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Simplicity of skew generalized power series rings

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Abstract. A skew generalized power series ring $R[[S,\omega]]$ consists of all functions from a strictly ordered monoid S to a ring R whose support contains neither infinite descending chains nor infinite antichains, with pointwise addition, and with multiplication given by convolution twisted by an action ω of the monoid S on the ring R. Special cases of the skew generalized power series ring construction are skew polynomial rings, skew Laurent polynomial rings, skew power series rings, skew Laurent series rings, skew monoid rings, skew group rings, skew Mal'cev–Neumann series rings, the "untwisted" versions of all of these, and generalized power series rings. In this paper we obtain necessary and sufficient conditions on R, S and ω such that the skew generalized power series ring $R[[S,\omega]]$ is a simple ring. As particular cases of our general results we obtain new theorems on skew monoid rings, skew Mal'cev–Neumann series rings and generalized power series rings, as well as known characterizations for the simplicity of skew Laurent polynomial rings, skew Laurent series rings and skew group rings.

Contents

1. Introduction

Given a ring R, a strictly ordered monoid $(S,\leq)$ and a monoid homomorphism $\omega : S \to \mathrm{End}(R)$, one can construct the skew generalized power

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series ring $R[[S,\omega]]$ (see Section 2 for details). Skew generalized power series rings are a common generalization of skew polynomial rings, skew power series rings, skew Laurent polynomial rings, skew Laurent series rings, skew monoid rings, skew group rings, skew Mal'cev–Neumann series rings, and of course the "untwisted" versions of all of these. Hence any result on skew generalized power series rings has its counterpart for each of these particular ring extensions, and these counterparts follow immediately from a single proof. This property makes skew generalized power series rings a useful tool for unifying results on the ring extensions listed above; such an approach was applied, e.g., in [1], [8], [9], [10], [15], [17], [19], [25].

Skew generalized power series rings that are division rings were studied in [13]. A more general class of rings than division rings is formed by *simple rings*, i.e., nonzero rings A such that the only ideals of A are the zero ideal (0) and the whole ring A. Because of the importance of simple rings in general theory of rings, it is natural to ask under what conditions on a ring R, a strictly ordered monoid $(S,\leq)$ and a monoid homomorphism

$$\omega : S \to \mathrm{End}(R),$$

the skew generalized power series ring $R[[S,\omega]]$ is simple. In this paper we study this problem, obtaining complete solutions for some quite general cases (e.g., when the monoid S is commutative or the order $\leq$ is total).

The paper is organized as follows. In Section 2 we recall the construction of a skew generalized power series ring $R[[S,\omega]]$ and show how the aforementioned ring extensions can be obtained as special cases of the construction. In Section 3 we prove that for the ring $R[[S,\omega]]$ to be simple it is necessary that R is (S,ω)-simple, i.e., (0) and R are the only ideals I of R with $\omega_s(I) \subseteq I$ for all $s \in S$; such an ideal I is said to be (S,ω)-invariant. In that section we study relationships between ideals of a skew generalized power series ring $R[[S,\omega]]$ and (S,ω)-invariant ideals of R. Since the center of a simple ring is a field, in Section 4 we focus on skew generalized power series rings whose central elements form a field. In particular, in Theorem 4.4 we show that if S is a torsion-free abelian group and R is (S,ω)-simple, then the center of the ring $R[[S,\omega]]$ is a field if and only if the elements $s \in S$ for which $\omega(s)$ is an inner automorphism of R satisfy some special property related to the order $\leq$ on S. In Section 5 we characterize the simplicity of a skew generalized power series ring $R[[S,\omega]]$, concentrating mainly on the cases where the order $\leq$ is total (Theorem 5.2) or the monoid S is commutative (Theorems 5.12, 5.15 and 5.17). As particular cases of our general results we obtain new theorems on skew monoid rings, skew Mal'cev–Neumann series rings and generalized power series rings. Moreover, special cases of our results are well-known characterizations of the simplicity of skew Laurent polynomial rings given by Jordan ([4]), skew Laurent series rings given by Tuganbaev ([24]), and skew group rings given by Crow ([2]) and Öinert ([18]) — in fact, these characterizations were the main motivations for this study.

Throughout this paper all rings are nonzero and with identity element. Monoid operation is written multiplicatively, except when we are dealing with special examples where an additive notation is more suitable. We assume that identity elements of rings and monoids are inherited by subrings and submonoids, and preserved under homomorphisms, but neither rings nor monoids are assumed to be commutative. We will denote by $\mathrm{End}(R)$ the monoid of endomorphisms of a ring R, and by $\mathrm{Aut}(R)$ the group of automorphisms of R.

If S is a monoid or a ring, then the group of invertible elements of S is denoted by $U(S)$. When we consider an order $\leq$ on a set S, then the word "an order" means "a partial order" unless otherwise stated. The order $\leq$ is *total* (respectively *trivial*) if any two different elements of S are comparable (respectively incomparable) with respect to $\leq$. The set of integers is denoted by $\mathbb{Z}$ and the set of positive integers by $\mathbb{N}$.

2. Preliminaries

In this section we recall the skew generalized power series ring construction (which was introduced in [13]) as well as some of its properties, which will be used in further sections. For this we need some definitions.

An ordered set $(S,\leq)$ is called *artinian* if every strictly decreasing sequence of elements of S is finite, and $(S,\leq)$ is called *narrow* if every subset of pairwise order-incomparable elements of S is finite. Thus $(S,\leq)$ is artinian and narrow if and only if every nonempty subset of S has at least one but only a finite number of minimal elements. An *ordered monoid* is a pair $(S,\leq)$ consisting of a monoid S and an order $\leq$ on S such that $a\leq b$ implies $ac\leq bc$ and $ca\leq cb$ for all $a,b,c\in S$. An ordered monoid $(S,\leq)$ is said to be *strictly ordered* if $a<b$ implies $ac<bc$ and $ca<cb$ for all $a,b,c\in S$.

For a ring R and a strictly ordered monoid $(S,\leq)$, in the 1990s Ribenboim defined the ring of generalized power series $R[[S]]$ consisting of all maps from S to R whose support is artinian and narrow, with the pointwise addition and the convolution multiplication (see [22]). This construction provided interesting examples of rings and it was extensively studied (e.g., in [3], [11], [16], [20], [21]).

In [13], the first author and Ziembowski introduced a "twisted" version of the Ribenboim construction. Now we recall the construction of the skew generalized power series ring introduced in [13]. Let R be a ring, $(S,\leq)$ a strictly ordered monoid, and $\omega : S\to \mathrm{End}(R)$ a monoid homomorphism. For $s\in S$, let ω_s denote the image of s under ω, that is $\omega_s=\omega(s)$. Let A be the set of all functions $f : S\to R$ such that the support

$$\mathrm{supp}(f)=\{s\in S : f(s)\neq 0\}$$

is artinian and narrow. Then for any $s\in S$ and $f,g\in A$ the set

$$X_s(f,g)=\{(x,y)\in \mathrm{supp}(f)\times \mathrm{supp}(g) : s=xy\}$$

is finite. Thus one can define the product $fg : S \to R$ of $f, g \in A$ as follows:

$$(fg)(s) = \sum_{(u,v)\in X_s(f,g)} f(u)\omega_u(g(v)) \quad \text{for any } s \in S$$

(by convention, a sum over the empty set is 0). With pointwise addition and multiplication as defined above, A becomes a ring, called the *ring of skew generalized power series* with coefficients in R and exponents in S (one can think of a map $f : S \to R$ as a formal series $\sum_{s\in S} r_s s$, where $r_s = f(s) \in R$) and denoted either by $R[[S, \omega, \leq]]$, or by $R[[S^{\leq}, \omega]]$, or by $R[[S, \omega]]$ if there is no ambiguity concerning the order $\leq$. We will use the same symbol 1 to denote the identity elements of the monoid S, the ring R, and the ring $R[[S, \omega]]$.

