New York Journal of Mathematics

New York J. Math. **23** (2017) 1357–1362.

# A remark on the Farrell–Jones conjecture

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Abstract. Assuming the classical Farrell–Jones conjecture we produce an explicit (commutative) group ring $R$ and a thick subcategory $\mathsf{C}$ of perfect $R$-complexes such that the Waldhausen $K$-theory space $\mathrm{K}(\mathsf{C})$ is equivalent to a rational Eilenberg-Maclane space.

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## 1. Introduction

Our main goal is to prove the following theorem

**Theorem 1.1** (Main result 3.5)**.** *There exists a commutative ring $R$ and a thick subcategory $\mathsf{C}$ of $\mathsf{Perf}(R)$ such that the space $\mathrm{K}(\mathsf{C})$ of Waldhausen $K$-theory is equivalent to an Eilenberg–MacLane space.*

In our opinion this theorem seems counterintuitive at the first glance. There are very few examples of rings for which the algebraic $K$-theory groups were computed in all degrees (e.g., the $K$-theory of finite fields computed by Quillen). Another source for such computations is the Farrell–Jones conjecture. We will compute explicitly the $K$-groups for some particular (commutative) group rings (Lemma 3.3).

**Conjecture 1.2** (Classical Farrell–Jones [Luck10])**.** *For any regular ring $k$ and any torsionfree group $G$, the assembly map*

$$\mathrm{H}_n(\mathrm{B}G; \mathbf{K}(k)) \longrightarrow \mathrm{K}_n(k[G])$$

*is an isomorphism for any $n \in \mathbb{Z}$.*

Received September 20, 2017.

2010 *Mathematics Subject Classification.* 19D50, 55P47, 55N20, 55P20, 18F25, 18E30.

*Key words and phrases.* Algebraic $K$-theory, Waldhausen $K$-theory, Farrell–Jones Conjecture, group algebra.

ISSN 1076-9803/2017

We refer to [Wal85] for the definition of the $K$-theory spectrum $\mathbf{K}(k)$ of a ring $k$. We recall that $\mathrm{B}G$ is the classifying space of the group $G$ and that $k[G]$ is the associated group ring with a natural augmentation $k[G] \to k$. We recall also that $\mathrm{H}_n(\mathrm{B}G; \mathbf{K}(k))$ is the same thing as the $n$-th stable homotopy group of the spectrum $\mathrm{B}G_+ \wedge \mathbf{K}(k)$. More precisely the assembly map is induced by the following map of spectra

$$\mathrm{B}G_+ \wedge \mathbf{K}(k) \to \mathbf{K}(k[G]).$$

Conjecture 1.2 admits a positive answer in the case where $k$ is regular ring and $G$ is a torsionfree abelian group: it is a particular case of the main result of [Weg15].

## 2. Fibre sequence for Waldhausen K-theory

**Notation 2.1.** We fix the following notations:

(1) Let $\mathcal{E}$ be any (differential graded) ring. Let $\mathsf{Mod}_{\mathcal{E}}$ denotes the (differential graded) model category of $\mathcal{E}$-complexes [Hov99]. And $\mathsf{Perf}(\mathcal{E})$ denotes the (differential graded) category of perfect (i.e., compact) $\mathcal{E}$-complexes.

(2) For any (differential graded) ring map $\mathcal{E} \to \mathcal{A}$, $\mathsf{Perf}(\mathcal{E}, \mathcal{A})$ denotes the thick subcategory of $\mathsf{Perf}(\mathcal{E})$ such that $M \in \mathsf{Perf}(\mathcal{E}, \mathcal{A})$ if and only if $M \otimes^{\mathbb{L}}_{\mathcal{E}} \mathcal{A} \simeq 0$, i.e., $M \otimes^{\mathbb{L}}_{\mathcal{E}} \mathcal{A}$ is quasi-isomorphic to 0. By the symbol $\otimes^{\mathbb{L}}_{\mathcal{E}}$ we do mean the derived tensor product over $\mathcal{E}$.

