

# $A_2$ colored polynomials of rigid vertex graphs

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ABSTRACT. The Kauffman–Vogel polynomials are three variable polynomial invariants of 4-valent rigid vertex graphs. A one-variable specialization of the Kauffman–Vogel polynomials for unoriented 4-valent rigid vertex graphs was given by using the Kauffman bracket and the Jones–Wenzl idempotent with the color 2. Bataineh, Elhamdadi and Hajij generalized it to any color with even positive integers. We give another generalization of the one-variable Kauffman–Vogel polynomial for oriented and unoriented 4-valent rigid vertex graphs by using the  $A_2$  bracket and the  $A_2$  clasps. These polynomial invariants are considered as the  $\mathfrak{sl}_3$  colored Jones polynomials for singular knots and links.

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## 1. Introduction

Kauffman considered an isotopy for embeddings of graphs in the 3-space in [Kau89]. It is called the vertex isotopy and he proved that the vertex isotopy for a 4-valent graph is generated by a generalization of the Reidemeister moves. Kauffman and Vogel defined polynomial invariants for

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regular vertex isotopy classes of 4-valent graphs in [KauV92]. These polynomial invariants are three-variable generalizations of polynomial invariants of knots: the Homfly polynomials for oriented knots and the Kauffman polynomials [Kau90] for unoriented knots. A one-variable specialization of the Kauffman–Vogel polynomial of a regular vertex isotopy class of an unoriented 4-valent graph was given by using the Kauffman bracket and the Jones–Wenzl idempotents (see, for example, Sect. 4.4 in [KauL94]). For an unoriented 4-valent graph  $G$ , this polynomial is defined by the value of the Kauffman bracket of a skein element obtained by coloring each edge of  $G$  with 2 and replacing 4-valent vertices by a certain type of skein element. In [BEH16], Bataineh, Elhamdadi and Hajij generalized the one-variable Kauffman–Vogel polynomials by changing the coloring from 2 to any even positive integer  $2n$ . There are many invariants of vertex isotopy classes of graphs, for example, Kauffman and Mishra [KauM13], Juyumaya and Lambropoulou [JL09] as an invariant of singular knots, Yamada [Yam89] as an invariant of spatial graphs, and so on. In [Wu12], Wu showed a relationship between the Kauffman–Vogel polynomial and the MOY graph polynomial [MOY98]. By using linear skein theory, some invariants of topological graphs were constructed, for example, in Yokota [Yok96] and Kawagoe [Kaw16].

In this paper, we will define one-variable polynomial invariants of the regular vertex isotopy classes of oriented and unoriented 4-valent graphs. These polynomial invariants are a skein theoretical generalization of the one-variable Kauffman–Vogel polynomials. We will construct the  $A_2$  colored polynomials using the  $A_2$  bracket and the  $A_2$  clasps instead of the Kauffman bracket and the Jones–Wenzl idempotents. These invariants are generalizations of the colored  $\mathfrak{sl}_3$  Jones polynomials. In fact, the rigid vertex isotopy class of an oriented (resp. unoriented) 4-valent graph  $G$  with no 4-valent vertices is an oriented (resp. unoriented) framed link. In this case, the  $A_2$  colored polynomials of  $G$  is the  $\mathfrak{sl}_3$  colored Jones polynomial of the oriented (resp. unoriented) framed link colored with one-row Young diagrams of even length (resp. two-row Young diagrams of the same length).

This paper is organized as follows. We introduce the definition of a 4-valent rigid vertex graph by diagrams on  $S^2$  and the generalized Reidemeister moves in Sect. 2. Next, we define the  $A_2$  bracket, the  $A_2$  clasps and show some useful formulas in Sect. 3. In Sect. 4, we define the polynomial invariants of oriented and unoriented 4-valent rigid vertex graphs. In Sect. 5, we compute these invariants for some 4-valent rigid vertex graphs.

## 2. Rigid vertex graphs

We will treat diagrammatically regular vertex isotopy classes of embeddings of oriented and unoriented 4-valent graphs in  $S^3$  through an equivalence class of 4-valent graph diagrams on  $S^2$ . We briefly explain the geometric definition of the rigid vertex graphs (see Kauffman [Kau89] for details.)

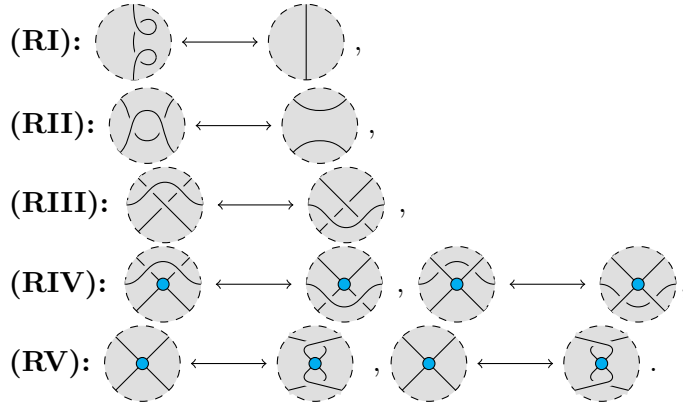



FIGURE 1. The Reidemeister moves for 4-valent graph diagrams

The rigid vertex means that the half-edges attaching to the vertex have a cyclic ordering. An embedding of a 4-valent rigid vertex graphs into  $S^3$  is an embedding of the underlying 4-valent graph into  $S^3$  with the following condition. Each embedded vertex  $v$  can be replaced by a small disk  $D_v$  in  $S^3$  and the half-edges at  $v$  are attached to  $\partial D_v$  such that the cyclic ordering coincides with the orientation of  $\partial D_v$ .

We deal with the regular isotopy classes of the above graphs in  $S^3$  as diagrams on  $S^2$  with an equivalence relation generated by Reidemeister moves (RI) – (RV).

**Definition 2.1.**

- A 4-valent graph diagram on  $S^2$  is an immersion of 4-valent graph into  $S^2$  whose intersection points are only transverse double points of edges. At each intersection point, two edges are equipped with crossing data .
- Two 4-valent graph diagrams  $G$  and  $G'$  are equivalent if  $G$  is related to  $G'$  by a finite sequence of Reidemeister moves (RI) – (RV) as in Fig. 2. This equivalence relation is called *regular vertex isotopy* in [Kau89].
- An oriented 4-valent graph diagram (see, for example [KauV92]) is a 4-valent graph diagram whose edges are oriented as one of the following:

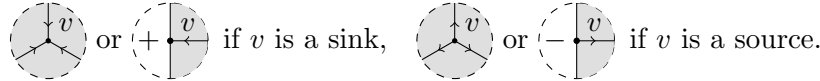


We call equivalence classes of 4-valent graph diagrams *4-valent rigid vertex graphs*.

