb{E}[h(yV_2) \cdot \mathbf{1}\{T_r > U_k\}] = \mathbb{E}_y[h(X_{U_1}) \cdot \mathbf{1}\{T_r > U_1\}].$$

It is time to use the fact that $h$ is harmonic on $L \wr G$. We write

$$\mathbb{E}_y[h(X_{U_1})\mathbf{1}\{T_r > U_1\}] = \mathbb{E}_y[h(X_{\min\{T_r, U_1\}})] - \mathbb{E}_y[h(X_{T_r})\mathbf{1}\{U_1 > T_r\}]$$

Since $h$ is harmonic, we have by the optional stopping theorem that

$$\mathbb{E}_y[h(X_{\min\{t, U_1, T_r\}})] = h(y) \qquad \forall t.$$

We now take $t \to \infty$ using the dominated convergence theorem, which we may because

$$\sup_t |h(X_{\min\{t, U_1, T_r\}})| \leq \max_{t \leq T_r} |h(X_t)|$$

$$\leq \max_{t \leq T_r} C \log(|X_t|) \leq C \log(|y| + T_r)$$

which is integrable, by Lemma 9. We get

$$\mathbb{E}_y[h(X_{\min\{U_1, T_r\}})] = h(y).$$

Inserting this into (10) gives

$$f(r) = \mathbb{E}_x\left[\left(h(y) - \mathbb{E}_y[h(X_{T_r})\mathbf{1}\{U_1 > T_r\}]\right)\mathbf{1}_{\neg \mathscr{A}}\right].$$

It will be convenient to add the condition that the height of the second-highest excursion is $\leq r$. We may do so because otherwise $|y| > r$, in the inner expectation the walker is stopped immediately ($T_r = 0$) and the inner expectation itself is exactly $h(y)$ and the term contributes zero. Denote $\mathscr{C} = \{$the second-highest excursion is $\leq r\} \setminus \mathscr{A}$. We write

$$f(r) = I + II \qquad I = \mathbb{E}_x[h(y) \cdot \mathbf{1}_{\mathscr{C}}] \qquad II = \text{the rest}$$

and bound these terms individually.

**Claim 15.** *As $r \to \infty$, $II \to 0$.*

**Proof.** We reverse the use of the Markov property and get

$$\mathbb{E}_y[h(X_{T_r})\mathbf{1}\{U_1 > T_r\}] = \mathbb{E}[h(yV_3)\mathbf{1}\{T_r < U_k\} \,|\, i, W, V_1],$$

where $V_3$ is the part of $V_2$ until the first time it exits the ball of radius $r$ in $G$. Recall that $y = \phi_\ell(W)V_1$ and that $V_1$ is a random walk on $L \wr G$ conditioned to be longer than all excursions in $W$ and stopped when it knows it is. Hence $V_1 V_3$ is simply a random walk conditioned to be longer than all excursions in $W$. Returning the integration over $W$ and $V_1$, we get

$$(11) \qquad II = -\mathbb{E}_x[h(\phi_\ell(W)V_1 V_3)\mathbf{1}_{\mathscr{D}}]$$

where $\mathscr{D}$ is the event that all excursions except for the longest did not exit the ball of radius $r$ (this part was $\mathscr{C}$), while the longest did exit it (this is $T_r < U_k$). Now write

$$|\phi_\ell(W)V_1 V_3| \overset{(7)}{\leq} |\ell| + |W| + |V_1 V_3|.$$

The expression $|W| + |V_1 V_3|$ allows us to get rid of the conditioning in $V_1 V_3$. We can now claim that $|W| + |V_1 V_3|$ is bounded by the sum of lengths of $k$ excursions exactly one of which exits the ball of radius $r$ in $G$. Denoting by

