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A note on Cartan isometries

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Abstract. We record a lifting theorem for the intertwiner of two S_Ω-isometries which are those subnormal operator tuples whose minimal normal extensions have their Taylor spectra contained in the Shilov boundary of a certain function algebra associated with Ω, Ω being a bounded convex domain in $\mathbb{C}^n$ containing the origin. The theorem captures several known lifting results in the literature and yields interesting new examples of liftings as a consequence of its being applicabile to Cartesian products Ω of classical Cartan domains in $\mathbb{C}^n$. Further, we derive intrinsic characterizations of S_Ω-isometries where Ω is a classical Cartan domain of any of the types I, II, III and IV, and we also provide a neat description of an S_Ω-isometry in case Ω is a finite Cartesian product of such Cartan domains.

Contents

1. Introduction

For $\mathcal{H}$ a complex infinite-dimensional separable Hilbert space, we use $\mathcal{B}(\mathcal{H})$ to denote the algebra of bounded linear operators on $\mathcal{H}$. An n-tuple $S = (S_1, \ldots, S_n)$ of commuting operators S_i in $\mathcal{B}(\mathcal{H})$ is said to be *subnormal* if there exist a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and an n-tuple $N = (N_1, \ldots, N_n)$ of commuting normal operators N_i in $\mathcal{B}(\mathcal{K})$ such that $N_i\mathcal{H} \subset \mathcal{H}$ and $N_i/\mathcal{H} = S_i$ for $1 \le i \le n$.

Suppose $S = (S_1, \ldots, S_n)$ is a tuple of commuting operators in $\mathcal{B}(\mathcal{H})$ and $T = (T_1, \ldots, T_n)$ a tuple of commuting operators in $\mathcal{B}(\mathcal{J})$. If there exists a

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bounded linear operator $X : \mathcal{H} \to \mathcal{J}$ such that $XS_i = T_iX$ for each i, then X is said to be an *intertwiner* (for S and T) and we denote this fact by $XS = TX$. If $X : \mathcal{H} \to \mathcal{J}$ and $Y : \mathcal{J} \to \mathcal{H}$ are two intertwiners for S and T such that $XS = TX$ and $YT = SY$, and both X and Y are injective and have dense ranges, then S is said to be *quasisimilar to* T. The operator tuple S is said to be *unitarily equivalent to* T if one can find a unitary intertwiner for S and T. Any subnormal operator tuple is known to admit a 'minimal' normal extension that is unique up to unitary equivalence (see [12]).

For a bounded domain Ω in $\mathbb{C}^n$, we let

$$A(\Omega) = \{f \in C(\bar{\Omega}) : f \text{ is holomorphic on } \Omega\},$$

where $C(\bar{\Omega})$ denotes the algebra of continuous functions on the closure $\bar{\Omega}$ of Ω. The *Shilov boundary* of $A(\Omega)$ (or Ω) is defined to be the smallest closed subset S_Ω of $\bar{\Omega}$ such that, for any $f \in A(\Omega)$,

$$\sup\{|f(z)| : z \in \bar{\Omega}\} = \sup\{|f(z)| : z \in S_\Omega\}.$$

Of special interest to us are domains Ω that are Cartesian products $\Omega_1 \times \cdots \times \Omega_m$ with $\Omega_i \subset \mathbb{C}^{n_i}$ being a classical Cartan domain of any of the four types I II, III and IV (refer to [7], [11], [13], [14]); any such domain Ω will be referred to as a *standard Cartan domain*. The open unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$ is a classical Cartan domain of type I with its Shilov boundary coinciding with the unit sphere in $\mathbb{C}^n$. The open unit polydisk $\mathbb{D}^n$ in $\mathbb{C}^n$ is a standard Cartan domain with its Shilov boundary coinciding with the unit polycircle in $\mathbb{C}^n$. The standard Cartan domains are special examples of bounded symmetric domains and are 'circled around the origin' in the sense that they contain the origin and are invariant under multiplication by $e^{\sqrt{-1}\theta}$, $\theta \in \mathbb{R}$. It follows from [9, Lemma 5.7] that the Shilov boundary S_Ω of any standard Cartan domain $\Omega = \Omega_1 \times \cdots \times \Omega_m$, where each Ω_i is a classical Cartan domain in $\mathbb{C}^{n_i}$, is given by $S_\Omega = S_{\Omega_1} \times \cdots \times S_{\Omega_m}$.

A subnormal tuple S will be referred to as an S_Ω*-isometry* if the Taylor spectrum $\sigma(N)$ of its minimal normal extension N is contained in the Shilov boundary S_Ω of Ω. We use $I_{\mathcal{H}}$ (resp. $0_{\mathcal{H}}$) to denote the identity operator (resp. the zero operator) on $\mathcal{H}$. An $S_{\mathbb{B}_n}$-isometry is precisely a *spherical isometry*, that is, an n-tuple S of commuting operators S_i in $\mathcal{B}(\mathcal{H})$ satisfying $\sum_{i=1}^n S_i^* S_i = I_{\mathcal{H}}$ (refer to [3, Proposition 2]). An $S_{\mathbb{D}^n}$-isometry is precisely a *toral isometry*, that is, an n-tuple S of commuting operators S_i in $\mathcal{B}(\mathcal{H})$ satisfying $S_i^* S_i = I_{\mathcal{H}}$ for each i (refer to [18, Proposition 6.2]). Any S_Ω-isomerty with Ω a standard Cartan domain will be referred to as a *Cartan isometry*.

We will say that a domain $\Omega \subset \mathbb{C}^n$ *satisfies the property (A)* if, for any positive regular Borel measure η supported on the Shilov boundary S_Ω of Ω, the triple $(A(\Omega)|S_\Omega, S_\Omega, \eta)$ is *regular* in the sense of [1], that is, for any positive continuous function ϕ defined on S_Ω, there exists a sequence of functions $\{\phi_m\}_{m\geq 1}$ in $A(\Omega)$ such that $|\phi_m| < \phi$ on S_Ω and $\lim_{m\to\infty} |\phi_m| = \phi$ η-almost everywhere.

