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On the spaces of bounded and compact multiplicative Hankel operators

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Abstract. A multiplicative Hankel operator is an operator with matrix representation $M(\alpha) = \{\alpha(nm)\}_{n,m=1}^{\infty}$, where α is the generating sequence of $M(\alpha)$. Let $\mathcal{M}$ and $\mathcal{M}_0$ denote the spaces of bounded and compact multiplicative Hankel operators, respectively. In this note it is shown that the distance from an operator $M(\alpha) \in \mathcal{M}$ to the compact operators is minimized by a nonunique compact multiplicative Hankel operator $N(\beta) \in \mathcal{M}_0$. Intimately connected with this result, it is then proven that the bidual of $\mathcal{M}_0$ is isometrically isomorphic to $\mathcal{M}$, $\mathcal{M}_0^{**} \simeq \mathcal{M}$. It follows that $\mathcal{M}_0$ is an M-ideal in $\mathcal{M}$. The dual space $\mathcal{M}_0^*$ is isometrically isomorphic to a projective tensor product with respect to Dirichlet convolution. The stated results are also valid for small Hankel operators on the Hardy space $H^2(\mathbb{D}^d)$ of a finite polydisk.

Contents

1. Introduction

Given a sequence $\alpha\colon \mathbb{N} \to \mathbb{C}$, we consider the corresponding multiplicative Hankel operator $m = M(\alpha)\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, defined by

$$\langle M(\alpha)a, b\rangle_{\ell^2(\mathbb{N})} = \sum_{n,m=1}^{\infty} a(n)\overline{b(m)}\alpha(nm), \quad a, b \in \ell^2(\mathbb{N}).$$

Initially, we consider this equality only for finite sequences a and b. It defines a bounded operator $M(\alpha)\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$, with matrix representation $\{\alpha(nm)\}_{n,m=1}^{\infty}$ in the standard basis of $\ell^2(\mathbb{N})$, if and only if there is a constant $C > 0$ such that

$$\left|\langle M(\alpha)a, b\rangle_{\ell^2(\mathbb{N})}\right| \le C\|a\|_{\ell^2(\mathbb{N})}\|b\|_{\ell^2(\mathbb{N})}, \quad a, b \text{ finite sequences.}$$

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Multiplicative Hankel operators are also known as Helson matrices, having been introduced by Helson in [14, 15].

There are two common alternative interpretations. One is in terms of Dirichlet series. Let $\mathcal{H}^2$ be the Hardy space of Dirichlet series, the Hilbert space with $(n^{-s})_{n=1}^{\infty}$ as a basis. Elements $f \in \mathcal{H}^2$ are holomorphic functions in the half-plane $\{s \in \mathbb{C} : \operatorname{Re} s > 1/2\}$. If

$$f(s) = \sum_{n=1}^{\infty} a(n)n^{-s},\ g(s) = \sum_{n=1}^{\infty} \overline{b(n)}n^{-s},\ \rho(s) = \sum_{n=1}^{\infty} \overline{\alpha(n)}n^{-s},$$

then

$$\langle M(\alpha)a, b\rangle_{\ell^2(\mathbb{N})} = \langle fg, \rho\rangle_{\mathcal{H}^2}.$$

Hence there is an isometric correspondence between Helson matrices and Hankel operators on $\mathcal{H}^2$, since the forms associated with the latter are precisely of the type $(f, g) \mapsto \langle fg, \rho\rangle_{\mathcal{H}^2}$.

The second interpretation is in terms of the Hardy space of the infinite polytorus $H^2(\mathbb{T}^\infty)$, the Hilbert space with basis $(z^\kappa)_\kappa$, where $z = (z_1, z_2, \ldots)$, and $\kappa = (\kappa_1, \kappa_2, \ldots)$ runs through the countably infinite, but finitely supported, multi-indices. Identify each integer n with a multi-index κ of this type through the factorization of n into the primes $p_1, p_2, \ldots$,

$$n \longleftrightarrow \kappa \text{ if and only if } n = \prod_{j=1}^{\infty} p_j^{\kappa_j}.$$

Under this equivalence, multiplicative Hankel operators correspond to additive Hankel operators on a countably infinite number of variables,

$$\langle M(\alpha)a, b\rangle_{\ell^2(\mathbb{N})} = \sum_{\kappa,\kappa'} a(\kappa)\overline{b(\kappa')}\alpha(\kappa + \kappa').$$

Hence the multiplicative Hankel operators correspond isometrically to small Hankel operators on $H^2(\mathbb{T}^\infty)$, since the matrix representations of the latter are of the form $\{\alpha(\kappa + \kappa')\}_{\kappa,\kappa'}$. See [14, 15] for details.

In particular, the Helson matrices generalize the small Hankel operators on the Hardy space of any finite polytorus $H^2(\mathbb{T}^d)$, $d < \infty$. In fact, the results in this note have analogous statements for small Hankel operators on $H^2(\mathbb{T}^d)$; every proof given remains valid verbatim after restricting the number of prime factors, that is, the number of variables.

The first result is the following. We denote by $\mathcal{B}(\ell^2(\mathbb{N}))$ and $\mathcal{K}(\ell^2(\mathbb{N}))$, respectively, the spaces of bounded and compact operators on $\ell^2(\mathbb{N})$.

Theorem 1.1. *Let $M(\alpha)$ be a bounded multiplicative Hankel operator. Then there exists a compact multiplicative Hankel operator $N(\beta)$ such that*

$$\|M(\alpha) - N(\beta)\|_{\mathcal{B}(\ell^2(\mathbb{N}))} = \inf\left\{\|M(\alpha) - K\|_{\mathcal{B}(\ell^2(\mathbb{N}))} : K \in \mathcal{K}(\ell^2(\mathbb{N}))\right\}. \quad (1)$$

The minimizer $N(\beta)$ is never unique, unless $M(\alpha)$ is compact.