$\mathscr{D}_i$ the event that the $i^{\text{th}}$ excursion is the one that exits the ball of radius $r$, we may write, under $\mathscr{D}_i$,

$$|W| + |V_1 V_3| \leq |x| + T_r + |U_k| - |U_{i+1}|$$

and then

$$\mathbb{E}[\log(|\phi_\ell(W) V_1 V_3|) \cdot \mathbf{1}\{\mathscr{D}_i\}]$$
$$\leq \mathbb{E}[\log(|x| + |\ell| + T_r) \cdot \mathbf{1}\{\mathscr{D}_i\}] + \mathbb{E}[\log(U_k - U_{i+1}) \cdot \mathbf{1}\{\mathscr{D}_i\}]$$

The first term may be estimated by Lemma 9 to be at most

$$Ci \log(r + |x| + |\ell|) \log(|x|) / \log r.$$

For the second term we note that the effect of $\mathscr{D}_i$ on the random walk after $T_{i+1}$ is just to prohibit exiting the ball of radius $r$ and then

$$\mathbb{E}[\log(U_k - U_{i+1}) \cdot \mathbf{1}\{\mathscr{D}_i\}]$$
$$\leq \mathbb{E}[\log(U_{k-i-1}) \cdot \mathbf{1}\{U_{k-i-1} < T_r\}] \cdot \mathbb{P}[T_r < U_{i+1}] \leq Ci.$$

where the last inequality estimates $\mathbb{E}[\cdot] \leq C \log r$ using Lemma 8 and

$$\mathbb{P}[\cdot] \leq Ci / \log r$$

by Lemma 7.

Combining both parts we may write

$$|II| \overset{(11)}{=} |\mathbb{E}_x[h(\phi_\ell(W) V_1 V_3) \mathbf{1}\{\mathscr{D}\}]|$$
$$\leq M(r) \mathbb{E}_x[\log(|\phi_\ell(W) V_1 V_3|) \mathbf{1}\{\mathscr{D}\}]$$
$$\leq M(r) \sum_{i=1}^{k} \mathbb{E}_x[\log(|\phi_\ell(W) V_1 V_3|) \mathbf{1}\{\mathscr{D}_i\}]$$
$$\overset{(*)}{\leq} M(r) \sum_{i=1}^{k} \left( \frac{Ci \log(r + |x| + |\ell|) \log(|x|)}{\log r} + Ci \right)$$
$$\leq C(x, \ell, k) M(r)$$

where in $(*)$ we applied the previous discussion. In particular $II \to 0$ as $r \to \infty$, as claimed. $\qquad \square$

We now move to the estimate of $I$. For this we denote by $\mathscr{E}_s$, $s > 2$ the event that the height of the second-heighest excursion is in $[s, s^2)$, and for $s = 2$ replace $[2, 4]$ with $[0, 4]$. Denote

$$I_s := \mathbb{E}_x[h(y) \cdot \mathbf{1}\{\mathscr{C}, \mathscr{E}_s\}]$$

Now, the event $\mathscr{E}_s$ has probability $\leq Ck^2 / \log^2 s$ since it requires two of the $k$ excursions to reach height $s$. On the other hand,

$$|y| \leq |x| + |\ell| + \text{the total time of the process.}$$

Define $S$ to be the sum of the lengths of the first two excursions to $s^2$, i.e., $S = S_3 - S_2 + S_1$, where

$$S_1 = T_{s^2},$$
$$S_2 = \min\{t > S_1 : g_t = 1_G\},$$
$$S_3 = \min\{t > S_2 : |g_t| > s^2\}.$$

Then, under $\mathscr{E}_s$, the total time of the process is bounded by $S$, and we may write

$$\mathscr{E}_s \Rightarrow |y| \leq |x| + |\ell| + S.$$

By Lemma 8 $\mathbb{E}(S) \leq s^4$ and $S$ has exponential concentration. Any positive variable with exponential concentration, when conditioned over an event of probability $p$, can "gain" no more than $|\log p|$ by the conditioning. Hence we get

$$(12) \qquad I_s \leq \mathbb{E}_x[C \log(|x| + |\ell| + U) \cdot \mathbf{1}\{\mathscr{C}, \mathscr{E}_s\}]$$
$$\leq \frac{Ck^2}{\log^2 s} \cdot \log(|x| + |\ell| + s^4) \left| \log\left(\frac{Ck^2}{\log^2 s}\right) \right|$$
$$\leq Ck^3 \log(|x| + |\ell|) \frac{\log \log s}{\log s}.$$

