On a special class of *A*-functions

Joseph Vade Burnett, Otto Vaughn Osterman and Reece Lewandowski

ABSTRACT. Narkewicz introduced a class of generalized arithmetical convolutions [Nar63]. Burnett and Osterman introduced the concept of an *A*-function to unify the treatment of these convolutions with generalized divisibility relations and generalizations of multiplicativity of arithmetical functions [BO19]. They also considered some sets of arithmetical functions that form groups under special convolutions. We generalize these results and prove necessary and sufficient conditions for these sets of arithmetical functions to form groups and rings under certain convolutions. Specifically, we introduce a special class of *A*-functions, which we call perfect, and prove that they correspond to rings of arithmetical functions with respect to the *A*-convolution and standard pointwise function multiplication. Finally, we introduce analogues of several of our results in the context of Davison *K*-convolutions that help illuminate possible future work.

CONTENTS

1. Introduction

In a recent paper by Burnett and Osterman [BO19], the authors investigated special cases of a class of arithmetical convolutions defined by Narkiewicz [Nar63] that permitted Abelian groups to be formed from certain sets of arithmetical functions. The authors used objects called *A*-**functions** to unify the treatment of divisibility relations, Narkiewicz-type arithmetical convolutions, and special sets of arithmetical functions determined by generalized multiplicativity rules:

Definition 1.1. *An A-**function** A is any map $A : \mathbb{N} \to 2^{\mathbb{N}}$ such that for all $n \in \mathbb{N}$, $A(n) \subseteq D(n)$, where $D(n)$ denotes the set of divisors of n.*

We have slightly augmented the definition of *A*-functions from that presented in [BO19] so that the *A*-functions correspond exactly to the convolutions of Narkiewicz type [Nar63]. We denote by $\mathbb{A}$ the set of all *A*-functions. For $A \in \mathbb{A}$ and $n \in \mathbb{N}$, we say d *A*-**divides** n if $d \in A(n)$. Furthermore, if f and g are arithmetical functions, then

$$(f *_A g)(n) = \sum_{d \in A(n)} f(d) g\left(\frac{n}{d}\right) \tag{1.1}$$

is called the *A*-**convolution** of f and g. Given $A \in \mathbb{A}$, we may talk about several relevant properties an *A*-function A may possess:

- If $1 \in A(n)$ for all $n \in \mathbb{N}$, then we call A **simple**.
- If $n \in A(n)$ for all $n \in \mathbb{N}$, then we call A **reflexive**.
- If $(m, n) = 1$ implies $A(m) \cdot A(n) = A(mn)$ for all m and n, then we call A **multiplicative**.
- If A is multiplicative and for all primes p and q, $p^b \in A(p^a)$ iff $q^b \in A(q^a)$, then we call A **homogeneous**.
- If A corresponds to a regular convolution, then we call A **regular**.
- If $d \in A(n)$ implies $\frac{n}{d} \in A(n)$, then we call A **symmetric**.
- If $d \in A(n)$ implies $A(d) \subseteq A(n)$, then we call A **transitive**.

1.1. Iteration of *A*-functions. We begin with the following definition from [BO19]:

Definition 1.2. *Let $A \in \mathbb{A}$ be simple. Then A^1 is the A-function determined by $A^1(n) := \{d \in D(n) \mid A(d) \cap A(\frac{n}{d}) = \{1\}\}$. We call A^1 the **iterate** of A.*

If A is simple, then we denote the k-fold application of the iterate procedure to A by A^k. Specifically, $A^k = (A^{k-1})^1$ for $k > 1$.

The following results are easily proven:

Proposition 1.3 (Properties of the iterate). *Let $A \in \mathbb{A}$ be simple. Then:*

- *A^1 is simple, reflexive, and symmetric.*
- *If A is multiplicative, then A^1 is multiplicative.*

Remark 1.4. *From this point on, unless otherwise noted, all A-functions will be assumed to be simple and reflexive.*

Example 1.5. D^1 *corresponds to the unitary divisibility relations. For all* $k \in \mathbb{N}$*, the* k*-ary divisibility relation, as origianally defined by* [Coh90]*, is* D^k*.*

For a detailed discussion of the k-ary divisibility relation and some of its relevant properties, we refer the reader to [BGSVB18, CH93, Coh90, Hau00]. The following generalizes a result from [BGSVB18] on the k-ary divisors:

Lemma 1.6 (Alternating Lemma)**.** *If A and B are simple A-functions such that $A \subseteq B$, then $A^1 \supseteq B^1$.*

Proof. For A and B as above, $m \in B^1(mn)$ iff $B(m) \cap B(n) = \{1\}$, but since $A \subseteq B$, $B(m) \cap B(n) = \{1\}$ implies $A(m) \cap A(n) = \{1\}$, we conclude that $m \in A^1(mn)$. $\qquad\square$

Cohen [Coh90] also introduced what he called the "infinitary divisibility relation" as a kind of limit process of the iterate. We denote this by Θ and give a characterization of Θ in Section 1.2. In fact, as we will see later, Θ plays a special role due to the property that $\Theta^1 = \Theta$, implicitly proven in [Coh90] and further investigated in [BGSVB18, BO19].