The quantity on the right-hand side of (1) is known as the essential norm of $M(\alpha)$. For classical Hankel operators on $H^2(\mathbb{T})$, this result was proven by Axler, Berg, Jewell, and Shields in [6], and can be viewed as a limiting case of the theory of Adamjan, Arov, and Krein [1]. The demonstration of Theorem 1.1 requires only a minor modification of the arguments in [6], the main point being that a characterization of the class of bounded multiplicative Hankel operators is not necessary for the proof.

On $H^2(\mathbb{T})$, Nehari's theorem [21] states that the class of bounded Hankel operators can be isometrically identified with $L^\infty(\mathbb{T})/H^\infty(\mathbb{T})$, where $L^\infty(\mathbb{T})$ and $H^\infty(\mathbb{T})$ denote the spaces of bounded and bounded analytic functions on $\mathbb{T}$, respectively. By Hartman's theorem [13], the class of compact Hankel operators is isometrically isomorphic to $(H^\infty(\mathbb{T}) + C(\mathbb{T}))/H^\infty(\mathbb{T})$, where $C(\mathbb{T})$ denotes the space of continuous functions on $\mathbb{T}$. Note that the spaces L^∞, H^∞, and $H^\infty + C$ are all algebras, as proven by Sarason [26].

Luecking [20] observed, through a very illustrative argument relying on function algebra techniques, that the compact Hankel operators form an M-ideal in the space of bounded Hankel operators. The concept of an M-ideal will be defined shortly, but let us note for now that M-ideality implies proximinality; the distance from a bounded Hankel operator to the compact Hankel operators has a minimizer. Thus Luecking reproved some of the results of [6]. Since

$$((H^\infty + C)/H^\infty)^{**} \simeq L^\infty/H^\infty,$$

it follows that the bidual of the space of compact Hankel operators is isometrically isomorphic to the space of bounded Hankel operators. Spaces which are M-ideals in their biduals are said to be M-embedded.

The multiplicative Hankel operators, on the other hand, have thus far resisted all attempts to characterize their boundedness. It has been shown that a Nehari-type theorem cannot exist [22], and positive results only exist in special cases [14, 24]. In spite of this, the main theorem shows that Luecking's result holds for multiplicative Hankel operators.

Let

$$\mathcal{M}_0 = \{m = M(\alpha) \,:\, M(\alpha)\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) \text{ compact}\}$$

and

$$\mathcal{M} = \{m = M(\alpha) \,:\, M(\alpha)\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) \text{ bounded}\}.$$

Equipped with the operator norm, $\mathcal{M}_0$ and $\mathcal{M}$ are closed subspaces of $\mathcal{K}(\ell^2(\mathbb{N}))$ and $\mathcal{B}(\ell^2(\mathbb{N}))$, respectively. For a Banach space Y, we denote by ι_Y the canonical embedding $\iota_Y\colon Y \to Y^{**}$,

$$\iota_Y y(y^*) = y^*(y), \quad y \in Y,\ y^* \in Y^*.$$

Theorem 1.2. *There is a unique isometric isomorphism $U\colon \mathcal{M}_0^{**} \to \mathcal{M}$ such that $U\iota_{\mathcal{M}_0} m = m$ for every $m \in \mathcal{M}_0$. Furthermore, $\mathcal{M}_0$ is an M-ideal in $\mathcal{M}$.*

Remark. As pointed out earlier, Theorem 1.2 is also true when stated for small Hankel operators on $H^2(\mathbb{T}^d)$, $d < \infty$. The biduality has in this case been demonstrated isomorphically in [18], with an argument based on the non-isometric Nehari-type theorems proven in [10, 17].

The M-ideal property means the following: there is an (onto) projection $L\colon \mathcal{M}^* \to \mathcal{M}_0^\perp$ such that

$$\|m^*\|_{\mathcal{M}^*} = \|Lm^*\|_{\mathcal{M}^*} + \|m^* - Lm^*\|_{\mathcal{M}^*}, \quad m^* \in \mathcal{M}^*,$$

where $\mathcal{M}_0^\perp$ denotes the space of functionals $m^* \in \mathcal{M}^*$ which annihilate $\mathcal{M}_0$. M-ideals were introduced by Alfsen and Effros [3] as a Banach space analogue of closed two-sided ideals in C^*-algebras. Very loosely speaking, the fact that $\mathcal{M}_0$ is an M-ideal in $\mathcal{M}$ implies that the norm of $\mathcal{M}$ resembles a maximum norm and, in this analogy, that $\mathcal{M}_0$ is the subspace of elements vanishing at infinity. The book [12] comprehensively treats M-structure theory and its applications.

We will make use of the following consequences of Theorem 1.2. Proximinality of $\mathcal{M}_0$ in $\mathcal{M}$ was already mentioned, but the M-ideal property also implies that the minimizer is never unique [16]. It also ensures that $\mathcal{M}_0^*$ is a strongly unique predual of $\mathcal{M}$ [12, Proposition III.2.10]. This means that every isometric isomorphism of $\mathcal{M}$ onto Y^*, Y a Banach space, is weak*-weak* continuous, that is, arises as the adjoint of an isometric isomorphism of Y onto $\mathcal{M}_0^*$. On the other hand, $\mathcal{M}_0^*$ has infinitely many different preduals [11, Theoreme 27].

The predual of $\mathcal{M}$ is well known to have an almost tautological characterization as a projective tensor product with respect to Dirichlet convolution,

$$\mathcal{X} = \ell^2(\mathbb{N}) \mathbin{\hat{\star}} \ell^2(\mathbb{N}).$$

The space $\mathcal{X}$ is also referred to as a weak product space. We defer the precise definition to the next section – after establishing the main theorems, we essentially show, following [25], that all reasonable definitions of $\mathcal{X}$ coincide.