This shows that

$$I = \sum_{n=0}^{\infty} I_{2^{2^n}} \overset{(12)}{\leq} \sum_{n=0}^{\infty} Ck^3 \log(|x| + |\ell|) \frac{n}{2^n}$$

and in particular $I$ is bounded by $Ck^3 \log(|x| + |\ell|)$ independently of $r$. Furthermore, it is clear that $I_{2^{2^n}}$ is independent of $r$ as long as $r > 2^{2^{n+1}}$, as the event that the second-highest excursion is not larger than $2^{2^{n+1}}$ implies that neither $y$ nor $\mathscr{C}$ depend on $r$. So we get that $I$ converges as $r \to \infty$, and its limit is bounded by $Ck^3 \log(|x| + |\ell|)$. As $II \to 0$ (by Claim 15) we get that $f(r)$ converges and its limit is bounded by $Ck^3 \log(|x| + |\ell|)$. By Claim 13, the proposition follows. □

We now complete the proof of Theorem 2.

**Proof of Theorem 2.** Recall that we are given a measure $\mu$ on the lamp group $L$ such that there are no nonconstant sublogarithmic $\mu$-harmonic functions on $L$. Denote by $P$ the operator $P : L^1(L) \to L^1(L)$ which applies a single step of the $\mu$-random walk to its input. In a formula,

$$P(\psi)(\ell) = \sum_{m \in L} \psi(m)\mu(m^{-1}l).$$

Let $Q = \frac{1}{2}(I + P)$ be the corresponding operator for lazy random walk on $L$.

Fix $x = (\omega, g) \in L \wr G$ and define a function $\psi : L \to \mathbb{R}$ by $\psi(\ell) = h(\phi_\ell(x))$. We wish to relate $\psi$ and $f_k$ (from Definition 10 with the same $x$ and $h$). Let

therefore $k \geq 1$ and $r$ sufficiently large and examine $h(X_{U_k})$ under the event $U_k < T_r$. At every visit to $1_G$, the lamp there has probability $\frac{1}{2}$ to move. Hence, at $U_k$ the distribution of the lamp is exactly that of lazy random walk on $L$ after $k$ steps. This means that, conditioning on everything that the walk does outside $1_G$,

$$h(X_{U_k}) \sim Q^k(h(\phi_{\omega(1_G)}(X_{U_k})))$$

(the operand of $Q$ above, i.e., $h(\phi_{\omega(1_G)}(X_{U_k}))$, is considered as a function of $\omega(1_G)$, i.e., of the status of the lamp at $1_G$ of the starting point). Taking expectations (still under $U_k < T_r$) we get

$$\mathbb{E}_x[h(X_{U_k})\mathbf{1}\{U_k < T_r\}] = Q^k(\mathbb{E}_x[h(\phi_{\omega(1_G)}(X_{U_k}))\mathbf{1}\{U_k < T_r\}]).$$

Because $h$ is sub-logarithmic, we know that $\mathbb{E}[h(X_{T_r}) \cdot \mathbf{1}\{T_r < U_k\}]$ goes to 0 as $r \to \infty$ (recall the proof of Claim 12). So we get for the left-hand side,

$$\lim_{r \to \infty} \mathbb{E}_x[h(X_{U_k})\mathbf{1}\{U_k < T_r\}] = \lim_{r \to \infty} \mathbb{E}_x[h(X_{\min\{T_r, U_k\}})] = h(x)$$

where in the last equality we used that $h$ is harmonic on $L \wr G$. For the right-hand side, we get

$$\lim_{r \to \infty} \mathbb{E}_x[h(\phi_\ell(X_{U_k}))\mathbf{1}_{\{U_k < T_r\}}] = \lim_{r \to \infty} \mathbb{E}_x[h(\phi_\ell(X_{\min\{U_k, T_r\}}))] = f_k(\ell).$$

This gives the sought-after relation,

$$\psi = Q^k f_k.$$

Using Proposition 5 and the facts that $\psi$ grows sub-logarithmically and $\mathrm{har}(L) \succeq \log(\cdot)$, we obtain that $\psi$ is constant. Therefore, $h(\phi_\ell(x)) = h(x)$ for all $\ell$ and all $x$; that is, $h$ does not depend on the status of the lamp at $1_G$.

