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Reducibility and unitary equivalence for a class of truncated Toeplitz operators on model spaces

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Abstract. In this paper we give a necessary and sufficient condition for the reducibility of a truncated Toeplitz operator on model spaces induced by a Blaschke product with two zeros. If the truncated Toeplitz operator is reducible, its restriction on a non-trivial reducing subspace is unitarily equivalent to another truncated Toeplitz operator induced by z.

Contents

1. Introduction

Let $L^2(\mathbb{T})$ and $L^\infty(\mathbb{T})$ denote the usual Lebesgue spaces on the unit circle $\mathbb{T}$. Let H^2 denote the Hardy space on the open unit disk $\mathbb{D}$ and H^∞ denote the space of bounded analytic function on $\mathbb{D}$. For φ in $L^\infty(\mathbb{T})$, the Toeplitz operator T_φ on H^2 is defined by

$$T_\varphi f = P(\varphi f),$$

where P is the orthogonal projection on $L^2(\mathbb{T})$ with range H^2.

To each non-constant inner function θ we associate the model space

$$K_\theta^2 = H^2 \ominus \theta H^2,$$

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which is a reproducing kernel Hilbert space whose kernel is given by

$$k_\lambda^\theta(z) = \frac{1-\overline{\theta(\lambda)}\theta(z)}{1-\overline{\lambda}z},\ \lambda, z \in \mathbb{D}.$$

The truncated Toeplitz operator on K_θ^2 with symbol $\varphi \in L^\infty(\mathbb{T})$ is the operator A_φ defined by

$$A_\varphi f = P_\theta(\varphi f),$$

where P_θ is the orthogonal projection on $L^2(\mathbb{T})$ with range K_θ^2. Truncated Toeplitz operators on model spaces have been formally introduced by Sarason in [9] and this area has undergone vigorous development during the past several years; see [1, 4, 5] and references therein.

Let T be a bounded linear operator on a Hilbert space H. A closed subspace M of H is called a reducing subspace of T if $TM \subset M$ and $T^*M \subset M$. If T has a proper reducing subspace, we say that T is reducible. The classification of invariant subspaces or reducing subspaces of various operators on function space has proven to be a very rewarding research problem in analysis. A lot of nice and deep work on the reducing subspaces of multiplication operators on the Bergman space induced by finite Blaschke products can be found in [11, 7, 3, 6].

It is well known that A_z is irreducible, see for example [5]. However, Ronald G. Douglas and C. Foias [2] showed that A_{z^2} is reducible if and only if either

$$\theta(z) \equiv \theta(-z), \qquad z \in \mathbb{D} \tag{1}$$

or there exists a $\lambda \in \mathbb{D}$ such that

$$\theta(z) = \varphi_\lambda(z)u(z), \tag{2}$$

where $\varphi_\lambda(z) = \frac{\lambda - z}{1-\overline{\lambda}z}$ and $u \in H^\infty$ satisfies

$$u(z) \equiv u(-z), \qquad z \in \mathbb{D}.$$

If θ satisfies (1), we call θ an even function for convenience. Therefore, we see that reducing subspaces of truncated Toeplitz operators display different pictures when compared to the case of Toeplitz operators on the Bergman space.

The main result of this paper is to give a necessary and sufficient condition for the reducibility of A_φ^θ, where φ is a Blaschke product with two zeros. The proof is function theoretical and interesting by itself. If A_φ^θ is reducible, the restriction of A_φ^θ on a proper reducing subspace is unitarily equivalent to A_z^ϕ for some inner function ϕ and we give a complete description of such inner functions ϕ. We need more concepts and notations to state the results.

The space K_θ^2 carries a natural conjugation (an isometric, conjugate-linear, involution, see [9]) defined by $Cf = \theta\overline{zf}$. Then we write

$$\tilde{k}_\lambda^\theta(z) = (Ck_\lambda^\theta)(z) = \frac{\theta(z)-\theta(\lambda)}{z-\lambda}.$$

In particular, $\tilde{k}_0^\theta = T_z^*\theta$. Let M_θ and $M_{\overline{\theta}}$ denote the multiplication operators on $L^2(\mathbb{T})$ induced by θ and $\overline{\theta}$, respectively. We have

$$P_\theta = P - M_\theta P M_{\overline{\theta}}.$$

Set $Q_\theta = M_\theta P M_{\overline{\theta}}$. Then Q_θ is the orthogonal projection on $L^2(\mathbb{T})$ with range θH^2. If it is necessary to specify the inner function θ, we will write A_φ^θ instead of A_φ. It is clear that $A_\varphi^* = A_{\overline{\varphi}}$. In particular, if $\varphi \in H^\infty$, K_θ^2 is an invariant subspace for T_φ^*. It follows that $A_\varphi^* = T_\varphi^* \mid_{K_\theta^2}$. For $\varphi, \psi \in H^\infty$, it is known that $T_\psi^* T_\varphi^* = T_{\varphi\psi}^*$. Thus $A_\psi^* A_\varphi^* = A_{\varphi\psi}^*$ and $A_\varphi A_\psi = A_{\varphi\psi}$.

For $\lambda \in \mathbb{D}$, let φ_λ be the Blaschke factor defined by

$$\varphi_\lambda(z) = \frac{\lambda - z}{1 - \overline{\lambda} z}, \ z \in \mathbb{D}.$$

In what follows, we will assume that $\dim K_\theta^2 \geq 2$, which will be used throughout the paper. The hyperbolic metric on $\mathbb{D}$ is defined by

$$\rho(z, \omega) = \frac{1}{2} \log \frac{1 + |\varphi_z(\omega)|}{1 - |\varphi_z(\omega)|}, \qquad z, \omega \in \mathbb{D}.$$

The hyperbolic metric is invariant under the action of the disk automorphisms. For any two points z and ω in $\mathbb{D}$, the geodesic between z and ω is of minimum length between z and ω in the hyperbolic metric. Furthermore, given any two different points z and ω in $\mathbb{D}$, there exists a unique circle γ through z and ω that is perpendicular to the unit circle. The geodesic between z and ω is then the arc of γ between z and ω that lies inside $\mathbb{D}$. Now given two points a and b in $\mathbb{D}$, there exists a unique geodesic

$$\gamma : (-\infty, +\infty) \mapsto \mathbb{D}$$

such that $\gamma(0) = a$ and $\gamma(1) = b$. The point $\gamma(\frac{1}{2})$ will be called the geodesic midpoint between a and b (see [11] or [12] for more detail).

Definition 1.1. For λ in $\mathbb{C}$, we write

$$M_\lambda^\theta = \overline{\operatorname{span}} \left\{ T_{z^{2n+1}}^* \theta + \lambda T_{z^{2n+2}}^* \theta) : n \geq 0 \right\}.$$

If $\lambda = \infty$, then we define

$$M_\infty^\theta = \overline{\operatorname{span}} \left\{ T_{z^{2n+2}}^* \theta : n \geq 0 \right\}.$$

We can now state the main results of the paper.