Theorem 1.3. *There is an isometric isomorphism $L\colon \mathcal{X} \to \mathcal{M}_0^*$ such that $L^*U^{-1}\colon \mathcal{M} \to \mathcal{X}^*$ is the canonical isometric isomorphism of $\mathcal{M}$ onto $\mathcal{X}^*$, where $U\colon \mathcal{M}_0^{**} \to \mathcal{M}$ is the isometric isomorphism of Theorem 1.2.*

Informally stated, $\mathcal{M}_0^* \simeq \mathcal{X}$ and $\mathcal{X}^* \simeq \mathcal{M}$. Theorem 1.3 follows at once from Theorem 1.2 and the uniqueness of the predual of $\mathcal{M}$, but we also supply a direct proof. While the duality $\mathcal{X}^* \simeq \mathcal{M}$ is a rephrasing of the definition of $\mathcal{M}$, it is difficult to identify a common approach to dualities of the type $\mathcal{M}_0^* \simeq \mathcal{X}$ in the existing literature. Often, the latter duality is deduced (isomorphically) via a concrete description of $\mathcal{M}$. For a small selection of relevant examples, see [4, 8, 12, 18, 19, 23, 28].

The idea behind this note is that the direct view of $\mathcal{M}$ as a subspace of $\mathcal{B}(\ell^2(\mathbb{N}))$ already provides sufficient information to prove Theorems 1.1, 1.2, and 1.3. In this direction, Wu [28] worked with an embedding into the space of bounded operators to deduce duality results for certain Hankel-type forms on Dirichlet spaces.

The proofs of the results only have two main ingredients. The first is a device to approximate elements of $\mathcal{M}$ by elements of $\mathcal{M}_0$ (Lemma 2.1). Such an approximation property is necessary, because if $\mathcal{M}_0^{**} \simeq \mathcal{M}$, then the unit ball of $\mathcal{M}_0$ is weak* dense in the unit ball of $\mathcal{M}$. The second ingredient is an inclusion of $\mathcal{M}$ into a reflexive space; in our case, $\ell^2(\mathbb{N})$. Analogous theorems could be proven for many other linear spaces of bounded and compact operators using the same technique.

2. Results

For a sequence a and $0 < r < 1$, let

$$D_r a(n) = r^{\sum_{j=1}^{\infty} j\kappa_j} a(n), \text{ where } n = \prod_{j=1}^{\infty} p_j^{\kappa_j}.$$

Note that

$$\sum_{\kappa} r^{2\sum_{j=1}^{\infty} j\kappa_j} = \prod_{j=1}^{\infty} \frac{1}{1-r^{2j}} < \infty.$$

Hence it follows by the dominated convergence theorem that $D_r \colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is a compact operator. Furthermore, D_r is self-adjoint and contractive, $\|D_r\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \le 1$. The dominated convergence theorem also implies that $D_r \to \mathrm{id}_{\ell^2(\mathbb{N})}$ in the strong operator topology (SOT) as $r \to 1$, that is, $\lim_{r\to 1} D_r a = a$ in $\ell^2(\mathbb{N})$, for every $a \in \ell^2(\mathbb{N})$. A study of the operators D_r in the context of Hardy spaces of the infinite polytorus can be found in [2].

The Dirichlet convolution of two sequences a and b is the new sequence $a \star b$ given by

$$(a \star b)(n) = \sum_{k|n} a(k)\overline{b(n/k)}, \quad n \in \mathbb{N}.$$

If a and b are two finite sequences, then

$$\langle M(\alpha)a, b\rangle_{\ell^2(\mathbb{N})} = (\alpha, a \star b), \tag{2}$$

where $(a, b) = \sum_{n=1}^{\infty} a(n)b(n)$ denotes the bilinear pairing between $a, b \in \ell^2(\mathbb{N})$. Note also that, for $0 < r < 1$,

$$D_r(a \star b) = D_r a \star D_r b. \tag{3}$$

The following simple lemma is key.

Lemma 2.1. *Let $M(\alpha)$ be a bounded multiplicative Hankel operator, $M(\alpha) \in \mathcal{M}$. For $0 < r < 1$, let $\alpha_r = D_r\alpha$. Then $M_{\alpha_r} \in \mathcal{M}_0$,*

$$\|M_{\alpha_r}\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \le \|M_\alpha\|_{\mathcal{B}(\ell^2(\mathbb{N}))},$$

and $M_{\alpha_r} \to M_\alpha$ and $M_{\alpha_r}^* \to M_\alpha^*$ SOT as $r \to 1$.

Proof. By (2) and (3), it holds for finite sequences a and b that

$$\langle M(\alpha_r)a, b\rangle_{\ell^2(\mathbb{N})} = \langle M_\alpha D_r a, D_r b\rangle_{\ell^2(\mathbb{N})}.$$

Hence $M_{\alpha_r} = D_r M_\alpha D_r$. We conclude that M_{α_r} is compact, $\|M_{\alpha_r}\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \leq \|M_\alpha\|_{\mathcal{B}(\ell^2(\mathbb{N}))}$, and $M_{\alpha_r} \to M_\alpha$ SOT as $r \to 1$. Similarly, $M_{\alpha_r}^* = M_{\overline{\alpha}_r} \to M_{\overline{\alpha}} = M_\alpha^*$ SOT as $r \to 1$. $\square$

The following is a recognizable consequence, cf. [27, Theorem 1]. Note that if S_n and T_n are operators such that $S_n \to S$ and $T_n \to T$ SOT, and if C is a compact operator, then $S_n C T_n^* \to SCT^*$ in operator norm.