1.2. Structured A-functions. In [BO19], the authors introduced the concept of a **structured** A-function as a generalization of the infinitary divisibility relation. This was done in two different ways: The first made use of certain types of unique factorization properties of the integers, while the second utilized what they called "Cohen Triangles." We will present the second definition here.

Definition 1.7. *Let A be multiplicative. For each prime number p, the* ***Cohen Triangle of*** A ***at*** p *is the infinite array $C_{A,p}$ (indexed from 0) such that the (a,b) entry of $C_{A,p}$ is 1 if $p^b \in A(p^a)$, and 0 otherwise.*

From the definitions of A-functions and the Cohen triangle, we observe that $C_{A,p}$ is a lower-triangular binary matrix. If A is homogeneous, we may simply refer to *the* Cohen triangle of A and write C_A, since it will be independent of p. However, we may write C_A in general whenever the specified prime in understood in context.

The definition of Cohen Triangles was motivated by Cohen's introductory paper on the k-ary divisibility relations [Coh90] in which several images of lower-triangular arrays were printed, displaying the inspiration behind the infinitary divisibility relation. The authors have found this graphical representation of multiplicative (homogeneous) A-functions to be invaluable with regards to the results of this manuscript.

Definition 1.8. *Let $m \in \mathbb{N}$. We call an $m \times m$ lower-triangular binary matrix a* **structure** *of size m.*

Where it is relevant, we will use properties defined for A-functions to describe structures, with the mediary for these properties being the obvious

connection between Cohen triangles and structures. For example, if we say a structure is simple, this means the first column of the structure is all 1s, and if a structure is reflexive, it means the diagonal of the structure is all 1s. In [BO19], the authors used structures to create so-called **structured A-functions**. We now present a generalization of this process.

Definition 1.9. *Let A be a multiplicative A-function and γ be a simple structure. Define A^γ to be the multiplicative A-function such that for all primes p, $C_{A^\gamma,p} = C_{A,p} \otimes \gamma$, the tensor product of $C_{A,p}$ with γ. Equivalently, if γ is of size m, $0 \leqslant r < m$, and $0 \leqslant s < m$, then $p^{mb+s} \in A(p^{ma+r})$ if a and only if $p^b \in A(p^a)$ and $\gamma_{r,s} = 1$.*

To make the interface between structures and A-functions easier, we may write $b \in \gamma(a)$ if $\gamma_{a,b} = 1$ and $b \notin \gamma(a)$ otherwise.

Example 1.10. *Let $\Delta = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, using notation from [BO19]. Then it can be easily deduced from the Cohen Triangle of Θ that $\Theta^\Delta = \Theta$.*

Definition 1.11. *A structure γ is **full** if every entry on and below the diagonal of γ is 1. We denote the (unique) $m \times m$ full structure by $\tilde{\gamma}_m$.*

We now give an equivalent definition of structured A-functions, as defined in [BO19]:

Definition 1.12. *Let B be a homogeneous multiplicative A-function. We say B is **structured** if there exists a finite or infinite sequence of A-functions $\{A_k\}_{k=0}^N$, where $N \in \mathbb{N}_0 \cup \{\infty\}$, such that $A_0 = B$ and for all k, there is some m_k such that $A_k = (A_{k+1})^{\tilde{\gamma}_{m_k}}$, and with $A_N = D$ if A is finite. We say B is **finitely** or **infinitely structured** according to whether N is finite or infinite.*

A structured A-function B can then be viewed as a finite or infinite tensor product of full structures. Note that if $N = 0$ in the definition above, then $A = D$.