We may repeat this argument for the lamps at other elements of $G$ by translating $h$. We conclude that $h$ is a function depending only on the position of the lamplighter, and not on the lamp configuration. Thus, $h$ can be viewed as a sub-linear (in fact sub-logarithmic) harmonic function on $G$. Since $G$ is recurrent, it implies that $h$ is constant.      □

**2.3. Harmonic growth with $\mathbb{Z}^2$ base.** Complementing Theorem 2, we show that when the base group is $\mathbb{Z}^2$ then the harmonic growth is logarithmic. For simplicity of the presentation, we show this for the group $G := (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^2$, other cases are similar.

Due to Theorem 2, it suffices to construct a logarithmically growing non-constant harmonic function on $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^2$.

Let $a : \mathbb{Z}^2 \to \mathbb{R}$ be the *potential kernel*, that is, $a$ is harmonic on $\mathbb{Z}^2 \setminus \{0\}$ and $\frac{1}{4} \sum_{w \sim 0} a(w) = a(0) + 1$ (see, e.g., [14, §4.4] for its construction). The standard normalisation is $a(0) = 0$, but we will normalise it instead by $a(0) = \frac{1}{2}$. Further, $a(x) = c \log|x| + O(1)$ as $|x| \to \infty$ for some constant $c > 0$ [14, Theorem 4.4.3].

We may now define our harmonic function. We define $h(\sigma, z) = (-1)^{\sigma(0)} \cdot a(z)$ for any $(\sigma, z) \in G$. The logarithmic growth is clear from the growth of $a$ so we need only show that $h$ is indeed harmonic.

Recall that the neighbours of $(\sigma, z)$ in $G$ are $(\sigma, z \pm e_j)$ and $(\sigma^z, z)$ where $e_1 = (1, 0), e_2 = (0, 1)$ and $\sigma^z$ is the configuration $\sigma$ with the state of the lamp at $z$ flipped. Assume that the random walk goes to $\sigma^z$ with probability $\frac{1}{2}$ and to each of the neighbours in $\mathbb{Z}^2$ with probability $\frac{1}{8}$.

We have that if $z \neq 0$,

$$\tfrac{1}{2}h(\sigma^z, z) + \tfrac{1}{8}\sum_{w \sim z} h(\sigma, w) = \tfrac{1}{2}(-1)^{\sigma(0)} \cdot \left(a(z) + \tfrac{1}{4}\sum_{w \sim z} a(w)\right)$$
$$= (-1)^{\sigma(0)}a(z) = h(\sigma, z),$$

and if $z = 0$,

$$\tfrac{1}{2}h(\sigma^0, 0) + \tfrac{1}{8}\sum_{w \sim 0} h(\sigma, w) = \tfrac{1}{2}(-1)^{\sigma(0)} \cdot \left(-\tfrac{1}{2} + \tfrac{1}{4}\sum_{w \sim 0} a(w)\right)$$
$$= \tfrac{1}{2}(-1)^{\sigma(0)} = h(\sigma, 0).$$

So $h$ is harmonic on $G$.

## 3. Iterated wreath products

**Proposition 16.** *For any $k \geq 1$, there exists a group $G$ and a finitely-supported random walk $X_n$ on it such that $H(X_n) \geq c(k)n/\log^{(k)} n$ but every sublogarithmic $X_n$-harmonic function is constant.*

**Proof.** Define $G_1 = (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}^2$ and $G_{k+1} = G_k \wr \mathbb{Z}^2$. Let $\mu_0$ be the measure on $\mathbb{Z}/2\mathbb{Z}$ giving equal weight to both elements, and let $\nu$ be the measure on $\mathbb{Z}^2$ giving weight $\frac{1}{4}$ to the 4 neighbours of the identity, i.e., to $(\pm 1, 0)$ and $(0, \pm 1)$. Let $\mu_k$ be the measure on $G_k$ given by $\mu_k = \mu_{k-1} \wr \nu$ (recall the definition of the move-or-switch generators in §1.3). On the one hand, Theorem 2 tells us that any sublogarithmic $\mu_k$-harmonic function on $G_k$ is constant.