The discussion in Section 5 of [9] shows that any bounded symmetric domain circled around the origin satisfies the property (A).

In Section 2, we state a lifting result for the intertwiner of certain S_Ω-isometries of which Cartan isometries are special examples. In Section 3 we provide an intrinsic characterization of S_Ω-isometries for Cartan domains Ω of type IV and then characterize S_Ω-isometries for Ω a Cartesian product of the open unit balls and Cartan domains of type IV (see Theorem 3.5). In Section 4, we characterize S_Ω-isometries for Cartan domains of type I and observe that Theorem 3.5 holds with the open unit balls replaced by Cartan domains of type I. Finally, in Section 5 we characterize S_Ω-isometries for Cartan domains of type II and of type III and end up with a substantial generalization of Theorem 3.5. For basic facts pertaining to classical Cartan domains and bounded symmetric domains in general, the reader is referred to [11], [13] and [14]. It may be noted that Shilov boundaries are referred to as 'characteristic manifolds' in [11].

2. A lifting theorem for certain S_Ω-isometries

The proof of Theorem 2.1 below is similar to the proofs of [4, Theorem 3.2] and [5, Proposition 4.6]; however, unlike there, it circumvents using the Taylor functional calculus of [19]. Also, unlike in [4] and [5], the Shilov boundary S_Ω of Ω may not coincide with the topological boundary $\partial\Omega$ of Ω.

Theorem 2.1. Let Ω be a bounded convex domain in $\mathbb{C}^n$ containing the origin and satisfying the property (A) of Section 1. Let $S = (S_1, \ldots, S_n) \in \mathcal{B}(\mathcal{H})^n$ and $T = (T_1, \ldots, T_n) \in \mathcal{B}(\mathcal{J})^n$ be S_Ω-isometries, and let $M = (M_1, \ldots, M_n) \in \mathcal{B}(\tilde{\mathcal{H}})^n$ and $N = (N_1, \ldots, N_n) \in \mathcal{B}(\tilde{\mathcal{J}})^n$ respectively be the minimal normal extensions of S and T. If $X : \mathcal{H} \to \mathcal{J}$ is an intertwiner for S and T, then X lifts to a (unique) intertwiner $\tilde{X} : \tilde{\mathcal{H}} \to \tilde{\mathcal{J}}$ for M and N; moreover, $\|\tilde{X}\| = \|X\|$.

Proof. Let $f \in A(\Omega)$. For any positive integer $m \geq 2$, f_m defined by $f_m(z) = f((1 - \frac{1}{m})z)$ is holomorphic on an open neighborhood of $\bar{\Omega}$. Since $\bar{\Omega}$ is polynomially convex, f_m is the uniform limit (on $\bar{\Omega}$) of a sequence $\{p_{m,k}\}_k$ of polynomials by the Oka-Weil approximation theorem (see [16], Chapter VI, Theorem 1.5). If X intertwines S and T, then one clearly has $Xp_{m,k}(S) = p_{m,k}(T)X$. If ρ_M and ρ_N are respectively the spectral measures of M and N (supported on S_Ω), then $\rho_S = P_{\mathcal{H}}\rho_M|\mathcal{H}$ and $\rho_T = P_{\mathcal{J}}\rho_N|\mathcal{J}$ are respectively the semi-spectral measure of S and T with $P_{\mathcal{H}}$ and $P_{\mathcal{J}}$ being appropriate projections, and for any $u \in \mathcal{H}$ and any $v \in \mathcal{K}$ one has

$$\|p_{m,k}(S)u\|^2 = \int_{S_\Omega} |p_{m,k}(z)|^2 d\langle \rho_S(z)u, u\rangle$$

and

$$\|p_{m,k}(T)v\|^2 = \int_{S_\Omega} |p_{m,k}(z)|^2 d\langle \rho_T(z)v, v\rangle.$$

Choosing $v = Xu$ and using $Xp_{m,k}(S) = p_{m,k}(T)X$, one has

$$\int_{S_\Omega} |p_{m,k}(z)|^2 d\langle \rho_T(z)Xu, Xu\rangle \leq \|X\|^2 \int_{S_\Omega} |p_{m,k}(z)|^2 d\langle \rho_S(z)u, u\rangle.$$

Letting first k tend to infinity and then m tend to infinity, one obtains

$$\int_{S_\Omega} |f(z)|^2 d\langle \rho_T(z)Xu, Xu\rangle \leq \|X\|^2 \int_{S_\Omega} |f(z)|^2 d\langle \rho_S(z)u, u\rangle.$$

Consider $\eta(\cdot) = \langle \rho_T(\cdot)Xu, Xu\rangle + \langle \rho_S(\cdot)u, u\rangle$. Since $(A(\Omega)|S_\Omega, S_\Omega, \eta)$ is a regular triple, for any positive continuous function ϕ on S_Ω there exists a sequence of functions $\{\phi_m\}_{m\geq 1}$ in $A(\Omega)$ such that $|\phi_m| < \sqrt{\phi}$ on S_Ω and $\lim_{m\to\infty} |\phi_m| = \sqrt{\phi}$ η-almost everywhere. Replacing f by ϕ_m in the last integral inequality and letting m tend to infinity, one obtains

$$\int_{S_\Omega} \phi(z) d\langle \rho_T(z)Xu, Xu\rangle \leq \|X\|^2 \int_{S_\Omega} \phi(z) d\langle \rho_S(z)u, u\rangle.$$

That yields $\langle \rho_T(\cdot)Xu, Xu\rangle \leq \|X\|^2 \langle \rho_S(\cdot)u, u\rangle$ for every u in $\mathcal{H}$. The desired conclusion now follows by appealing to [15, Lemma 4.1]. □

In so far as the function algebra $A(\Omega)$ is concerned, Theorem 2.1 is an improvement over [15, Theorem 5.1] by virtue of its using the more widely applicable property (A) in place of the property ‘approximating in modulus’ as required of a function algebra in [15].