Theorem 1.2. *Let $\varphi = \varphi_{\lambda_0}\varphi_{\lambda_1}$ be a Blaschke product with two zeros λ_0 and λ_1 in $\mathbb{D}$. Let p be the geodesic midpoint between λ_0 and λ_1. Then A_φ^θ is reducible on K_θ^2 if and only if $\theta \circ \varphi_p$ satisfies either* (1) *or* (2). *Moreover, if A_φ^θ is reducible and M is a proper reducing subspace of A_φ^θ, then*

(1) *if $\theta \circ \varphi_p$ is even, then there is $\omega \in \hat{\mathbb{C}}$, where $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, such that*

$$M = \left\{ \sqrt{\varphi_p'} \, (f \circ \varphi_p) : f \in M_\omega^{\theta \circ \varphi_p} \right\};$$

(2) *if* $\theta\circ\varphi_p=\varphi_\lambda u$, *where* $\lambda\in\mathbb{D}$ *and* u ($u\in H^\infty$) *is even, we have*

$$M=\left\{\sqrt{\varphi_p'}\,(f\circ\varphi_p):f\in M_\lambda^{\theta\circ\varphi_p}\right\},$$

or

$$M=\left\{\sqrt{\varphi_p'}\,(f\circ\varphi_p):f\in M_{-\frac{1}{\bar\lambda}}^{\theta\circ\varphi_p}\right\}.$$

Theorem 1.3. *Let* φ, p *be defined as above. If* A_φ^θ *is reducible and* M *is any proper reducing subspace of* A_φ^θ, *then there is an inner function* ϕ *such that* $A_\varphi^\theta|_M$ *is unitarily equivalent to* A_z^ϕ. *Moreover,*

(1) *if* $\theta\circ\varphi_p(z)=u(z^2)$ *for some inner function* u, *then* $\phi(z)=u\circ\varphi_0\circ\varphi_{-a}(z)$;
(2) *if* $\theta\circ\varphi_p(z)=\varphi_\lambda(z)u(z^2)$ *for some* $\lambda\in\mathbb{D}$ *and some inner function* u, *then either* $\phi(z)=(\varphi_{\lambda^2}u)\circ\varphi_0\circ\varphi_{-a}(z)$ *or* $\phi(z)=u\circ\varphi_0\circ\varphi_{-a}(z)$, *where* $a=\varphi_p^2(\lambda_0)$.

Since A_z is irreducible, it follows from the Theorem 1.3 that the proper reducing subspaces mentioned in Theorem 1.2 are all minimal.

2. Reducing subspaces of A_{z^2}

Using model theory for C_0 operators (see [10]), Douglas and Foias have given a necessary and sufficient condition for the reducibility of A_{z^2}. The purpose of this section is to present a function theoretic proof of the reducibility of A_{z^2}. Moreover, as a result of this analysis, we obtain a complete description of the reducing subspaces of A_{z^2}.

To start with, we need some new notations.

Definition 2.1. For φ in $L^\infty(\mathbb{T})$, define

$$X_\varphi:\theta H^2\mapsto K_\theta^2,\ X_\varphi f=P_\theta\varphi f,$$

$$Y_\varphi:K_\theta^2\mapsto\theta H^2,\ Y_\varphi f=Q_\theta\varphi f,$$

and

$$D_\varphi:\theta H^2\mapsto\theta H^2,\ D_\varphi f=Q_\theta\varphi f.$$

In particular, if $\varphi\in H^\infty$, then $D_\varphi f=\varphi f$ for $f\in\theta H^2$ and $X_\varphi=0$. Therefore, we have the following decomposition for T_{z^2}:

$$T_{z^2}=\begin{pmatrix}A_{z^2} & 0\\ Y_{z^2} & D_{z^2}\end{pmatrix}\begin{matrix}K_\theta^2\\ \theta H^2.\end{matrix}$$

Lemma 2.2. *For any nonnegative integer* n *and* $f\in K_\theta^2$, *we have*

$$A_{z^n}f=z^nf-\sum_{k=1}^{n}\langle A_{z^{n-k}}f,\tilde{k}_0^\theta\rangle z^{k-1}\theta.$$

Proof. It can be seen in [8, pp.3] for $n = 1$. Now for any nonnegative integer n, since $\{z^k\theta : k \geq 0\}$ is an orthonormal basis for θH^2 and $H^2 = K^2_\theta \oplus \theta H^2$, it follows that for any $f \in K^2_\theta$,

$$\begin{aligned} A_{z^n} f &= z^n f - \sum_{k=0}^{\infty} \langle z^n f, z^k\theta\rangle z^k\theta \\ &= z^n f - \sum_{k=0}^{n-1} \langle z^n f, z^k\theta\rangle z^k\theta \\ &= z^n f - \sum_{k=0}^{n-1} \langle z^{n-k} f, \theta\rangle z^k\theta. \end{aligned}$$

Note that $\tilde{k}^\theta_0 = T^*_z\theta$. So we have

$$A_{z^n} f = z^n f - \sum_{k=1}^{n} \langle z^{n-k} f, \tilde{k}^\theta_0\rangle z^{k-1}\theta,$$

as desired. □

Proposition 2.3. *Suppose* $\dim K^2_\theta \geq 2$. *Then*

$$Y_{z^2} = \theta \otimes A^*_z \tilde{k}^\theta_0 + z\theta \otimes \tilde{k}^\theta_0$$

and $\dim(Y_{z^2} K^2_\theta) = 2$.

Proof. For any $f \in K^2_\theta$, by Lemma 2.2, we have

$$\begin{aligned} Y_{z^2} f &= (T_{z^2} - A_{z^2}) f \\ &= \langle f, A^*_z \tilde{k}^\theta_0\rangle\theta + \langle f, \tilde{k}^\theta_0\rangle z\theta \\ &= (\theta \otimes A^*_z \tilde{k}^\theta_0 + z\theta \otimes \tilde{k}^\theta_0) f. \end{aligned}$$

To prove $\dim(Y_{z^2} K^2_\theta) = 2$, observe that

$$\dim(Y_{z^2} K^2_\theta) \leq 2.$$

If $\dim(Y_{z^2} K^2_\theta) < 2$, then there exist $\lambda_1, \lambda_2 \in \mathbb{C}$, not all zeros, such that for every $f \in K^2_\theta$,

$$\langle Y_{z^2} f, \lambda_1\theta + \lambda_2 z\theta\rangle = 0,$$

that is,

$$\langle f, \lambda_1 A^*_z \tilde{k}^\theta_0 + \lambda_2 \tilde{k}^\theta_0\rangle = 0.$$

Therefore,

$$\lambda_1 A^*_z \tilde{k}^\theta_0 + \lambda_2 \tilde{k}^\theta_0 = 0.$$

Recall that $\tilde{k}^\theta_0$ is a cyclic vector of A^*_z (see [9, Lemma 2.3]). It follows that $\dim K^2_\theta \leq 1$. This is a contradiction and the proof is complete. □

Lemma 2.4. *If M is a proper reducing subspace of A_{z^2}, then for any $f \in M$ and $g \in M^\perp = K^2_\theta \ominus M$ we have $\langle Y_{z^2} f, Y_{z^2} g\rangle = 0$.*

Proof. Since both M and $M^\perp$ are proper reducing subspaces of A_{z^2}, it follows from the fact $T^*_{z^2}|_{K^2_\theta} = A^*_{z^2}$ that

$$\begin{aligned}\langle Y_{z^2}f, Y_{z^2}g\rangle &= \langle (T_{z^2} - A_{z^2})f, (T_{z^2} - A_{z^2})g\rangle \\ &= \langle z^2 f, z^2 g\rangle - \langle T_{z^2}f, A_{z^2}g\rangle - \langle A_{z^2}f, T_{z^2}g\rangle + \langle A_{z^2}f, A_{z^2}g\rangle \\ &= -\langle f, A^*_{z^2}A_{z^2}g\rangle - \langle A^*_{z^2}A_{z^2}f, g\rangle \\ &= 0.\end{aligned}$$

□

Lemma 2.5. *If M is a proper reducing subspace of A_{z^2}, then* $\dim(Y_{z^2}M) = 1$.