On the other hand, let $L$ be a group and $G = L \wr \mathbb{Z}^2$. Let $H_G(n)$ be the entropy of the $n^{\text{th}}$ step of a random walk on $G$. Then we claim that for some constant $c > 0$ (which depends only on the degree of the Cayley graph chosen for $L$),

$$(13) \qquad\qquad H_G(n) \geq c\frac{n}{\log n} H_L(\log n).$$

To see (13), let $(X_n = (\sigma_n, z_n))_n$ be a random walk on $G$. For $z \in \mathbb{Z}^2$, let $K_n(z)$ be the number of times the lamplighter was at $z$ up to time $n$. For each $z \in \mathbb{Z}^2$, $\sigma_n(z)$ is a lazy random walk on $L$ that has taken $K_n(z)$ steps. Thus, if $Y$ denotes the lazy random walk on $L$,

$$H_G(n) = H((\sigma_n, z_n)) \geq H(\sigma_n) \geq H(\sigma_n|K_n) = \mathbb{E}\left[\sum_{z \in \mathbb{Z}^2} H(Y_{K_n(z)})\right]$$

$$\geq \mathbb{E}\left[|\{z \in \mathbb{Z}^2 : K_n(z) \geq \log n\}|\right] \cdot H_G(\log n).$$

The walk $(z_n)$ is a lazy random walk on $\mathbb{Z}^2$. Known estimates [14] give that $\mathbb{E}\left[|\{z \in \mathbb{Z}^2 : K_n(z) \geq \log n\}|\right] > cn/\log n$ for some universal constant $c > 0$ small enough. Equation (13) follows readily.

The estimate above enables us to relate $G_k$ to $G_{k-1}$. Iterating until $G_1$, we find that

$$H_{G_k}(n) > c^k \frac{n}{\log^{(k)} n},$$

where as usual $\log^{(k)}$ is iteration of $\log k$ times.                               $\square$

We remark that in fact $H_{G_k}(n) < Cn/\log^{(k)} n$ as well, which can be proved using the same calculation, bounding the error terms.

The contrapositive of the above is that if $f$ is some monotone function such that for all groups $\mathrm{har}(G) \preceq f(n/H(X_n))$ then $f$ must grow faster than $\exp \exp \cdots \exp n$ for any number of iterations of exponentials.

## 4. Open questions

Let us list some of the many natural open problems that arise in the context of unbounded harmonic functions on Cayley graphs (discrete groups).

(1) A major open question is whether the Liouville property is a group property or not, or in other words, if it is independent of the choice of generators. A generalisation of this question is: Does the harmonic growth of the group depend on the finite generating set? That is, given a group $G$ with $G = \langle S \rangle = \langle S' \rangle$ for finite symmetric sets $S, S'$, is it true that the harmonic growth is the same for both Cayley graphs?

(2) This paper only focuses on the smallest growing nonconstant harmonic functions. One may also consider larger growth harmonic functions. It is well-known that on $\mathbb{Z}^d$ the smallest nonconstant harmonic functions are linear, and the second-smallest are quadratic (see, e.g., [15]). Do other groups (of nonpolynomial volume growth) admit such a "forbidden gap" in the growth of nonconstant harmonic functions? See [1, 11, 17] for precise results in the case of polynomial growth. We have a proof that the lamplighter group $(\mathbb{Z}/2) \wr \mathbb{Z}$ does not admit such a gap. This proof will be presented elsewhere.

(3) Gromov's theorem states that a group with polynomial growth is virtually nilpotent [10]. A key ingredient in Kleiner's new proof of Gromov's Theorem [13, 20] is the fact that on a group of polynomial volume growth, the space of Lipschitz harmonic functions is finite dimensional. The following question is therefore natural:

Let $G$ be a finitely generated group, and consider the space of Lipschitz harmonic functions on $G$. Suppose this space is finite dimensional. Does it follow that $G$ is virtually nilpotent?