To each $r \in R$ and $s \in S$ we associate elements $\mathsf{c}_r, \mathsf{e}_s \in R[[S, \omega]]$ defined by

$$\mathsf{c}_r(x) = \begin{cases} r & \text{if } x = 1, \\ 0 & \text{if } x \in S \setminus \{1\} \end{cases} \quad \text{and} \quad \mathsf{e}_s(x) = \begin{cases} 1 & \text{if } x = s, \\ 0 & \text{if } x \in S \setminus \{s\} \end{cases}$$

(i.e., c_r and e_s are the power series $\mathsf{c}_r = r1$ and $\mathsf{e}_s = 1s$). It is clear that $r \mapsto \mathsf{c}_r$ is a ring embedding of R into $R[[S, \omega]]$ and $s \mapsto \mathsf{e}_s$ is a monoid embedding of S into the multiplicative monoid of the ring $R[[S, \omega]]$, and $\mathsf{e}_s\mathsf{c}_r = \mathsf{c}_{\omega_s(r)}\mathsf{e}_s$.

As promised, below we show how the classical constructions mentioned in Section 1 can be viewed as special cases of the skew generalized power series ring construction (the next three paragraphs are taken from [10, Section 1], with some small changes).

Let R be a ring and α an endomorphism of R. Then for the additive monoid $S = \mathbb{N} \cup \{0\}$ of nonnegative integers, the map $\omega : S \to \mathrm{End}(R)$ given by

$$\omega(n) = \alpha^n \text{ for any } n \in S, \tag{2.1}$$

is a monoid homomorphism. If furthermore α is an automorphism of R, then (2.1) defines also a monoid homomorphism $\omega : S \to \mathrm{Aut}(R)$ for $S = \mathbb{Z}$, the additive monoid of integers. We can consider two different orders on each of the monoids $\mathbb{N} \cup \{0\}$ and $\mathbb{Z}$: the trivial order and the natural linear order. In both cases these monoids are strictly ordered, and thus in each of the cases we can construct the skew generalized power series ring $R[[S, \omega]]$. As a result, we obtain the following extensions of the ring R:

(1) If $S = \mathbb{N} \cup \{0\}$ and $\leq$ is the trivial order, then the ring $R[[S, \omega]]$ is isomorphic to the skew polynomial ring $R[x, \alpha]$.
(2) If $S = \mathbb{N} \cup \{0\}$ and $\leq$ is the natural linear order, then $R[[S, \omega]]$ is isomorphic to the skew power series ring $R[[x, \alpha]]$.
(3) If $S = \mathbb{Z}$ and $\leq$ is the trivial order, and α is an automorphism of R, then $R[[S, \omega]]$ is isomorphic to the skew Laurent polynomial ring $R[x, x^{-1}; \alpha]$.

(4) If $S = \mathbb{Z}$ and $\leq$ is the natural linear order, and α is an automorphism of R, then $R[[S,\omega]]$ is isomorphic to the skew Laurent series ring $R[[x,x^{-1};\alpha]]$.

By applying the above points (1)–(4) to the case where α is the identity map of R, we can see that also the following ring extensions are special cases of the skew generalized power series ring construction: the ring of polynomials $R[x]$, the ring of power series $R[[x]]$, the ring of Laurent polynomials $R[x,x^{-1}]$, and the ring of Laurent series $R[[x,x^{-1}]]$.

Furthermore, any monoid S is a strictly ordered monoid with respect to the trivial order on S. Hence if R is a ring, and S is a monoid, and $\omega : S \to \mathrm{End}(R)$ is a monoid homomorphism, then we can impose the trivial order on S and construct the skew generalized power series ring $R[[S,\omega]]$, which in this case will be denoted by $R[S,\omega]$. It is clear that the ring $R[S,\omega]$ is isomorphic to the classical skew monoid ring built from R and S using the action ω of S on R. If ω is trivial (i.e., ω sends every element of S to the identity endomorphism of R), we write $R[S]$ instead of $R[S,\omega]$. Obviously the ring $R[S]$ is isomorphic to the ordinary monoid ring of S over R.

If S is a commutative monoid, then the skew generalized power series ring $R[[S,\omega]]$ is the same as the twisted generalized power series ring related to ω, introduced by Liu in [7].

If $(T,\leq)$ is a totally ordered set, then a nonempty subset $X \subseteq T$ is artinian and narrow if and only if X is well-ordered. Hence, if $(S,\leq)$ is a totally ordered group, then the generalized power series ring $R[[S]]$ is the same as the Mal'cev–Neumann series ring $R((S))$, and if $\omega : S \to \mathrm{Aut}(R)$ is a group homomorphism, then the skew generalized power series ring $R[[S,\omega]]$ coincides with the skew Mal'cev–Neumann series ring $R((S,\omega))$ (see [5, §14]).

We now recall some facts about units of skew generalized power series rings, which will be used later on in this paper.

Recall from [22] that an order $\leq$ on a monoid S is said to be *subtotal* if for any $s,t \in S$ there exists $n \in \mathbb{N}$ such that $s^n \leq t^n$ or $t^n \leq s^n$. Subtotal orders appear naturally in the context of fields of generalized power series (see Corollary 5.10). A total order on a monoid is clearly subtotal, but the converse need not be true (see, e.g., [14, Example 3.8] or [22, p. 371]). If $(S,\leq)$ is an ordered abelian group, then the order $\leq$ is subtotal if and only if for every $s \in S$ there exists $n \in \mathbb{N}$ such that $s^n \geq 1$ or $s^n \leq 1$.

Recall that a monoid S is said to be *torsion-free* if for any $n \in \mathbb{N}$ and $s,t \in S$, $s^n = t^n$ implies $s = t$. It is easy to see that if $(S,\leq)$ is an ordered torsion-free commutative monoid such that $\leq$ is subtotal, then the binary relation $\preceq$ on S defined by

$$s \preceq t \quad \text{if and only if} \quad s^n \leq t^n \quad \text{for some} \quad n \in \mathbb{N}$$

is a total order on S and $(S,\preceq)$ is a strictly ordered monoid. The order $\preceq$ will be called the *total order associated with* $\leq$. Clearly, $s \leq t$ implies $s \preceq t$ for any $s,t \in S$, and thus by [13, Proposition 1.1], if a subset T of S is artinian and narrow with respect to $\leq$, then T is well-ordered with respect to $\preceq$. Hence for any $f \in R[[S,\omega]] \setminus \{0\}$ there exists a smallest element s_0 of $\mathrm{supp}(f)$ with respect to $\preceq$.

To characterize skew generalized power series rings that are simple, we will need the following results on units of such rings.

Proposition 2.1 ([13, Lemma 2.5])**.** *Let R be a ring, $(S,\leq)$ an ordered abelian torsion-free group such that $\leq$ is subtotal, $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism, and $\preceq$ the total order associated with $\leq$. Let $A = R[[S,\omega]]$. If $f \in A \setminus \{0\}$ and for the smallest element s_0 of* $\mathrm{supp}(f)$ *with respect to $\preceq$ we have $f(s_0) \in U(R)$, then $f \in U(A)$.*

Proposition 2.2 ([13, Proposition 2.2])**.** *Let R be a ring, $(S,\leq)$ a strictly ordered monoid, $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism and*

$$A = R[[S,\omega]].$$

Let $f \in A$ and assume that there exists a smallest element s_0 in $\mathrm{supp}(f)$*. If $s_0 \in U(S)$ and $f(s_0) \in U(R)$, then $f \in U(A)$.*

3. (S,ω)-invariant ideals and (S,ω)-simple rings

The purpose of this paper is to find sufficient and necessary conditions on a ring R, a strictly ordered monoid $(S,\leq)$ and a monoid homomorphism $\omega : S \to \mathrm{End}(R)$ under which the skew generalized power series ring $R[[S,\omega]]$ is simple, i.e., has no proper nonzero ideals. For this it is natural to start with taking a closer look at connections between ideals of $R[[S,\omega]]$ and those of R (for clarity, we will usually use the symbol I to denote an ideal of R, and J for an ideal of $R[[S,\omega]]$).

