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Representations of surface groups with finite mapping class group orbits

Indranil Biswas, Thomas Koberda, Mahan Mj and Ramanujan Santharoubane

Abstract. Let $(S, *)$ be a closed oriented surface with a marked point, let G be a fixed group, and let $\rho\colon \pi_1(S) \longrightarrow G$ be a representation such that the orbit of ρ under the action of the mapping class group $\mathrm{Mod}(S,*)$ is finite. We prove that the image of ρ is finite. A similar result holds if $\pi_1(S)$ is replaced by the free group F_n on $n \geq 2$ generators, and where $\mathrm{Mod}(S,*)$ is replaced by $\mathrm{Aut}(F_n)$. We show that if G is a linear algebraic group and if the representation variety of $\pi_1(S)$ is replaced by the character variety, then there are infinite image representations which are fixed by the whole mapping class group.

Contents

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1. Introduction

Let G and Γ be groups, and let

$$\mathcal{R}(\Gamma, G) := \mathrm{Hom}(\Gamma, G)$$

be the *representation variety* of Γ. The automorphism group $\mathrm{Aut}(\Gamma)$ acts on $\mathcal{R}(\Gamma, G)$ by precomposition.

Let $\Gamma = \pi_1(S)$, where S is a closed, orientable surface of genus at least two with a base-point $*$. The Dehn–Nielsen–Baer Theorem (see [FM]) implies that the mapping class group $\mathrm{Mod}(S, *)$ of S which preserves $*$ is identified with an index two subgroup of $\mathrm{Aut}(\Gamma)$. In this note, we show that if $\rho \in \mathcal{R}(\Gamma, G)$ has a finite $\mathrm{Mod}(S, *)$-orbit, then the image of ρ is finite. We show that the same conclusion holds if Γ is the free group F_n of finite rank $n \geq 2$, and $\mathrm{Mod}(S, *)$ is replaced by $\mathrm{Aut}(F_n)$.

1.1. Main results.

In the sequel, we assume that S is a closed, orientable surface of genus $g \geq 2$ and that F_n is a free group of rank at least two, unless otherwise stated explicitly.

Theorem 1.1. *Let $\Gamma = \pi_1(S)$ or F_n, and let G be an arbitrary group. Suppose that $\rho \in \mathcal{R}(\Gamma, G)$ has a finite orbit under the action of* $\mathrm{Aut}(\Gamma)$. *Then $\rho(\Gamma)$ is finite.*

Note that if $\Gamma = \pi_1(S)$ then ρ has a finite orbit under $\mathrm{Aut}(\Gamma)$ if and only if it has a finite orbit under $\mathrm{Mod}(S, *)$, since $\mathrm{Mod}(S, *)$ is a subgroup of $\mathrm{Aut}(\Gamma)$ of finite index. Note also that if the homomorphism ρ has finite image then the orbit of ρ for the action of $\mathrm{Aut}(\Gamma)$ on $\mathcal{R}(\Gamma, G)$ is finite, because Γ is finitely generated. We remark that the principal content of Theorem 1.1 is the passage from a fixed point to a finite orbit. It is rather straightforward to establish the conclusion of the main result for a fixed point of $\mathrm{Aut}(\Gamma)$, and the main difficulties involve the generalization to a nontrivial finite orbit.

We will show by example that Theorem 1.1 fails for a general group Γ. Moreover, Theorem 1.1 fails if Γ is a linear algebraic group with the representation variety of Γ being replaced by the character variety $\mathrm{Hom}(\Gamma, G)/G$, as follows fairly easily from a result of the second and fourth authors:

Proposition 1.2. *Let $\Gamma = \pi_1(S)$ and let*

$$\mathcal{X}(\Gamma, \mathrm{GL}_n(\mathbb{C})) := \mathrm{Hom}(\Gamma, \mathrm{GL}_n(\mathbb{C})) /\!\!/ \mathrm{GL}_n(\mathbb{C})$$

be its $\mathrm{GL}_n(\mathbb{C})$ character variety. For $n \gg 0$, there exists a point $\chi \in \mathcal{X}(\Gamma, \mathrm{GL}_n(\mathbb{C}))$ such that χ is the character of a representation with infinite image, and such that the action of $\mathrm{Mod}(S, *)$ *on $\mathcal{X}(\Gamma, \mathrm{GL}_n(\mathbb{C}))$ fixes χ.*

Proposition 1.2 resolves a well–known question of M. Kisin.