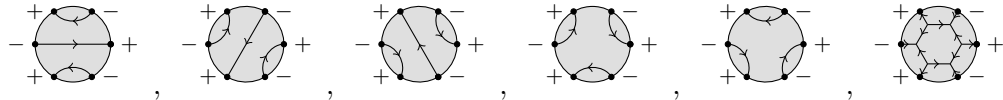
### 3. The $A_2$ bracket and some formulas

We will construct invariants of oriented and unoriented 4-valent rigid vertex graphs using the linear skein theory corresponding to the quantum  $A_2$  representation. In this section, we introduce the  $A_2$  web spaces, the  $A_2$  bracket, and the  $A_2$  clasps defined by Kuperberg [Kup94, Kup96]. Special skein elements called the  $A_2$  clasps play an important role in construction of the colored  $A_2$  polynomials for 4-valent rigid vertex graphs.

Let  $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m)$  be an  $m$ -tuple of signs  $+$  or  $-$ . Let  $D_\varepsilon$  denote the unit disk with signed marked points  $\{\exp(2\pi\sqrt{-1}/m)^{j-1} \mid j = 1, 2, \dots, m\}$  on its boundary. The sign of  $\exp(2\pi\sqrt{-1}/m)^{j-1}$  is given by  $\varepsilon_j$  for  $j = 1, 2, \dots, m$ . A *bipartite uni-trivalent graph*  $G$  is a directed graph such that each vertex is either trivalent or univalent and the vertices are divided into the sinks and the sources. A sink (resp. source) is a vertex such that all edges adjoining to the vertex point into (resp. away from) it. A *bipartite trivalent graph*  $G$  in  $D_\varepsilon$  is an embedding of a uni-trivalent graph into  $D_\varepsilon$  such that any vertex  $v$  has the following neighborhoods:



An  $A_2$  *basis web* is the boundary-fixing isotopy class of a bipartite trivalent graph  $G$  in  $D_\varepsilon$ , where any internal face of  $D_\varepsilon \setminus G$  has at least six sides. Let us denote  $B_\varepsilon$  the set of  $A_2$  basis webs in  $D_\varepsilon$ . For example,  $B_{(+, -, +, -, +, -, -)}$  has the following  $A_2$  basis webs:



The  $A_2$  *web space*  $W_\varepsilon$  is the  $\mathbb{Q}(q^{\frac{1}{6}})$ -vector space spanned by  $B_\varepsilon$ . A *tangled trivalent graph diagram* in  $D_\varepsilon$  is an immersed bipartite uni-trivalent graph in  $D_\varepsilon$  whose intersection points are only transverse double points of edges with crossing data or . Tangled trivalent graph diagrams  $G$  and  $G'$  are regularly isotopic if  $G$  is obtained from  $G'$  by a finite sequence of boundary-fixing isotopies and Reidemeister moves, see Figure 2, with some direction of edges.

*Tangled trivalent graphs* in  $D_\varepsilon$  are regular isotopy classes of tangled trivalent graph diagrams in  $D_\varepsilon$ . We denote  $T_\varepsilon$  the set of tangled trivalent graphs in  $D_\varepsilon$ .

**Definition 3.1** (The  $A_2$  bracket [Kup96]). We define a  $\mathbb{Q}(q^{\frac{1}{6}})$ -linear map  $\langle \cdot \rangle_3 : \mathbb{Q}(q^{\frac{1}{6}})T_\varepsilon \rightarrow W_\varepsilon$  by the following.

- $\langle \text{crossing with } + \rangle_3 = q^{\frac{1}{3}} \langle \text{sink} \rangle_3 - q^{-\frac{1}{6}} \langle \text{source} \rangle_3,$
- $\langle \text{crossing with } - \rangle_3 = q^{-\frac{1}{3}} \langle \text{sink} \rangle_3 - q^{\frac{1}{6}} \langle \text{source} \rangle_3,$

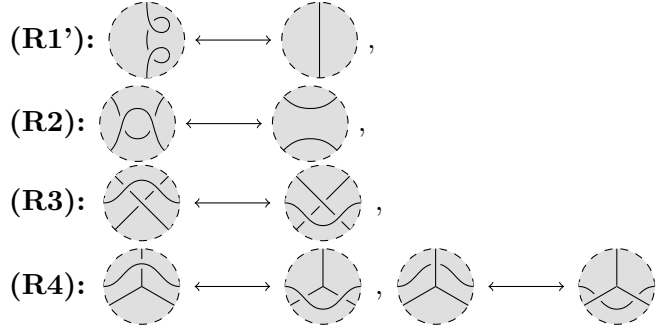


FIGURE 2. The Reidemeister moves for tangled trivalent graph diagrams

- $\langle \text{[Diagram 1]} \rangle_3 = \langle \text{[Diagram 2]} \rangle_3 + \langle \text{[Diagram 3]} \rangle_3,$
- $\langle \text{[Diagram 4]} \rangle_3 = [2] \langle \text{[Diagram 5]} \rangle_3,$
- $\langle G \sqcup \text{[Diagram 6]} \rangle_3 = [3] \langle G \rangle_3,$

where  $[n] = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$  is a quantum integer.

We remark that this map is invariant under the Reidemeister moves for tangled trivalent graphs.

We next consider the  $A_2$  web space  $W_{n^+ + n^-} = W_{(+, +, \dots, +, -, -, \dots, -)}$ . The  $n$  marked points with sign  $+$  lie in the right side and the  $n$  marked points with sign  $-$  in left side. We define  $A_2$  clasps  $\text{[Diagram 7]}^n \in W_{n^+ + n^-}$  inductively by the following.

**Definition 3.2.** (The  $A_2$  clasps)

$$\begin{aligned} & \text{[Diagram 8]}^1 = \text{[Diagram 9]}^1 \in W_{1^+ + 1^-} \\ (1) \quad & \text{[Diagram 10]}^n = \left\langle \text{[Diagram 11]}^{n-1} \right\rangle_3 - \frac{[n-1]}{[n]} \left\langle \text{[Diagram 12]}^{n-1} \right\rangle_3 \in W_{n^+ + n^-} \end{aligned}$$

A strand decorated by a nonnegative integer  $n$  means  $n$  parallelization of the strand. For example,  $\text{[Diagram 13]}^n = \text{[Diagram 14]}^n$ ,  $\text{[Diagram 15]}^n = n \left\{ \text{[Diagram 16]} \right\}$ , and

$$\text{[Diagram 17]}^l \text{[Diagram 18]}^k = \text{[Diagram 19]}^{l,k}.$$

$A_2$  clasps have the following properties.