Let us remark that this cannot be deduced directly from Kleiner's proof. This is due to the fact that Kleiner's proof contains an inductive step where one reduces the question to a subgroup. The property of having polynomial growth can be carried from a group to a subgroup, but for the property of having a finite-dimensional space of harmonic functions, we do not know a-priori if this carries over to a subgroup.

See [16] for a treatment of this question in the solvable case. Also related is Tointon's result characterizing virtually $\mathbb{Z}$ groups as those with the space of all harmonic functions (i.e., without any growth restrictions) being finite dimensional, see [21].

Even if we cannot deduce that $G$ is virtually nilpotent, we might still be able to deduce some properties of random walk on it. A weaker conjecture would be:

Suppose $G$ has a finite-dimensional space of Lipschitz harmonic functions. Does it follow that the random walk on $G$ is diffusive?

(4) Another interesting question is to characterise those groups for which there do not exist sub-linear nonconstant harmonic functions. As noted above, all groups with polynomial volume growth are such, but also $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$. See [4] for interesting new examples of groups with no sub-linear harmonic functions.

## Appendix A. Entropy Bound

**A.1. Entropy.** For background on entropy see, e.g., [5].

Let $\mu, \nu$ be probability measures supported on a finite set $\Omega$. Define

$$H(\mu) = \sum_\omega \mu(\omega) \log \mu(\omega),$$

$$D(\mu||\nu) = \sum_\omega \mu(\omega) \log(\tfrac{\mu(\omega)}{\nu(\omega)}),$$

where $x \log \frac{x}{0}$ is interpreted as $\infty$ (so $D(\mu||\nu)$ is finite only if $\mu$ is absolutely continuous with respect to $\nu$). Let $P$ be a probability measure on $\Omega \times \Omega'$, where $\Omega, \Omega'$ are finite with marginal probability measures $\mu$ and $\nu$ on $\Omega$ and $\Omega'$ respectively. Define

$$I(\mu, \nu) = \sum_{(\omega, \omega') \in \Omega \times \Omega'} P(\omega, \omega') \log \left( \tfrac{P(\omega, \omega')}{\mu(\omega)\nu(\omega')} \right).$$

($I$ depends on $P$ but we omit it in the notation). If $X, Y$ are random variables in some probability space, taking finitely many values, then we define $H(X)$, $D(X||Y)$ and $I(X, Y)$ using the corresponding induced measures on the value space.

For two random variables $X$ and $Y$, the *conditional entropy* of $X$ conditioned on $Y$ is defined as $H(X|Y) = H(X,Y) - H(Y)$. If $p(x,y)$ is the probability that $(X,Y) = (x,y)$ and $p(x|y) = p(x,y)/p(y)$, then

$$H(X|Y) = \mathbb{E}[-\log p(X|Y)] = - \sum_{y \,:\, p(y)>0} p(y) \sum_{x \,:\, p(x|y)>0} p(x|y) \log p(x|y).$$

It may also be easily checked that

$$I(X,Y) = H(X) - H(X|Y) = H(X) + H(Y) - H(X,Y).$$

**Lemma 17.** *Let $X$ and $Y$ be two random variables on some probability space, taking finitely many values. Let $f$ be some real valued function defined on the range of $X$ and $Y$. Then,*

$$(\mathbb{E}[f(X)] - \mathbb{E}[f(Y)])^2 \le 2D(X||Y) \cdot (\mathbb{E}[f(X)^2] + \mathbb{E}[f(Y)^2]).$$

**Proof.** Define the following distance between variables

$$d_{\mathrm{BTV}}(X,Y) = \sum_z \frac{(\mathbb{P}[X=z] - \mathbb{P}[Y=z])^2}{\mathbb{P}[X=z] + \mathbb{P}[Y=z]}.$$