Proof. By Proposition 2.3 and Lemma 2.4, it suffices to prove that for any proper reducing subspace M of A_{z^2} we have $\dim(Y_{z^2}M) \neq 0$. Otherwise, assume that $\dim(Y_{z^2}M) = 0$. Then for any $f \in M$,

$$Y_{z^2}f = \langle f, A^*_z \tilde{k}^\theta_0\rangle\theta + \langle f, \tilde{k}^\theta_0\rangle z\theta = 0,$$

which implies that $\langle f, A^*_z\tilde{k}^\theta_0\rangle = 0$ and $\langle f, \tilde{k}^\theta_0\rangle = 0$. Hence both $A^*_z\tilde{k}^\theta_0$ and $\tilde{k}^\theta_0$ are in $M^\perp$. Moreover, since $M^\perp$ reduces $A^*_{z^2}$, we have

$$A^*_{z^{2n+1}}\tilde{k}^\theta_0, \quad A^*_{z^{2n}}\tilde{k}^\theta_0 \in M^\perp$$

for all nonnegative integer n. Since $\tilde{k}^\theta_0$ is a cyclic vector for A^*_z (see [9, Lemma 2.3]), it follows that $M^\perp = K^2_\theta$ and hence $M = \{0\}$. This is a contradiction. □

If A_{z^2} is reducible and M is a proper reducing subspace of A_{z^2}, it follows from Proposition 2.3 and Lemma 2.5 that there exist $\lambda_1, \lambda_2 \in \mathbb{C}$, not all zeros, such that

$$Y_{z^2}M = \overline{\text{span}}\left\{\lambda_1 z\theta + \lambda_2\theta\right\}. \tag{3}$$

Similarly, since $M^\perp$ is also the proper reducing subspace of A_{z^2}, there exist $\lambda_3, \lambda_4 \in \mathbb{C}$, not all zeros, such that

$$Y_{z^2}M^\perp = \overline{\text{span}}\left\{\lambda_3 z\theta + \lambda_4\theta\right\}. \tag{4}$$

Moreover, Lemma 2.4 yields

$$\lambda_1\overline{\lambda}_3 + \lambda_2\overline{\lambda}_4 = 0. \tag{5}$$

Clearly, $\lambda_1 \neq 0$ or $\lambda_3 \neq 0$. Without loss of generality, we assume that $\lambda_1 \neq 0$. By (5), $\lambda_4 \neq 0$, since if $\lambda_4 = 0$, then $\lambda_3 = 0$, this is impossible. Let $\omega = \lambda_2/\lambda_1$. Together, (3), (4) and (5) give that

$$\begin{cases} Y_{z^2}M = \overline{\text{span}}\{z\theta + \omega\theta\}, \\ Y_{z^2}M^\perp = \overline{\text{span}}\{-\overline{\omega}z\theta + \theta\}. \end{cases}$$

Again, from Lemma 2.4, it follows that

$$\begin{cases} \langle Y_{z^2} f, -\overline{\omega} z\theta + \theta \rangle = 0, & f \in M, \\ \langle Y_{z^2} g, z\theta + \omega\theta \rangle = 0, & g \in M^\perp. \end{cases}$$

A calculation shows that

$$\begin{cases} \langle f, -\overline{\omega}\tilde{k}_0^\theta + A_z^* \tilde{k}_0^\theta \rangle = 0, & f \in M, \\ \langle g, \tilde{k}_0^\theta + \omega A_z^* \tilde{k}_0^\theta \rangle = 0, & g \in M^\perp. \end{cases}$$

This implies that

$$\begin{cases} \tilde{k}_0^\theta + \omega A_z^* \tilde{k}_0^\theta \in M, \\ -\overline{\omega}\tilde{k}_0^\theta + A_z^* \tilde{k}_0^\theta \in M^\perp. \end{cases}$$

Therefore,

$$M \supset M_\omega^\theta,\ M^\perp \supset M^\theta_{-\frac{1}{\overline{\omega}}}.$$

Note that for each nonnegative integer k,

$$\begin{cases} A^*_{z^{2k}} \tilde{k}_0^\theta = \frac{1}{1+|\omega|^2} A^*_{z^{2k}} (\tilde{k}_0^\theta + \omega A_z^* \tilde{k}_0^\theta) - \frac{\omega}{1+|\omega|^2} A^*_{z^{2k}} (-\overline{\omega}\tilde{k}_0^\theta + A_z^* \tilde{k}_0^\theta), \\ A^*_{z^{2k+1}} \tilde{k}_0^\theta = \frac{\overline{\omega}}{1+|\omega|^2} A^*_{z^{2k}} (\tilde{k}_0^\theta + \omega A_z^* \tilde{k}_0^\theta) + \frac{1}{1+|\omega|^2} A^*_{z^{2k}} (-\overline{\omega}\tilde{k}_0^\theta + A_z^* \tilde{k}_0^\theta). \end{cases}$$

Thus we have

$$A^*_{z^k} \tilde{k}_0^\theta \in M_\omega^\theta + M^\theta_{-\frac{1}{\overline{\omega}}} \tag{6}$$

for all nonnegative integer k. Furthermore, since $\tilde{k}_0^\theta$ is a cyclic vector of A_z^*, we get that

$$M = M_\omega^\theta,\ M^\perp = M^\theta_{-\frac{1}{\overline{\omega}}}. \tag{7}$$

Proposition 2.6. *A_{z^2} is reducible on K_θ^2 if and only if there exists $\omega \in \mathbb{C} \cup \{\infty\}$ such that for all nonnegative integers m and n,*

$$\langle A^*_{z^{2n}} (\tilde{k}_0^\theta + \omega A_z^* \tilde{k}_0^\theta), A^*_{z^{2m}} (-\overline{\omega}\tilde{k}_0^\theta + A_z^* \tilde{k}_0^\theta) \rangle = 0.$$

Proof. For the case $\omega = \infty$, the equations

$$\langle A^*_{z^{2n}} (\tilde{k}_0^\theta + \omega A_z^* \tilde{k}_0^\theta), A^*_{z^{2m}} (-\overline{\omega}\tilde{k}_0^\theta + A_z^* \tilde{k}_0^\theta) \rangle = 0,\ n, m \geq 0$$

mean that

$$\langle A^*_{z^{2n+1}} \tilde{k}_0^\theta, A^*_{z^{2m}} \tilde{k}_0^\theta \rangle = 0,\ n, m \geq 0,$$

which is the same as the case $\omega = 0$. Therefore we only need to prove the case $\omega \in \mathbb{C}$.