1.2. Punctured surfaces.

If S is not closed then $\pi_1(S)$ is a free group, and the group $\mathrm{Mod}(S, *)$ is identified with a subgroup of $\mathrm{Aut}(\pi_1(S))$, though this subgroup does not have finite index. The conclusion of Theorem 1.1

fails for surfaces with punctures unless the mapping class group is replaced by the automorphism group of the free group. See Proposition 4.2.

2. Aut(Γ)-invariant representations

We first address the question in the special case where the Aut(Γ)-orbit of $\rho : \Gamma \longrightarrow G$ on $\mathcal{R}(\Gamma, G)$ consists of a single point.

Lemma 2.1. *Let* Γ *be any group, and suppose that* $\rho \in \mathcal{R}(\Gamma, G)$ *is* Aut(Γ)*-invariant. Then* $\rho(\Gamma)$ *is abelian.*

Proof. Since $\rho \in \mathcal{R}(\Gamma, G)$ is invariant under the normal subgroup $\mathrm{Inn}(\Gamma) < \mathrm{Aut}(\Gamma)$ consisting of inner automorphisms,

$$\rho(ghg^{-1}) = \rho(h)$$

for all $g, h \in G$. Hence we have $\rho(g)\rho(h) = \rho(h)\rho(g)$. □

Lemma 2.2. *Let* $\Gamma = \pi_1(S)$ *or* $\Gamma = F_n$*, and let* $\rho \colon \Gamma \longrightarrow G$ *be* Aut(Γ)*-invariant. Then* $\rho(\Gamma)$ *is trivial.*

Proof. Without loss of generality, assume that $\rho(\Gamma) = G$. By Lemma 2.1, the group $\rho(\Gamma)$ is abelian. Hence ρ factors as

$$\rho = \rho^{ab} \circ A \,, \tag{1}$$

where

$$A : \Gamma \longrightarrow \Gamma/[\Gamma, \Gamma] = H_1(\Gamma, \mathbb{Z})$$

is the abelianization map, and $\rho^{ab} \colon H_1(\Gamma, \mathbb{Z}) \longrightarrow G$ is the induced representation of $H_1(\Gamma, \mathbb{Z})$

We first suppose that $\Gamma = \pi_1(S)$, where S is closed of genus g, so that the rank of $H_1(\Gamma, \mathbb{Z})$ is $2g$. Fix a symplectic basis

$$\{a_1, \dots, a_g, b_1, \dots, b_g\}$$

of $H_1(\Gamma, \mathbb{Z})$. The group of automorphisms of $H_1(\Gamma, \mathbb{Z})$ preserving the symplectic form is identified with $\mathrm{Sp}_{2g}(\mathbb{Z})$, and it is a standard fact that the natural action of $\mathrm{Mod}(S, *)$ on $H_1(\Gamma, \mathbb{Z})$ induces a surjection to $\mathrm{Sp}_{2g}(\mathbb{Z})$ [FM].

Consider the group $\rho(\Gamma)$, and consider the action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $H_1(\Gamma, \mathbb{Z})$, induced by the action of the mapping class group $\mathrm{Mod}(S, *)$ on $H_1(\Gamma, \mathbb{Z})$. There is an element of $\mathrm{Sp}_{2g}(\mathbb{Z})$ taking a_i to $a_i + b_i$. Therefore, from the assumption that the action of $\mathrm{Mod}(S, *)$ on ρ has a trivial orbit, it follows that $\rho^{ab}(a_i) = \rho^{ab}(a_i + b_i)$, and hence $\rho^{ab}(b_i) = 0$. Exchanging the roles of a_i and of b_i, we have $\rho^{ab}(a_i) = 0$. Thus ρ^{ab} is a trivial representation, and hence ρ is also trivial by (1).