As we show in Proposition 3.2(a) below, for any right ideal I of R the set

$$I[[S,\omega]] = \{f \in R[[S,\omega]] : \ f(s) \in I \textit{ for any } s \in S\},$$

i.e., the set of power series $f \in R[[S,\omega]]$ with all coefficients in I, is a right ideal of $R[[S,\omega]]$. However, for $I[[S,\omega]]$ to be an ideal of $R[[S,\omega]]$ it may not be enough that I is an ideal of R; as we show in Proposition 3.2(b) the following property of I is crucial.

Definition 3.1. Let R be a ring, S a monoid, and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. An ideal I of R is said to be (S,ω)-*invariant* if $\omega_s(I) \subseteq I$ for any $s \in S$.

Keeping the notation of Definition 3.1, let us notice that if S is a group, then each ω_s is an automorphism of R and thus in this case I is (S,ω)-invariant if and only if $\omega_s(I) = I$ for any $s \in S$.

Proposition 3.2. *Let R be a ring, $(S,\leq)$ a strictly ordered monoid, and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. Then:*

(a) *If I is a right ideal of R, then $I[[S,\omega]]$ is a right ideal of $R[[S,\omega]]$.*

(b) *If I is an ideal of R, then $I[[S,\omega]]$ is an ideal of $R[[S,\omega]]$ if and only if I is (S,ω)-invariant.*

Proof. (a) Assume that I is a right ideal of R. Then $0 \in I[[S,\omega]]$, so $I[[S,\omega]] \neq \emptyset$. Let $f,g \in I[[S,\omega]]$ and $h \in R[[S,\omega]]$. Then $f(x), g(x) \in I$ for any $x \in S$ and thus for any $s \in S$ we have

$$(f+g)(s) = f(s) + g(s) \in I$$

and

$$(fh)(s) = \sum_{(x,y)\in X_s(f,h)} f(x)\omega_x(h(y)) \in I.$$

Hence $f+h, fh \in I[[S,\omega]]$ and (a) follows.

(b) Let I be an ideal of R. Assume that $I[[S,\omega]]$ is an ideal of $R[[S,\omega]]$ and let $s \in S$ and $a \in I$. Then $\mathsf{c}_a \in I[[S,\omega]]$, and since $I[[S,\omega]]$ is an ideal, also $\mathsf{e}_s\mathsf{c}_a \in I[[S,\omega]]$. Hence $\omega_s(a) = (\mathsf{e}_s\mathsf{c}_a)(s) \in I$, which shows that $\omega_s(I) \subseteq I$. Thus I is (S,ω)-invariant.

To complete the proof of (b), assume that I is (S,ω)-invariant and let $f \in I[[S,\omega]]$. Then for any $x,y \in S$ we have $\omega_x(f(y)) \in \omega_x(I) \subseteq I$. Hence for any $h \in R[[S,\omega]]$ and $s \in S$ we obtain

$$(hf)(s) = \sum_{(x,y)\in X_s(h,f)} h(x)\omega_x(f(y)) \in I.$$

Thus $hf \in I[[S,\omega]]$, which together with (a) shows that $I[[S,\omega]]$ is an ideal of $R[[S,\omega]]$. $\square$

Let R be a ring, $(S,\leq)$ a strictly ordered monoid, and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. As we already know from Proposition 3.2(b), each (S,ω)-invariant ideal of R leads to an ideal of the skew generalized power series ring $R[[S,\omega]]$. Now we focus on the opposite direction. Namely, we will show how (S,ω)-invariant ideals of R can be produced from ideals of $R[[S,\omega]]$. In particular, we will make use of the order $\leq$ on S to determine some (S,ω)-invariant ideals of R. For this we introduce the following notation: if $f \in R[[S,\omega]] \setminus \{0\}$, then we write $\mu(f)$ to denote the set of minimal elements of $\mathrm{supp}(f)$. In the proof of Proposition 3.4 we will need the following properties of the set $\mu(f)$.

Lemma 3.3. *Let R be a ring, $(S,\leq)$ a strictly ordered monoid, and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. Then the following inclusions hold for any $f,g \in R[[S,\omega]]$ and $r \in R$:*

(a) $\mu(f) \cap \mu(g) \cap \mathrm{supp}(f+g) \subseteq \mu(f+g)$.
(b) $\mu(f) \cap supp(\mathsf{c}_r f) \subseteq \mu(\mathsf{c}_r f)$.
(c) $\mu(f) \cap supp(f\mathsf{c}_r) \subseteq \mu(f\mathsf{c}_r)$.

Proof. To prove (a), consider an element $t \in \mu(f) \cap \mu(g) \cap \mathrm{supp}(f+g)$ and suppose that $t \notin \mu(f+g)$. Then $t \in \mathrm{supp}(f+g) \setminus \mu(f+g)$ and thus there exists $v \in \mathrm{supp}(f+g)$ such that $v < t$. Since $v < t$ and $t \in \mu(f) \cap \mu(g)$, it must be $f(v) = 0$ and $g(v) = 0$. Hence $(f + g)(v) = f(v) + g(v) = 0$, so $v \notin \mathrm{supp}(f + g)$. This contradiction completes the proof of (a). Parts (b) and (c) can be proved in a similar fashion. $\square$

We are now ready to prove the following result which provides important examples of (S, ω)-invariant ideals. If S is a monoid or a ring, then $Z(S)$ denotes the *center* of S, i.e.,

$$Z(S) = \{s \in S :\ st = ts \ for\ any\ t \in S\}.$$

An element $s \in S$ is said to be *central* if $s \in Z(S)$.

Proposition 3.4. *Let R be a ring, $(S, \leq)$ an ordered group, and*

$$\omega : S \to \mathrm{End}(R)$$

a monoid homomorphism. Let J be a nonempty subset of $R[[S, \omega]]$ such that for any $f, g \in J, r \in R$ and $s \in S$ we have

$$f + g,\ \ \mathsf{c}_r f,\ \ f\mathsf{c}_r,\ \ \mathsf{e}_s f \mathsf{e}_{s^{-1}} \in J \tag{3.1}$$

(for instance, J can be an ideal of $R[[S, \omega]]$). Then:

(a) *For any $t \in Z(S)$ the sets*

$$I_1 = \{f(t) :\ f \in J\} \quad and \quad I_2 = \{f(t) :\ f \in J,\ t \in \mu(f)\} \cup \{0\}$$

are (S, ω)-invariant ideals of R.

(b) *If the group S is abelian, then for any $t \in S$ and $h \in R[[S, \omega]]$ the set*

$$I_3 = \{f(t) :\ f \in J,\ \mathrm{supp}(f) \subseteq \mathrm{supp}(h)\}$$

is an (S, ω)-invariant ideal of R.

Proof. (a) Let $t \in Z(S)$. To prove that I_1 is an (S, ω)-invariant ideal of R, first observe that for any $f, g \in J$ and $r \in R$ we have

$$f(t)+g(t) = (f+g)(t), \quad r\cdot f(t) = (\mathsf{c}_r f)(t), \quad f(t)\cdot r = (f\mathsf{c}_{\omega_{t^{-1}}(r)})(t). \tag{3.2}$$

Combining (3.1) with (3.2), we can see that I_1 is an ideal of R. To show that the ideal I_1 is (S, ω)-invariant, consider any $s \in S$. Notice that for any $f \in R[[S, \omega]]$ and $v \in S$ the following equality holds:

$$(\mathsf{e}_s f \mathsf{e}_{s^{-1}})(v) = \omega_s(f(s^{-1}vs)). \tag{3.3}$$

Since $t \in Z(S)$, it follows from (3.3) that for any $f \in J$ we have

$$\omega_s(f(t)) = (\mathsf{e}_s f \mathsf{e}_{s^{-1}})(t), \tag{3.4}$$

and furthermore $\mathsf{e}_s f \mathsf{e}_{s^{-1}} \in J$ by (3.1), which implies that $\omega_s(I_1) \subseteq I_1$, as desired.