Proposition 2.2. *Let $M(\alpha) \in \mathcal{M}$. Then $M(\alpha) \in \mathcal{M}_0$ if and only if*

$$\lim_{r\to 1} \|M(\alpha_r) - M(\alpha)\|_{\mathcal{B}(\ell^2(\mathbb{N}))} = 0. \tag{4}$$

Proof. If (4) holds, then $M(\alpha) \in \mathcal{M}_0$, since $M(\alpha_r)$ is compact for every $0 < r < 1$. If $M(\alpha) \in \mathcal{M}_0$, then (4) holds, since $M(\alpha_r) = D_r M(\alpha) D_r = D_r M(\alpha) D_r^*$ and $D_r \to \mathrm{id}_{\ell^2(\mathbb{N})}$ SOT as $r \to 1$. $\square$

Recall next the main tool from [6].

Theorem 2.3 ([6])**.** *Let $T\colon \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be a non-compact operator and (T_n) a sequence of compact operators such that $T_n \to T$ SOT and $T_n^* \to T^*$ SOT. Then there exists a sequence (c_n) of non-negative real numbers such that $\sum_n c_n = 1$ for which the compact operator*

$$J = \sum_n c_n T_n$$

satisfies

$$\|T - J\|_{\mathcal{B}(\ell^2(\mathbb{N}))} = \inf\left\{\|T - K\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \,:\, K \in \mathcal{K}(\ell^2(\mathbb{N}))\right\}.$$

Lemma 2.1 and Theorem 2.3 immediately yield the existence part of Theorem 1.1.

Proof of Theorem 1.1. Let $M(\alpha)$ be a bounded multiplicative Hankel operator and let (r_k) be a sequence such that $0 < r_k < 1$ and $r_k \to 1$. Then $M(\alpha)$ has a best compact approximant of the form

$$N = \sum_k c_k M(\alpha_{r_k}).$$

But then $N = N(\beta)$ is a multiplicative Hankel operator, $\beta = \sum_k c_k \alpha_{r_k}$.

The non-uniqueness of $N(\beta)$ follows immediately once we have established Theorem 1.2, by general M-ideal results [16]. In fact, if $M(\alpha) \notin \mathcal{M}_0$, then the set of minimizers $N(\beta)$ is so large that it spans $\mathcal{M}_0$. $\square$

Note that

$$\|M(\alpha)\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \geq \varlimsup_{N\to\infty} \frac{1}{\|(\alpha(n))_{n=1}^N\|_{\ell^2(\mathbb{N})}} \sum_{n=1}^N |\alpha(n)|^2 = \|\alpha\|_{\ell^2(\mathbb{N})}.$$

Therefore the inclusion $I\colon \mathcal{M}_0 \to \ell^2(\mathbb{N})$ is a contractive operator, $Im = I(M(\alpha)) = \alpha$. We can state Theorem 1.2 slightly more precisely in terms of I.

Theorem 1.2. *Consider the bitranspose $U = I^{**}\colon \mathcal{M}_0^{**} \to \ell^2(\mathbb{N})$. Then $U\mathcal{M}_0^{**} = \mathcal{M}$, viewing $\mathcal{M}$ as a (non-closed) subspace of $\ell^2(\mathbb{N})$. Furthermore,*

$$U\iota_{\mathcal{M}_0} m = m, \quad m \in \mathcal{M}_0,$$

and

$$\|Um^{**}\|_{\mathcal{B}(\ell^2(\mathbb{N}))} = \|m^{**}\|_{\mathcal{M}_0^{**}}, \quad m^{**} \in \mathcal{M}_0^{**}.$$

If $V\colon \mathcal{M}_0^{**} \to \mathcal{M}$ is another isometric isomorphism such that $V\iota_{\mathcal{M}_0} m = m$ for all $m \in \mathcal{M}_0$, then $V = U$. Furthermore, $\mathcal{M}_0$ is an M-ideal in $\mathcal{M}$.

Proof. We identify $\left(\ell^2(\mathbb{N})\right)^* \simeq \ell^2(\mathbb{N})$ linearly through the pairing $(a, b) = \sum_{n=1}^{\infty} a(n)b(n)$ between $a, b \in \ell^2(\mathbb{N})$. With this convention, $I^*\colon \ell^2(\mathbb{N}) \to \mathcal{M}_0^*$ is also contractive, and

$$I^*a(m) = (\alpha, a), \quad a \in \ell^2(\mathbb{N}),\ m = M(\alpha) \in \mathcal{M}_0.$$

Since I is injective, I^* has dense range. In particular, $\mathcal{M}_0^*$ is separable. Furthermore, $I^{**}\colon \mathcal{M}_0^{**} \to \ell^2(\mathbb{N})$ is injective. By the reflexivity of $\ell^2(\mathbb{N})$, we have that $I^{**}\iota_{\mathcal{M}_0} = I$, since

$$(I^{**}\iota_{\mathcal{M}_0} m, a) = \iota_{\mathcal{M}_0} m(I^*a) = (\alpha, a) = (Im, a)$$

for every $m = M(\alpha) \in \mathcal{M}_0$ and $a \in \ell^2(\mathbb{N})$. The interpretation, viewing $\mathcal{M}$ as a non-closed subspace of $\ell^2(\mathbb{N})$, is that $I^{**}\iota_{\mathcal{M}_0} m = m$, for all $m \in \mathcal{M}_0$.