A similar argument works if we set $\Gamma = F_n$. Instead of $\mathrm{Sp}_{2g}(\mathbb{Z})$, we have an action of $\mathrm{GL}_n(\mathbb{Z})$ on $H_1(\Gamma, \mathbb{Z})$ after choosing a basis $\{a_1, \dots, a_n\}$ for $H_1(\Gamma, \mathbb{Z})$. Then for each $1 \le j \le n$ and $i \ne j$, there exists an element of $\mathrm{GL}_n(\mathbb{Z})$ that takes a_i to $a_i + a_j$. This implies that $\rho^{ab}(a_j) = 0$ as before. □

The following is an immediate generalization of Lemma 2.2 whose proof is identical to the one given:

Lemma 2.3. *Let* Γ *be group, let*

$$H_1(\Gamma, \mathbb{Z})_{\mathrm{Out}(\Gamma)} = H_1(\Gamma, \mathbb{Z})/\langle \phi(v) - v \mid v \in H_1(\Gamma, \mathbb{Z}) \text{ and } \phi \in \mathrm{Out}(\Gamma)\rangle$$

be the module of co-invariants of the $\mathrm{Out}(\Gamma)$ *action on* $H_1(\Gamma, \mathbb{Z})$*, and let* $\rho \in \mathcal{R}(\Gamma, G)$ *be an* $\mathrm{Aut}(\Gamma)$*-invariant representation of* Γ*. If* $H_1(\Gamma, \mathbb{Z})_{\mathrm{Out}(\Gamma)} = 0$ *then* $\rho(\Gamma)$ *is trivial. If* $H_1(\Gamma, \mathbb{Z})_{\mathrm{Out}(\Gamma)}$ *is finite then* $\rho(\Gamma)$ *is finite as well.*

Corollary 2.4. *Let* Γ *be a closed surface group or a finitely generated free group, and let* $H < \mathrm{Aut}(\Gamma)$ *be a finite index subgroup. Then the module of* H*–co-invariants for* $H_1(\Gamma, \mathbb{Z})$ *is finite.*

Proof. Since $H < \mathrm{Aut}(\Gamma)$ has finite index, there exists an integer N such that for each $\phi \in \mathrm{Aut}(\Gamma)$, we have $\phi^N \in H$. In particular, the N^{th} multiples of the transvections occurring in the proof of Lemma 2.2 lie in H, whence the N^{th} multiples of elements of a basis for $H_1(\Gamma, \mathbb{Z})$ must be trivial. Consequently, the module of H–co-invariants is finite. □

3. Representations with a finite orbit

3.1. Central extensions of finite groups.

Lemma 3.1. *Let* Γ *be any group, and let* $\rho\colon \Gamma \longrightarrow G$ *be a representation. Suppose that the orbit of* ρ *under the action of* $\mathrm{Aut}(\Gamma)$ *on* $\mathcal{R}(\Gamma, G)$ *is finite. Then* $\rho(\Gamma)$ *is a central extension of a finite group.*

Proof. By restricting the action of $\mathrm{Aut}(\Gamma)$ to the action of $\mathrm{Inn}(\Gamma)$, we have that the ρ-orbit of the action of $\mathrm{Inn}(\Gamma)$ on $\mathcal{R}(\Gamma, G)$ is finite. Consequently, there exists a finite index subgroup Γ_1 of Γ that fixes ρ under the inner action. Hence by the same argument as in Lemma 2.1, the group $\rho(\Gamma_1)$ commutes with $\rho(\Gamma)$, so the center of $\rho(\Gamma)$ contains $\rho(\Gamma_1)$. Since Γ_1 is of finite index in Γ, the result follows. □

Lemma 3.2. *Let* ρ *be as in Lemma 3.1 and let* $H = \mathrm{Stab}(\rho) < \mathrm{Aut}(\Gamma)$ *be the stabilizer of* ρ*. Then the center* Z *of* $\rho(\Gamma)$ *is isomorphic to the module of co-invariants* Z_H *under the* H*-action.*