As shown above, we have proved the necessity of the proposition. For the sufficiency, let

$$N_1 = M_\omega^\theta,\ N_2 = M^\theta_{-\frac{1}{\overline{\omega}}}.$$

It is not hard to see that $N_i \neq \{0\}$ for $i = 1, 2$, since if not, then $\tilde{k}_0^\theta + \omega A_z^* \tilde{k}_0^\theta = 0$ or $-\overline{\omega}\tilde{k}_0^\theta + A_z^* \tilde{k}_0^\theta = 0$. Since $\tilde{k}_0^\theta$ is cyclic vector of A_z^*, we have that $\dim K_\theta^2 \leq 1$ and this is a contradiction.

Since for any nonnegative integers m and n,

$$\langle A^*_{z^{2n}}(\tilde{k}^\theta_0 + \omega A^*_z \tilde{k}^\theta_0), A^*_{z^{2m}}(-\overline{\omega}\tilde{k}^\theta_0 + A^*_z \tilde{k}^\theta_0)\rangle = 0,$$

we conclude that $N_1 \perp N_2$. Moreover, (6) and the fact that $\tilde{k}^\theta_0$ is a cyclic vector of A^*_z give that $N_1 \oplus N_2 = K^2_\theta$. Clearly, both N_1 and N_2 are invariant subspaces of $A^*_{z^2}$. Therefore, A_{z^2} is reducible. □

Proposition 2.6 can be improved as follows.

Corollary 2.7. *A_{z^2} is reducible on K^2_θ if and only if there is a complex number ω, $|\omega| \le 1$, such that for any nonnegative integers m and n,*

$$\langle T^*_{z^{2n+2}}(z+\omega)\theta, T^*_{z^{2m+2}}(-\overline{\omega}z+1)\theta\rangle = 0. \tag{8}$$

Proof. Sufficiency is obvious. Conversely, if A_{z^2} is reducible on K^2_θ, then by Proposition 2.6, there is $\lambda \in \mathbb{C} \cup \{\infty\}$ such that for all nonnegative integers m and n,

$$\langle T^*_{z^{2n+2}}(z+\lambda)\theta, T^*_{z^{2m+2}}(-\overline{\lambda}z+1)\theta\rangle = 0.$$

Case 1: if $|\lambda| \le 1$, then the proof is complete.

Case 2: if $|\lambda| > 1$, then let $\omega = -\frac{1}{\lambda}$ ($\omega = 0$ if $\lambda = \infty$) and $|\omega| < 1$. Hence

$$\langle T^*_{z^{2n+2}}(-\overline{\omega}z+1)\theta, T^*_{z^{2m+2}}(z+\omega)\theta\rangle = 0,$$

for all nonnegative integers m and n. This proves the desired result. □

To deal with equation (8), we need the following lemma.

Lemma 2.8. *Let $\theta(z) = \sum_{k=0}^\infty a_k z^k$ be an inner function. Then for any positive integer n,*

$$\sum_{k=0}^\infty a_k \overline{a}_{k+n} = 0.$$

Proof. It follows easily from the fact that $\langle z^n\theta, \theta\rangle = 0$ for any positive integer n. □

Now we can characterize the inner functions θ such that (8) is valid.

Proposition 2.9. *Let $\omega \in \overline{\mathbb{D}}$. We have that*

(1) *if $|\omega| = 1$, then (8) is valid if and only if θ is even.*
(2) *if $|\omega| < 1$, then (8) is valid if and only if θ is even or $\theta = \varphi_\omega u$, where $u \in H^\infty$ is even.*

Proof. Let $\omega \in \overline{\mathbb{D}}$ and

$$\theta(z) = \sum_{k=0}^\infty a_k z^k.$$

For convenience, let

$$A(n,m) = \sum_{k=0}^\infty a_{k+n}\overline{a}_{k+m}.$$

Now assume that (8) is valid, that is, for all nonnegative integers m and n,

$$\langle T^*_{z^{2n+2}}(z+\omega)\theta, T^*_{z^{2m+2}}(-\overline{\omega}z+1)\theta\rangle = 0.$$

From Taylor expansion, it follows that this formula is equivalent to

$$\begin{aligned}&-\omega A(2n+1,2m+1)-\omega^2 A(2n+2,2m+1)\\&+A(2n+1,2m+2)+\omega A(2n+2,2m+2)=0,\end{aligned}$$

which can be written as

$$-\omega a_{2n+1}\overline{a}_{2m+1}-\omega^2 A(2n+2,2m+1)+A(2n+1,2m+2)=0. \tag{9}$$

Interchanging m with n and taking conjugates, we get

$$-\overline{\omega}a_{2n+1}\overline{a}_{2m+1}-\overline{\omega}^2 A(2n+1,2m+2)+A(2n+2,2m+1)=0. \tag{10}$$

$(9)+\omega^2\times(10)$ gives that

$$-\omega a_{2n+1}\overline{a}_{2m+1}+(1-|\omega|^2)A(2n+1,2m+2)=0. \tag{11}$$

Replace m by $m-1$. Then

$$-\omega a_{2n+1}\overline{a}_{2m-1}+(1-|\omega|^2)A(2n+1,2m)=0. \tag{12}$$

Similarly, $\overline{\omega}^2\times(9)+(10)$ gives that

$$-\overline{\omega}a_{2n+1}\overline{a}_{2m+1}+(1-|\omega|^2)A(2n+2,2m+1)=0. \tag{13}$$

Combining (12), (13) and the fact that

$$A(2n+2,2m+1)=A(2n+1,2m)-a_{2n+1}\overline{a}_{2m},$$

we obtain

$$a_{2n+1}(-\omega\overline{a}_{2m-1}+(1-|\omega|^2)\overline{a}_{2m}+\overline{\omega a}_{2m+1})=0,\ n\ge 0 \tag{14}$$

for $m>0$. For (13), in particular, the condition $m=0$ and Lemma 2.8 yield

$$a_{2n+1}(\overline{\omega a}_1+(1-|\omega|^2)\overline{a}_0)=0,\ n\ge 0. \tag{15}$$

For (1), since $|\omega|=1$, by (11), we conclude that

$$-\omega a_{2n+1}\overline{a}_{2m+1}=0$$

for all nonnegative integers m and n. Let $m=n$. Then $a_{2n+1}=0$ for any n, which implies that θ is even. Conversely, if θ is even, then clearly, (9) is valid and hence (8) is valid. Thus (1) is proved.

Now we consider (2) and let $|\omega|<1$.

Case I: $a_{2n+1}=0$ for all nonnegative integers n, then θ is even.

Case II: There is a nonnegative integer n such that $a_{2n+1}\neq 0$. Then (14) and (15) give

$$\begin{cases}-\overline{\omega}a_{2t-1}+(1-|\omega|^2)a_{2t}+\omega a_{2t+1}=0,\ \text{for } t\ge 1,\\(1-|\omega|^2)a_0+\omega a_1=0,\end{cases} \tag{16}$$

which can be written as

$$\begin{cases} \overline{\omega}(a_{2t-1}+\omega a_{2t}) = a_{2t}+\omega a_{2t+1}, \text{ for } t\geq 1, \\ -(\overline{\omega}a_{2t-1}-a_{2t}) = \omega(\overline{\omega}a_{2t}-a_{2t+1}), \text{ for } t\geq 1, \\ a_0+\omega a_1 = |\omega|^2 a_0. \end{cases} \tag{17}$$

((17) will be used in section 4).