Corollary 2.2. Let Ω be any bounded symmetric domain circled around the origin (so that Ω can in particular be a standard Cartan domain). Let $S = (S_1, \ldots, S_n) \in \mathcal{B}(\mathcal{H})^n$ and $T = (T_1, \ldots, T_n) \in \mathcal{B}(\mathcal{J})^n$ be S_Ω-isometries, and let $M = (M_1, \ldots, M_n) \in \mathcal{B}(\tilde{\mathcal{H}})^n$ and $N = (N_1, \ldots, N_n) \in \mathcal{B}(\tilde{\mathcal{J}})^n$ respectively be the minimal normal extensions of S and T. If $X : \mathcal{H} \to \mathcal{J}$ is an intertwiner for S and T, then X lifts to a (unique) intertwiner $\tilde{X} : \tilde{\mathcal{H}} \to \tilde{\mathcal{J}}$ for M and N; moreover, $\|\tilde{X}\| = \|X\|$.

Proof. Any bounded symmetric domain circled around the origin is convex by [14, Corollary 4.6] and, as noted in Section 1, satisfies the property (A). □

Remark 2.3. Letting Ω to be the open unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$, Corollary 2.2 captures [3, Proposition 8] which is a lifting result for the intertwiner of spherical isometries. Letting Ω to be the open unit polydisk $\mathbb{D}^n$ in $\mathbb{C}^n$, Corollary 2.2 captures [15, Proposition 5.2] which is a lifting result for the intertwiner of toral isometries. In [5], the author introduced a class $\Omega^{(n)}$ of convex domains Ω_p in $\mathbb{C}^n$ that satisfy the property (A); for $n \geq 2$, the class $\Omega^{(n)}$ happens to be distinct from the class of strictly pseudoconvex domains and the class of bounded symmetric domains in $\mathbb{C}^n$. Letting Ω to be Ω_p, Theorem 2.1 (but not Corollary 2.2) captures [5, Proposition 4.6]. A variant of Theorem 2.1 that is valid for (not necessarily convex) strictly pseudoconvex bounded domains Ω with C^2 boundary was proved in [4]; however, Theorem 2.1 does apply to strictly pseudoconvex bounded domains that are convex since any strictly pseudoconvex bounded domain Ω is known to satisfy the property (A) (refer to [1] and [9]).

Remark 2.4. Arguing as in [15, Theorem 5.2], one can establish the following facts in the context of Theorem 2.1: If X is isometric, then so is $\tilde{X}$; if X has dense range, then so has $\tilde{X}$; if X is bijective, then so is $\tilde{X}$. Also, it follows from [3, Lemma 1] that if S and T of Theorem 2.1 are quasisimilar, then the minimal normal extensions of S and T are unitarily equivalent (cf. [3, Proposition 9]).

3. Lie sphere isometries: S_Ω-isometries for Cartan domains Ω of type IV

The Lie ball $Ł_n$ in $\mathbb{C}^n$ is defined by

$$Ł_n = \left\{ z \in \mathbb{C}^n : \left(\|z\|^2 + \sqrt{\|z\|^4 - |\langle z, \bar{z}\rangle|^2} \right)^{1/2} < 1 \right\}.$$

Lie balls $Ł_n$ are classical Cartan domains $\Omega_{IV}(n)$. We note that $Ł_1 = \mathbb{D}^1 = \mathbb{B}_1$. The Shilov boundary $S_{Ł_n}$ of $Ł_n$ (also referred to as the *Lie sphere*) is given by

$$S_{Ł_n} = \{(z_1, \ldots, z_n) : z_i = x_i e^{\sqrt{-1}\theta},\ \theta \in \mathbb{R},\ x_i \in \mathbb{R},\ x_1^2 + \cdots + x_n^2 = 1\}.$$

We will refer to an $S_{Ł_n}$-isometry as a *Lie sphere isometry*; thus Lie sphere isometries are S_Ω-isometries for classical Cartan domains Ω of type IV. It should be noted that $S_{Ł_n}$ is contained in $S_{\mathbb{B}_n}$ so that any Lie sphere isometry is a spherical isometry! We plan to provide an intrinsic characterization of a Lie sphere isometry, and for that purpose we need Lemma 3.1 below. (A result more general than that of Lemma 3.1 is present in the unpublished work [8]; we present here a direct proof for the reader's convenience).

Lemma 3.1. Let $S = (S_1, \ldots, S_n) \in \mathcal{B}(\mathcal{H})^n$ be a subnormal tuple with the minimal normal extension $N = (N_1, \ldots, N_n) \in \mathcal{B}(\mathcal{K})^n$. If $S_i^* S_j = S_j^* S_i$ (so that $S_i^* S_j$ is self-adjoint) for some i and j, then $N_i^* N_j = N_j^* N_i$ (so that $N_i^* N_j$ is also self-adjoint).

Proof. For arbitrary non-negative integers k_i and l_i $(1 \le i \le n)$, consider

$$\langle (N_i^* N_j - N_j^* N_i)(N_1^{*k_1} \cdots N_n^{*k_n} x), (N_1^{*l_1} \cdots N_n^{*l_n} y)\rangle \quad (x, y \in \mathcal{H}).$$

Using that N_p and N_q^* commute for all p and q and $N_p|\mathcal{H} = S_p$ for every p, it is easy to see that this inner product reduces to

$$\langle (S_i^* S_j - S_j^* S_i)({S_1}^{l_1} \cdots {S_n}^{l_n} x), ({S_1}^{k_1} \cdots {S_n}^{k_n} y)\rangle.$$

Since $\mathcal{K}$ is the closed linear span of vectors of the type $N_1^{*k_1} \cdots N_n^{*k_n} x$, the desired result is obvious. □

Theorem 3.2. For an n-tuple $S = (S_1, \ldots, S_n)$ of operators S_i in $\mathcal{B}(\mathcal{H})$, (a) and (b) below are equivalent.
(a) S is a Lie sphere isometry.
(b) S is a spherical isometry and $S_i^* S_j$ is self-adjoint for every i and j.