Proof. This is immediate, since $\rho(\Gamma)$ is invariant under the action of H. Thus, we have that $\rho(\phi(g)) = \rho(g)$ for all $\phi \in H$ and all $g \in \Gamma$. Restricting to elements $z \in \Gamma$ such that $\rho(z) \in Z$, we see that the subgroup of Z generated by elements of the form $\rho(\phi(z)) - \rho(z)$ is trivial, so that the natural quotient map $Z \to Z_H$ is an isomorphism. □

3.2. Homology of finite index subgroups. Let Γ be a finitely generated group, and let $\rho \in \mathcal{R}(\Gamma, G)$ be a representation whose orbit under the action of $\mathrm{Aut}(\Gamma)$ is finite. By Lemma 3.1, we have that $\rho(\Gamma)$ fits into a central extension:

$$1 \longrightarrow Z \longrightarrow \rho(\Gamma) \longrightarrow F \longrightarrow 1\,,$$

where F is a finite group and Z is a (possibly trivial) finitely generated torsion–free abelian group lying in (though not necessarily equal to) the center of $\rho(\Gamma)$.

Consider the group $N = \rho^{-1}(Z) < \Gamma$. This is a finite index subgroup of Γ, since Z has finite index in $\rho(\Gamma)$. By replacing N by a further finite index subgroup of Γ if necessary, we may assume that N is characteristic in Γ and hence N is invariant under automorphisms of Γ.

Since Z is an abelian group, we have that the restriction of ρ to N factors through the abelianization $H_1(N, \mathbb{Z})$. As before, we write

$$\rho^{ab} : \, H_1(N, \mathbb{Z}) \longrightarrow Z$$

for the corresponding map, and we write $Q = \Gamma/N$. The group Γ acts by conjugation on N and on $H_1(N, \mathbb{Z})$, and on Z via conjugation by the image of ρ, thus turning both $H_1(N, \mathbb{Z})$ and Z into $\mathbb{Z}[\Gamma]$-modules. Observe that the Γ-action on $H_1(N, \mathbb{Z})$ turns this group into a $\mathbb{Z}[Q]$-module, and that the $\mathbb{Z}[\Gamma]$-module structure on Z is trivial. Note that the map ρ^{ab} is a homomorphism of $\mathbb{Z}[\Gamma]$-modules. Summarizing this discussion, we have that following diagram commutes Γ-equivariantly:

$$\begin{array}{ccc} N & \xrightarrow{A} & H_1(N,\mathbb{Z}) \\ \downarrow{\scriptstyle \rho} & \swarrow{\scriptstyle \rho^{ab}} & \\ Z & & \end{array}$$

3.3. Chevalley–Weil Theory. Let Γ be a group, and let $N < \Gamma$ be a finite index normal subgroup with quotient group

$$Q := \Gamma/N \,. \tag{2}$$

When Γ is a closed surface group or a finitely generated free group, it is possible to describe $H_1(N, \mathbb{Q})$ as a $\mathbb{Q}[Q]$-module. We address closed surface groups first:

Theorem 3.3 (Chevalley–Weil Theory for surface groups, [CWH], [GLLM, Ko])**.** *Let $S = S_g$ be a closed surface of genus g, and let $\Gamma = \pi_1(S)$. Then there is an isomorphism of $\mathbb{Q}[Q]$-modules*

$$H_1(N, \mathbb{Q}) \xrightarrow{\sim} \rho_{reg}^{2g-2} \oplus \rho_0^2 \,,$$

where ρ_{reg} is the regular representation of Q and ρ_0 is the trivial representation of Q. Moreover, the invariant subspace of $H_1(N, \mathbb{Q})$ is $\mathrm{Aut}(\Gamma)$*-equivariantly isomorphic to $H_1(\Gamma, \mathbb{Q})$ via the transfer map.*

The corresponding statement for finitely generated free groups was also observed by Gaschütz, and is identical to the statement for surface groups, *mutatis mutandis*:

Theorem 3.4 (Chevalley–Weil Theory for free groups, [CWH], [GLLM, Ko])**.** *Let $\Gamma = F_n$ be a free group of rank n. Then there is an isomorphism of $\mathbb{Q}[Q]$-modules*

$$H_1(N, \mathbb{Q}) \xrightarrow{\sim} \rho_{reg}^{n-1} \oplus \rho_0 \,,$$

where ρ_{reg} is the regular representation of Q and ρ_0 is the trivial representation of Q. Moreover, the invariant subspace of $H_1(N, \mathbb{Q})$ is $\mathrm{Aut}(\Gamma)$-equivariantly isomorphic to $H_1(\Gamma, \mathbb{Q})$ via the transfer map.

Tensoring with $\mathbb{Q}$, we have a map

$$\rho^{ab} \otimes \mathbb{Q} \colon H_1(N, \mathbb{Q}) \longrightarrow Z \otimes \mathbb{Q}$$

which is a homomorphism of $\mathbb{Q}[\Gamma]$-modules since the natural map

$$\rho^{ab} \colon H_1(N, \mathbb{Z}) \longrightarrow Z$$

is Γ-equivariant. We decompose $H_1(N, \mathbb{Q}) = V_0 \bigoplus \big(\oplus_\chi V_\chi\big)$ according to its structure as a $\mathbb{Q}[\Gamma]$-module, where V_0 is the invariant subspace and χ ranges over nontrivial irreducible characters of Q.

Note that since N is characteristic in Γ, $\mathrm{Aut}(\Gamma)$ acts on $H_1(N, \mathbb{Q})$ and this action preserves V_0. Moreover, Theorems 3.3 and 3.4 imply that the $\mathrm{Aut}(\Gamma)$-action on V_0 is canonically isomorphic to the $\mathrm{Aut}(\Gamma)$ action on $H_1(\Gamma, \mathbb{Q})$, by the naturality of the transfer map.

We are now ready to prove the main result of this note:

Proof of Theorem 1.1. Recall that if $\rho \colon \Gamma \longrightarrow G$ has a finite orbit under the action of $\mathrm{Aut}(\Gamma)$, then $\rho(\Gamma)$ is a central extension of the form

$$1 \longrightarrow Z \longrightarrow \rho(\Gamma) \longrightarrow F \longrightarrow 1 \,,$$

where F is finite. Clearly, it suffices to prove that the vector space $Z \otimes \mathbb{Q}$ is trivial.

As discussed above, we have that $Z \otimes \mathbb{Q}$ is a quotient of $H_1(N, \mathbb{Q})$ for a suitable finite index characteristic subgroup $N < \Gamma$, where this quotient map is equivariant with respect to the conjugation action of Γ on itself. We now apply Chevalley–Weil Theory to $H_1(N, \mathbb{Q})$. Considering the image of each irreducible Γ/N–representation $V_\chi \subset H_1(N, \mathbb{Q})$ under $\rho^{ab} \otimes \mathbb{Q}$, it follows from Schur's Lemma that either V_χ is in the kernel of $\rho^{ab} \otimes \mathbb{Q}$ or it is mapped isomorphically onto its image. Since $\rho^{ab} \otimes \mathbb{Q}$ is a $\mathbb{Q}[\Gamma]$-module homomorphism and since $Z \otimes \mathbb{Q}$ is a trivial $\mathbb{Q}[\Gamma]$-module, we have that $V_\chi \subset \ker \rho^{ab} \otimes \mathbb{Q}$ whenever χ is a nontrivial irreducible character of $Q = \Gamma/N$. It follows that $Z \otimes \mathbb{Q}$ is a quotient of V_0, the submodule of $H_1(N, \mathbb{Q})$ on which Q acts trivially.