In what follows, we shall show that in this case, $\theta=\varphi_\omega u$ for some even inner function u. Write

$$\theta(z)=\theta_1(z^2)+z\theta_2(z^2),\ z\in\mathbb{D}. \tag{18}$$

Obviously,

$$\theta_1(z)=\frac{\theta(\sqrt{z})+\theta(-\sqrt{z})}{2},\ z\in\mathbb{D}$$

and

$$\theta_2(z)=\begin{cases}\frac{\theta(\sqrt{z})-\theta(-\sqrt{z})}{2\sqrt{z}}, & 0\neq z\in\mathbb{D}\\ \theta'(0), & z=0.\end{cases}$$

Then (16) is equivalent to

$$-\overline{\omega}z\theta_2+(1-|\omega|^2)\theta_1+\omega\theta_2=0,\qquad z\in\mathbb{D}.$$

This can be written as

$$\theta(z)(z+\omega)(1-\overline{\omega}z)-\theta(-z)(\omega-z)(1+\overline{\omega}z)=0.$$

Since $|\omega|<1$, we have

$$\theta(z)\varphi_{-\omega}(z)=\theta(-z)\varphi_\omega(z).$$

If $\omega=0$, then $\theta(z)+\theta(-z)=0$ and hence θ is an odd inner function. If $0<|\omega|<1$, then $\theta(\omega)=0$ and let $\theta=\varphi_\omega u$ for some inner function u, we can check that u is even.

For the other direction, assume that θ is even or $\theta=\varphi_\omega u$, where $u\in H^\infty$ is even and $|\omega|<1$.

If θ is even. It is easy to check that (9) is valid and hence (8) is valid.

If $\theta=\varphi_\omega u$, where $u\in H^\infty$ is even and $|\omega|<1$. Obviously, u is an inner function and $K_u^2\subset K_\theta^2$, then we have

$$\begin{aligned} T_z^*\theta &= \frac{\theta-\theta(0)}{z} \\ &= \frac{\varphi_\omega u-\omega u+\omega u-\omega u(0)}{z} \\ &= -\frac{1-|\omega|^2}{1-\overline{\omega}z}u+T_z^*u. \end{aligned} \tag{19}$$

Similarly,

$$T_{z^2}^*\theta=-(1-|\omega|^2)(\frac{\overline{\omega}u}{1-\overline{\omega}z}+T_z^*u)+\omega T_{z^2}^*u.$$

Therefore,

$$T_z^*\theta + \omega T_{z^2}^*\theta = -(1-|\omega|^4)\frac{u}{1-\overline{\omega}z} + \omega|\omega|^2 T_z^* u + \omega^2 T_{z^2}^* u, \tag{20}$$

and

$$-\overline{\omega}T_z^*\theta + T_{z^2}^*\theta = -T_z^* u + \omega T_{z^2}^* u. \tag{21}$$

Note that

$$T_z^*\frac{u}{1-\overline{\omega}z} = \frac{\overline{\omega}u}{1-\overline{\omega}z} + T_z^* u.$$

Hence for any nonnegative integer n,

$$\begin{aligned} T_{z^{2n}}^*\frac{u}{1-\overline{\omega}z} &= \frac{\overline{\omega}^{2n}u}{1-\overline{\omega}z} + \sum_{k=0}^{2n-1}\overline{\omega}^{2n-1-k}T_{z^{k+1}}^* u \\ &= \frac{\overline{\omega}^{2n}u}{1-\overline{\omega}z} + \sum_{j=0}^{n-1}\overline{\omega}^{2n-2j-2}T_{z^{2j+2}}^*(\overline{\omega}z+1)u. \end{aligned} \tag{22}$$

When u is even, for each $\omega \in \mathbb{D}$, it follows fromTaylor expansion that

$$\langle T_{z^{2n+2}}^*(-z+\omega)u, T_{z^{2m+2}}^*(\overline{\omega}z+1)u\rangle = 0,$$

for all nonnegative integers m and n. Hence (20)-(22) and the fact $\frac{u}{1-\overline{\omega}z} \in uH^2$ yield that

$$\langle T_{z^{2n+2}}^*(z+\omega)u, T_{z^{2m+2}}^*(-\overline{\omega}z+1)u\rangle = 0,$$

for all nonnegative integers m and n. This completes the proof of proposition. □

We are now ready to state the main result of this section.

Theorem 2.10. *A_{z^2} is reducible on K_θ^2 if and only if θ satisfies either* (1) *or* (2). *Moreover, if A_{z^2} is reducible and M is a proper reducing subspace of A_{z^2}, then*

(1) *if θ is even, there is $\omega \in \hat{\mathbb{C}}$ ($\hat{\mathbb{C}} = \mathbb{C}\cup\{\infty\}$) such that $M = M_\omega^\theta$.*
(2) *if $\theta = \varphi_\lambda u$, where $u \in H^\infty$ is even and $\lambda \in \mathbb{D}$, then $M = M_\lambda^\theta$ or $M = M_{-\frac{1}{\overline{\lambda}}}^\theta$.*

Proof. This follows from Corollary 2.7, Proposition 2.9, and (7). □

3. Reducing subspaces of $A_{\varphi_{\lambda_0}\varphi_{\lambda_1}}^\theta$

In this section, by using Theorem 2.10, we will prove the Theorem 1.2.

For $\lambda \in \mathbb{D}$, recall that φ_λ is defined by

$$\varphi_\lambda(z) = \frac{\lambda - z}{1-\overline{\lambda}z}, \quad z \in \mathbb{D}.$$

If we set

$$U_{\varphi_\lambda} : K_\theta^2 \to K_{\theta\circ\varphi_\lambda}^2,\ U_{\varphi_\lambda} f = \sqrt{\varphi_\lambda'}(f\circ\varphi_\lambda),\ \sqrt{\varphi_\lambda'(z)} = i\frac{\sqrt{1-|z|^2}}{1-\overline{\lambda}z},$$

then a calculation (for example, see [1, Proposition 4.1]) shows that U_{φ_λ} is a unitary transformation and for $g \in K^2_{\theta\circ\varphi_\lambda}$,

$$\begin{aligned} U^*_{\varphi_\lambda} g &= \sqrt{(\varphi_\lambda^{-1})'}(g \circ \varphi_\lambda^{-1}) \\ &= \sqrt{\varphi'_\lambda}(g \circ \varphi_\lambda). \end{aligned} \tag{23}$$

Furthermore, if $\varphi \in L^\infty$,

$$U_{\varphi_\lambda} A^\theta_\varphi U^*_{\varphi_\lambda} = A^{\theta\circ\varphi_\lambda}_{\varphi\circ\varphi_\lambda}.$$

In particular,

$$U_{\varphi_\lambda} A^\theta_{\varphi_\lambda} U^*_{\varphi_\lambda} = A^{\theta\circ\varphi_\lambda}_z.$$

To prove Theorem 1.2, we need the following lemma.