Proof. Suppose (a) holds so that $S = (S_1, \ldots, S_n) \in \mathcal{B}(\mathcal{H})^n$ is a Lie sphere isometry. Then the minimal normal extension $N = (N_1, \ldots, N_n) \in \mathcal{B}(\mathcal{K})^n$ of S has its Taylor spectrum $\sigma(N)$ contained in $S_{\text{Ł}_n}$. Since for any $(z_1, \ldots, z_n) \in S_{\text{Ł}_n}$ the equalities $|z_1|^2 + \cdots + |z_n|^2 = 1$ and $\bar{z}_i z_j - \bar{z}_j z_i = 0$ $(1 \le i, j \le n)$ hold, one has $N_1^* N_1 + \cdots + N_n^* N_n = I_{\mathcal{K}}$ and $N_i^* N_j - N_j^* N_i = 0_{\mathcal{K}}$ $(1 \le i, j \le n)$. Compressing these equations to $\mathcal{H}$, (b) is seen to hold.

Conversely, suppose (b) holds. Since one has $\sum_i S_i^* S_i = I_{\mathcal{H}}$, [4, Proposition 2] gives that S is a subnormal tuple with the Taylor spectrum $\sigma(N)$ of its minimal normal extension N contained in the unit sphere $S_{\mathbb{B}_n}$. The condition that $S_i^* S_j$ is self-adjoint for every i and j guarantees, by Lemma 3.1, that $N_i^* N_j - N_j^* N_i = 0_{\mathcal{K}}$ for every i and j. It follows then from the spectral theory for N that the Taylor spectrum of N is contained in the set $\{z \in S_{\mathbb{B}_n} : \bar{z}_i z_j - \bar{z}_j z_i = 0 \text{ for every } i \text{ and } j\}$ which, as an elementary verification using polar coordinates shows, is the set $S_{\text{Ł}_n}$. □

At this stage we introduce a notational convention that will be convenient to use in the sequel. For a complex polynomial $p(z, w) = \sum_{\alpha,\beta} a_{\alpha,\beta} z^\alpha w^\beta$ in the variables $z, w \in \mathbb{C}^n$ and for any n-tuple S of commuting operators S_i in $\mathcal{B}(\mathcal{H})$, $p(z, w)(S, S^*)$ is to be interpreted as $\sum_{\alpha,\beta} a_{\alpha,\beta} {S^*}^\beta S^\alpha$. Thus S is a spherical isometry if and only if $(1 - \sum_{i=1}^n z_i w_i)(S, S^*) = 0_{\mathcal{H}}$. A *contraction* is an operator S in $\mathcal{B}(\mathcal{H})$ for which $(I - S^* S) \equiv (1 - zw)(S, S^*) \ge 0_{\mathcal{H}}$. As proved in [2], an n-tuple S of commuting contractions S_i in $\mathcal{B}(\mathcal{H})$ is subnormal if and only if $\Pi_{i=1}^n (1 - z_i w_i)^{k_i}(S, S^*) \ge 0_{\mathcal{H}}$ for all non-negative integers k_i. Further, with $p(z, w)$ as here and with S a subnormal tuple, the proof of Lemma 3.1 goes through with $S_i^* S_j - S_j^* S_i$ there replaced by $p(z, w)(S, S^*)$. We state this generalization (due to Chavan) of [6, Proposition 8] as Lemma 3.3.

Lemma 3.3 [8]. Let $S \in \mathcal{B}(\mathcal{H})^n$ be a subnormal tuple with the minimal normal extension $N \in \mathcal{B}(\mathcal{K})^n$. If $p(z, w)(S, S^*) = 0_{\mathcal{H}}$, then $p(z, w)(N, N^*) =$

$0_{\mathcal{K}}$.

Lemma 3.4. Let $S = (S_1, \ldots, S_n)$ be a tuple of commuting operators in $\mathcal{B}(\mathcal{H})$ such that each S_i is a coordinate of a subtuple of S that is a spherical isometry. Then S is subnormal.

Proof. Suppose for each i there exist positive integers $j(i,1), \ldots, j(i,p_i)$, with $j(i,k) = i$ for some k, such that $(S_{j(i,1)}, \ldots, S_{j(i,k)} = S_i, \ldots, S_{j(i,p_i)})$ is a spherical isometry. It is clear that each S_i is then a contraction. We need to verify that $\Pi_{i=1}^n (1 - z_i w_i)^{k_i}(S, S^*) \geq 0$ for all non-negative integers k_i. The verification results by writing each factor $(1 - z_i w_i)$ as $(1 - z_i w_i) = (\{1 - \sum_{l=1}^{p_i} z_{j(i,l)} w_{j(i,l)}\} + \sum_{\substack{l=1 \\ l \neq k}}^{p_i} z_{j(i,l)} w_{j(i,l)})$. □

We are now in a position to characterize S_Ω-isometries in case Ω is a Cartesian product of the open unit balls and the Lie balls. A substantial generalization of Theorem 3.5 below will be achieved in Section 5; however, the essential ingredients of the relevant argument are present in the proof of Theorem 3.5 and occur at this stage without the clutter of too many ideas.

Theorem 3.5. Let $\Omega = \Omega_1 \times \cdots \times \Omega_m \subset \mathbb{C}^n$ where each Ω_i is either the open unit ball in $\mathbb{C}^{n_i}$ or the Lie ball in $\mathbb{C}^{n_i}$ (and where $n = n_1 + \cdots + n_m$). Let $S_i = (S_{i,1}, \ldots, S_{i,n_i})$ be an n_i-tuple of operators in $\mathcal{B}(\mathcal{H})$ for $1 \leq i \leq m$ and let the operator coordinates of the n-tuple $S = (S_1; \ldots; S_m)$ commute with each other. Then S is an S_Ω-isometry if and only if each S_i is an S_{Ω_i}-isometry.