**Lemma 3.3** (Properties of  $A_2$  clasps). *For any positive integer  $n$ ,*

- $\left\langle - \left[ \begin{array}{c} n \\ \rightarrow \\ \left[ \begin{array}{c} n \\ \rightarrow \end{array} \right] \end{array} \right] \right\rangle_3 = - \left[ \begin{array}{c} n \\ \rightarrow \end{array} \right],$
- $\left\langle \rightarrow \left[ \begin{array}{c} n-k-2 \\ \rightarrow \\ \left[ \begin{array}{c} n-k-2 \\ \rightarrow \\ \left[ \begin{array}{c} n-k-2 \\ \rightarrow \end{array} \right] \end{array} \right] \end{array} \right] \right\rangle_3 = 0 \quad (k = 0, 1, \dots, n-2).$

We also define the  $A_2$  clasp of type  $(n, m)$  according to Ohtsuki and Yamada [OY97].

**Definition 3.4** (the  $A_2$  clasp of type  $(n, m)$ ).

$$\left\langle \begin{array}{c} n \\ \left[ \begin{array}{c} \left[ \begin{array}{c} n \\ \leftarrow \end{array} \right] \\ \left[ \begin{array}{c} n \\ \rightarrow \end{array} \right] \end{array} \right] \\ m \end{array} \right\rangle_3 = \sum_{k=0}^{\min\{m,n\}} (-1)^k \frac{[n][m]}{[n+m+1][k]} \left\langle \begin{array}{c} n \\ \left[ \begin{array}{c} \left[ \begin{array}{c} n-k \\ \leftarrow \end{array} \right] \\ \left[ \begin{array}{c} n-k \\ \rightarrow \end{array} \right] \end{array} \right] \\ m \end{array} \right\rangle_3$$

where

$$\left[ \begin{array}{c} n \\ k \end{array} \right] = \frac{[n]!}{[k]! [n-k]!} = \frac{\{n\}!}{\{k\}! \{n-k\}!}$$

for  $k \leq n$ .

**Lemma 3.5** (Property of  $A_2$  clasps of type  $(m, n)$ ).

$$\left\langle \begin{array}{c} \left[ \begin{array}{c} \left[ \begin{array}{c} n-1 \\ \leftarrow \end{array} \right] \\ \left[ \begin{array}{c} n-1 \\ \rightarrow \end{array} \right] \end{array} \right] \\ m-1 \end{array} \right\rangle_3 = 0.$$

We use the following graphical notations to represent certain  $A_2$  webs.

**Definition 3.6.** For positive integers  $n$  and  $m$ ,

$$\begin{array}{c} \uparrow m \\ \left[ \begin{array}{c} n \\ \rightarrow \\ \left[ \begin{array}{c} n \\ \rightarrow \end{array} \right] \\ \downarrow m \end{array} \right] \in W_{n^+ + m^+ + n^- + m^-} \end{array}$$

is defined as follows:  $\begin{array}{c} \uparrow n \\ \left[ \begin{array}{c} n \\ \rightarrow \\ \left[ \begin{array}{c} n \\ \rightarrow \end{array} \right] \\ \downarrow n \end{array} \right] = \begin{array}{c} \rightarrow \\ \vdots \\ \rightarrow \end{array} \begin{array}{c} \uparrow \\ \vdots \\ \uparrow \end{array} \in W_{n^+ + 1^+ + n^- + 1^-}$  for  $m = 1$ ,

$\begin{array}{c} \uparrow m \\ \left[ \begin{array}{c} n \\ \rightarrow \\ \left[ \begin{array}{c} n \\ \rightarrow \end{array} \right] \\ \downarrow m \end{array} \right] = \begin{array}{c} m-1 \uparrow \\ \left[ \begin{array}{c} n \\ \rightarrow \\ \left[ \begin{array}{c} n \\ \rightarrow \end{array} \right] \\ \downarrow m-1 \end{array} \right] \left[ \begin{array}{c} n \\ \rightarrow \\ \left[ \begin{array}{c} n \\ \rightarrow \end{array} \right] \\ \downarrow n \end{array} \right] \end{array}$  for  $m > 1$ . We also define  $\begin{array}{c} \downarrow m \\ \left[ \begin{array}{c} n \\ \rightarrow \\ \left[ \begin{array}{c} n \\ \rightarrow \end{array} \right] \\ \downarrow m \end{array} \right] \in W_{n^+ + m^- + n^- + m^+}$

in the same way.

**Definition 3.7.** For a positive integer  $n$ ,

$$\begin{array}{c} \left[ \begin{array}{c} n \\ \leftarrow \\ \left[ \begin{array}{c} n \\ \leftarrow \\ \left[ \begin{array}{c} n \\ \leftarrow \end{array} \right] \\ \downarrow n \end{array} \right] \end{array} \right] \in W_{n^+ + n^+ + n^+} \end{array}$$

is defined as follows:  $\begin{array}{c} \left[ \begin{array}{c} n \\ \leftarrow \\ \left[ \begin{array}{c} n \\ \leftarrow \\ \left[ \begin{array}{c} n \\ \leftarrow \end{array} \right] \\ \downarrow n \end{array} \right] \end{array} \right] \text{ for } n = 1, \begin{array}{c} \left[ \begin{array}{c} n \\ \leftarrow \\ \left[ \begin{array}{c} n \\ \leftarrow \\ \left[ \begin{array}{c} n \\ \leftarrow \end{array} \right] \\ \downarrow n \end{array} \right] \end{array} \right] = \begin{array}{c} 1 \\ \leftarrow \\ \left[ \begin{array}{c} n-1 \\ \leftarrow \\ \left[ \begin{array}{c} n-1 \\ \leftarrow \end{array} \right] \\ \downarrow n-1 \end{array} \right] \end{array} \text{ for } n > 1.$

We also define  $\begin{array}{c} \left[ \begin{array}{c} n \\ \leftarrow \\ \left[ \begin{array}{c} n \\ \leftarrow \\ \left[ \begin{array}{c} n \\ \leftarrow \end{array} \right] \\ \downarrow n \end{array} \right] \end{array} \right] \in W_{n^- + n^- + n^-}$  in the same way.

We sometimes omit orientations of  $A_2$  webs when the orientations obviously turn out from the previous  $A_2$  webs.

We review some formulas for the  $A_2$  bracket.