Consider any $m^{**} \in \mathcal{M}_0^{**}$, and let $\alpha = I^{**}m^{**} \in \ell^2(\mathbb{N})$. Since $\mathcal{M}_0^*$ is separable, the weak* topology of the unit ball $B_{\mathcal{M}_0^{**}}$ of $\mathcal{M}_0^{**}$ is metrizable. As is the case for every Banach space, $\iota_{\mathcal{M}_0}(B_{\mathcal{M}_0})$ is weak* dense in $B_{\mathcal{M}_0^{**}}$. Hence there is a sequence $(m_n)_{n=1}^{\infty}$ in $\mathcal{M}_0$ such that $\iota_{\mathcal{M}_0} m_n \to m^{**}$ weak* and $\|m_n\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \leq \|m^{**}\|_{\mathcal{M}_0^{**}}$. Suppose that $m_n = M(\alpha_n)$ and let $a, b \in \ell^2(\mathbb{N})$ be two finite sequences. Then, since $\iota_{\mathcal{M}_0} m_n \to m^{**}$ weak*,

$$\langle M(\alpha_n)a, b\rangle_{\ell^2(\mathbb{N})} = (\alpha_n, a \star b) = I^*(a \star b)(m_n) \to m^{**}(I^*(a \star b)) = (\alpha, a \star b),$$

as $n \to \infty$. It follows that

$$\begin{aligned} |\langle M(\alpha)a, b\rangle_{\ell^2(\mathbb{N})}| = |(\alpha, a \star b)| &\leq \varlimsup_{n\to\infty} \|m_n\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \|a\|_{\ell^2(\mathbb{N})} \|b\|_{\ell^2(\mathbb{N})} \\ &\leq \|m^{**}\|_{\mathcal{M}_0^{**}} \|a\|_{\ell^2(\mathbb{N})} \|b\|_{\ell^2(\mathbb{N})}. \end{aligned}$$

Since a, b were arbitrary finite sequences, it follows that $M(\alpha) \in \mathcal{M}$ and

$$\|M(\alpha)\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \leq \|m^{**}\|_{\mathcal{M}_0^{**}}.$$

Since $\alpha = I^{**}m^{**}$ this proves that I^{**} maps $\mathcal{M}_0^{**}$ contractively into $\mathcal{M}$.

Conversely, suppose that $m = M(\alpha) \in \mathcal{M}$. By Lemma 2.1, for $0 < r < 1$, $M(\alpha_r) \in \mathcal{M}_0$, $\|M(\alpha_r)\| \leq \|M(\alpha)\|$, and $\alpha_r \to \alpha$ in $\ell^2(\mathbb{N})$ as $r \to 1$. Define $m^{**} \in \mathcal{M}_0^{**}$ by

$$m^{**}(I^*a) := (\alpha, a) = \lim_{r\to 1}(\alpha_r, a) = \lim_{r\to 1} I^*a(M(\alpha_r)), \quad a \in \ell^2(\mathbb{N}). \tag{5}$$

This specifies an element $m^{**} \in \mathcal{M}_0^{**}$ since I^* has dense range in $\mathcal{M}_0^*$ and

$$|m^{**}(I^*a)| \leq \varlimsup_{r\to 1} \|M(\alpha_r)\|_{\mathcal{B}(\ell^2(\mathbb{N}))}\|I^*a\|_{\mathcal{M}_0^*} \leq \|M(\alpha)\|_{\mathcal{B}(\ell^2(\mathbb{N}))}\|I^*a\|_{\mathcal{M}_0^*}.$$

From this inequality we also see that

$$\|m^{**}\|_{\mathcal{M}_0^{**}} \leq \|m\|_{\mathcal{B}(\ell^2(\mathbb{N}))}. \tag{6}$$

Furthermore, since

$$(I^{**}m^{**}, a) = m^{**}(I^*a) = (\alpha, a), \quad a \in \ell^2(\mathbb{N}),$$

we have that $I^{**}m^{**} = \alpha$. Hence I^{**} maps $\mathcal{M}_0^{**}$ bijectively and contractively onto $\mathcal{M}$. By (6), $I^{**}\colon \mathcal{M}_0^{**} \to \mathcal{M}$ is also expansive, and hence it is an isometric isomorphism.

Recall that $\mathcal{K}(\ell^2(\mathbb{N}))$ is an M-ideal in $\mathcal{B}(\ell^2(\mathbb{N}))$ [9] – indeed, $\mathcal{K}(\ell^2(\mathbb{N}))$ is a two-sided closed ideal in $\mathcal{B}(\ell^2(\mathbb{N}))$. It is well known that there is an isometric isomorphism $E\colon \mathcal{K}(\ell^2(\mathbb{N}))^{**} \to \mathcal{B}(\ell^2(\mathbb{N}))$ such that $E\iota_{\mathcal{K}(\ell^2(\mathbb{N}))}K = K$ for all $K \in \mathcal{K}(\ell^2(\mathbb{N}))$. Thus $\mathcal{K}(\ell^2(\mathbb{N}))$ is M-embedded. Since $\mathcal{M}_0$ is a closed subspace of $\mathcal{K}(\ell^2(\mathbb{N}))$, $\mathcal{M}_0$ is also M-embedded [12, Theorem III.1.6]. Hence, since we have shown that $I^{**}\colon \mathcal{M}_0^{**} \to \mathcal{M}$ is an isometric isomorphism for which $I^{**}\iota_{\mathcal{M}_0}m = m$ for all $m \in \mathcal{M}_0$, it follows that $\mathcal{M}_0$ is an M-ideal in $\mathcal{M}$.

Finally, if $V\colon \mathcal{M}_0^{**} \to \mathcal{M}$ is another isometric isomorphism such that $V\iota_{\mathcal{M}_0}m = m$, $m \in \mathcal{M}_0$, then $F = V^{-1}I^{**}\colon \mathcal{M}_0^{**} \to \mathcal{M}_0^{**}$ is an isometric isomorphism such that $F\iota_{\mathcal{M}_0} = \iota_{\mathcal{M}_0}$. However, since $\mathcal{M}_0$ is M-embedded, F must be obtained as the bitranspose, $F = G^{**}$, of an isometric isomorphism $G\colon \mathcal{M}_0 \to \mathcal{M}_0$ [12, Proposition III.2.2]. But then $G = \mathrm{id}_{\mathcal{M}_0}$, since

$$m^*(Gm) = G^*m^*(m) = F\iota_{\mathcal{M}_0}m(m^*) = m^*(m), \quad m \in \mathcal{M}_0,\ m^* \in \mathcal{M}_0^*.$$