To prove that I_2 is an (S, ω)-invariant ideal of R, we start by showing that if $a, b \in I_2$, then also $a + b \in I_2$. The case where $a = 0$, or $b = 0$, or $a + b = 0$ is clear. Hence we assume that $a, b, a + b$ are all nonzero. Then $a = f(t)$ and $b = g(t)$ for some $f, g \in J$ such that $t \in \mu(f)$ and $t \in \mu(g)$. Since furthermore $(f + g)(t) = f(t) + g(t) = a + b \neq 0$, it follows that

$t \in \mu(f) \cap \mu(g) \cap \operatorname{supp}(f+g)$ and thus $t \in \mu(f+g)$ by Lemma 3.3. Hence (3.1) implies that $a+b = (f+g)(t) \in I_2$, which shows that $I_2 + I_2 \subseteq I_2$. Using (3.1), (3.2) and Lemma 3.3 one can easily verify that also $rI_2 \subseteq I_2$ and $I_2 r \subseteq I_2$ for any $r \in R$, and thus I_2 is an ideal of R. To complete the proof of (a), we have to show that if $s \in S$ and $a \in I_2$, then $\omega_s(a) \in I_2$. The case where $\omega_s(a) = 0$ is clear. Thus we assume that $\omega_s(a) \neq 0$. Then $a = f(t)$ for some $f \in J$ such that $t \in \mu(f)$. We have $\omega_s(a) = (\mathsf{e}_s f \mathsf{e}_{s^{-1}})(t)$ by (3.4), and $\mathsf{e}_s f \mathsf{e}_{s^{-1}} \in J$ by (3.1). Hence to show that $\omega_s(a) \in I_2$ it suffices to show that $t \in \mu(\mathsf{e}_s f \mathsf{e}_{s^{-1}})$. Suppose that $t \notin \mu(\mathsf{e}_s f \mathsf{e}_{s^{-1}})$. Since $(\mathsf{e}_s f \mathsf{e}_{s^{-1}})(t) = \omega_s(a) \neq 0$, $t \in \operatorname{supp}(\mathsf{e}_s f \mathsf{e}_{s^{-1}}) \backslash \mu(\mathsf{e}_s f \mathsf{e}_{s^{-1}})$ and thus there exists $v \in \operatorname{supp}(\mathsf{e}_s f \mathsf{e}_{s^{-1}})$ such that $v < t$. Since $t \in Z(S)$, we get $s^{-1}vs < s^{-1}ts = t$ and thus $f(s^{-1}vs) = 0$. Hence, applying (3.3), we obtain

$$0 \neq (\mathsf{e}_s f \mathsf{e}_{s^{-1}})(v) = \omega_s(f(s^{-1}vs)) = \omega_s(0) = 0.$$

This contradiction completes the proof of (a).

(b) To prove that I_3 is an (S,ω)-invariant ideal of R, it suffices to use (3.1), (3.2), (3.4) and the following inclusions, which hold for any $f, g \in R[[S,\omega]]$, $r \in R$ and $s \in S$ (we recall that, by assumption, S is abelian):

$$\operatorname{supp}(f+g) \subseteq \operatorname{supp}(f) \cup \operatorname{supp}(g),$$
$$\operatorname{supp}(\mathsf{c}_r f) \cup \operatorname{supp}(f \mathsf{c}_r) \cup \operatorname{supp}(\mathsf{e}_s f \mathsf{e}_{s^{-1}}) \subseteq \operatorname{supp}(f). \qquad \square$$

We know from Proposition 3.2(b) that for a ring R, a strictly ordered monoid $(S, \leq)$ and a monoid homomorphism $\omega : S \to \operatorname{End}(R)$, if I is an (S,ω)-invariant ideal of R, then $I[[S,\omega]]$ is an ideal of $R[[S,\omega]]$. Hence, if the ring $R[[S,\omega]]$ is simple, then it must be that

$$I[[S,\omega]] = (0) \quad \text{or} \quad I[[S,\omega]] = R[[S,\omega]],$$

and thus $I = (0)$ or $I = R$. Therefore, the property of R used in the following definition is a necessary condition for $R[[S,\omega]]$ to be a simple ring.

Definition 3.5. Let R be a ring, S a monoid, and $\omega : S \to \operatorname{End}(R)$ a monoid homomorphism. The ring R is said to be (S,ω)-*simple* if (0) and R are the only (S,ω)-invariant ideals of R.

In the case when S is a group, the notion of an (S,ω)-simple ring appeared in [2] and [18], where the simplicity of skew group rings was studied. The notion of an (S,ω)-simple ring is also a generalization of the well-known notion of an α-simple ring, which appears in the literature in the context of the simplicity of skew Laurent series rings (see, e.g., [5, p. 46]). Recall that if α is an endomorphism of a ring R, then the ring R is said to be α-*simple* if R contains no nonzero proper ideal I with $\alpha(I) \subseteq I$. Obviously, R is α-simple if and only if R is α^n-simple for every nonnegative integer n. Hence R is α-simple if and only if R is (S,ω)-simple, where S is the additive monoid of nonnegative integers and $\omega : S \to \operatorname{End}(R)$ is defined by $\omega(n) = \alpha^n$.

Keeping the notation of Definition 3.5, let us observe that if R is a simple ring, then R is (S,ω)-simple. However, the opposite implication need not be true. To see this, it suffices to take any simple ring Q and consider the direct sum $R = Q \oplus Q$ of two copies of Q, the additive monoid S of nonnegative integers, and the monoid homomorphism $\omega : S \to \mathrm{End}(R)$ defined by $\omega_1(x,y) = (y,x)$ for any $(x,y) \in R$ (cf. [23, Example 4.10]). Indeed, the only nonzero proper ideals of R are $Q \oplus (0)$ and $(0) \oplus Q$, and none of them is (S,ω)-invariant.

Let us notice that in the definition of an (S,ω)-simple ring no order on S is required. However, when a skew generalized power series ring $R[[S,\omega]]$ is considered, then S is necessarily endowed with a strict order $\leq$. The following proposition characterizes (S,ω)-simple rings in terms of properties of the ring $R[[S,\omega]]$ in the case where $(S,\leq)$ is an ordered group. We will use this result in Section 5 to characterize the simplicity of $R[[S,\omega]]$.

Proposition 3.6. *Let R be a ring, $(S,\leq)$ an ordered group, and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. Then the following statements are equivalent:*

(1) *R is an (S,ω)-simple ring.*

(2) *If J is a nonempty subset of $R[[S,\omega]]$ such that $J \neq \{0\}$ and for any $g,h \in J, r \in R$ and $s,t \in S$ we have*

$$g+h, \quad \mathsf{c}_r g, \quad g\mathsf{c}_r, \quad \mathsf{e}_s g \mathsf{e}_t \in J$$

(for instance, if J is a nonzero ideal of $R[[S,\omega]]$), then there exists $f \in J$ such that $f(1) = 1$ and 1 is a minimal element of $\mathrm{supp}(f)$.