**Lemma 3.8** ([Yua17]).

$$\begin{aligned}
 (1) \quad & \begin{array}{ccc} m_1 & m_2 & m_1+m_2 \\ \uparrow & \uparrow & \uparrow \\ n \rightarrow \square & \square \rightarrow n & n \rightarrow \square \\ \downarrow & \downarrow & \downarrow \\ m_1 & m_2 & m_1+m_2 \end{array} = \begin{array}{ccc} & & n \\ & & \uparrow \\ n \rightarrow \square & \square \rightarrow n & \\ & & \downarrow \\ & & m_1+m_2 \end{array}, \\
 (2) \quad & \begin{array}{ccc} & & n \\ & & \uparrow \\ n \rightarrow \square & \square \rightarrow n & \\ \downarrow & \downarrow & \\ n & n & \end{array}, \\
 (3) \quad & \begin{array}{ccc} \rightarrow \swarrow & \leftarrow m & \\ \rightarrow \square & \rightarrow n & \\ \leftarrow \swarrow & \leftarrow m & \end{array} = \begin{array}{ccc} \rightarrow \swarrow & \leftarrow m & \\ \rightarrow \square & \rightarrow n & \\ \leftarrow \swarrow & \leftarrow m & \end{array}, \\
 (4) \quad & \left\langle \begin{array}{ccc} 1 & \rightarrow & \\ \rightarrow \square & \square \rightarrow & \\ \downarrow & \downarrow & \\ n & n & \end{array} \right\rangle_3 = n \begin{array}{ccc} \rightarrow & & \\ \rightarrow \square & \square \rightarrow & \\ \downarrow & \downarrow & \\ n & n & \end{array} + \sum_{i=0}^{n-1} \begin{array}{ccc} \rightarrow & & n-i-1 \\ \rightarrow \square & \square \rightarrow & \\ \downarrow & \downarrow & \\ n & n & \end{array}, \\
 (5) \quad & \left\langle \begin{array}{ccc} n+m & \rightarrow & \leftarrow m \\ \rightarrow \square & \square \rightarrow & \\ \leftarrow \square & \square \leftarrow & \\ \downarrow & \downarrow & \\ n & n & \end{array} \right\rangle_3 = \left\langle \begin{array}{ccc} n+m & \rightarrow & \leftarrow m \\ \rightarrow \square & \square \rightarrow & \\ \leftarrow \square & \square \leftarrow & \\ \downarrow & \downarrow & \\ n & n & \end{array} \right\rangle_3, \\
 (6) \quad & \left\langle \begin{array}{ccc} n & \rightarrow & \leftarrow n \\ \rightarrow \square & \square \rightarrow & \\ \leftarrow \square & \square \leftarrow & \\ \downarrow & \downarrow & \\ n & n & \end{array} \right\rangle_3 = \left\langle \begin{array}{ccc} n & \rightarrow & \leftarrow n \\ \rightarrow \square & \square \rightarrow & \\ \leftarrow \square & \square \leftarrow & \\ \downarrow & \downarrow & \\ n & n & \end{array} \right\rangle_3.
 \end{aligned}$$

The above equations also hold for the opposite orientations.

**Lemma 3.9.** For  $k = 0, 1, \dots, n$ ,

$$\begin{aligned}
 (1) \quad & \left\langle \begin{array}{ccc} n & \rightarrow & \leftarrow k \\ \rightarrow \square & \square \rightarrow & \\ \leftarrow \square & \square \leftarrow & \\ \downarrow & \downarrow & \\ n & n & \end{array} \right\rangle_3 = q^{\frac{k(n-k)}{3}} \left\langle \begin{array}{ccc} & & n \\ \rightarrow \square & \square \rightarrow & \\ \downarrow & \downarrow & \\ n & n & \end{array} \right\rangle_3 \\
 & \text{and } \left\langle \begin{array}{ccc} n & \rightarrow & \leftarrow k \\ \rightarrow \square & \square \rightarrow & \\ \leftarrow \square & \square \leftarrow & \\ \downarrow & \downarrow & \\ n & n & \end{array} \right\rangle_3 = q^{-\frac{k(n-k)}{3}} \left\langle \begin{array}{ccc} & & n \\ \rightarrow \square & \square \rightarrow & \\ \downarrow & \downarrow & \\ n & n & \end{array} \right\rangle_3, \\
 (2) \quad & \left\langle \begin{array}{ccc} n & \rightarrow & \\ \rightarrow \square & \square \rightarrow & \\ \leftarrow \square & \square \leftarrow & \\ \downarrow & \downarrow & \\ n & n & \end{array} \right\rangle_3 = \frac{[n+1][n+2]}{[n-k+1][n-k+2]} \left\langle \begin{array}{ccc} & & n-k \\ \rightarrow \square & \square \rightarrow & \\ \downarrow & \downarrow & \\ n & n & \end{array} \right\rangle_3, \\
 (3) \quad & \left\langle \begin{array}{ccc} n & \rightarrow & \\ \rightarrow \square & \square \rightarrow & \\ \leftarrow \square & \square \leftarrow & \\ \downarrow & \downarrow & \\ n & n & \end{array} \right\rangle_3 = q^{\frac{n^2+3n}{3}} \left\langle \begin{array}{ccc} & & n \\ \rightarrow \square & \square \rightarrow & \\ \downarrow & \downarrow & \\ n & n & \end{array} \right\rangle_3, \quad \left\langle \begin{array}{ccc} n & \rightarrow & \\ \rightarrow \square & \square \rightarrow & \\ \leftarrow \square & \square \leftarrow & \\ \downarrow & \downarrow & \\ n & n & \end{array} \right\rangle_3 = q^{-\frac{n^2+3n}{3}} \left\langle \begin{array}{ccc} & & n \\ \rightarrow \square & \square \rightarrow & \\ \downarrow & \downarrow & \\ n & n & \end{array} \right\rangle_3.
 \end{aligned}$$