Hence $F = \mathrm{id}_{\mathcal{M}_0^{**}}$ and so $V = I^{**}$. $\square$

The predual of a space of Hankel operators usually has an abstract description as a projective tensor product [5, 7, 10]. In the present context, let

$$X = \left\{ c : c = \sum_{\text{finite}} a_k \star b_k,\ a_k, b_k \text{ finite sequences} \right\},$$

and equip X with the norm

$$\|c\|_X = \inf \sum_{\text{finite}} \|a_k\|_{\ell^2(\mathbb{N})}\|b_k\|_{\ell^2(\mathbb{N})},$$

where the infimum is taken over all **finite** representations of c. By writing $c = c \star (1, 0, 0, \ldots)$ it is clear that $\|c\|_X \leq \|c\|_{\ell^2(\mathbb{N})}$ for $c \in X$.

We define the projective tensor product space $\mathcal{X} = \ell^2(\mathbb{N}) \hat{\star} \ell^2(\mathbb{N})$ with respect to Dirichlet convolution as the Banach space completion of X. It is essentially definition that $\mathcal{X}^* \simeq \mathcal{M}$.

Lemma 2.4. *For $m = M(\alpha) \in \mathcal{M}$, let*

$$Jm(c) = (\alpha, c), \quad c \in X.$$

Then Jm extends to a bounded functional on $\mathcal{X}$ for every $m \in \mathcal{M}$, and $J\colon \mathcal{M} \to \mathcal{X}^*$ is an isometric isomorphism.

Proof. Let $m \in \mathcal{M}$. If $c \in X$ and $\varepsilon > 0$, choose a representation $c = \sum_{k=1}^N a_k \star b_k$, where a_k and b_k are finite sequences for every k, and

$$\sum_{k=1}^N \|a_k\|_{\ell^2(\mathbb{N})}\|b_k\|_{\ell^2(\mathbb{N})} < \|c\|_X + \varepsilon.$$

Then

$$|Jm(c)| = \left|\sum_{k=1}^N \langle M(\alpha)a_k, b_k\rangle_{\ell^2(\mathbb{N})}\right| \leq \|m\|_{\mathcal{B}(\ell^2(\mathbb{N}))}(\|c\|_X + \varepsilon).$$

Hence $\|Jm\|_{\mathcal{X}^*} \leq \|m\|_{\mathcal{B}(\ell^2(\mathbb{N}))}$. Choosing finite sequences a and b such that $\|a\|_{\ell^2(\mathbb{N})} = \|b\|_{\ell^2(\mathbb{N})} = 1$ and $\langle M(\alpha)a, b\rangle_{\ell^2(\mathbb{N})} > \|m\|_{\mathcal{B}(\ell^2(\mathbb{N}))} - \varepsilon$, and letting $c = a \star b$ gives that

$$\|m\|_{\mathcal{B}(\ell^2(\mathbb{N}))} - \varepsilon < \|Jm\|_{\mathcal{X}^*}\|c\|_X \leq \|Jm\|_{\mathcal{X}^*}.$$

Hence J is an isometry.

The inclusion of finite sequences into X extends to a contractive map $E\colon \ell^2(\mathbb{N}) \to \mathcal{X}$. Let $\ell \in \mathcal{X}^*$ and let $c \in X$. Then $\ell(c) = (\alpha, c)$, where $\alpha = E^*\ell \in \ell^2(\mathbb{N})$. Then $m = M(\alpha) \in \mathcal{M}$, since $\ell \in \mathcal{X}^*$. Clearly $Jm = \ell$ and thus J is onto. □

Theorem 1.3. *For every $c \in X$, let*

$$Lc(m) = (\alpha, c), \quad m = M(\alpha) \in \mathcal{M}_0.$$

Then L extends to an isometric isomorphism $L\colon \mathcal{X} \to \mathcal{M}_0^*$, and

$$L^*U^{-1} = J\colon \mathcal{M} \to \mathcal{X}^*$$

is the isometric isomorphism of Lemma 2.4. Here $U\colon \mathcal{M}_0^{**} \to \mathcal{M}$ is the isometric isomorphism of Theorem 1.2.

Proof. The quickest proof proceeds by noting that $\mathcal{M}_0^*$ is a strongly unique predual of $\mathcal{M}_0^{**}$, since $\mathcal{M}_0$ is M-embedded. This implies that the isometric isomorphism $JU\colon \mathcal{M}_0^{**} \to \mathcal{X}^*$ is the adjoint of an isometric isomorphism $E\colon \mathcal{X} \to \mathcal{M}_0^*$, $E^* = JU$. But then, for $c \in X$ and $m = M(\alpha) \in \mathcal{M}_0$,

$$\begin{aligned} Ec(m) &= \iota_{\mathcal{M}_0}m(Ec) = E^*\iota_{\mathcal{M}_0}m(c) = JU\iota_{\mathcal{M}_0}m(c) \\ &= Jm(c) = (\alpha, c) = Lc(m). \end{aligned} \tag{7}$$

Hence $L = E$, and thus L is an isometric isomorphism.

Alternatively, the weak*-weak* continuity of JU can be proven by hand. L clearly extends to a contractive operator $L\colon \mathcal{X} \to \mathcal{M}_0^*$. The computation (7) shows that $JU\iota_{\mathcal{M}_0} = L^*\iota_{\mathcal{M}_0}$. Let $m^{**} \in \mathcal{M}_0^{**}$ and let $M(\alpha) = Um^{**}$.