**Lemma 2.2.** *Let $\mathcal{E} \to \mathcal{A}$ be a morphism of (differential graded) rings such that $\mathcal{A} \otimes^{\mathbb{L}}_{\mathcal{E}} \mathcal{A} \simeq \mathcal{A}$, then*

$$\mathrm{K}(\mathcal{E}, \mathcal{A}) \to \mathrm{K}(\mathcal{E}) \to \mathrm{K}(\mathcal{A})$$

*is a fibre sequence of (infinite loop) spaces where* $\mathrm{K}(\mathcal{E}, \mathcal{A}) := \mathrm{K}(\mathsf{Perf}(\mathcal{E}, \mathcal{A}))$.

**Proof.** Let $\mathbf{w}$ be the class of equivalences in $\mathsf{Mod}_{\mathcal{E}}$ defined as follows: a map $P \to P'$ is $\mathbf{w}$-equivalence if and only if $\mathcal{A} \otimes^{\mathbb{L}}_{\mathcal{E}} P \to \mathcal{A} \otimes^{\mathbb{L}}_{\mathcal{E}} P'$ is a quasi-isomorphism (**q.i.**).

The left Bousfield localization [Hir09] of the model category $\mathsf{Mod}_{\mathcal{E}}$ with respect to the class $\mathbf{w}$ exists and it is denoted by $\mathrm{L}_{\mathbf{w}}\mathsf{Mod}_{\mathcal{E}}$. Since $\mathcal{A} \otimes^{\mathbb{L}}_{\mathcal{E}} \mathcal{A} \simeq \mathcal{A}$ we obtain a Quillen equivalence

$$\mathrm{L}_{\mathbf{w}}\mathsf{Mod}_{\mathcal{E}} \underset{U}{\overset{\mathcal{A}\otimes_{\mathcal{E}} -}{\rightleftarrows}} \mathsf{Mod}_{\mathcal{A}}$$

More precisely, for any $M \in \mathsf{Mod}_{\mathcal{A}}$ the (derived) counit map

$$\mathcal{A} \otimes^{\mathbb{L}}_{\mathcal{E}} U(M) \to M$$

is a quasi-isomorphism (because it is a quasi-isomorphism for $\mathcal{A} = M$, the functor $\mathcal{A} \otimes^{\mathbb{L}}_{\mathcal{E}} -$ commutes with homotopy colimits and $\mathcal{A}$ is a generator for the homotopy category of $\mathsf{Mod}_{\mathcal{A}}$). On another hand, the derived unit map $P \to \mathcal{A} \otimes^{\mathbb{L}}_{\mathcal{E}} U(P)$ is an equivalence in $\mathrm{L}_{\mathbf{w}}\mathsf{Mod}_{\mathcal{E}}$ for any $P \in \mathsf{Mod}_{\mathcal{E}}$ by definition. In particular the subcategory of compact objects in $\mathrm{L}_{\mathbf{w}}\mathsf{Mod}_{\mathcal{E}}$ is equivalent to $\mathsf{Perf}(\mathcal{A})$. Thus, by [Sag04, theorem 3.3], we have an equivalence of the $K$-theory spaces

$$\mathrm{K}((\mathsf{Perf}(\mathcal{E}), \mathbf{w})) \simeq \mathrm{K}((\mathsf{Perf}(\mathcal{A}), \mathbf{q.i.})) := \mathrm{K}(\mathcal{A}).$$

By Waldhausen fundamental theorem [Wal85, Theorem 1.6.4], the sequence of Waldhausen categories

$$(\mathsf{Perf}(\mathcal{E})^{\mathbf{w}}, \mathbf{q.i.}) \to (\mathsf{Perf}(\mathcal{E}), \mathbf{q.i.}) \to (\mathsf{Perf}(\mathcal{E}), \mathbf{w})$$

induces a fibre sequence of $K$-theory spaces

$$\mathrm{K}((\mathsf{Perf}(\mathcal{E})^{\mathbf{w}}, \mathbf{q.i.})) \to \mathrm{K}(\mathcal{E}) \to \mathrm{K}(\mathcal{A})$$