**Proof.** It is easy to prove (1)–(3). See, for example, [OY97]. □

**Lemma 3.10.**

$$\begin{aligned}
 (1) \quad & \left\langle \begin{array}{ccc} n & \rightarrow & \leftarrow n \\ \rightarrow \square & \square \rightarrow & \\ \leftarrow \square & \square \leftarrow & \\ \downarrow & \downarrow & \\ n & n & \end{array} \right\rangle_3 = (-1)^n q^{-\frac{n^2}{6}} \left\langle \begin{array}{ccc} n & \rightarrow & \leftarrow n \\ \rightarrow \square & \square \rightarrow & \\ \leftarrow \square & \square \leftarrow & \\ \downarrow & \downarrow & \\ n & n & \end{array} \right\rangle_3 \\
 & \text{and } \left\langle \begin{array}{ccc} n & \rightarrow & \leftarrow n \\ \rightarrow \square & \square \rightarrow & \\ \leftarrow \square & \square \leftarrow & \\ \downarrow & \downarrow & \\ n & n & \end{array} \right\rangle_3 = (-1)^n q^{\frac{n^2}{6}} \left\langle \begin{array}{ccc} n & \rightarrow & \leftarrow n \\ \rightarrow \square & \square \rightarrow & \\ \leftarrow \square & \square \leftarrow & \\ \downarrow & \downarrow & \\ n & n & \end{array} \right\rangle_3
 \end{aligned}$$

$$(2) \left\langle n \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \right\rangle_3 = (-1)^n q^{-\frac{n^2+3n}{6}} \left\langle n \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \right\rangle_3$$

and  $\left\langle n \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \right\rangle_3 = (-1)^n q^{\frac{n^2+3n}{6}} \left\langle n \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \right\rangle_3.$

The above equations also hold for the opposite orientations.

**Proof.** (1) is derived by Lemma 3.5 and the colored  $A_2$  skein relation in [Yua17] (see Theorem 5.1 in Sect. 5). We only prove the first equation of (2) by induction. It is proven by straightforward calculation for  $n = 1$ . Set  $C_n = (-1)^n q^{-\frac{n^2+3n}{6}}$ ,

$$\begin{aligned} \left\langle n \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \right\rangle_3 &= \left\langle n \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \right\rangle_3 \\ &= C_1 \left\langle n \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \right\rangle_3 = C_1 q^{-\frac{2}{3}(n-1)} \left\langle n \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \right\rangle_3 \\ &= C_1 q^{-\frac{2}{3}(n-1)} (-q^{\frac{1}{6}})^{n-1} \left\langle n \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \right\rangle_3 \\ &= C_1 q^{-\frac{2}{3}(n-1)} (-q^{\frac{1}{6}})^{n-1} C_{n-1} \left\langle n \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \right\rangle_3 \\ &= C_1 q^{-\frac{2}{3}(n-1)} (-q^{\frac{1}{6}})^{n-1} C_{n-1} (-q^{\frac{1}{6}})^{n-1} \left\langle n \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \right\rangle_3 \end{aligned}$$

The last equation is easily derived by using the  $A_2$  skein relation  $n - 1$  times at the crossing. We applied the following calculation to the second line of the above equation.

$$\begin{aligned} \left\langle \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \right\rangle_3 &= \left\langle \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \right\rangle_3 = q^{-\frac{1}{3}} \left\langle \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \right\rangle_3 - q^{\frac{1}{6}} \left\langle \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \right\rangle_3 \\ &= -q^{\frac{1}{6}} \left\langle \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \right\rangle_3 = \dots = (-q^{\frac{1}{6}})^{n-2} \left\langle \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \right\rangle_3 \end{aligned}$$





**Theorem 4.2.**  $[G]_{2n}$  is invariant under the Reidemeister moves (RI) – (RV).

**Lemma 4.3.**

$$\begin{aligned} \bullet \left\langle \begin{array}{c} n \quad n \\ \diagdown \quad \diagup \\ \diagup \quad \diagdown \\ n \quad n \end{array} \right\rangle_3 &= (-1)^n q^{\frac{n^2+3n}{6}} \left\langle \begin{array}{c} \text{trivalent graph} \\ n \quad n \\ n \quad n \end{array} \right\rangle_3 = \left\langle \begin{array}{c} \text{trivalent graph} \\ n \quad n \\ n \quad n \end{array} \right\rangle_3 \\ \bullet \left\langle \begin{array}{c} n \quad n \\ \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ n \quad n \end{array} \right\rangle_3 &= (-1)^n q^{-\frac{n^2+3n}{6}} \left\langle \begin{array}{c} \text{trivalent graph} \\ n \quad n \\ n \quad n \end{array} \right\rangle_3 = \left\langle \begin{array}{c} \text{trivalent graph} \\ n \quad n \\ n \quad n \end{array} \right\rangle_3 \end{aligned}$$

**Proof.** By using the Reidemeister moves for tangled trivalent graph diagrams and Lemma 3.10 (2),

$$\begin{aligned} \left\langle \begin{array}{c} \text{trivalent graph} \\ n \quad n \\ n \quad n \end{array} \right\rangle_3 &= \left\langle \begin{array}{c} \text{trivalent graph} \\ n \quad n \\ n \quad n \end{array} \right\rangle_3 = \left\langle \begin{array}{c} \text{trivalent graph} \\ n \quad n \\ n \quad n \end{array} \right\rangle_3 \\ &= (-1)^n q^{-\frac{n^2+3n}{6}} \left\langle \begin{array}{c} \text{trivalent graph} \\ n \quad n \\ n \quad n \end{array} \right\rangle_3 = (-1)^n q^{-\frac{n^2+3n}{6}} \left\langle \begin{array}{c} \text{trivalent graph} \\ n \quad n \\ n \quad n \end{array} \right\rangle_3 . \end{aligned}$$

The other identities are also proven in the same way. □

**Proof of Theorem 4.2.** The invariance under (RI) – (RIV) is guaranteed by the invariance of  $A_2$  webs under the Reidemeister moves (R1) – (R4) for tangled trivalent graph diagrams. Thus we show the invariance under the first move of (RV):

$$\left[ \begin{array}{c} \text{trivalent graph} \\ \text{blue dot} \end{array} \right]_{2n} = \left[ \begin{array}{c} \text{trivalent graph} \\ \text{blue dot} \end{array} \right]_{2n} , \left[ \begin{array}{c} \text{trivalent graph} \\ \text{blue dot} \end{array} \right]_{2n} = \left[ \begin{array}{c} \text{trivalent graph} \\ \text{blue dot} \end{array} \right]_{2n} , \text{ and } \left[ \begin{array}{c} \text{trivalent graph} \\ \text{blue dot} \end{array} \right]_{2n} = \left[ \begin{array}{c} \text{trivalent graph} \\ \text{blue dot} \end{array} \right]_{2n} .$$

Other cases can be obtained by changing the orientation of the edges or the over/under information at the crossing points in the above diagrams. These cases can be proven in the same way as the proof of the above equations. Therefore, we only show the above three equations. Let us denote the first equation of (2) in Lemma 3.10 by  $C_n = (-1)^n q^{-\frac{n^2+3n}{6}}$ .