Lemma 3.1. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic function. Then any reducing subspace of A^θ_φ is also a reducing subspace of $A^\theta_{\varphi_\lambda\circ\varphi}$.*

Proof. Let X be a reducing subspace of A^θ_φ. By Taylor expansion for φ_λ, each invariant subspace of A^θ_φ is also invariant under $A^\theta_{\varphi_\lambda\circ\varphi}$. This means that both X and $X^\perp$ are invariant under $A^\theta_{\varphi_\lambda\circ\varphi}$ and the proof is complete. □

We will now prove Theorem 1.2.

If $\lambda_1 = -\lambda_0$, then the geodesic midpoint between λ_0 and λ_1 is 0, so $p = 0$ and

$$\varphi(z) = -\varphi_{\lambda_0^2}(z^2), \qquad z^2 = \varphi_{\lambda_0^2}(-\varphi(z)).$$

By Lemma 3.1, A^θ_φ and $A^\theta_{z^2}$ have the same reducing subspaces.

Now for any λ_0 and λ_1, define $\phi = \varphi \circ \varphi_p$. By Lemma 9 of [11], there exists a unimodulus constant c such that

$$\phi = \varphi \circ \varphi_p = c\varphi_{\varphi_p(\lambda_0)}\varphi_{\varphi_p(\lambda_1)}.$$

Since p is the geodesic midpoint between λ_0 and λ_1 and the hyperbolic metric is invariant under disk automorphisms (see [12, pp.67]), the geodesic midpoint between $\varphi_p(\lambda_0)$ and $\varphi_p(\lambda_1)$ is $\varphi_p(p) = 0$. It is well known that any geodesic through the origin is a diameter. Hence $\varphi_p(\lambda_0) = -\varphi_p(\lambda_1)$. As shown above, $A^{\theta\circ\varphi_p}_\phi$ and $A^{\theta\circ\varphi_p}_{z^2}$ have the same reducing subspaces. By Theorem 2.10, we conclude that $A^{\theta\circ\varphi_p}_\phi$ is reducible on $K^2_{\theta\circ\varphi_p}$ if and only if $\theta \circ \varphi_p$ satisfies either (1) or (2). Furthermore,

Case I: if $\theta \circ \varphi_p$ is even, the proper reducing subspace M of $A^{\theta\circ\varphi_p}_\phi$ is of the form $M^{\theta\circ\varphi_p}_\omega$ for some $\omega \in \hat{\mathbb{C}}$.

Case II: if $\theta \circ \varphi_p = \varphi_\lambda u$, where $u \in H^\infty$ is even and $\lambda \in \mathbb{D}$, then $A^{\theta\circ\varphi_p}_\phi$ has only two proper reducing subspaces which are $M^{\theta\circ\varphi_p}_\lambda$ and $M^{\theta\circ\varphi_p}_{-\frac{1}{\bar\lambda}}$.

Recall that

$$U_{\varphi_p} A^\theta_\varphi U^*_{\varphi_p} = A^{\theta\circ\varphi_p}_{\varphi\circ\varphi_p} = A^{\theta\circ\varphi_p}_\phi.$$

It follows that A_{φ}^{θ} is reducible on K_{θ}^{2} if and only if $A_{\phi}^{\theta\circ\varphi_p}$ is reducible on $K_{\theta\circ\varphi_p}^{2}$. Moreover,

Case I: $\theta \circ \varphi_p$ is even. The proper reducing subspace M of A_{φ}^{θ} is

$$U_{\varphi_p}^{*} M_{\omega}^{\theta\circ\varphi_p} = \left\{ \sqrt{\varphi_p'}(f \circ \varphi_p) : f \in M_{\omega}^{\theta\circ\varphi_p} \right\},$$

for some $\omega \in \hat{\mathbb{C}}$.

Case II: $\theta \circ \varphi_p = \phi_\lambda u$, where u ($u \in H^\infty$) is even and $\lambda \in \mathbb{D}$. Then A_{φ}^{θ} has only two proper reducing subspaces:

$$U_{\varphi_p}^{*} M_{\lambda}^{\theta\circ\varphi_p} = \left\{ \sqrt{\varphi_p'}(f \circ \varphi_p) : f \in M_{\lambda}^{\theta\circ\varphi_p} \right\},$$

and

$$U_{\varphi_p}^{*} M_{-\frac{1}{\bar{\lambda}}}^{\theta\circ\varphi_p} = \left\{ \sqrt{\varphi_p'}(f \circ \varphi_p) : f \in M_{-\frac{1}{\bar{\lambda}}}^{\theta\circ\varphi_p} \right\}.$$

This completes the proof of Theorem 1.2.

4. Unitary equivalence

In this section, we shall prove Theorem 1.3 which rests on the following proposition.

Proposition 4.1. *Let $\varphi = \varphi_a\varphi_{-a}$ for some $a \in \mathbb{D}$. If A_{φ}^{θ} is reducible and M is a reducing subspace of A_{φ}^{θ}, Then there is an inner function ϕ such that $A_{\varphi}^{\theta}\mid_M$ is unitarily equivalent to $A_{-\varphi_{a^2}}^{\phi}$.*

Proof. It suffices to show that there is an inner function ϕ such that $(A_{\varphi}^{\theta})^{*}\mid_M$ is unitarily equivalent to $(A_{-\varphi_{a^2}}^{\phi})^{*}$. First of all, note that φ is even, so let φ and θ have Taylor expansions

$$\varphi(z) = \sum_{k=0}^{\infty} c_{2k} z^{2k}, qquad \theta(z) = \sum_{k=0}^{\infty} a_k z^k,\ z \in \mathbb{D}.$$

Clearly,

$$-\varphi_{a^2}(z) = \varphi(\sqrt{z}) = \sum_{k=0}^{\infty} c_{2k} z^k.$$

It follows from the proof of Theorem 1.2 that A_{φ}^{θ} and $A_{z^2}^{\theta}$ have the same reducing subspaces. Therefore, by Theorem 2.10, A_{φ}^{θ} is reducible if and only if θ satisfies either (1) or (2), and the reducing subspaces of A_{φ}^{θ} are of the following form:

1. if θ is even, then there is $\omega \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ such that $M = M_{\omega}^{\theta}$.
2. if $\theta = \varphi_\lambda u$, where u ($u \in H^\infty$) is even and $\lambda \in \mathbb{D}$, then $M = M_{\lambda}^{\theta}$ or $M = M_{-\frac{1}{\bar{\lambda}}}^{\theta}$.

For an inner function ϕ, write

$$F_\omega^\theta = \begin{cases} T_{z^2}^*(z+\omega)\theta, & \omega \in \mathbb{C}, \\ T_{z^2}^*\theta, & \omega = \infty, \end{cases} \quad G_\omega^\phi = \begin{cases} (1+|\omega|^2)^{\frac{1}{2}} T_z^*\phi, & \omega \in \mathbb{C}, \\ T_z^*\phi, & \omega = \infty. \end{cases}$$

Then for each nonnegative integer n, we define

$$U_\omega : (A_{z^{2n}}^\theta)^* F_\omega^\theta \mapsto (A_{z^n}^\phi)^* G_\omega^\phi. \tag{24}$$

We claim that if the map U_ω from M_ω^θ to K_ϕ^2, defined densely as in (24), is an isometry, then it is a unitary such that

$$U_\omega (A_\varphi^\theta)^* U_\omega^* = (A_{-\varphi_{a^2}}^\phi)^*.$$

In fact, we see that U_ω is a unitary by the facts that $\{(A_{z^{2n}}^\theta)^* F_\omega^\theta : n \geq 0\}$ is dense in M_ω^θ and $\{(A_{z^n}^\phi)^* G_\omega^\phi : n \geq 0\}$ is dense in K_ϕ^2.