(3) *For any nonzero ideal J of $R[[S,\omega]]$ there exists $f \in J$ such that $f(1) = 1$.*

Proof. (1) $\Rightarrow$ (2) Assume (1) and let J be a subset of $A = R[[S,\omega]]$ such that J satisfies all the conditions given in (2). Then there exists $g \in J \setminus \{0\}$. Choose any $t \in \mu(g)$ and denote $h = g\mathsf{e}_{t^{-1}}$. Then $h = \mathsf{e}_1 g \mathsf{e}_{t^{-1}} \in J$ and $h(1) = (g\mathsf{e}_{t^{-1}})(1) = g(t) \neq 0$. Hence $1 \in \mathrm{supp}(h)$. We show that $1 \in \mu(h)$. Otherwise there exists $v \in S$ such that $v < 1$ and $h(v) \neq 0$. Since $v < 1$, we obtain $vt < t$ and since $t \in \mu(g)$, it must be $g(vt) = 0$. But then

$$0 \neq h(v) = (g\mathsf{e}_{t^{-1}})(v) = g(vt) = 0,$$

and this contradiction shows that $1 \in \mu(h)$. Hence $h(1)$ is a nonzero element of the set

$$I = \{f(1) : f \in J,\ 1 \in \mu(f)\} \cup \{0\}.$$

Thus, by Proposition 3.4(a), I is a nonzero (S,ω)-invariant ideal of R, so (1) implies that $I = R$. Hence $1 \in I$ and thus (2) holds.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) Assume (3) and let I be a nonzero (S,ω)-invariant ideal of R. Then by Proposition 3.2(b) the set $I[[S,\omega]]$ is a nonzero ideal of $R[[S,\omega]]$. Hence by (3) there exists $f \in I[[S,\omega]]$ such that $f(1) = 1$. Since $f \in I[[S,\omega]]$,

we have $f(t) \in I$ for any $t \in S$. Hence $1 = f(1) \in I$, and thus $I = R$, which shows that R is (S, ω)-simple. $\square$

4. Skew generalized power series rings whose center is a field

It is well known that the center of any simple ring is a field (see, e.g., [6, p. 22]). Since the aim of this paper is to characterize simple skew generalized power series rings, in this section we concentrate on skew generalized power series rings whose center is a field. We start with a necessary condition for this phenomenon. Recall that an endomorphism α of a ring R is said to be an inner automorphism of R induced by a unit u of R if $\alpha(x) = uxu^{-1}$ holds for all $x \in R$.

Proposition 4.1. *Let R be a ring, $(S, \leq)$ a strictly ordered monoid, and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism such that the center of the ring $R[[S, \omega]]$ is a field. Let s be a central and cancellative element of S. If ω_s is an inner automorphism of R induced by a unit u of R such that $\omega_t(u) = u$ for any $t \in S$, then there are positive integers m, n such that $m \neq n$ and $s^m \leq s^n$.*

Proof. The case where $s = 1$ is clear. Hence we assume that $s \in S \setminus \{1\}$ is a central and cancellative element such that $\omega_s(x) = uxu^{-1}$ for any $x \in R$, where u is a unit of R with the property that $\omega_t(u) = u$ for any $t \in S$ (hence also $\omega_t(u^{-1}) = u^{-1}$, which will be used for a moment). We show that $\mathsf{c}_{u^{-1}}\mathsf{e}_s$ is in the center of $R[[S, \omega]]$. For this, consider any $g \in R[[S, \omega]]$ and $v \in S$. Assume first that there exists $t \in S$ such that $v = ts$. Then, since s is cancellative and central, such an element t is unique and $v = st$, and thus

$$\begin{aligned}(g\mathsf{c}_{u^{-1}}\mathsf{e}_s)(v) &= g(t)\omega_t(u^{-1}) = g(t)u^{-1} = u^{-1}\left(ug(t)u^{-1}\right) \\ &= u^{-1}\omega_s(g(t)) = (\mathsf{c}_{u^{-1}}\mathsf{e}_s g)(v).\end{aligned}$$

We are left with the case where there is no $t \in S$ with $v = ts$. Then $X_v(g, \mathsf{c}_{u^{-1}}\mathsf{e}_s) = \emptyset = X_v(\mathsf{c}_{u^{-1}}\mathsf{e}_s, g)$, and thus $(g\mathsf{c}_{u^{-1}}\mathsf{e}_s)(v)$ and $(\mathsf{c}_{u^{-1}}\mathsf{e}_s g)(v)$ are both equal to 0. Hence in any case we have $(g\mathsf{c}_{u^{-1}}\mathsf{e}_s)(v) = (\mathsf{c}_{u^{-1}}\mathsf{e}_s g)(v)$, which shows that $\mathsf{c}_{u^{-1}}\mathsf{e}_s$ is in the center of $R[[S, \omega]]$. Thus $1 + \mathsf{c}_{u^{-1}}\mathsf{e}_s$ is in the center of $R[[S, \omega]]$ as well, and since $s \neq 1$, the central element $1 + \mathsf{c}_{u^{-1}}\mathsf{e}_s$ is nonzero. By hypothesis, the center of $R[[S, \omega]]$ is a field and thus $1 + \mathsf{c}_{u^{-1}}\mathsf{e}_s$ is a unit of $R[[S, \omega]]$. Now by combining [12, Lemma 2.1] and [13, Lemma 1.4] we conclude that there exist different positive integers m and n such that $s^m \leq s^n$. $\square$

In the following two lemmas we focus on central elements of a skew generalized power series ring $R[[S, \omega]]$ in the case where S is a commutative cancellative monoid.

Lemma 4.2. *Let R be a ring, $(S, \leq)$ an ordered commutative cancellative monoid, and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. Then the following statements are equivalent:*

(1) $f \in Z(R[[S,\omega]])$.
(2) $\mathsf{c}_r\mathsf{e}_s f = f\mathsf{c}_r\mathsf{e}_s$ *for any* $s \in S$ *and* $r \in R$.
(3) *For any* $s \in S$ *the following conditions are satisfied:*
 (i) $\omega_t(f(s)) = f(s)$ *for any* $t \in S$.
 (ii) $r \cdot f(s) = f(s)\omega_s(r)$ *for any* $r \in R$.

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3) Let $s \in S$. By (2), for any $t \in S$ we have

$$\mathsf{e}_t f = \mathsf{c}_1\mathsf{e}_t f = f\mathsf{c}_1\mathsf{e}_t = f\mathsf{e}_t.$$

Hence $\omega_t(f(s)) = (\mathsf{e}_t f)(ts) = (f\mathsf{e}_t)(st) = f(s)$, i.e., (3i) holds. To prove (3ii), let r be any element of R. Then $\mathsf{c}_r f = \mathsf{c}_r\mathsf{e}_1 f = f\mathsf{c}_r\mathsf{e}_1 = f\mathsf{c}_r$ by (2), and thus $r \cdot f(s) = (\mathsf{c}_r f)(s) = (f\mathsf{c}_r)(s) = f(s)\omega_s(r)$, as desired.

(3) $\Rightarrow$ (1) Assume (3) and let $g \in R[[S,\omega]]$. Then for any $v \in S$ we obtain

$$\begin{aligned}(fg)(v) &= \sum_{(s,t)\in X_v(f,g)} f(s)\omega_s(g(t)) = \sum_{(t,s)\in X_v(g,f)} g(t)f(s)\\ &= \sum_{(t,s)\in X_v(g,f)} g(t)\omega_t(f(s)) = (gf)(v),\end{aligned}$$

which shows that $fg = gf$. Hence f belongs to the center of $R[[S,\omega]]$. $\square$

Lemma 4.3. *Let* R *be a ring,* $(S,\leq)$ *an ordered commutative cancellative monoid, and* $\omega : S \to \mathrm{End}(R)$ *a monoid homomorphism such that the ring* R *is* (S,ω)*-simple. Then the following statements are equivalent:*

(1) $f \in Z(R[[S,\omega]])$.
(2) *For any* $s \in \mathrm{supp}(f)$ *we have that* $f(s) \in U(R)$*,* ω_s *is the inner automorphism of* R *induced by* $u = f(s)^{-1}$*, and* $\omega_t(u) = u$ *for any* $t \in S$.