where $\mathsf{Perf}(\mathcal{E})^{\mathbf{w}}$ is the full subcategory of $\mathsf{Perf}(\mathcal{E})$ such that $E \in \mathsf{Perf}(\mathcal{E})^{\mathbf{w}}$ if and only if $\mathcal{A} \otimes^{\mathbb{L}}_{\mathcal{E}} E \simeq 0$. It is obvious by definition that

$$\mathsf{Perf}(\mathcal{E})^{\mathbf{w}} = \mathsf{Perf}(\mathcal{E}, \mathcal{A}).$$

Hence

$$\mathrm{K}(\mathcal{E}, \mathcal{A}) \to \mathrm{K}(\mathcal{E}) \to \mathrm{K}(\mathcal{A})$$

is a homotopy fibre sequence of spaces. □

A similar result can be found in [NR04, Theorem 0.5] and in [CX12, Lemma 5.1].

## 3. Farrell–Jones conjecture

**Notation 3.1.** We fix the following notations:

(1) $k = \mathbb{F}_2$ is the finite field with two elements.
(2) $R$ is the group algebra $k[\mathbb{Q}]$, where $\mathbb{Q}$ is the additive abelian group of rational numbers.

**Proposition 3.2.** *If $\mathbb{V}$ is a rational vector space and $A$ is a finite abelian group then*

$$\mathrm{H}_*(\mathrm{B}\mathbb{V}; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{V} & \text{if } n = 1 \\ 0 & \text{else} \end{cases}$$

*and*

$$\mathrm{H}_*(\mathrm{B}\mathbb{V}; A) = \begin{cases} A & \text{if } n = 0 \\ 0 & \text{else.} \end{cases}$$

**Lemma 3.3.**

$$\pi_n \mathrm{K}(R) := \mathrm{K}_n(R) = \begin{cases} \mathrm{K}_n(k) & \text{if } n \neq 1 \\ \mathbb{Q} & \text{if } n = 1. \end{cases}$$

**Proof.** By Quillen theorem [Quil72], the algebraic $K$-theory of the finite field $k$ is given by

$$\mathrm{K}_n(k) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{if } n \text{ even } > 0 \\ \mathbb{Z}/(2^j - 1) & \text{if } n = 2j-1 \text{ and } j > 0. \end{cases}$$

Since $\mathbb{Q}$ is a rational vector space and $\mathrm{K}_n(k)$ are finite abelian groups (for $n > 0$) then by Proposition 3.2 we have that

$$\mathrm{H}_p(\mathrm{B}\mathbb{Q}; \mathrm{K}_q(k)) = \begin{cases} \mathbb{Q} & \text{if } p = 1 \text{ and } q = 0 \\ \mathrm{K}_q(k) & \text{if } p = 0 \text{ and } q \geq 0 \\ 0 & \text{else.} \end{cases}$$

The second page $E^2_{p,q} = \mathrm{H}_p(\mathrm{B}\mathbb{Q}; \mathrm{K}_q(k))$ of the converging Atiyah–Hirzebruch spectral sequence [Luck10]

$$\mathrm{H}_p(\mathrm{B}\mathbb{Q}; \mathrm{K}_q(k)) \Longrightarrow \mathrm{H}_{p+q}(\mathrm{B}\mathbb{Q}; \mathbf{K}(k))$$

has graphically the shape shown in Figure 1, where the differentials

$$d^2 : E^2_{p,q} \to E^2_{p-2,q+1}$$

are obviously identical to 0. It means that the spectral sequence collapses, hence in our particular case it implies that

$$\mathrm{H}_p(\mathrm{B}\mathbb{Q}; \mathrm{K}_q(k)) = \mathrm{H}_{p+q}(\mathrm{B}\mathbb{Q}; \mathbf{K}(k)).$$