Note that φ is analytic on a neighborhood of $\overline{\mathbb{D}}$. Formal series manipulation gives that

$$\begin{aligned} U_\omega((A_\varphi^\theta)^*((A_{z^{2n}}^\theta)^* F_\omega^\theta)) &= \sum_{k=0}^\infty \overline{c}_{2k} U_\omega((A_{z^{2k+2n}}^\theta)^* F_\omega^\theta) \\ &= \sum_{k=0}^\infty \overline{c}_{2k} (A_{z^{k+n}}^\phi)^* G_\omega^\phi \\ &= (A_{-\varphi_{a^2}}^\phi)^* (A_{z^n}^\phi)^* G_\omega^\phi, \end{aligned}$$

which implies

$$U_\omega (A_\varphi^\theta)^* U_\omega^* = (A_{-\varphi_{a^2}}^\phi)^*.$$

Thus the claim is proved.

To end the proof, by the above claim, it suffices to construct an inner function ϕ such that U_ω is an isometry.

Case I: θ is even. Then we set

$$\phi(z) = \theta(\sqrt{z}), \ z \in \mathbb{D}.$$

For each nonnegative integer n,

$$\begin{aligned} (A_{z^{2n}}^\theta)^* F_\omega^\theta(z) &= \begin{cases} T_{z^{2n+1}}^*\theta + \omega T_{z^{2n+2}}^*\theta, & \omega \in \mathbb{C}, \\ T_{z^{2n+2}}^*\theta, & \omega = \infty, \end{cases} \\ &= \begin{cases} \sum\limits_{k=0}^\infty a_{2n+2k+2} z^{2k+1} + \omega \sum\limits_{k=0}^\infty a_{2n+2k+2} z^{2k}, & \omega \in \mathbb{C}, \\ \sum\limits_{k=0}^\infty a_{2n+2k+2} z^{2k}, & \omega = \infty, \end{cases} \end{aligned} \tag{25}$$

and

$$(A^{\phi}_{z^n})^* G^{\phi}_{\omega}(z) = \begin{cases} (1+|\omega|^2)^{\frac{1}{2}} T^*_{z^{n+1}}\phi, & \omega \in \mathbb{C}, \\ T^*_{z^{n+1}}\phi, & \omega = \infty, \end{cases}$$

$$= \begin{cases} (1+|\omega|^2)^{\frac{1}{2}} \sum_{k=0}^{\infty} a_{2n+2k+2} z^{2k}, & \omega \in \mathbb{C}, \\ \sum_{k=0}^{\infty} a_{2n+2k+2} z^{2k}, & \omega = \infty, \end{cases} \tag{26}$$

which implies that

$$\langle (A^{\theta}_{z^{2n}})^* F^{\theta}_{\omega}, (A^{\theta}_{z^{2m}})^* F^{\theta}_{\omega} \rangle = \langle (A^{\phi}_{z^n})^* G^{\phi}_{\omega}, (A^{\phi}_{z^m})^* G^{\phi}_{\omega} \rangle, \text{ for } m, n \geq 0, \tag{27}$$

and therefore U_ω is an isometry.

Case II: $\theta = \varphi_\lambda u$, where $u \in H^\infty$ is even and $\lambda \in \mathbb{D}$. Then A^{θ}_{φ} has only two proper reducing subspaces M^{θ}_{λ} and $M^{\theta}_{-\frac{1}{\overline{\lambda}}}$.

For M^{θ}_{λ}, we set

$$\begin{aligned} \phi(z) &= \lambda\theta_1(z) + z\theta_2(z) \\ &= \varphi_{\lambda^2}(z) u(\sqrt{z}), \ z \in \mathbb{D}. \end{aligned} \tag{28}$$

Obviously, ϕ is inner. Note that for $z \in \mathbb{D}$, by (17), we obtain

$$\begin{aligned} (A^{\theta}_{z^{2n}})^* F^{\theta}_{\lambda}(z) &= \sum_{k=0}^{\infty} a_{2n+1+k} z^k + \lambda \sum_{k=0}^{\infty} a_{2n+2+k} z^k \\ &= \sum_{k=0}^{\infty} (a_{2n+1+k} + \lambda a_{2n+2+k}) z^k \\ &= \sum_{k=0}^{\infty} (a_{2n+1+2k} + \lambda a_{2n+2+2k}) z^{2k} \\ &+ \sum_{k=0}^{\infty} (a_{2n+2+2k} + \lambda a_{2n+3+2k}) z^{2k+1} \\ &= \sum_{k=0}^{\infty} (a_{2n+1+2k} + \lambda a_{2n+2+2k}) z^{2k} \\ &+ \overline{\lambda} \sum_{k=0}^{\infty} (a_{2n+1+2k} + \lambda a_{2n+2+2k}) z^{2k+1}. \end{aligned} \tag{29}$$

A simple calculation shows that

$$\begin{aligned} (A^{\phi}_{z^n})^* G^{\phi}_{\lambda}(z) &= (1+|\lambda|^2)^{\frac{1}{2}} T^*_{z^{n+1}}(z\theta_2 + \lambda\theta_1)(z) \\ &= (1+|\lambda|^2)^{\frac{1}{2}} \sum_{k=0}^{\infty} (a_{2n+1+2k} + \lambda a_{2n+2+2k}) z^k. \end{aligned}$$

This means that (27) is also valid.

For $M_{-\frac{1}{\overline{\lambda}}}$, define

$$\phi(z) = u(\sqrt{z}),\ z \in \mathbb{D}.$$

Clearly, ϕ is inner.

Case 1): $\lambda = 0$. Then $\theta = -zu$. It follows that for all nonnegative integers n,

$$(A^{\theta}_{z^{2n}})^* F^{\theta}_{\infty} = \sum_{k=0}^{\infty} a_{2n+3+2k} z^{2k+1}, \tag{30}$$

and

$$(A^{\phi}_{z^{n}})^* G^{\phi}_{\infty} = -\sum_{k=0}^{\infty} a_{2n+3+2k} z^{k}. \tag{31}$$

Combining (30) with (31), it is not hard to see that (27) is valid.