Structured A-functions may also be defined as a type of unique factorization as in [BO19]. Specifically, for any p, the B**-primes**, denoted by $\mathbb{P}_B$, are prime powers of the form p^{t_k}, where $t_k = \prod_{i=0}^{k-1} m_i$, and the values of m_k are as in Definition 1.12. The **character** of this B-prime, as defined in [BO19], is $\chi_B\left(p^{t_k}\right) = m_k$. Then, any $n \in \mathbb{N}$ can be factored uniquely as

$$n = q_1^{a_1} q_2^{a_2} \cdots q_s^{a_s}, \tag{1.2}$$

where for all i, q_i is a B-prime and $1 \leqslant a_i < \chi_B(q_i)$. Then, $B(n)$ conststs of all positive integers d of the form

$$d = q_1^{b_1} q_2^{b_2} \cdots q_s^{b_s}, \tag{1.3}$$

where for all i, $0 \leqslant b_i \leqslant a_i$.

1.3. Structures and the iterate. It should be noted that the above definition for A^γ applies only to homogeneous structured A-functions. By this, we mean that it applies the same structure equally to each Cohen Triangle of A at p. A more general definition was described in [BO19] in which a different structure can be applied to each Cohen Triangle of A at p, but due to the awkwardness of representing this generalization compared to the ease of understanding exactly what it entails, we have decided not to elaborate extensively on it. In fact, our results will make no use of the homogeneity properties of A-functions, and those that use the homogeneous definition of A^γ can be readily extended in an obvious manner to a broader class of results.

In the discussion that follows, we define the iterate of a structure γ of size m in a natural manner: If A is an A-function such that the $m \times m$ upper-left submatrix of A is γ, then we define the iterate γ^1 to be the $m \times m$ upper-left submatrix of A^1. In particular, $b \in \gamma^1(a)$ if and only if row b and row a have only their first entry in common. We will denote this by $\gamma(b) \cap \gamma(a) = \{0\}$.

Lemma 1.13. *Suppose A is a multiplicative A-function and $m \in \mathbb{N}$. Then,* $(A^{\tilde\gamma_m})^1 = (A^1)^{\tilde\gamma_m^1}$.

Proof. Suppose A and $\tilde\gamma$ are as above. Since all of the A-functions in the statement of the Lemma, including A and A^1, are multiplicative, we only need to prove the Lemma for the prime powers p^a. We do this by demonstrating that the conditions under which $p^{bm+s} \in (A^1)^{\tilde\gamma^1}(p^{am+r})$ are the same as the conditions under which $p^{bm+s} \in (A^{\tilde\gamma})^1(p^{am+r})$. Again, $0 \leqslant r < m$ and $0 \leqslant s < m$.

$p^{bm+s} \in (A^1)^{\tilde\gamma^1}(p^{am+r})$ if and only if $p^b \in A^1(p^a)$ and $s \in \tilde\gamma^1(r)$, meaning $s = 0$ or r. On the other hand, $p^{bm+s} \in (A^{\tilde\gamma})^1(p^{am+r})$ if and only if $A^{\tilde\gamma}(p^{bm+s}) \cap A^{\tilde\gamma}(p^{cm+t}) = \{1\}$, where $c = a - b$ and $t = r - s$. If $t < 0$, corresponding to $r < s$, then we can conclude that $p^{bm+s} \notin (A^{\tilde\gamma})^1(p^{am+r})$, using the fact that $\tilde\gamma$ is full to see that $\tilde\gamma(s) \cap \tilde\gamma(t) \neq \{0\}$. For $s \leqslant r$, either $s = 0$ or $s = r$ is necessary for $p^{bm+s} \in (A^{\tilde\gamma})^1(p^{am+r})$, in addition to the condition that $p^b \in A^1(p^a)$. These two conditions are easily seen to be sufficient, and equivalent to the conditions under which $p^{bm+s} \in (A^1)^{\tilde\gamma^1}(p^{am+r})$. $\square$

Corollary 1.14. *For any structured A-function B, $B^1 \subseteq B$.*

Proof. This is a simple consequence of Lemma 1.13 applied to the sequence corresponding to this structured A-function as defined in Definition 1.12, together with the fact that for any full structure $\tilde\gamma_m$, $\tilde\gamma_m^1 \subseteq \tilde\gamma_m$. If B is infinitely structured, we can use the property that for any $a \in \mathbb{N}$, there is a sufficiently large N so that the first a rows of the Cohen triangle for B are unchanged by truncating the structure sequence to the first N terms. $\square$

Even if γ is not full, we have the following weaker version of Lemma 1.13:

Lemma 1.15. *Suppose A is a multiplicative A-function and γ is a simple structure. Then, $(A^\gamma)^1 \supseteq (A^1)^{\gamma^1}$.*

Proof. Here we may proceed exactly as in the proof of Lemma 1.13, with the exception that now we cannot rule out the case where $r < s$. However, the proof for the case where $s \leqslant r$ is still valid. $\qquad \square$