Proof. We illustrate the proof for the case $m = 2$, $n_1 = 2$, $n_2 = 3$, $\Omega_1 = \mathbb{B}_2$ and $\Omega_2 = Ł_3$. The general case is then no more than an exercise in notational book-keeping.

Suppose first that $S = (S_1; S_2) = (S_{1,1}, S_{1,2}; S_{2,1}, S_{2,2}, S_{2,3})$ is an $S_{\mathbb{B}_2 \times Ł_3}$-isometry so that S is subnormal and the Taylor spectrum $\sigma(N)$ of its minimal normal extension $N = (N_1; N_2) = (N_{1,1}, N_{1,2}; N_{2,1}, N_{2,2}, N_{2,3}) \in \mathcal{B}(\mathcal{K})^5$ is contained in $S_{\mathbb{B}_2 \times Ł_3} = S_{\mathbb{B}_2} \times S_{Ł_3}$. By the projection property of the Taylor spectrum (refer to [19]), the inclusions $\sigma(N_1) \subset S_{\mathbb{B}_2}$ and $\sigma(N_2) \subset S_{Ł_3}$ hold. While N_1 and N_2 may not be the minimal normal extensions of S_1 and S_2, they certainly satisfy the relations

$$\sum_{i=1}^{2} N_{1,i}^* N_{1,i} = I_{\mathcal{K}}, \sum_{j=1}^{3} N_{2,j}^* N_{2,j} = I_{\mathcal{K}}, N_{2,k}^* N_{2,l} = N_{2,l}^* N_{2,k}, 1 \leq k, l \leq 3.$$

Compressing these equations to $\mathcal{H}$, one obtains

$$\sum_{i=1}^{2} S_{1,i}^* S_{1,i} = I_{\mathcal{H}}, \sum_{j=1}^{3} S_{2,j}^* S_{2,j} = I_{\mathcal{H}}, S_{2,k}^* S_{2,l} = S_{2,l}^* S_{2,k}, 1 \leq k, l \leq 3.$$

Using our observations in Section 1 related to spherical isometries and appealing to Theorem 3.2, it follows that S_1 is an $S_{\mathbb{B}_2}$-isometry and S_2 is an $S_{\mathbb{L}_3}$-isometry.

Conversely, suppose $S_1 = (S_{1,1}, S_{1,2})$ is an $S_{\mathbb{B}_2}$-isometry and that $S_2 = (S_{2,1}, S_{2,2}, S_{2,3})$ an $S_{\mathbb{L}_3}$-isometry. Then the identities for S as recorded above hold so that

$$(1 - \sum_{i=1}^{2} z_i w_i)(S_1, S_1{}^*) = 0_{\mathcal{H}}$$

and

$$(1 - \sum_{j=1}^{3} z_j w_j)(S_2, S_2{}^*) = 0_{\mathcal{H}}, (z_l w_k - z_k w_l)(S_2, S_2{}^*) = 0_{\mathcal{H}}, 1 \le k, l \le 3.$$

While both S_1 and S_2 are subnormal, the crucial thing to verify is that $S = (S_1; S_2)$ is subnormal. But the subnormality of S is now a consequence of Lemma 3.4. Letting $N = (N_1; N_2)$ to be the minimal normal extension of $S = (S_1; S_2)$ and using Lemma 3.3, we see that N satisfies the same identities as S. That $\sigma(N)$ is contained in $S_{\mathbb{B}_2} \times S_{\mathbb{L}_3} = S_{\mathbb{B}_2 \times \mathbb{L}_3}$ is now a consequence of the spectral theory for N. □

4. S_Ω-isometries for Cartan domains Ω of type I

We use the symbol $\mathbb{M}(p, q)$ to denote the set of complex matrices of order $p \times q$ and the symbol I_n to denote the identity matrix of order n. The complex tranjugate of a matrix Z will be denoted by Z^* so that Z^* is the transpose $\bar{Z}^t$ of the complex conjugate $\bar{Z}$ of Z. The classical Cartan domain $\Omega_I(p, q)$ of type I in $\mathbb{C}^n$ is defined by the following conditions:

$$n = pq, \ 1 \le p \le q, \ \Omega_I(p, q) = \{Z \in \mathbb{M}(p, q) : I_p - ZZ^* \ge 0\}$$

The Shilov boundary of $\Omega_I(p, q)$ is given by

$$S_{\Omega_I(p,q)} = \{Z \in \mathbb{M}(p, q) : I_p - ZZ^* = 0\}.$$

It will be convenient to rewrite $\Omega_I(p, q)$ as

$$\{(z_{1,1}, \dots, z_{1,q}; z_{2,1}, \dots, z_{2,q}; \dots; z_{p,1}, \dots, z_{p,q}) \in \mathbb{C}^{pq} : I_p - (z_{i,j})(\overline{z_{j,i}}) \ge 0\}$$

and $S_{\Omega_I(p,q)}$ as

$$\{(z_{1,1}, \dots, z_{1,q}; z_{2,1}, \dots, z_{2,q}; \dots; z_{p,1}, \dots, z_{p,q}) \in \mathbb{C}^{pq} : I_p - (z_{i,j})(\overline{z_{j,i}}) = 0\}.$$

The conditions defining the Shilov boundary $S_{\Omega_I(p,q)}$ can be written as

$$\sum_{k=1}^{q} \overline{z_{j,k}} z_{i,k} = \delta_{i,j}, \quad 1 \le i \le j \le p.$$

Formally replacing $z_{i,j}$ by $S_{i,j}$ and $\overline{z_{i,j}}$ by $S^*_{i,j}$ (where $S_{i,j} \in \mathcal{B}(\mathcal{H})$), one is led to

$$\sum_{k=1}^{q} S^*_{j,k} S_{i,k} = \delta_{i,j} I_{\mathcal{H}}, \quad 1 \le i \le j \le p.$$