$$\left[ \begin{array}{c} \text{trivalent graph} \\ \text{blue dot} \end{array} \right]_{2n} = \left\langle \begin{array}{c} 2n \quad 2n \\ \text{trivalent graph} \\ n \quad n \\ 2n \quad 2n \end{array} \right\rangle_3 = \left\langle \begin{array}{c} \text{trivalent graph} \\ n \quad n \end{array} \right\rangle_3 = \left\langle \begin{array}{c} \text{trivalent graph} \\ n \quad n \end{array} \right\rangle_3$$

$$\begin{aligned}
 &= (q^{\frac{n^2}{3}})^2 (q^{-\frac{n^2}{3}})^2 \left\langle \begin{array}{c} \text{Diagram 1} \\ n \quad n \end{array} \right\rangle_3 = \left\langle \begin{array}{c} \text{Diagram 2} \\ n \quad n \end{array} \right\rangle_3 \\
 &= C_n C_n^{-1} \left\langle \begin{array}{c} \text{Diagram 3} \\ n \quad n \end{array} \right\rangle_3 = \left[ \text{Diagram 4} \right]_{2n}.
 \end{aligned}$$

We used Lemma 3.9 (1) substituting  $n$  for  $2n$  and  $k$  for  $n$  in the second line of the above identities.

$$\begin{aligned}
 \left[ \text{Diagram 4} \right]_{2n} &= \left\langle \begin{array}{c} \text{Diagram 5} \\ 2n \quad 2n \\ n \quad n \\ n \quad n \\ 2n \quad 2n \end{array} \right\rangle_3 = \left\langle \begin{array}{c} \text{Diagram 6} \\ n \quad n \\ n \quad n \end{array} \right\rangle_3 \\
 &= q^{\frac{n^2+3n}{3}} q^{-\frac{n^2+3n}{3}} \left\langle \begin{array}{c} \text{Diagram 7} \\ n \quad n \\ n \quad n \end{array} \right\rangle_3 = (q^{\frac{n^2}{3}})^2 (q^{-\frac{n^2}{3}})^2 \left\langle \begin{array}{c} \text{Diagram 8} \\ n \quad n \\ n \quad n \end{array} \right\rangle_3 \\
 &= \left\langle \begin{array}{c} \text{Diagram 9} \\ n \quad n \\ n \quad n \end{array} \right\rangle_3 = \left\langle \begin{array}{c} \text{Diagram 10} \\ 2n \quad 2n \\ n \quad n \\ n \quad n \\ 2n \quad 2n \end{array} \right\rangle_3 = \left[ \text{Diagram 4} \right]_{2n}.
 \end{aligned}$$

We used Lemma 3.9 (1), (3) and Lemma 4.3 in the second line.

$$\begin{aligned}
 \left[ \text{Diagram 4} \right]_{2n} &= \left\langle \begin{array}{c} \text{Diagram 11} \\ 2n \quad 2n \\ n \quad n \\ n \quad n \\ 2n \quad 2n \end{array} \right\rangle_3 = \left\langle \begin{array}{c} \text{Diagram 12} \\ n \quad n \\ n \quad n \end{array} \right\rangle_3 \\
 &= q^{\frac{n^2+3n}{3}} q^{-\frac{n^2+3n}{3}} \left\langle \begin{array}{c} \text{Diagram 13} \\ n \quad n \end{array} \right\rangle_3 = \left\langle \begin{array}{c} \text{Diagram 14} \\ n \quad n \end{array} \right\rangle_3 \\
 &= (q^{\frac{n^2}{3}})^2 (q^{-\frac{n^2}{3}})^2 \left\langle \begin{array}{c} \text{Diagram 15} \\ n \quad n \\ n \quad n \end{array} \right\rangle_3 = \left[ \text{Diagram 4} \right]_{2n}.
 \end{aligned}$$



□

**Proposition 4.4.**

- We can also define  $[G]_{2n}^{(k)}$  by replacing Definition 4.1 (3) with

$$(3-k) \quad \left[ \begin{array}{c} \text{crossing} \\ \text{blue dot} \end{array} \right]_{2n}^{(k)} = \left\langle \begin{array}{c} 2n \text{ } k \text{ } k \text{ } 2n \\ 2n-k \text{ } \text{ } \text{ } 2n-k \\ 2n \text{ } k \text{ } k \text{ } 2n \end{array} \right\rangle_3 \quad \text{and} \quad \left[ \begin{array}{c} \text{crossing} \\ \text{blue dot} \end{array} \right]_{2n}^{(k)} = \left\langle \begin{array}{c} 2n \text{ } n \text{ } 2n \\ n \text{ } \text{ } \text{ } n \\ 2n \text{ } n \text{ } 2n \end{array} \right\rangle_3,$$

for  $k = 0, 1, \dots, 2n$ . We note that the definition (3-n) agrees with Definition 4.1 (3).

- If  $G$  is a singular link, that is, all 4-valent vertices of  $G$  are , then there is no need to define the bracket for . In this case, we can define the invariant for any positive integer  $m$  and  $0 \leq k \leq m$  by replacing Definition 4.1 (3) by

$$(3-k)' \quad \left[ \begin{array}{c} \text{crossing} \\ \text{blue dot} \end{array} \right]_m^{(k)} = \left\langle \begin{array}{c} m \text{ } k \text{ } k \text{ } m \\ m-k \text{ } \text{ } \text{ } m-k \\ m \text{ } k \text{ } k \text{ } m \end{array} \right\rangle_3.$$

We denote this invariant for a singular link  $G$  by also  $[G]_m^{(k)}$ . This invariant is considered the  $\mathfrak{sl}_3$  colored Jones polynomials for singular links.

**Proof.** We only have to show

$$\left[ \begin{array}{c} \text{crossing} \\ \text{blue dot} \end{array} \right]_m^{(k)} = \left[ \begin{array}{c} \text{crossing} \\ \text{blue dot} \end{array} \right]_m^{(k)}, \quad \left[ \begin{array}{c} \text{crossing} \\ \text{blue dot} \end{array} \right]_m^{(k)} = \left[ \begin{array}{c} \text{crossing} \\ \text{blue dot} \end{array} \right]_m^{(k)}.$$

It is proved in a similar way to the proof of Theorem 4.2 as follows.