Proof. (1) $\Rightarrow$ (2) Let $f \in Z(R[[S,\omega]])$. Take any $s \in \mathrm{supp}(f)$ and denote $a = f(s)$. Applying Lemma 4.2 we obtain that

$$\omega_t(a) = a \quad \text{for any } t \in S \tag{4.1}$$

and

$$ra = a\omega_s(r) \quad \text{for any } r \in R. \tag{4.2}$$

It follows from (4.2) that $Ra \subseteq aR$, and thus aR is an ideal of R. Furthermore, (4.1) implies that the nonzero ideal aR is (S,ω)-invariant. Since R is (S,ω)-simple, $1 \in aR$ and thus $1 = au$ for some $u \in R$. Applying (4.1) and (4.2) we obtain that also $1 = \omega_s(1) = \omega_s(au) = \omega_s(a)\omega_s(u) = a\omega_s(u) = ua$ and thus $a = f(s)$ is a unit of R. Now for $u = a^{-1} = f(s)^{-1}$ we infer from (4.1) and (4.2) that $\omega_t(u) = u$ and $\omega_s(r) = uru^{-1}$ for any $t \in S$ and $r \in R$, which completes the proof of (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1) It is easy to see that condition (2) of Lemma 4.3 implies condition (3) of Lemma 4.2. Hence $f \in Z(R[[S,\omega]]$ by Lemma 4.2. $\square$

We close this section with the following result, which shows that if S is a torsion-free abelian group and R is an (S,ω)-simple ring, then whether or not the center of the ring $R[[S,\omega]]$ is a field depends on some special property of the elements $s \in S$ for which ω_s is an inner automorphism.

Theorem 4.4. *Let R be a ring, $(S,\leq)$ an ordered torsion-free abelian group, and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism such that the ring R is (S,ω)-simple. Then the following statements are equivalent:*

(1) *The center of the ring $R[[S,\omega]]$ is a field.*
(2) *For any $s \in S$, if ω_s is an inner automorphism of R induced by a unit u of R such that $\omega_t(u) = u$ for any $t \in S$, then there is a positive integer n such that $s^n \geq 1$ or $s^n \leq 1$.*

Proof. (1) $\Rightarrow$ (2) Assume (1) and consider any $s \in S$ for which ω_s is an inner automorphism of R induced by a unit u of R such that $\omega_t(u) = u$ for any $t \in S$. Then by Proposition 4.1 there exist $k, m \in \mathbb{N}$ such that $k < m$ and

$$s^m \geq s^k \quad \text{or} \quad s^m \leq s^k. \tag{4.3}$$

Since S is a group, s is invertible and thus for $n = m - k$ we infer from (4.3) that $s^n \geq 1$ or $s^n \leq 1$. Hence (2) holds.

(2) $\Rightarrow$ (1) Assume (2), denote $A = R[[S,\omega]]$, and consider any element $f \in Z(A) \setminus \{0\}$. To complete the proof it suffices to show that f is a unit of A. Let $s \in \mathrm{supp}(f)$. From Lemma 4.3 it follows that ω_s satisfies the condition described in (2), i.e.,

$$\text{(4.4)} \qquad \textit{ω_s is an inner automorphism of R induced by a unit u of R such that $\omega_t(u) = u$ for any $t \in S$.}$$

Therefore, if V is the subgroup of S generated by $\mathrm{supp}(f)$, then for any $v \in V$ the endomorphism ω_v satisfies condition (4.4) (with s replaced by v) and thus it follows from (2) that

$$\text{(4.5)} \qquad \textit{for any $v \in V$ there exists $n \in \mathbb{N}$ such that $v^n \geq 1$ or $v^n \leq 1$.}$$

Hence the order $\leq$ restricted to V is subtotal. Since S is a torsion-free abelian group, so is V. Thus if $\leq'$ denotes the order $\leq$ restricted to V, then $(V,\leq')$ is a subtotally ordered torsion-free abelian group. Furthermore, Lemma 4.3 implies that $f(s)$ is a unit of R for any $s \in \mathrm{supp}(f)$. Hence it follows from Proposition 2.1 that f is a unit of the ring $B = R[[V,\omega',\leq']]$, where ω' is the restriction of the monoid homomorphism $\omega : S \to \mathrm{End}(R)$ to V. Since B is a subring of A, f is also a unit of A, as desired. $\square$

5. Simple skew generalized power series rings

In this section we will characterize the simplicity of a skew generalized power series ring $R[[S,\omega]]$ under various assumptions on R, S and ω. In previous sections we found and studied two necessary conditions for the simplicity of $R[[S,\omega]]$: the ring R being (S,ω)-simple, and the center of $R[[S,\omega]]$ being a field. In the following lemma, to these two conditions we add some other necessary conditions for $R[[S,\omega]]$ to be a simple ring.

Lemma 5.1. *Let R be a ring, $(S,\leq)$ a strictly ordered monoid, and*

$$\omega : S \to \operatorname{End}(R)$$

a monoid homomorphism. If the ring $R[[S,\omega]]$ is simple, then:

(i) *S is the only ideal of the monoid S.*
(ii) *R is an (S,ω)-simple ring.*
(iii) *$Z(R[[S,\omega]])$ is a field.*
(iv) *For any $s \in Z(S)$, if ω_s is an inner automorphism of R induced by a unit u of R such that $\omega_t(u) = u$ for any $t \in S$, then there is a positive integer n such that $s^n \geq 1$ or $s^n \leq 1$.*
(v) *For any nonzero ideal J of $R[[S,\omega]]$ there exists $f \in J \setminus \{0\}$ such that $\operatorname{supp}(f)$ is finite.*
(vi) *$J \cap Z(R[[S,\omega]]) \neq \{0\}$ for any nonzero ideal J of $R[[S,\omega]]$.*

Proof. (i) Let $s \in S$. Since $A = R[[S,\omega]]$ is a simple ring, $A\mathsf{e}_sA = A$ and thus $1 \in A\mathsf{e}_sA$. Hence

$$1 = f_1\mathsf{e}_s g_1 + \cdots + f_n\mathsf{e}_s g_n \tag{5.1}$$

for some $f_1, g_1, \ldots, f_n, g_n \in A$. From (5.1) it follows that for some $i \leq n$ we have $1 \in \operatorname{supp}(f_i\mathsf{e}_s g_i)$ and thus there exist $t, v \in S$ such that $1 = tsv$. Hence $SsS = S$, which proves (i).

(ii) was proved in the paragraph preceding Definition 3.5.

(iii) follows from the well-known fact that the center of any simple ring is a field.

(iv) Let $s \in Z(S)$. Then (i) implies that s is an invertible element of S. Now (iv) follows from (iii) and Proposition 4.1.

(v) If J is a nonzero ideal of $R[[S,\omega]]$ and the ring $R[[S,\omega]]$ is simple, then $J = R[[S,\omega]]$ and thus $1 \in J$, which proves (v).

(vi) follows by the same argument as that used in the proof of (v), since $1 \in Z(R[[S,\omega]])$. $\square$

The following theorem characterizes simple rings of skew generalized power series with exponents in a strictly totally ordered monoid. As we will see in the five corollaries following this theorem, it provides a rich source of examples of simple rings.

Theorem 5.2. *Let R be a ring, $(S,\leq)$ a strictly totally ordered monoid, and $\omega : S \to \operatorname{End}(R)$ a monoid homomorphism. Then the following statements are equivalent:*

(1) *$R[[S,\omega]]$ is a simple ring.*
(2) (i) *S is a group.*
 (ii) *R is an (S,ω)-simple ring.*

Proof. (1) $\Rightarrow$ (2) Assume (1) and consider any element $s \in S$. It follows from Lemma 5.1(i) that $1 = tsv$ for some $t, v \in S$. Since $(S, \leq)$ is a strictly totally ordered monoid, it is cancellative and thus

$$1 = tsv \;\Rightarrow\; t = (tsv)t \;\Rightarrow\; t = t(svt) \;\Rightarrow\; 1 = s(vt),$$

Hence all elements of S are invertible and thus S is a group, proving (2i). Part (2ii) is an immediate consequence of Lemma 5.1(ii).

(2) $\Rightarrow$ (1) Assume (2) and let J be any nonzero ideal of $R[[S, \omega]]$. Then by Proposition 3.6 there exists $f \in J$ such that 1 is the smallest element of $\mathrm{supp}(f)$ and $f(1) = 1$. It follows from Proposition 2.2 that f is a unit of $R[[S, \omega]]$, and since $f \in J$, it must be $J = R[[S, \omega]]$. Hence (2) implies (1). $\square$

The following corollaries are immediate consequences of Theorem 5.2.