Since the $\mathrm{Aut}(\Gamma)$-actions on $H_1(\Gamma, \mathbb{Z})$ and on V_0 are isomorphic via the transfer map, Corollary 2.4 implies that the module of rational H-co-invariants for V_0 is trivial for any finite index subgroup $H < \mathrm{Aut}(\Gamma)$, meaning

$$V_0 / \langle \phi(v) - v \mid v \in V_0 \text{ and } \phi \in H \rangle = 0.$$

Let $H = \mathrm{Stab}(\rho) < \mathrm{Aut}(\Gamma)$ be the stabilizer of ρ, which has finite index in $\mathrm{Aut}(\Gamma)$ by assumption. Let $v_0 \in V_0$ be an arbitrary element. Since the module of H–co-invariants of V_0 is trivial, we have that

$$v_0 = \sum_{i=1}^{k} a_i(\phi_i(v_i) - v_i)$$

for suitable vectors $(v_1, \ldots, v_k) \in V_0^k$, rational numbers $(a_1, \ldots, a_k) \in \mathbb{Q}^k$, and automorphisms $(\phi_1, \ldots, \phi_k) \in H^k$. Applying $\rho^{ab} \otimes \mathbb{Q}$, we have

$$(\rho^{ab} \otimes \mathbb{Q})(v_0) = \sum_{i=1}^{k} a_i \cdot (\rho^{ab} \otimes \mathbb{Q})(\phi_i(v_i) - v_i).$$

Since ρ is H-invariant, we have that $(\rho^{ab} \otimes \mathbb{Q})(\phi(v_i) - v_i) = 0$, whence $(\rho^{ab} \otimes \mathbb{Q})(v_0) = 0$. Thus, $v_0 \in \ker \rho^{ab} \otimes \mathbb{Q}$, and consequently $Z \otimes \mathbb{Q} = 0$. □

4. Counterexamples for general groups

It is not difficult to see that Theorem 1.1 is false for general groups. We have the following easy proposition:

Proposition 4.1. *Let Γ be a finitely generated group such that Γ surjects to $\mathbb{Z}$ and such that $\mathrm{Out}(\Gamma)$ is finite. Then there exists a group G and a representation $\rho \in \mathcal{R}(\Gamma, G)$ such that ρ has infinite image and such that the $\mathrm{Aut}(\Gamma)$-orbit of ρ is finite.*

Proof. Set

$$G = \Gamma^{ab},$$

and let $\rho : \Gamma \longrightarrow G$ be the abelianization map. Since $\mathrm{Out}(\Gamma)$ is finite, we have that $\mathrm{Aut}(\Gamma)$ induces only finitely many distinct automorphisms of G, and hence ρ has a finite orbit under the $\mathrm{Aut}(\Gamma)$ action on $\rho \in \mathcal{R}(\Gamma, G)$. □

It is easy to see that Proposition 4.1 generalizes to the case where ρ has infinite abelian image with G being an arbitrary group.

There are many natural classes of groups which satisfy the hypotheses of Proposition 4.1. For instance, one can take a cusped finite volume hyperbolic 3–manifold or a closed hyperbolic 3–manifold with positive first Betti number; every closed hyperbolic 3–manifold has such a finite cover by the work of Agol [Ag]. The fundamental groups of these manifolds are finitely generated with infinite abelianization, and by Mostow Rigidity, their groups of outer automorphisms are finite.

Another natural class of groups satisfying the hypotheses of Proposition 4.1 is the class of random right-angled Artin groups, in the sense of Charney–Farber [CF]. Every right-angled Artin group has infinite abelianization, though many have infinite groups of outer automorphisms. Certain graph theoretic conditions which are satisfied by generic finite graphs in a suitable random model guarantee that the outer automorphism group is finite, however. An explicit right-angled Artin group with a finite group of outer automorphisms is the right-angled Artin group on the pentagon graph.

Let D_n denote the disk with $n \geq 2$ punctures. The mapping class group $\mathrm{Mod}(D_n, \partial D_n)$ is identified with the braid group B_n on n strands, and naturally sits inside of $\mathrm{Aut}(F_n) = \mathrm{Aut}(\pi_1(D_n))$. The following easy proposition illustrates another failure of Theorem 1.1 to generalize:

Proposition 4.2. *Let G be a group which contains an element of infinite order. Then there exists an infinite image representation $\rho \in \mathcal{R}(F_n, G)$ which is fixed by the action of $B_n < \mathrm{Aut}(F_n)$.*