Theorem 4.1. For $p \le q$, let $S_i = (S_{i,1}, \ldots, S_{i,q})$ be a q-tuple of operators in $\mathcal{B}(\mathcal{H})$ for $1 \le i \le p$ and let the operator coordinates of the pq-tuple $S = (S_1; \ldots; S_p)$ commute with each other. Then (a) and (b) below are equivalent.
(a) S is an $S_{\Omega_I(p,q)}$-isometry.
(b)

$$\sum_{k=1}^{q} S^*_{j,k} S_{i,k} = \delta_{i,j} I_{\mathcal{H}},\ 1 \le i \le j \le p.$$

Proof. Suppose S is an $S_{\Omega_I(p,q)}$-isometry. Then its minimal normal extension $N = (N_1; \ldots; N_p) \in \mathcal{B}(\mathcal{K})^{pq}$ (with $N_i = (N_{i,1}, \ldots, N_{i,q})$ for each i) has its Taylor spectrum $\sigma(N)$ contained in $S_{\Omega_I(p,q)}$. Since for any $z = (z_{1,1}, \ldots, z_{p,q}) \in S_{\Omega_I(p,q)}$ the equalities $\sum_{k=1}^{q} \overline{z_{j,k}} z_{i,k} = \delta_{i,j},\ 1 \le i \le j \le p$ hold, one has $\sum_{k=1}^{q} N^*_{j,k} N_{i,k} = \delta_{i,j} I_{\mathcal{K}},\ \ 1 \le i \le j \le p$. Compressing the last equations to $\mathcal{H}$, (b) is seen to hold.

Conversely, suppose (b) holds. The conditions in (b) corresponding to $1 \le i = j \le p$ guarantee that each S_i is a spherical isometry. It then follows from Lemma 3.4 that $S = (S_1; \ldots; S_p)$ is subnormal. If N in the notation used above is the minimal normal extension of S, then Lemma 3.3 yields the equalities $\sum_{k=1}^{q} N^*_{j,k} N_{i,k} = \delta_{i,j} I_{\mathcal{K}},\ 1 \le i \le j \le p$. The spectral theory for N now implies that $\sigma(N)$ is contained in $S_{\Omega_I(p,q)}$. □

Using Theorems 3.2 and 4.1 and arguing as in the proof of Theorem 3.5, one can now establish Theorem 4.2 below.

Theorem 4.2. Let $\Omega = \Omega_1 \times \cdots \times \Omega_m \subset \mathbb{C}^n$ where each Ω_i is a classical Cartan domain of any of the types I and IV in $\mathbb{C}^{n_i}$ (and where $n = n_1 + \cdots + n_m$). Let $S_i = (S_{i,1}, \ldots, S_{i,n_i})$ be an n_i-tuple of operators in $\mathcal{B}(\mathcal{H})$ for $1 \le i \le m$ and let the operator coordinates of the n-tuple $S = (S_1; \ldots; S_m)$ commute with each other. Then S is an S_Ω-isometry if and only if each S_i is an S_{Ω_i}-isometry.

Remark 4.3. Since $\Omega_{1,n}$ is the open unit ball in $\mathbb{C}^n$, Theorem 4.1 generalizes the well-known characterization of an $S_{\mathbb{B}_n}$-isometry as a spherical isometry, the case $n = 1$ of course yielding the identification of an $S_{\mathbb{B}_1}$-isometry with an isometry. Also, Theorem 4.2 generalizes Theorem 3.5 and, with Ω_i chosen to be the unit disk $\mathbb{D}^1 = \mathbb{B}_1$ in $\mathbb{C}$ for each i, yields the well-known characterization of an $S_{\mathbb{D}^n}$-isometry as a toral isometry.

5. S_Ω-isometries for Cartan domains Ω of type II and of type III

Let $\mathcal{S}(p) = \{Z \in \mathbb{M}(p,p) : Z^t = Z\}$ and let $\mathcal{A}(p) = \{Z \in \mathbb{M}(p,p) : Z^t = -Z\}$. The classical Cartan domain $\Omega_{II}(p)$ of type II in $\mathbb{C}^n$ is defined by the following conditions:

$$n = p(p+1)/2,\ p \geq 1,\ \Omega_{II}(p) = \{Z \in \mathcal{S}(p) : I_p - ZZ^* \geq 0\}$$

The classical Cartan domain $\Omega_{III}(p)$ of type III in $\mathbb{C}^n$ is defined by the following conditions:

$$n = p(p-1)/2,\ p \geq 2,\ \Omega_{III}(p) = \{Z \in \mathcal{A}(p) : I_p - ZZ^* \geq 0\}$$

(Some authors may refer to type II domains as type III domains and vice versa).

The Shilov boundary of $\Omega_{II}(p)$ is given by

$$S_{\Omega_{II}(p)} = \{Z \in \mathcal{S}(p) : I_p - ZZ^* = 0\}$$

and the Shilov boundary of $\Omega_{III}(2p)$ is given by

$$S_{\Omega_{III}(2p)} = \{Z \in \mathcal{A}(2p) : I_{2p} - ZZ^* = 0\}.$$

(We will comment on $S_{\Omega_{III}(2p+1)}$ later.)