From (5) we deduce that $m_r^{**} = \iota_{\mathcal{M}_0} M(\alpha_r) \to m^{**}$ weak* in $\mathcal{M}_0^{**}$. Hence $L^* m_r^{**} \to L^* m^{**}$ weak* in $\mathcal{X}^*$. On the other hand, for $c \in X$,

$$\begin{aligned} JUm^{**}(c) &= (\alpha, c) = \lim_{r\to 1} (\alpha_r, c) = \lim_{r \to 1} JUm_r^{**}(c) \\ &= \lim_{r\to 1} L^* m_r^{**}(c) = L^* m^{**}(c). \end{aligned}$$

This shows that $JU = L^*$, and hence L is an isometric isomorphism. □

Remark. In the notation of Theorem 1.2, $I^* c = Lc$ for $c \in X$. Theorem 1.3 hence completes the picture of Theorem 1.2 by giving an interpretation of the operator I^*.

Suppose that we had instead defined the projective tensor product space $\ell^2(\mathbb{N}) \hat{\star} \ell^2(\mathbb{N})$ as the sequence space

$$\mathcal{Y} = \left\{ c \,:\, c = \sum_{k=1}^{\infty} a_k \star b_k,\ a_k, b_k \in \ell^2(\mathbb{N}),\ \sum_{k=1}^{\infty} \|a_k\|_{\ell^2(\mathbb{N})} \|b_k\|_{\ell^2(\mathbb{N})} < \infty \right\},$$

normed by

$$\|c\|_{\mathcal{Y}} = \inf \sum_{k=1}^{\infty} \|a_k\|_{\ell^2(\mathbb{N})} \|b_k\|_{\ell^2(\mathbb{N})},$$

where the infimum is taken over all representations of c. One would like to know that $\mathcal{Y} = \mathcal{X}$. Indeed, it is not a priori clear that $\mathcal{X}$ is a sequence space; or if $\mathcal{X}$ is identifiable with a space of Dirichlet series, if considering multiplicative Hankel operators in that context. For $\mathcal{Y}$ these properties are immediate.

Lemma 2.5. *$\mathcal{Y}$ is a Banach space.*

Proof. Since $|(a \star b)(n)| \leq \|a\|_{\ell^2(\mathbb{N})} \|b\|_{\ell^2(\mathbb{N})}$ it is clear that

$$e_n(c) = c(n), \quad c \in \mathcal{Y},$$

defines an element $e_n \in \mathcal{Y}^*$, for every $n \in \mathbb{N}$. It follows that $\|c\|_{\mathcal{Y}} = 0$ if and only if $c = 0$.

Suppose that $\sum_{k=1}^{\infty} c_k$ is an absolutely convergent series in $\mathcal{Y}$. Then there are double sequences $(a_{k,j})$ and $(b_{k,j})$ such that $c_k = \sum_{j=1}^{\infty} a_{k,j} \star b_{k,j}$ for every k and

$$\sum_{k,j=1}^{\infty} \|a_{k,j}\|_{\ell^2(\mathbb{N})} \|b_{k,j}\|_{\ell^2(\mathbb{N})} < \infty.$$

Then $c = \sum_{k,j=1}^{\infty} a_{k,j} b_{k,j}$ is an element of $\mathcal{Y}$ and

$$\|c - \sum_{k=1}^{N} c_k\|_{\mathcal{Y}} \leq \sum_{k=N+1}^{\infty} \sum_{j=1}^{\infty} \|a_{k,j}\|_{\ell^2(\mathbb{N})} \|b_{k,j}\|_{\ell^2(\mathbb{N})} \to 0, \quad N \to \infty.$$

Hence $\sum_{k=1}^{\infty} c_k$ converges in $\mathcal{Y}$ to c. Thus $\mathcal{Y}$ is complete. □

We now prove that $\mathcal{Y} = \mathcal{X}$. The details are similar to those of [25], where projective tensor products of spaces of holomorphic functions were considered. Note that X is contractively contained in $\mathcal{Y}$.

Proposition 2.6. *The inclusion $V\colon X \to \mathcal{Y}$ extends to an isometric isomorphism $V\colon \mathcal{X} \to \mathcal{Y}$.*

Proof. We make the following preliminary observation. Since for every $0 < r < 1$,

$$D_r(a \star b) = D_r a \star D_r b, \quad \|D_r\|_{\mathcal{B}(\ell^2(\mathbb{N}))} \leq 1,$$

D_r defines a bounded operator $D_r\colon \mathcal{X} \to \mathcal{X}$,

$$\|D_r\|_{B(\mathcal{X})} \leq 1.$$

Furthermore, since $D_r \to \mathrm{id}_{\ell^2(\mathbb{N})}$ SOT on $\ell^2(\mathbb{N})$ as $r \to 1$, it follows that $\|D_r c - c\|_X \leq \|D_r c - c\|_{\ell^2(\mathbb{N})} \to 0$ as $r \to 1$ for every $c \in X$. Hence $D_r \to \mathrm{id}_{\mathcal{X}}$ SOT on $\mathcal{X}$ as $r \to 1$.

As in Lemma 2.5, for each $n \in \mathbb{N}$,

$$e_n(c) = c(n), \quad c \in X,$$

extends to a functional $e_n \in \mathcal{X}^*$ with $\|e_n\|_{\mathcal{X}^*} \leq 1$. We show now that (e_n) is a complete sequence in $\mathcal{X}^*$ with respect to the weak* topology. Suppose that $c \in \mathcal{X}$ and that $e_n(c) = 0$ for all n. Pick a sequence (c_k) in X such that $c_k \to c$ in $\mathcal{X}$. Then for fixed $r < 1$,

$$\begin{aligned}\|D_r c\|_{\mathcal{X}} &\leq \varlimsup_{k\to\infty} \left(\|D_r(c - c_k)\|_{\mathcal{X}} + \|D_r c_k\|_{\mathcal{X}}\right) \\ &= \varlimsup_{k\to\infty} \|D_r c_k\|_X \leq \varlimsup_{k\to\infty} \|D_r c_k\|_{\ell^2(\mathbb{N})}.\end{aligned}$$

Since $c_k \to c$ in $\mathcal{X}$ and $e_n \in \mathcal{X}^*$, we have that $\lim_{k\to\infty} c_k(n) = e_n(c) = 0$ for every n. Furthermore, $|c_k(n)| \leq \|e_n\|_{\mathcal{X}^*}\|c_k\|_X \leq \|c_k\|_X$ is uniformly bounded in k and n. Hence it follows by the dominated convergence theorem that $\varlimsup_{k\to\infty} \|D_r c_k\|_{\ell^2(\mathbb{N})} = 0$ and thus that $D_r c = 0$. Since $D_r c \to c$ in $\mathcal{X}$ as $r \to 1$ we conclude that $c = 0$. Therefore (e_n) is complete.