$$\begin{aligned} \left[ \begin{array}{c} \text{crossing} \\ \text{blue dot} \end{array} \right]_m^{(k)} &= \left\langle \begin{array}{c} m \text{ } m \\ m-k \text{ } m-k \\ m \text{ } m \end{array} \right\rangle_3 = \left\langle \begin{array}{c} m-k \text{ } m-k \\ m-k \text{ } m-k \end{array} \right\rangle_3 \\ &= (q^{\frac{k(m-k)}{3}})^2 (q^{-\frac{k(m-k)}{3}})^2 \left\langle \begin{array}{c} m-k \text{ } m-k \\ m-k \text{ } m-k \end{array} \right\rangle_3 = \left\langle \begin{array}{c} m-k \text{ } m-k \\ m-k \text{ } m-k \end{array} \right\rangle_3 \\ &= C_k C_k^{-1} \left\langle \begin{array}{c} m-k \text{ } m-k \\ m-k \text{ } m-k \end{array} \right\rangle_3 = \left[ \begin{array}{c} \text{crossing} \\ \text{blue dot} \end{array} \right]_m^{(k)}. \\ \left[ \begin{array}{c} \text{crossing} \\ \text{blue dot} \end{array} \right]_m^{(k)} &= \left\langle \begin{array}{c} m \text{ } k \text{ } k \text{ } m \\ k \text{ } \text{ } \text{ } k \\ m \text{ } k \text{ } k \text{ } m \end{array} \right\rangle_3 = \left\langle \begin{array}{c} k \text{ } k \\ k \text{ } k \end{array} \right\rangle_3 \end{aligned}$$

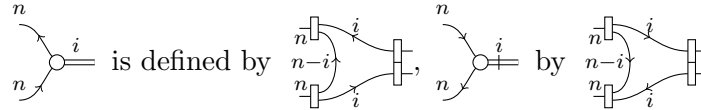
$$\begin{aligned}
 &= q^{\frac{(m-k)^2+3(m-k)}{3}} q^{-\frac{(m-k)^2+3(m-k)}{3}} \left\langle \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right\rangle_3 \\
 &= \left( q^{\frac{k(m-k)}{3}} \right)^2 \left( q^{-\frac{k(m-k)}{3}} \right)^2 \left\langle \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right\rangle_3 = \left\langle \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right\rangle_3 = \left[ \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} \right]_m^{(k)}.
 \end{aligned}$$

We used Lemma 3.9 (1), (3) and Lemma 4.3 in the second line. □

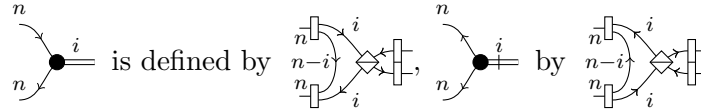
**4.2. Invariant of unoriented 4-valent rigid vertex graphs.** For unoriented 4-valent rigid vertex graph, we will define the invariant using the colored trivalent graphs. Firstly, we represent two types of clasped  $A_2$  web using colored trivalent graphs with white and black vertices. In general, a diagrammatic expression of a colored trivalent graph for a  $A_2$  web is given by Kim [Kim06].

We denote  $\begin{array}{c} n \\ \text{---} \\ n \end{array}$  by  $\text{---}^n$  and  $\begin{array}{c} n \\ \text{---} \\ n \end{array}$  by  $\text{---}_n$ .

**Definition 4.5.** Let  $n$  be a nonnegative integer. For  $0 \leq i \leq n$ , we define two types of trivalent vertices as follows.



and



Let  $\bar{G}$  be an unoriented 4-valent rigid vertex graph diagram.

**Definition 4.6.** We define a polynomial  $[\bar{G}]_{(n,n)}$  by the following rules:

- (1)  $[\text{---}]_{(n,n)} = \left\langle \text{---}^n \right\rangle_3$ ,
- (2)  $\left[ \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right]_{(n,n)} = \left\langle \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right\rangle_3$ ,
- (3)  $\left[ \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right]_{(n,n)} = \left\langle \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right\rangle_3 + \left\langle \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} \right\rangle_3$ .

**Theorem 4.7.**  $[\bar{G}]_{(n,n)}$  is invariant under the Reidemeister moves (RI) – (RV).

**Proof.** We show the invariance under the Reidemeister move (RV).

$$\begin{aligned}
 \left\langle \begin{array}{c} n \\ \text{---} \\ n \\ \text{---} \\ n \end{array} \right\rangle_3 &= \left\langle \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right\rangle_3 = \left\langle \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right\rangle_3 = \left\langle \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right\rangle_3 \\
 &= ((-1)^n q^{\frac{n^2}{6}})^2 ((-1)^n q^{-\frac{n^2}{6}})^2 \left\langle \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right\rangle_3 \\
 &= \left\langle \begin{array}{c} n \\ \text{---} \\ n \\ \text{---} \\ n \end{array} \right\rangle_3
 \end{aligned}$$

The above calculation is similar to the final calculation of the proof of Theorem 4.2. We used Lemma 3.10 (1) in the second line. In the same way, we can show

$$\left\langle \begin{array}{c} n \\ \text{---} \\ n \\ \text{---} \\ n \end{array} \right\rangle_3 = \left\langle \begin{array}{c} n \\ \text{---} \\ n \\ \text{---} \\ n \end{array} \right\rangle_3 .$$

These two identities imply

$$\left\langle \begin{array}{c} n \\ \text{---} \\ n \\ \text{---} \\ n \end{array} \right\rangle_3 = \left\langle \begin{array}{c} n \\ \text{---} \\ n \\ \text{---} \\ n \end{array} \right\rangle_3$$

and

$$\left\langle \begin{array}{c} n \\ \text{---} \\ n \\ \text{---} \\ n \end{array} \right\rangle_3 = \left\langle \begin{array}{c} n \\ \text{---} \\ n \\ \text{---} \\ n \end{array} \right\rangle_3 .$$

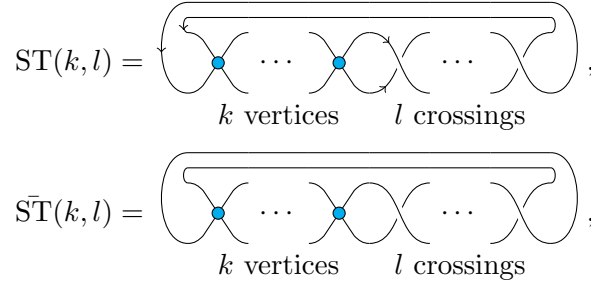
Consequently,

$$\left[ \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \right]_{(n,n)} = \left\langle \begin{array}{c} n \\ \text{---} \\ n \\ \text{---} \\ n \end{array} \right\rangle_3 + \left\langle \begin{array}{c} n \\ \text{---} \\ n \\ \text{---} \\ n \end{array} \right\rangle_3$$

is invariant under the Reidemeister move (RV). □

### 5. Computing the A<sub>2</sub> colored Kauffman–Vogel polynomials

We define the oriented 4-valent rigid vertex graph ST(k, l) and the unoriented 4-valent rigid vertex graph S̄T(k, l) as follows:



Elhamdadi and Hajij computed the one-variable Kauffman–Vogel invariant for the Kauffman bracket of S̄T(k, l) in [EH17a]. We only compute the one-variable Kauffman–Vogel invariant for the A<sub>2</sub> bracket in easy cases.