2. Classes of arithmetical functions and A-functions

We begin by introducing a generalization of multiplicativity of arithmetical functions. This was first considered by Yocom [Yoc73] in the context of Narkiewicz regular A-functions and more recently in [BO19] using the same terminology as below:

Definition 2.1. *Let $B \in \mathbb{A}$ and $f : \mathbb{N} \to \mathbb{R}$ be not identically zero. If $m \in B(mn)$ implies $f(m)f(n) = f(mn)$, then we say f is **class-**B and write $f \in c(B)$, the set of all class-B arithmetical functions.*

*Let $A \in \mathbb{A}$. If $m \in B(mn)$ implies $A(m) \cdot A(n) = A(mn)$, then we say A is **class-**B and write $A \in C(B)$, the set of all class-B A-functions.*

Note that, without loss of generality, we may assume B is symmetric. In fact, for the remainder of this section, we will assume this whenever we write $c(B)$ or $C(B)$. Furthermore, if A is reflexive and $A \in C(B)$, then $m \in B(mn)$ implies $m \in A(mn)$.

Example 2.2. *$c(D^1)$ is the set of all multiplicative arithmetical functions, while $c(D)$ is the set of all completely multiplicative arithmetical functions. The set of so-called "I-multiplicative functions" of Cohen [CH93] is $c(\Theta)$.*

Up to this point, the authors have not seen an A-function explored in the literature, either implicitly or explicitly, that is not multiplicative. Curiously, $\Theta \in C(\Theta)$. We will discuss the implications of this condition below.

We generally use the notation that capital Latin and Greek letters are, or pertain directly to, A-functions, whereas lowercase letters pertain to arithmetical functions and integers, with the context always specified. The following list of properties should serve to familiarize oneself with the functionality of classes of arithmetical functions and A-functions:

Proposition 2.3 (Properties of classes of arithmetic functions and A-functions)**.**
Let $A, B, B_1, B_2 \in \mathbb{A}$ with B_1, B_2 symmetric. Then:

(1) *If $B_1 \subseteq B_2$, then $c(B_1) \supseteq c(B_2)$ and $C(B_1) \supseteq C(B_2)$.*
(2) *If $B_1 \subseteq B_2$ and $B_1 \in C(B_2)$, then $B_1 = B_2$.*
(3) *If $f \in c(B)$ for some $B \in \mathbb{A}$, then $f(1) = 1$.*
(4) *For every arithmetical function f such that $f(1) = 1$, there exists a unique symmetric A-function B such that $m \in B(mn)$ is equivalent to $f(m)f(n) = f(mn)$.*
(5) *If $A \in C(A)$, then A is transitive.*
(6) *If $\tau_A \in c(B^1)$, then for all completely multiplicative f, $f *_A f \in c(B^1)$.*
(7) *For every A-function $A \in \mathbb{A}$, there exists a symmetric A-function B such that $m \in B(mn)$ is equivalent to $A(m) \cdot A(n) = A(mn)$.*

Definition 2.4. *Suppose B is a symmetric A-function and $f \in \mathcal{A}$ is such that $m \in B(mn)$, $a \in B(m)$, and $b \in B(n)$ together imply $f(ab) = f(a)f(b)$ for all m, n, a, b. We call f B-**split**. If under the same assumptions we have $B(ab) = B(a)B(b)$, then we say B is **split**.*

Observe that if B is split, then every $f \in c(B)$ is B-split as well. We remark that these definitions of being "split" have not appeared in the literature before, and introduce one more novel definition:

Definition 2.5. *Suppose $A, B \in \mathbb{A}$ are symmetric A-functions and $f : \mathbb{N} \to \mathbb{R}$ is an arithmetical function such that $m \in B(mn)$, $a \in A(m)$, and $b \in A(n)$ together imply $f(ab) = f(a)f(b)$. We call f (A, B)-**split**. If, under the same conditions, for $H \in \mathbb{A}$ we have $H(ab) = H(a)H(b)$, then we say H is (A, B)-**split**.*

Example 2.6. *D^1 and D are both split. In general, if B is a structured A-function of [BO19], then both B and B^1 are split.*

Example 2.7. *For any multiplicative $A \in \mathbb{A}$ and multiplicative $f : \mathbb{N} \to \mathbb{R}$, f is (A, D^1)-split.*

Lemma 2.8. *If B is split and $B \in C(B)$, then B is (B, B)-split.*

Proof. Since B is split, we have that for all $m \in B(mn)$, $a \in B(m)$, and $b \in B(n)$, it follows that $a \in B(ab)$. But since $B \in C(B)$, $a \in B(ab)$ implies that $B(ab) = B(a)B(b)$. $\qquad\square$

As in the classical case of the Dirichlet convolution of two multiplicative functions being multiplicative, we may speak of an A-convolution $*_A$ as "preserving" some class of arithmetical functions $c(B)$, in the sense that if $f, g \in c(B)$, then $f *_A g \in c(B)$.