If there exists $z$ such that $\mathbb{P}[X=z] > \mathbb{P}[Y=z] = 0$, then $D(X||Y) = \infty$ and there is nothing to prove. Let us now assume that for all $z$, $\mathbb{P}[Y=z] = 0$ implies $\mathbb{P}[X=z] = 0$. Hence we can always write

$$p(z) := \mathbb{P}[X=z]/\mathbb{P}[Y=z]$$

and

$$d_{\mathrm{BTV}}(X,Y) = \sum_z \mathbb{P}[Y=z] \cdot \frac{(1-p(z))^2}{1+p(z)}.$$

Consider the function $f(\xi) = \xi \log \xi$ (with $f(0) = 0$). We have that

$$f'(\xi) = \log \xi + 1, \quad f''(\xi) = 1/\xi.$$

Thus, expanding around 1 we have that for all $\xi > 0$,

$$\xi \log \xi - \xi + 1 \ge \frac{(\xi - 1)^2}{2(1+\xi)}.$$

This implies

$$d_{\mathrm{BTV}}(X,Y) \le 2\sum_z \mathbb{P}[Y=z](1-p(z)) + 2D(X||Y) = 2D(X||Y).$$

Using Cauchy–Schwarz, one obtains

$$|\mathbb{E}[f(X)] - \mathbb{E}[f(Y)]| = \sum_z |\mathbb{P}[X=z] - \mathbb{P}[Y=z]| \cdot |f(z)|$$

$$\le \sqrt{d_{\mathrm{BTV}}(X,Y)} \cdot \sqrt{\mathbb{E}[f(X)^2] + \mathbb{E}[f(Y)^2]}$$

$$\le \sqrt{2D(X||Y)} \cdot \sqrt{\mathbb{E}[f(X)^2] + \mathbb{E}[f(Y)^2]}. \qquad \square$$

**Corollary 18.** *Let $X, Y$ be random variables on some probability space, taking finitely many values. Let $f$ be some real valued function on the range of $X$. Then,*

$$\mathbb{E}\left[\left|\mathbb{E}[f(X)|Y] - \mathbb{E}[f(X)]\right|\right] \leq 2\sqrt{I(X,Y)}\sqrt{\mathbb{E}[f(X)^2]}.$$

**Proof.** Define $X|y$ to be the random variable whose density is

$$\mathbb{P}[X|y = x] = \mathbb{P}[X = x|Y = y].$$

By Lemma 17 applied to $X$ and $f(X|y)$,

$$\left|\mathbb{E}[f(X|y)] - \mathbb{E}[f(X)]\right|^2 \leq 2D\left(X|y\big\|X\right) \cdot (\mathbb{E}[(f(X|y)^2] + \mathbb{E}[f(X)^2]))$$

$$\leq 2D\left(X|y\big\|X\right) \cdot (\mathbb{E}[f(X|y)^2] + \mathbb{E}[f(X)^2]).$$

Summing (after weighting by $\mathbb{P}[Y = y]$) the previous equation for every $y$, and observing that

$$I(X,Y) = \sum_y \mathbb{P}[Y = y]D(X|y\|X)$$

and $\sum_y \mathbb{P}[Y = y]\,\mathbb{E}[f(X|y)^2] = \mathbb{E}[f(X)^2]$, the Cauchy–Schwarz inequality implies

$$\mathbb{E}\left|\mathbb{E}[f(X)|Y] - \mathbb{E}[f(X)]\right|$$

$$= \sum_y \mathbb{P}[Y = y] \cdot \left|\mathbb{E}[f(X|y)] - \mathbb{E}[f(X)]\right|$$

$$\leq \sum_y \mathbb{P}[Y = y]\sqrt{2D\left(X|y\big\|X\right) \cdot (\mathbb{E}[f(X|y)^2] + \mathbb{E}[f(X)^2])}$$

$$\leq \sqrt{2}\sqrt{I(X,Y)} \cdot \sqrt{2\,\mathbb{E}[f(X)^2]}. \qquad \square$$

The next proposition is our main tool, relating harmonic functions and the incremental entropy of a random walk.