Case 2): $0 < |\lambda| < 1$. A routine computation shows that

$$\begin{aligned}\phi(z) &= u(\sqrt{z})\\ &= \frac{-\overline{\lambda} z\theta_2(z) + \theta_1(z)}{\lambda},\ z \in \mathbb{D}.\end{aligned}$$

For $z \in \mathbb{D}$, combining (17) with (29) (λ is replaced by $-\frac{1}{\overline{\lambda}}$), we have

$$\begin{aligned}(A^{\theta}_{z^{2n}})^* F^{\theta}_{-\frac{1}{\overline{\lambda}}}(z) &= \sum_{k=0}^{\infty}(a_{2n+1+2k} - \frac{1}{\overline{\lambda}} a_{2n+2+2k}) z^{2k}\\ &\quad - \frac{1}{\lambda}\sum_{k=0}^{\infty}(a_{2n+1+2k} - \frac{1}{\overline{\lambda}} a_{2n+2+2k}) z^{2k+1}.\end{aligned}$$

On a different note,

$$\begin{aligned}(A^{\phi}_{z^{n}})^* G^{\phi}_{-\frac{1}{\overline{\lambda}}}(z) &= (1 + \frac{1}{|\lambda|^2})^{\frac{1}{2}} T^*_{z^{n+1}}(\frac{-\overline{\lambda} z\theta_2(z) + \theta_1(z)}{\lambda})(z)\\ &= -(1 + \frac{1}{|\lambda|^2})^{\frac{1}{2}} \frac{\overline{\lambda}}{\lambda}\sum_{k=0}^{\infty}(a_{2n+1+2k} - \frac{1}{\overline{\lambda}} a_{2n+2+2k}) z^{k},\ z \in \mathbb{D}.\end{aligned}$$

Therefore, (27) is valid and thus we complete the proof of Proposition 4.1. □

We proceed to prove Theorem 1.3.

Let p be the geodesic midpoint between λ_0 and λ_1. By the proof of the Theorem 1.2, there exists a unimodulus constant c such that

$$\varphi \circ \varphi_p = c\varphi_{\varphi_p(\lambda_0)}\varphi_{-\varphi_p(\lambda_0)},$$

which leads to

$$U_{\varphi_p} A^{\theta}_{\varphi} U^*_{\varphi_p} = A^{\theta\circ\varphi_p}_{\varphi\circ\varphi_p} = cA^{\theta\circ\varphi_p}_{\varphi_{\varphi_p(\lambda_0)}\varphi_{-\varphi_p(\lambda_0)}}.$$

Therefore, $A_{\varphi}^{\theta}\mid_M$ is unitarily equivalent to $A^{\theta\circ\varphi_p}_{\varphi_{\varphi_p(\lambda_0)}\varphi_{-\varphi_p(\lambda_0)}}\mid_{U_{\varphi_p}M}$ which is unitarily equivalent to $A^{\psi}_{-\varphi_{(\varphi_p(\lambda_0))^2}}$ for some inner function ψ by Proposition 4.1.

Let

$$a = (\varphi_p(\lambda_0))^2$$

and

$$\phi = \psi \circ \varphi_0 \circ \varphi_{-a}.$$

Note that

$$-\varphi_a \circ \varphi_0 \circ \varphi_{-a}(z) = z.$$

It follows from

$$U_{\varphi_{-a}} U_{\varphi_0} A^{\psi}_{-\varphi_a} U^*_{\varphi_0} U^*_{\varphi_{-a}} = A_z^{\psi\circ\varphi_0\circ\varphi_{-a}} = A_z^{\phi}$$

that $A_{\varphi}^{\theta}\mid_M$ is unitarily equivalent to A_z^{ϕ} and Theorem 1.3 is proved.

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References

[1] Cima, Joseph A.; Garcia, Stephan Ramon; Ross, William T.; Wogen, Warren R. Truncated Toeplitz operators: spatial isomorphism, unitary equivalence, and similarity. *Indiana Univ. Math. J.* **59** (2010), no. 2, 595–620. MR2648079, Zbl 1215.47024, arXiv:0907.2489, doi: 10.1512/iumj.2010.59.4097. 930, 940

[2] Douglas, Ronald G.; Foias, Ciprian. On the structure of the square of a $C_0(1)$ operator. *Modern operator theory and applications*, 75–84, Oper. Theory Adv. Appl., 170. *Birkhäuser, Basel,* 2007. MR2279383, Zbl 1119.47010, arXiv:math/0508347. 930

[3] Douglas, Ronald G.; Putinar, Mihai; Wang, Kai. Reducing subspaces for analytic multipliers of the Bergman space. *J. Funct. Anal.* **263** (2012), no. 6, 1744–1765. MR2948229, Zbl 1275.47071, arXiv:1110.4920, doi: 10.1016/j.jfa.2012.06.008. 930

[4] Garcia, Stephan Ramos; Ross, William T. Recent progress on truncated Toeplitz operators. *Blaschke products and their applications*, 275–319, Fields Inst. Commun., 65. *Springer, New York*, 2013. MR3052299, Zbl 1277.47040, arXiv:1108.1858, doi: 10.1007/978-1-4614-5341-3_15. 930

[5] Garcia, Stephan Ramon; Ross, William T.; Wogen, Warren R. C^*-algebras generated by truncated Toeplitz operators. *Concrete operators, spectral theory, operators in harmonic analysis and approximation*, 181–192, Oper. Theory Adv. Appl., 236. *Birkhäuser/Springer, Basel*, 2014. MR3203060, Zbl 1323.47082, arXiv:1203.2412, doi: 10.1007/978-3-0348-0648-0_11. 930

[6] Guo, Kunyu; Huang, Hansong. Multiplication operators on the Bergman space. Lecture Notes in Mathematics, 2145. *Springer, Heidelberg*, 2015. viii+322 pp. ISBN: 978-3-662-46844-9; 978-3-662-46845-6. MR3363367, Zbl 1321.47002, doi: 10.1007/978-3-662-46845-6. 930

[7] Guo, Kunyu; Sun, Shunhua; Zheng, Dechao; Zhong, Changyong. Multiplication operators on the Bergman space via the Hardy space of the bidisk. *J. Reine Angew. Math.* **628** (2009), 129–168. MR2503238, Zbl 1216.47055, doi: 10.1515/CRELLE.2009.021. 930

[8] Sarason, Donald. Angular derivatives via Hilbert space. *Complex Variables Theory Appl.* **10** (1988), no. 1, 1–10. MR0946094, Zbl 0635.30024, doi: 10.1080/17476938808814282. 933

[9] Sarason, Donald. Algebraic properties of truncated Toeplitz operators. *Oper. Matrices* **1** (2007), no. 4, 491–526. MR2363975, Zbl 1144.47026, doi: 10.7153/oam-01-29. 930, 933, 934

[10] Sz.-Nagy, Béla; Foias, Ciprian; Bercovici, Hari; Kérchy, László. Harmonic analysis of operators on Hilbert space. Second edition. Revised and enlarged edition. Universitext. *Springer, New York,* 2010. xiv+474 pp. ISBN: 978-1-4419-6093-1. MR2760647, Zbl 1234.47001, doi: 10.1007/978-1-4419-6094-8. 932

[11] Zhu, Kehe. Reducing subspaces for a class of multiplication operators. *J. London Math. Soc.* (2) **62** (2000), no. 2, 553–568. MR1783644, Zbl 1158.47309, doi: 10.1112/S0024610700001198. 930, 931, 940

[12] Zhu, Kehe. Operator theory in function spaces. Second edition. *Mathematical Surveys and Monographs*, 138. *American Mathematical Society, Providence, RI*, 2007. xvi+348 pp. ISBN: 978-0-8218-3965-2. MR2311536, Zbl 1123.47001, doi: 10.1090/surv/138. 931, 940

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