We let

$$z_{\mathcal{S}(p)} = (z_{1,1}, \ldots, z_{1,p}; z_{2,2}, \ldots, z_{2,p}; \ldots; z_{p,p})$$

and

$$z_{\mathcal{A}(p)} = (z_{1,2}, \ldots, z_{1,p}; z_{2,3}, \ldots, z_{2,p}; \ldots; z_{p-1,p}).$$

It will be convenient to rewrite $\Omega_{II}(p)$ as

$$\{z_{\mathcal{S}(p)} \in \mathbb{C}^{p(p+1)/2} : \text{With } z_{j,i} := z_{i,j} \text{ for } i \leq j, I_p - (z_{i,j})(\overline{z_{j,i}}) \geq 0\}$$

and $\Omega_{III}(p)$ as

$$\{z_{\mathcal{A}(p)} \in \mathbb{C}^{p(p-1)/2} : \text{With } z_{j,i} := -z_{i,j} \text{ for } i \leq j, I_p - (z_{i,j})(\overline{z_{j,i}}) \geq 0\}.$$

The conditions defining the Shilov boundary $S_{\Omega_{II}(p)}$ can be written as follows:

$$\text{With } z_{j,i} := z_{i,j} \text{ for } i \leq j,\ \sum_{k=1}^{p} \overline{z_{j,k}} z_{i,k} = \delta_{i,j}, \quad 1 \leq i \leq j \leq p$$

Also, the conditions defining the Shilov boundary $S_{\Omega_{III}(2p)}$ can be written as follows:

$$\text{With } z_{j,i} := -z_{i,j} \text{ for } i \leq j,\ \sum_{k=1}^{2p} \overline{z_{j,k}} z_{i,k} = \delta_{i,j}, \quad 1 \leq i \leq j \leq 2p$$

Formally replacing $z_{i,j}$ by $S_{i,j}$ and $\overline{z_{i,j}}$ by $S^*_{i,j}$ (where $S_{i,j} \in \mathcal{B}(\mathcal{H})$), one is led to formulate Theorems 5.1 and 5.2 below.

Theorem 5.1. Let $S = (S_{1,1}, \ldots, S_{1,p}; S_{2,2}, \ldots, S_{2,p}; \ldots; S_{p,p})$ be a $\frac{p(p+1)}{2}$-tuple of commuting operators in $\mathcal{B}(\mathcal{H})$. Then (a) and (b) below are equivalent.
(a) S is an $S_{\Omega_{II}(p)}$-isometry.
(b) With $S_{j,i} := S_{i,j}$ for $i \leq j$,

$$\sum_{k=1}^{p} S_{j,k}^* S_{i,k} = \delta_{i,j} I_{\mathcal{H}}, \quad 1 \leq i \leq j \leq p.$$

Proof. The necessity of the conditions (b) is by now obvious. For the sufficiency part we note that the conditions in (b) corresponding to $1 \leq i = j \leq p$ guarantee that each $S_{l,m}$, with $l \leq m$, is an operator coordinate of a p-tuple that is a spherical isometry so that Lemma 3.4 applies. One can then argue as in the proof of Theorem 4.1. □

Theorem 5.2. Let $S = (S_{1,2}, \ldots, S_{1,2p}; S_{2,3}; \ldots, S_{2,2p}; \ldots; S_{2p-1,2p})$ be a $p(2p-1)$-tuple of commuting operators in $\mathcal{B}(\mathcal{H})$. Then (a) and (b) below are equivalent.
(a) S is an $S_{\Omega_{III}(2p)}$-isometry.
(b) With $S_{j,i} := -S_{i,j}$ for $i \leq j$,

$$\sum_{k=1}^{2p} S_{j,k}^* S_{i,k} = \delta_{i,j} I_{\mathcal{H}}, \quad 1 \leq i \leq j \leq 2p.$$

Proof. The necessity of the conditions (b) is obvious. For the sufficiency part we note that the conditions in (b) corresponding to $1 \leq i = j \leq 2p$ guarantee that each $S_{l,m}$, with $l < m$, is an operator coordinate of a $(2p-1)$-tuple that is a spherical isometry so that Lemma 3.4 applies. One can then argue as in the proof of Theorem 4.1. □

Remark 5.3. In view of Theorems 3.2, 4.1, 5.1 and 5.2, it is clear that the argument in the proof of Theorem 3.5 can be pushed through to accommodate the domains $\Omega_{II}(p)$ and $\Omega_{III}(2p)$ as well and the statement of Theorem 4.2 stands generalized by way of letting each Ω_i to be any of $\Omega_I(p,q)$, $\Omega_{IV}(n)$, $\Omega_{II}(p)$ and $\Omega_{III}(2p)$.

We now turn our attention to the domains $\Omega_{III}(2p+1)$. The Shilov boundary $S_{\Omega_{III}(2p+1)}$ is the set

$\{z_{\mathcal{A}(2p+1)} \in \mathbb{C}^{p(2p+1)}$: With $z_{j,i} := -z_{i,j}$ for $i \leq j,\ (z_{i,j}) = UKU^t$ for some unitary matrix $U\}$

where

$$K = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{p \text{ summands}} \oplus [0].$$

The matrix $Z := (z_{i,j}) = UKU^t$ is such that Z^*Z has 0 as a characteristic value of multiplicity 1 and 1 as a characteristic value of multiplicity $2p$.

For $p(2p+1)$-tuples $z_{\mathcal{A}(2p+1)}$ and $w_{\mathcal{A}(2p+1)}$, we let $z_{j,i} = -z_{i,j}$, $w_{j,i} = -w_{i,j}$ for $i \leq j$ and, for the $(2p+1) \times (2p+1)$ antisymmetric matrices $Z = (z_{i,j})$ and $W = (w_{i,j})$, we let $q(\lambda; Z, W)$ denote the characteristic polynomial $\det(\lambda I_{2p+1} - W^t Z)$ of $W^t Z$. We write $q(\lambda; Z, W)$ as

$$q(\lambda; Z, W) = q_0(Z, W) + q_1(Z, W)\lambda + \cdots + q_{2p+1}(Z, W)\lambda^{2p+1}.$$

Any $q_k(Z, W)$ is a polynomial in the $2p(2p+1)$ variables $z_{1,2}, \cdots, z_{2p,2p+1}$ and $w_{1,2}, \cdots, w_{2p,2p+1}$.