Hence $\mathcal{X}$ is a space of sequences. More precisely, since every evaluation e_n is a bounded functional on $\mathcal{Y}$ as well, the extension $V\colon \mathcal{X} \to \mathcal{Y}$ of the inclusion map is given by

$$Vc = (e_n(c))_{n=1}^{\infty}, \quad c \in \mathcal{X}. \tag{8}$$

The completeness of (e_n) implies that V is injective.

We next prove that V is onto. The argument is precisely as in [25], but we include it for completeness. For a sequence a and $m \in \mathbb{N}$, let $a^m = (a(1), \ldots, a(m), 0, \ldots)$. Given $a \in \ell^2(\mathbb{N})$ and $\delta > 0$, choose a sequence $(m_1, m_2, \ldots)$ such that $\|a - a^{m_k}\|_{\ell^2(\mathbb{N})} \leq 2^{-k}$. Let $a_k = a^{m_{k+1}} - a^{m_k}$. Then, for sufficiently large K,

$$a = a^{m_K} + \sum_{k=K}^{\infty} (a^{m_{k+1}} - a^{m_k}), \quad \sum_{k=K}^{\infty} \|a^{m_{k+1}} - a^{m_k}\|_{\ell^2(\mathbb{N})} < \delta.$$

Hence we can write $a = \sum_{j=1}^{\infty} a_j$, where each a_j is a finite sequence and $\sum_j \|a_j\|_{\ell^2(\mathbb{N})} < \|a\|_{\ell^2(\mathbb{N})} + \delta$.

Given $c \in \mathcal{Y}$ and $\varepsilon > 0$, choose $(a_k)_{k=1}^{\infty}$ and $(b_k)_{k=1}^{\infty}$ such that

$$c = \sum_{k=1}^{\infty} a_k \star b_k, \quad \sum_{k=1}^{\infty} \|a_k\|_{\ell^2(\mathbb{N})} \|b_k\|_{\ell^2(\mathbb{N})} < \|c\|_{\mathcal{Y}} + \varepsilon.$$

For each k, write, as in the preceding paragraph, $a_k = \sum_{j=1}^{\infty} a_{k,j}$, $b_k = \sum_{j=1}^{\infty} b_{k,j}$, where each $a_{k,j}$ and $b_{k,j}$ is a finite sequence and

$$\sum_{j=1}^{\infty} \|a_{k,j}\|_{\ell^2(\mathbb{N})} < \|a_k\|_{\ell^2(\mathbb{N})} + \delta_k, \quad \sum_{j=1}^{\infty} \|b_{k,j}\|_{\ell^2(\mathbb{N})} < \|b_k\|_{\ell^2(\mathbb{N})} + \delta_k.$$

Here the δ_k are chosen so that

$$\sum_{k=1}^{\infty} (\|a_k\|_{\ell^2(\mathbb{N})} + \delta_k)(\|b_k\|_{\ell^2(\mathbb{N})} + \delta_k) < \sum_{k=1}^{\infty} \|a_k\|_{\ell^2(\mathbb{N})} \|b_k\|_{\ell^2(\mathbb{N})} + \epsilon.$$

Then $c = \sum_{k,j,l=1}^{\infty} a_{k,j} \star b_{k,l}$, and

$$\sum_{k,j,l=1}^{\infty} \|a_{k,j}\|_{\ell^2(\mathbb{N})} \|b_{k,l}\|_{\ell^2(\mathbb{N})} < \sum_{k=1}^{\infty} \|a_k\|_{\ell^2(\mathbb{N})} \|b_k\|_{\ell^2(\mathbb{N})} + \epsilon < \|c\|_{\mathcal{Y}} + 2\varepsilon.$$

Relabeling, we have a representation $c = \sum_{n=1}^{\infty} a_n \star b_n$ where a_n and b_n are finite sequences and $\sum_n \|a_n\|_{\ell^2(\mathbb{N})} \|b_n\|_{\ell^2(\mathbb{N})} < \|c\|_{\mathcal{Y}} + 2\varepsilon$. Let $c_N = \sum_{n=1}^{N} a_n \star b_n$. Then $c_N \to c$ in $\mathcal{Y}$, and furthermore (c_N) is a Cauchy sequence in X, hence has a limit $\tilde{c}$ in $\mathcal{X}$. By continuity of the functionals e_n on both $\mathcal{Y}$ and $\mathcal{X}$, we find in view of (8) that $V\tilde{c} = c$. Hence V is onto.

Furthermore, since V is contractive,

$$\|c\|_{\mathcal{Y}} \leq \|\tilde{c}\|_{\mathcal{X}} = \lim_{N \to \infty} \|c_N\|_X < \|c\|_{\mathcal{Y}} + 2\varepsilon.$$

We already showed that V is injective, so that $\tilde{c}$ is uniquely defined by c. On the other hand, ε is arbitrary. We conclude that $\|c\|_{\mathcal{Y}} = \|\tilde{c}\|_{\mathcal{X}}$. It follows that V is an isometric isomorphism. □

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