Corollary 5.3. *Let R be a ring, let $(S, \leq)$ be a totally ordered group, and let $\omega : S \to \mathrm{Aut}(R)$ be a group homomorphism. Then the skew Mal'cev–Neumann series ring $R((S, \omega))$ is simple if and only if R is (S, ω)-simple.*

Corollary 5.4 ([24, Theorem 11.8])**.** *Let R be a ring and α an automorphism of R. Then the skew Laurent series ring $R[[x, x^{-1}; \alpha]]$ is simple if and only if R is α-simple.*

Corollary 5.5. *Let R be a ring and $(S, \leq)$ a strictly totally ordered monoid. Then the generalized power series ring $R[[S]]$ is simple if and only if S is a group and the ring R is simple.*

Corollary 5.6. *Let R be a ring and $(S, \leq)$ a totally ordered group. Then the Mal'cev–Neumann series ring $R((S))$ is simple if and only if R is simple.*

Corollary 5.7. *Let R be a ring. Then the Laurent series ring $R[[x, x^{-1}]]$ is simple if and only if R is simple.*

The following theorem provides a characterization of the simplicity of a skew generalized power series ring $R[[S, \omega]]$ in the case where S is a torsion-free commutative monoid and the order $\leq$ is subtotal.

Theorem 5.8. *Let R be a ring, $(S, \leq)$ a strictly ordered torsion-free commutative monoid, and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. Assume that the order $\leq$ is subtotal. Then the following statements are equivalent:*

(1) *$R[[S, \omega]]$ is a simple ring.*
(2) (i) *S is a group.*
    (ii) *R is an (S, ω)-simple ring.*

Proof. (1) $\Rightarrow$ (2) It is easy to see that every strictly subtotally ordered torsion-free commutative monoid is cancellative. Now (1) $\Rightarrow$ (2) follows by arguments similar to those used in the first part of the proof of Theorem 5.2.

(2) $\Rightarrow$ (1) Assume (2) and let $\preceq$ be the total order associated with $\leq$. Then $A = R[[S, \omega]] = R[[S, \omega, \leq]]$ is a subring of the ring $B = R[[S, \omega, \preceq]]$.

To get (1) we need to show that if J is a nonzero ideal of A, then $J = A$. Since R is (S,ω)-simple, Proposition 3.6 applied to the ring B implies that there exists $f \in J$ such that $f(1) = 1$ and 1 is the smallest element of $\mathrm{supp}(f)$ with respect to $\preceq$. Hence we deduce from Proposition 2.1 that f is a unit of A, which implies that $J = A$, as desired. $\square$

The following result characterizes simple rings among rings of generalized power series with exponents in a commutative monoid.

Corollary 5.9. *Let R be a ring and let $(S,\leq)$ be a strictly ordered commutative monoid. Then the following statements are equivalent:*

(1) *$R[[S]]$ is a simple ring.*
(2) (i) *S is a torsion-free group.*
 (ii) *$\leq$ is subtotal.*
 (iii) *R is a simple ring.*

Proof. (1) $\Rightarrow$ (2) Assume that $R[[S]]$ is a simple ring. Then parts (i) and (ii) of Lemma 5.1 imply that S is a group and R is a simple ring. Furthermore, since we consider the generalized power series ring $R[[S]]$, for any $s \in S$ the endomorphism ω_s is by definition the identity map of R and thus each ω_s is the inner automorphism of R induced by $1 \in R$. Hence by Lemma 5.1(iv), if $s \in S$, then there exists a positive integer n such that $s^n \geq 1$ or $s^n \leq 1$. Thus the order $\leq$ is subtotal. It remains to show that the group S is torsion-free. Let $s \in S \setminus \{1\}$ and suppose that there exists $m \in \mathbb{N}$ such that $s^m = 1$. Then $m > 1$ and we may assume m to be the smallest such integer. Since S is commutative, $\mathsf{e}_s - 1 \in Z(R[[S]])$, and since (1) implies that $Z(R[[S]])$ is a field, it follows that $\mathsf{e}_s - 1$ is a unit of $R[[S]]$. From $s^m = 1$ we infer that

$$(\mathsf{e}_s - 1)(\mathsf{e}_{s^{m-1}} + \mathsf{e}_{s^{m-2}} + \cdots + \mathsf{e}_s + 1) = 0.$$

Since $\mathsf{e}_s - 1$ is a unit, it follows that $\mathsf{e}_{s^{m-1}} + \mathsf{e}_{s^{m-2}} + \cdots + \mathsf{e}_s + 1 = 0$, which however is impossible, since by the minimality of m we have $s^k \neq 1$ for any $1 \leq k \leq m-1$. Hence the group S is torsion-free.

(2) $\Rightarrow$ (1) is an immediate consequence of Theorem 5.8. $\square$

Applying Corollary 5.9 to commutative rings, we obtain the following result of Elliott and Ribenboim which characterizes generalized power series rings that are fields.

Corollary 5.10 ([3, Theorem 1])**.** *Let R be a commutative ring and $(S,\leq)$ a strictly ordered commutative monoid. Then the following statements are equivalent:*

(1) *$R[[S]]$ is a field.*
(2) *R is a field, S is a torsion-free group and $\leq$ is subtotal.*

In [18, Proposition 4.2] Öinert proved that if G is an abelian group, R is a ring, and $\omega : G \to \mathrm{Aut}(R)$ is a group homomorphism such that R is (G,ω)-simple, then every nonzero ideal of the skew group ring $R[G,\omega]$ contains a nonzero central element. In the following lemma we extend Öinert's result to skew generalized power series rings. This lemma will be used in the proof of Theorem 5.12.

Lemma 5.11. *Let R be a ring, $(S, \leq)$ an ordered abelian group, and*

$$\omega : S \to \mathrm{End}(R)$$

a monoid homomorphism such that the ring R is (S,ω)-simple. Let J be an ideal of the ring $A = R[[S,\omega]]$ such that $\mathrm{supp}(f)$ is finite for some $f \in J \backslash \{0\}$. Then $J \cap Z(A) \neq \{0\}$.

Proof. By assumption, in J there exists a nonzero element whose support is finite. We choose $k \in J \setminus \{0\}$ such that the number of elements of $\mathrm{supp}(k)$, say n, is minimal. Choose any $t \in \mathrm{supp}(k)$ and set $h = k\mathsf{e}_{t^{-1}}$. Then $h(1) = k(t) \neq 0$ and $h \in J$, and thus by Proposition 3.4(b) the set

$$I = \{f(1) : \ f \in J, \ \mathrm{supp}(f) \subseteq \mathrm{supp}(h)\}$$

is a nonzero (S,ω)-invariant ideal of R. Since R is (S,ω)-simple, it follows that $I = R$ and thus $1 \in I$. Hence there exists $f \in J$ such that $f(1) = 1$ and $\mathrm{supp}(f) \subseteq \mathrm{supp}(h)$. In particular we have the following relations between cardinalities of the supports of f, h and k:

$$|\mathrm{supp}(f)| \leq |\mathrm{supp}(h)| \leq |\mathrm{supp}(k)| = n.$$

Hence the minimality of n implies that $|\mathrm{supp}(f)| = n$. Observe that if $r \in R$ and $s \in S$, then $\mathrm{supp}(\mathsf{c}_r\mathsf{e}_s f \mathsf{e}_{s^{-1}} - f\mathsf{c}_r) \subseteq \mathrm{supp}(f)$ and since

$$(\mathsf{c}_r\mathsf{e}_s f \mathsf{e}_{s^{-1}})(1) = r = (f\mathsf{c}_r)(1),$$

we conclude that $1 \in \mathrm{supp}(f) \setminus \mathrm{supp}(\mathsf{c}_r\mathsf{e}_s f \mathsf{e}_{s^{-1}} - f\mathsf{c}_r)$. Hence

$$\mathrm{supp}(\mathsf{c}_r\mathsf{e}_s f \mathsf{e}_{s^{-1}} - f\mathsf{c}_r) \subsetneq \mathrm{supp}(f),$$

and thus the minimality of n forces that $\mathsf{c}_r\mathsf{e}_s f \mathsf{e}_{s^{-1}} = f\mathsf{c}_r$. Hence

$$\mathsf{c}_r\mathsf{e}_s f = f\mathsf{c}_r\mathsf{e}_s.$$

Since $r \in R$ and $s \in S$ are arbitrary, it follows from Lemma 4.2 that $f \in Z(A)$. Hence $f \in J \cap Z(A) \setminus \{0\}$, which shows that $J \cap Z(A) \neq \{0\}$. □

The following result characterizes the simplicity of skew generalized power series rings $R[[S,\omega]]$ in the case where $(S, \leq)$ is a strictly ordered commutative monoid.