**Proposition 19.** *Let $G$ be a group. Let $(X_n)_{n \geq 0}$ be a random walk on $G$. Let $h : G \to \mathbb{R}$ be a harmonic function. Then,*

$$(\mathbb{E}_z |h(X_1) - h(z)|)^2 \leq 4\,\mathbb{E}_z[|h(X_n) - h(z)|^2] \cdot (H(X_n) - H(X_{n-1})).$$

**Proof.** Since $h$ is harmonic we have that

$$|h(X_1) - h(z)| = |\mathbb{E}_z[h(X_n)|X_1] - \mathbb{E}_z[h(X_n)]|.$$

Using Corollary 18 (with $X$ being $X_n$, $Y$ being $X_1$ and $f(x) = h(x) - h(z)$), we find that

$$\mathbb{E}_z |h(X_1) - h(z)| \leq 2 \cdot \sqrt{I(X_n, X_1)} \cdot \sqrt{\mathbb{E}_z[(h(X_n) - h(z))^2]}.$$

Since $G$ is transitive, we have that $H(X_n|X_1) = H(X_{n-1})$. Thus,

$$I(X_n, X_1) = H(X_n) + H(X_1) - H(X_n, X_1)$$

$$= H(X_n) - H(X_n|X_1) = H(X_n) - H(X_{n-1}),$$

which implies the claim readily. □

This inequality provides a quantitative estimate on the growth of harmonic functions, which quantifies one direction of the Kaimanovich-Vershik criterion.

**Theorem 20.** *Let $G$ be a group. Let $(X_n)_{n \geq 0}$ be a random walk on $G$. Then,*

$$\operatorname{har}(G) \succeq \sqrt{n/H(X_n)}.$$

**Proof.** Note that by the Markov property, for any $k$,

$$H(X_1|X_k) = H(X_1|X_k, X_{k+1}, \ldots).$$

Thus,

$$H(X_k) - H(X_{k-1}) = H(X_k) - H(X_k|X_1) = H(X_1) - H(X_1|X_k)$$

is a decreasing sequence in $k$. Thus,

$$H(X_n) = \sum_{k=1}^{n} H(X_k) - H(X_{k-1}) \geq n \cdot \big(H(X_n) - H(X_{n-1})\big).$$

Let $h : G \to \mathbb{R}$ be a nonconstant harmonic function. Let $x \sim y$ be vertices such that $h(x) \neq h(y)$. By Lemma 19

$$(\mathbb{E}_x |h(X_1) - h(x)|)^2 \leq 4 \mathbb{E}_x[h(X_n)^2]H(X_n)/n \leq 4M_h(x,n)^2 H(X_n)/n.$$

Since $\mathbb{E}_x |h(X_1) - h(x)|$ is a positive constant, we have that

$$M_h(x,n) \succeq \sqrt{n/H(X_n)}. \qquad □$$

**Corollary 21** (Avez, Kaimanovich–Vershik [12])**.** *Let $G$ be a group. Let $(X_n)_{n \geq 0}$ be a random walk on $G$. If $H(X_n)/n$ tends to 0, then $G$ is Liouville (meaning that any bounded harmonic function is constant).*

**Remark 22.** One can also use Proposition 19 to show that for groups with polynomial growth, the harmonic growth is linear. This is done (in a slightly different context) in [3]. Since it is so short, let us repeat the argument here.

The only required fact is that for a group of polynomial growth the random walk is diffusive. So if $h$ is a sub-linear harmonic function then $\mathbb{E}_x[h(X_n)^2] = o(n)$ as $n \to \infty$. Since the group is of polynomial growth, $H(X_n) = O(\log n)$, and so there are infinitely many $n$ for which

$$H(X_n) - H(X_{n-1}) = O(n^{-1}).$$

Along this infinite sequence of $n$, we have by Lemma 19

$$(\mathbb{E}_x |h(X_1) - h(x)|)^2 \leq o(n) \cdot O(n^{-1}) = o(1),$$

so $h(X_1) = h(x)$ a.s. Since this holds for all $x$, it must be that $h$ is constant.

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