Proof. Small loops about the punctures of D_n can be connected to a basepoint on the boundary of D_n in order to obtain a free basis for $\pi_1(D_n)$. Since the braid group consists of isotopy classes of homeomorphisms of D_n, we have that B_n acts on the homology classes of these loops by permuting them. Therefore, we may let ρ be the homomorphism $F_n \longrightarrow \mathbb{Z}$ obtained by taking the exponent sum of a word in the chosen free basis for $\pi_1(D_n)$, and then sending a generator for $\mathbb{Z}$ to an infinite order element of G. It is clear from this construction that ρ is B_n-invariant and has infinite image. □

5. Character varieties

In this section we prove Proposition 1.2, which relies on one of the results in [KS].

Theorem 5.1 (cf. [KS], Corollary 4.3)**.** *Let S be a closed surface of genus $g \geq 2$. Then there exists a linear representation*

$$\rho : \ \mathrm{Mod}(S, *) \longrightarrow \mathrm{PGL}_n(\mathbb{C})$$

such that the restriction of ρ to $\pi_1(S)$ has infinite image.

The basic idea behind Theorem 5.1 is to consider the SO(3)–TQFT representations of a mapping class group $\mathrm{Mod}(S, *)$. Recall that if S is an orientable surface with negative Euler characteristic then the Birman Exact Sequence furnishes a normal copy of $\pi_1(S)$ inside of the pointed mapping class group $\mathrm{Mod}(S, *)$ (see for instance [Bi, FM]), called the point–pushing subgroup. The conjugation action of $\mathrm{Mod}(S, *)$ on this copy of $\pi_1(S)$ is by the natural action by automorphisms. The TQFT representations give rise to a family of linear representations of $\mathrm{Mod}(S, *)$, and in [KS] it was proved that the image of the point–pushing subgroup is infinite for sufficiently complicated representations in this family.

We remark that in Theorem 5.1, it can be arranged for the image of $\pi_1(S)$ under ρ to have a free group in its image, as discussion in [KS]. Theorem 5.1 implies Proposition 1.2 without much difficulty.

Proof of Proposition 1.2. Let a representation

$$\rho : \ \mathrm{Mod}(S, *) \longrightarrow \mathrm{PGL}_n(\mathbb{C})$$

be given as in Theorem 5.1. Choose an arbitrary embedding of $\mathrm{PGL}_n(\mathbb{C})$ into $\mathrm{GL}_m(\mathbb{C}) = G$ for some $m \geq n$, and let σ be the corresponding representation of $\mathrm{Mod}(S, *)$ obtained by composing ρ with the embedding. We will write χ for the character of σ, and we claim that χ satisfies the conclusions of the proposition.

That χ corresponds to a representation of $\pi_1(S)$ with infinite image is immediate from the construction. Note that χ is actually the character of a representation of $\mathrm{Mod}(S, *)$, and that $\mathrm{Inn}(\mathrm{Mod}(S, *))$ acts trivially on the character variety $\mathcal{X}(\mathrm{Mod}(S, *), G)$. It follows that $\mathrm{Inn}(\mathrm{Mod}(S, *))$ fixes χ even when χ is viewed as a character of $\pi_1(S)$, since

$$\pi_1(S) \ < \ \mathrm{Mod}(S, *)$$

is normal. The conjugation action of $\mathrm{Mod}(S, *)$ on $\pi_1(S)$ is by automorphisms via the natural embedding

$$\mathrm{Mod}(S, *) < \mathrm{Aut}(\pi_1(S))\,.$$

It follows that χ is invariant under the action of $\mathrm{Mod}(S, *)$, the desired conclusion. □

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(Indranil Biswas) School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India
indranil@math.tifr.res.in

(Thomas Koberda) Department of Mathematics, University of Virginia, Charlottesville, VA 22904-4137, USA
thomas.koberda@gmail.com

(Mahan Mj) School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India
mahan@math.tifr.res.in

(Ramanujan Santharoubane) Department of Mathematics, University of Virginia, Charlottesville, VA 22904-4137, USA
ramanujan.santharoubane@gmail.com

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