Theorem 5.12. *Let R be a ring, $(S, \leq)$ a strictly ordered commutative monoid, and $\omega : S \to \mathrm{End}(R)$ a monoid homomorphism. Then the following statements are equivalent:*

(1) *$R[[S,\omega]]$ is a simple ring.*
(2) (i) *S is a group.*
  (ii) *R is an (S,ω)-simple ring.*
  (iii) *$Z(R[[S,\omega]])$ is a field.*

(iv) *For any nonzero ideal* J *of* $R[[S,\omega]]$ *there exists* $f \in J \setminus \{0\}$ *such that* $\mathrm{supp}(f)$ *is finite.*

(3) (i) S *is a group.*

(ii) R *is an* (S,ω)*-simple ring.*

(iii) $Z(R[[S,\omega]])$ *is a field.*

(iv) $J \cap Z(R[[S,\omega]]) \neq \{0\}$ *for any nonzero ideal* J *of* $R[[S,\omega]]$.

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are immediate consequences of Lemmas 5.1 and 5.11, respectively. To prove the implication (3) $\Rightarrow$ (1), assume (3) and let J be a nonzero ideal of $A = R[[S,\omega]]$. By part (3iv) of (3), there exists $f \in J \setminus \{0\}$ such that $f \in Z(A)$. Hence (3iii) implies that f is a unit of A and thus $J = A$, which proves that A is a simple ring. $\square$

The following two corollaries are immediate consequences of the equivalence of (1) and (2) in Theorem 5.12.

Corollary 5.13. *Let* R *be a ring,* S *a commutative monoid, and*

$$\omega : S \to \mathrm{End}(R)$$

a monoid homomorphism. Then the following statements are equivalent:

(1) *The skew monoid ring* $R[S,\omega]$ *is simple.*

(2) S *is a group,* R *is an* (S,ω)*-simple ring and* $Z(R[S,\omega])$ *is a field.*

Corollary 5.14 ([18, Theorem 1.2(c)])**.** *Let* R *be a ring,* S *an abelian group, and* $\omega : S \to \mathrm{Aut}(R)$ *a group homomorphism. Then the skew group ring* $R[S,\omega]$ *is simple if and only if* R *is* (S,ω)*-simple and* $Z(R[S,\omega])$ *is a field.*

Below we provide a characterization of simple skew generalized power series rings $R[[S,\omega]]$ in the case where $(S,\leq)$ is a strictly ordered torsion-free commutative monoid.

Theorem 5.15. *Let* R *be a ring,* $(S,\leq)$ *a strictly ordered torsion-free commutative monoid, and* $\omega : S \to \mathrm{End}(R)$ *a monoid homomorphism. Then the following statements are equivalent:*

(1) $R[[S,\omega]]$ *is a simple ring.*

(2) (i) S *is a group.*

(ii) R *is an* (S,ω)*-simple ring.*

(iii) *For any* $s \in S$*, if* ω_s *is an inner automorphism of* R *induced by a unit* u *of* R *such that* $\omega_t(u) = u$ *for any* $t \in S$*, then there is a positive integer* n *such that* $s^n \geq 1$ *or* $s^n \leq 1$.

(iv) *For any nonzero ideal* J *of* $R[[S,\omega]]$ *there exists* $f \in J \setminus \{0\}$ *such that* $\mathrm{supp}(f)$ *is finite.*

(3) (i) S *is a group.*

(ii) R *is an* (S,ω)*-simple ring.*

(iii) *For any* $s \in S$*, if* ω_s *is an inner automorphism of* R *induced by a unit* u *of* R *such that* $\omega_t(u) = u$ *for any* $t \in S$*, then there is a positive integer* n *such that* $s^n \geq 1$ *or* $s^n \leq 1$.

(iv) $J \cap Z(R[[S,\omega]]) \neq \{0\}$ *for any nonzero ideal* J *of* $R[[S,\omega]]$.

Proof. For the proof it suffices to note that, by Theorem 4.4, if S is torsion-free, then in Theorem 5.12 the condition (2iii) of part (2) can be replaced by the condition (2iii) stated in Theorem 5.15. Similarly, condition (3iii) of part (3) in Theorem 5.12 can be replaced by the condition (3iii) stated in Theorem 5.15. □

The following well-known result is a consequence of Theorem 5.15.

Corollary 5.16 ([4, Theorem 1], [5, Theorem 3.18])**.** *Let* α *be an automorphism of a ring* R. *Then the following statements are equivalent:*

(1) $R[x,x^{-1};\alpha]$ *is a simple ring.*
(2) R *is* α*-simple and there is no positive integer* n *for which* α^n *is an inner automorphism of* R.
(3) R *is* α*-simple and there is no positive integer* n *for which* α^n *is an inner automorphism of* R *induced by a unit of* R *fixed by* α.

Proof. (1) $\Leftrightarrow$ (3) is an immediate consequence of the equivalence of (1) and (2) in Theorem 5.15. (2) $\Rightarrow$ (3) is obvious. For the proof of (3) $\Rightarrow$ (2) see [5, Proof of Theorem 3.13]. □

We close this paper by extending to skew generalized power series rings a characterization of skew group rings of abelian groups due to Crow (see Corollary 5.18).

Let $\omega : S \to \mathrm{End}(R)$ be an action of a monoid S on a ring R (i.e., ω is a monoid homomorphism). If the identity of S is the only element of S that maps to an inner automorphism of R, then the action ω is said to be *outer* (cf. [2, p. 127]).

Theorem 5.17. *Let* R *be a ring,* $(S,\leq)$ *a strictly ordered commutative monoid, and* $\omega : S \to \mathrm{End}(R)$ *an outer action of the monoid* S *on the ring* R. *Then the following conditions are equivalent:*

(1) $R[[S,\omega]]$ *is a simple ring.*
(2) (i) S *is a group.*
 (ii) R *is an* (S,ω)*-simple ring.*
 (iii) *For any nonzero ideal* J *of* $R[[S,\omega]]$ *there exists* $f \in J \setminus \{0\}$ *such that* $\mathrm{supp}(f)$ *is finite.*
(3) (i) S *is a group.*
 (ii) R *is an* (S,ω)*-simple ring.*
 (iii) $J \cap Z(R[[S,\omega]]) \neq \{0\}$ *for any nonzero ideal* J *of* $R[[S,\omega]]$.

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) follow immediately from Lemmas 5.1 and 5.11, respectively. To prove the implication (3) $\Rightarrow$ (1), assume (3) and let J be a nonzero ideal of $A = R[[S,\omega]]$. By part (3iii) of (3), there exists $f \in J \setminus \{0\}$ such that $f \in Z(A)$. Since ω is outer, Lemma 4.3 implies that $\mathrm{supp}(f) = \{1\}$ and $f(1) \in U(R)$. Hence it follows from Proposition 2.2 that f is a unit of A. Thus $J = A$, which proves the simplicity of A. □

As an immediate corollary of the equivalence (1) $\Leftrightarrow$ (2) in Theorem 5.17 we obtain the aforementioned result of Crow.

Corollary 5.18 ([2, Proposition 2.1])**.** *If S is an abelian group with outer action ω on a ring R, then the skew group ring $R[S,\omega]$ is simple if and only if R is (S,ω)-simple.*

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