We use the following formulas to calculate the invariants for some examples. Let us denote a q-Pochhammer symbol by

$$(q)_k = \prod_{l=1}^k (1 - q^l)$$

and a q-binomial coefficient by

$$\binom{n}{k}_q = \frac{(q)_n}{(q)_k (q)_{n-k}}$$

for k ≤ n. We also define a q-multinomial coefficient as

$$\binom{n}{n_1, n_2, \dots, n_m}_q = \frac{(q)_n}{(q)_{n_1} (q)_{n_2} \cdots (q)_{n_m}},$$

where n<sub>1</sub>, n<sub>2</sub>, ..., n<sub>m</sub> are nonnegative integers such that

$$n_1 + n_2 + \cdots + n_m = n.$$

**Theorem 5.1** ([Yua17, Theorem 3.11]). *Let n be a positive integer.*

- (1)  $\left\langle \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right\rangle_3 = \sum_{k=0}^n (-1)^k q^{\frac{2n^2 - 6nk + 3k^2}{6}} \binom{n}{k}_q \left\langle \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right\rangle_3,$
- (2)  $\left\langle \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} \right\rangle_3 = \sum_{k=0}^n (-1)^k q^{\frac{-2n^2 + 3k^2}{6}} \binom{n}{k}_q \left\langle \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array} \right\rangle_3,$
- (3)  $\left\langle \begin{array}{c} \text{Diagram 9} \\ \text{Diagram 10} \end{array} \right\rangle_3 = \sum_{k=0}^n \left\langle \begin{array}{c} \text{Diagram 11} \\ \text{Diagram 12} \end{array} \right\rangle_3,$





**A computation of  $[\overline{\text{ST}}(\mathbf{1}, \mathbf{2l})]_{(n,n)}$ .** We prepare an easy lemma for colored trivalent graphs.

**Lemma 5.4.** For  $0 \leq i \leq n$ ,

$$\left\langle \begin{array}{c} n \\ \bullet \\ n \end{array} \begin{array}{c} i \\ \text{---} \\ i \end{array} \begin{array}{c} n \\ \bullet \\ n \end{array} \right\rangle_3 = \left\langle \begin{array}{c} n \\ \circ \\ n \end{array} \begin{array}{c} i \\ \text{---} \\ i \end{array} \begin{array}{c} n \\ \circ \\ n \end{array} \right\rangle_3 \quad \text{and} \quad \left\langle \begin{array}{c} n \\ \bullet \\ n \end{array} \begin{array}{c} i \\ \text{---} \\ i \end{array} \begin{array}{c} n \\ \bullet \\ n \end{array} \right\rangle_3 = \left\langle \begin{array}{c} n \\ \circ \\ n \end{array} \begin{array}{c} i \\ \text{---} \\ i \end{array} \begin{array}{c} n \\ \circ \\ n \end{array} \right\rangle_3.$$

**Proof.** Thus Lemma follows from

$$\left\langle \begin{array}{c} i \\ \text{---} \\ i \end{array} \begin{array}{c} i \\ \text{---} \\ i \end{array} \begin{array}{c} i \\ \text{---} \\ i \end{array} \begin{array}{c} i \\ \text{---} \\ i \end{array} \right\rangle_3 = \left\langle \begin{array}{c} i \\ \text{---} \\ i \end{array} \begin{array}{c} i \\ \text{---} \\ i \end{array} \begin{array}{c} i \\ \text{---} \\ i \end{array} \begin{array}{c} i \\ \text{---} \\ i \end{array} \right\rangle_3 = \left\langle \begin{array}{c} i \\ \text{---} \\ i \end{array} \begin{array}{c} i \\ \text{---} \\ i \end{array} \right\rangle_3.$$

The first equation is obtained by applying Theorem 5.1 (1) and (2) to the center tangle. We expand the clasp of type  $(i, i)$  in the center tangle by using Definition 3.4 and use Lemme 3.9 (1) and (3). Thus, we obtain the second equation.  $\square$

As in the same computation to the proof of Theorem 4.7, we can see

$$\left\langle \begin{array}{c} n \\ \text{---} \\ n \end{array} \begin{array}{c} n \\ \text{---} \\ n \end{array} \right\rangle_3 = q^{-\frac{2n^2+3n}{3}} \left\langle \begin{array}{c} n \\ \bullet \\ n \end{array} \begin{array}{c} n \\ \text{---} \\ n \end{array} \right\rangle_3$$

and

$$\left\langle \begin{array}{c} n \\ \bullet \\ n \end{array} \begin{array}{c} n \\ \text{---} \\ n \end{array} \right\rangle_3 = q^{-\frac{2n^2+3n}{3}} \left\langle \begin{array}{c} n \\ \circ \\ n \end{array} \begin{array}{c} n \\ \text{---} \\ n \end{array} \right\rangle_3$$

The above equations and Theorem 5.2 derives:

$$\begin{aligned} & \left\langle \begin{array}{c} n \\ \text{---} \\ n \end{array} \begin{array}{c} n \\ \text{---} \\ n \end{array} \dots \begin{array}{c} n \\ \text{---} \\ n \end{array} \right\rangle_3 \\ & \quad \quad \quad 2l \text{ crossings} \\ & = q^{-\frac{2l}{3}(2n^2+3n)} \left\langle \begin{array}{c} n \\ \bullet \\ n \end{array} \begin{array}{c} n \\ \text{---} \\ n \end{array} \dots \begin{array}{c} n \\ \bullet \\ n \end{array} \begin{array}{c} n \\ \text{---} \\ n \end{array} \right\rangle_3 \\ & \quad \quad \quad 2l \text{ crossings} \\ & = q^{-2l(n^2+2n)} \sum_{0 \leq k_1 \leq \dots \leq k_l \leq n} q^{n-k_l} q^{\sum_{i=1}^l (k_i^2+2k_i)} \\ & \quad \times \frac{(q)_n}{(q)_{k_l}} \binom{n}{k'_1, k'_2, \dots, k'_l, k_l}_q \left\langle \begin{array}{c} n \\ \text{---} \\ n \end{array} \begin{array}{c} k_l \\ \text{---} \\ k_l \end{array} \right\rangle_3. \end{aligned}$$





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