Since the Farrell–Jones conjecture is true in the case of torsionfree abelian groups [Weg15], we obtain that

$$\mathrm{K}_n(R) \cong \mathrm{H}_n(\mathrm{B}\mathbb{Q}; \mathbf{K}(k)) = \begin{cases} \mathrm{K}_n(k) & \text{if } n \neq 1 \\ \mathbb{Q} & \text{if } n = 1. \end{cases}$$ □

**Lemma 3.4.** *There is a fibre sequence of Waldhausen $K$-theory spaces given by*

$$\mathrm{K}(R, k) \to \mathrm{K}(R) \to \mathrm{K}(k)$$

**Proof.** Since $k$ is a finite field (in particular a finite abelian group) and $\mathbb{Q}$ is a rational vector space, it follows by Proposition 3.2 that

$$\mathrm{H}_n(\mathrm{B}\mathbb{Q}; k) = \mathrm{Tor}^R_n(k, k) = \begin{cases} k & \text{if } n = 0 \\ 0 & \text{else.} \end{cases}$$

therefore $k \otimes^{\mathbb{L}}_R k \simeq k$. The conclusion follows from Lemma 2.2 when $k = \mathcal{A}$ and $R = \mathcal{E}$. □

**Theorem 3.5.** *With the same notation, the $K$-theory space of the thick subcategory* $\mathsf{Perf}(R, k)$ *is equivalent to the Eilenberg–MacLane space* $\mathrm{B}\mathbb{Q}$.

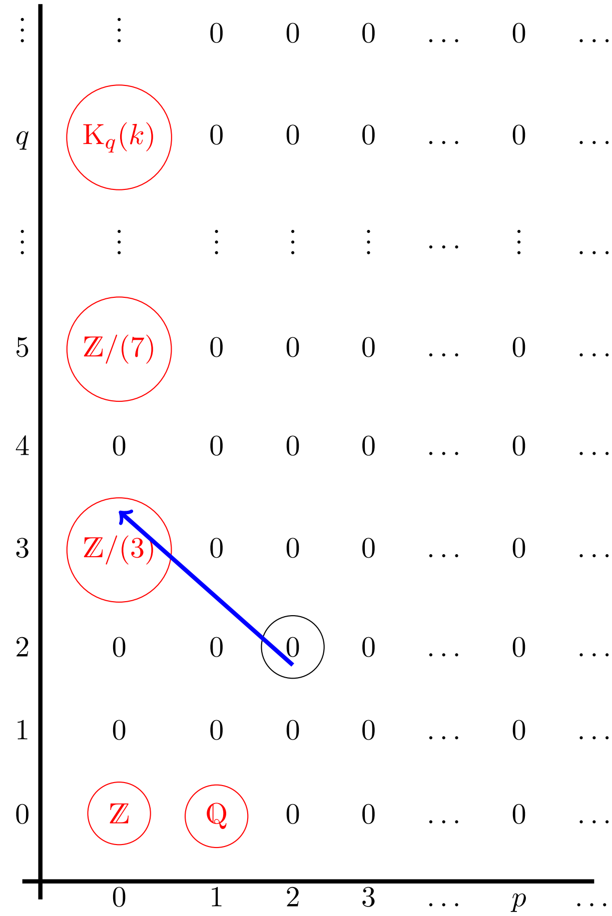


Figure 1. $E^2$ page of the Atiyah–Hirzebruch spectral sequence.

**Proof.** Since the Farrell–Jones conjecture is true for $G = \mathbb{Q}$. Combining Lemma 3.4 and Lemma 3.3, we have by Serre's long exact sequence that the homotopy groups of the homotopy fibre $\mathrm{K}(R, k)$ of $\mathrm{K}(R) \to \mathrm{K}(k)$ are given by

$$\mathrm{K}_n(R, k) = \begin{cases} \mathbb{Q} & \text{if } n = 1 \\ 0 & \text{else} \end{cases}$$

and by definition $\mathrm{K}(R, k) := \mathrm{K}(\mathsf{Perf}(R, k))$, hence we have proved the main theorem 1.1. □

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This paper is available via http://nyjm.albany.edu/j/2017/23-61.html.