Theorem 5.4. Let $S = (S_{1,2}, \ldots, S_{1,2p+1}; S_{2,3}, \ldots, S_{2,2p+1}; \ldots; S_{2p,2p+1})$ be a $p(2p+1)$-tuple of commuting operators in $\mathcal{B}(\mathcal{H})$. Then (a) and (b) below are equivalent.
(a) S is an $S_{\Omega_{III}(2p+1)}$-isometry.
(b)

$$q_0(Z, W)(S, S^*) = 0_{\mathcal{H}};$$

$$q_m(Z, W)(S, S^*) = (-1)^{m-1} \binom{2p}{m-1} I_{\mathcal{H}}, \ 1 \leq m \leq 2p+1.$$

Proof. Suppose S is an $S_{\Omega_{III}(2p+1)}$-isometry. Then the Taylor spectrum $\sigma(N)$ of the minimal normal extension

$$N = (N_{1,2}, \ldots, N_{1,2p+1}; N_{2,3}, \ldots, N_{2,2p+1}; \ldots; N_{2p,2p+1}) \in \mathcal{B}(\mathcal{K})^{p(2p+1)}$$

of S is contained in $S_{\Omega_{III}(2p+1)}$. Since for any $z_{\mathcal{A}(2p+1)} \in S_{\Omega_{III}(2p+1)}$ the matrix Z^*Z has 0 as a characteristic value of multiplicity 1 and 1 as a characteristic value of multiplicity $2p$, the characteristic polynomial $q(\lambda; Z, \bar{Z})$ of Z^*Z coincides with $\lambda(\lambda-1)^{2p}$ and the scalar equalities

$$q_0(Z, \bar{Z}) = 0; \ q_m(Z, \bar{Z}) = (-1)^{m-1} \binom{2p}{m-1}, \ 1 \leq m \leq 2p+1$$

hold. The operator equalities

$$q_0(Z, W)(N, N^*) = 0_{\mathcal{K}}$$

and

$$q_m(Z, W)(N, N^*) = (-1)^{m-1} \binom{2p}{m-1} I_{\mathcal{K}}, \ 1 \leq m \leq 2p+1$$

follow. Compressing the last equations to $\mathcal{H}$, (b) is seen to hold.

Conversely, suppose (b) holds. The condition $q_{2p}(Z,W)(S,S^*) = -2pI_{\mathcal{H}}$ gives

$$S_{1,2}^*S_{1,2} + \cdots + S_{2p,2p+1}^*S_{2p,2p+1} = pI_{\mathcal{H}}$$

so that $(1/\sqrt{p})S$ is a spherical isometry. It follows that $(1\sqrt{p})S$ and hence S is subnormal. Let N in the notation used above be the minimal normal extension of S. Now Lemma 3.3 yields

$$q_0(Z,W)(N,N^*) = 0_{\mathcal{K}}$$

and

$$q_m(Z,W)(N,N^*) = (-1)^{m-1}\binom{2p}{m-1}I_{\mathcal{K}},\ 1 \le m \le 2p+1.$$

By the spectral theory for N, the scalar equalities

$$q_0(Z,\bar{Z}) = 0;\ q_m(Z,\bar{Z}) = (-1)^{m-1}\binom{2p}{m-1},\ 1 \le m \le 2p+1$$

hold for any $z_{\mathcal{A}(2p+1)}$ in the Taylor spectrum $\sigma(N)$ of N. But then the characteristic polynomial $q(\lambda; Z, \bar{Z})$ of Z^*Z coincides with $\lambda(\lambda-1)^{2p}$ so that Z^*Z has 0 as a characteristic value of multiplicity 1 and 1 as a characteristic value of multiplicity $2p$. At this stage, we invoke a result originally due to Hua [10] (see also [17, THEOREM 1]) to assert the existence of a unitary matrix U such that $UZU^t = K$. But this clearly implies $z_{\mathcal{A}(2p+1)} \in S_{\Omega_{III}(2p+1)}$. □

Remark 5.5. As observed in the proof of Theorem 5.5, any $S_{\Omega_{III}(2p+1)}$-isometry S is such that $(1/\sqrt{p})S$ is a spherical isometry. This necessitates, for our purposes, that the following elementary observation be made: Suppose S_i is an n_i-tuple of operators in $\mathcal{B}(\mathcal{H})$ for $1 \le i \le m$ with $S = (S_1; \ldots; S_m)$ being an $(n_1 + \cdots + n_m)$-tuple of commuting operators. If the set $\{1, \ldots, m\}$ can be partitioned into sets $\{p_1, \ldots, p_k\}$ and $\{q_1, \ldots, q_l\}$ such that each S_{p_i} satisfies the hypotheses of Lemma 3.4 and each S_{q_j} is such that $(1/m_j)S_{q_j}$ is a spherical isometry for some positive number m_j, then S is subnormal. Indeed, the tuple S' consisting of S_{p_i} and $(1/m_j)S_{q_j}$ satisfies the hypotheses of Lemma 3.4 and hence admits a normal extension N with commuting coordinates N_{p_i} and N_{q_j}; the tuple N with the coordinates N_{p_i} and $m_jN_{q_j}$ is then a normal extension of S.

Using Theorems 3.2, 4.1, 5.1, 5.2, 5.4, Remark 5.5 and arguing as in the proof of Theorem 3.5, one can now establish Theorem 5.6 below.

Theorem 5.6. Let $\Omega = \Omega_1 \times \cdots \times \Omega_m \subset \mathbb{C}^n$ where each Ω_i is a classical Cartan domain of any of the types I, II, III and IV in $\mathbb{C}^{n_i}$ (and where $n = n_1 + \cdots + n_m$). Let $S_i = (S_{i,1}, \ldots, S_{i,n_i})$ be an n_i-tuple of operators in $\mathcal{B}(\mathcal{H})$ for $1 \le i \le m$ and let the operator coordinates of the n-tuple $S = (S_1; \ldots; S_m)$ commute with each other. Then S is an S_Ω-isometry if and only if each S_i is an S_{Ω_i}-isometry.

It is interesting to note how the "stars-on-the-left" functional calculus, in conjunction with the known characterization of an $S_{\mathbb{B}_n}$-isometry as a spherical isometry, facilitates our arguments in Sections 3, 4 and 5.

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