Theorem 2.9. *Suppose $A, B \in \mathbb{A}$ are symmetric A-functions. Then, $*_A$ preserves $c(B)$ if and only if*

(1) *$A \in C(B)$,*
(2) *$\tau_A \in c(B)$, and*
(3) *for every $f \in c(B)$, f is (A, B)-split.*

Proof. Suppose all three conditions hold, and let $m \in B(mn)$ and $f, g \in c(B)$. Then,

$$(f *_A g)(mn) = \sum_{d \in A(mn)} f(d)g\left(\frac{mn}{d}\right) = \sum_{d \in A(m) \cdot A(n)} f(d)g\left(\frac{mn}{d}\right),$$

using the fact that $A \in C(B)$. Furthermore, $\tau_A \in c(B)$, so $|A(mn)| = \tau_A(mn) = \tau_A(m)\tau_A(n) = |A(m)||A(n)|$. Since $A(mn) = A(m) \cdot A(n)$, this means that each $d \in A(mn)$ may be uniquely represented as $d = ab$ for $a \in A(m)$ and $b \in A(n)$. Since f and g are (A, B)-split, this means the above sum is equal to

$$\sum_{a \in A(m)} \sum_{b \in A(n)} f(ab)g\left(\frac{mn}{ab}\right) = \sum_{a \in A(m)} \sum_{b \in A(n)} f(a)f(b)g\left(\frac{m}{a}\right)g\left(\frac{n}{b}\right) = (f *_A g)(m)(f *_A g)(n).$$

To show the converse, first note that $\tau_A \in c(B)$ is necessary since $\tau_A = u *_A u$, and $u \in c(B)$. Now consider $\sigma_{A,r} := \mathrm{id}^r *_A u$ for real r, with $\mathrm{id}^r(n) := n^r$. Since id^r and u are completely multiplicative (and hence in $c(B)$), $\sigma_{A,r} \in c(B)$. Furthermore, for all $r \in \mathbb{R}$,

$$\sum_{d \in A(mn)} d^r = \left(\sum_{a \in A(m)} a^r \right) \left(\sum_{b \in A(n)} b^r \right).$$

This can only happen if each d is equal to some ab, and each product ab is equal to some d. This means that $A \in C(B)$. Finally, suppose $f \in c(B)$ is not (A, B)-split and let $f_r := f \cdot \mathrm{id}^r$. Since $f \in c(B)$ and $\mathrm{id} \in c(B)$, $f_r \in c(B)$. Notice that $f_r(mn) - f_r(m)f_r(n)$ cannot be identically zero by the same reasoning as above. Thus, condition 3 is necessary as well. $\square$

Corollary 2.10 (Theorem 10 of [CH93])**.** *If f and g are in $c(\Theta)$, then $f *_\Theta g \in c(\Theta)$.*

Proof. Given the formula for τ_Θ given in [Coh90], it is not difficult to show that $\tau_\Theta \in c(\Theta)$. Furthermore, the fact that $\Theta \in C(\Theta)$ was demonstrated in [BO19] using $\Theta^1 = \Theta$ together with their Lemma 4.4. Finally, by Lemma 2.8, Θ is (Θ, Θ)-split, so every $f \in c(\Theta)$ is (Θ, Θ)-split, giving us our result. $\square$

Corollary 2.11. *If $*_A$ is associative in addition to the conditions of Theorem 2.9 being satisfied for $A, B \in \mathbb{A}$, then $(c(B), *_A)$ forms an Abelian group.*

Proof. As proven in [Nar63], $*_A$ is associative if and only if the conditions $d \in A(m)$ and $m \in A(n)$ are together equivalent to $d \in A(n)$ and $\frac{m}{d} \in A\left(\frac{n}{d}\right)$. By Theorem 2.9, $c(B)$ is closed under $*_A$. Moreover, the identity under $*_A$ is ι, defined by $\iota(1) = 1$ and for all $n > 1$, $\iota(n) = 0$. $\iota \in c(B)$ for any $B \in \mathbb{A}$. Finally, since A is reflexive, an inverse function to f may be defined.

To show that this inverse function is class-B, it suffices to prove an analogue of Theorem 2.15 of Apostol [Apo13], which can be easily done in the general case. Since A is symmetric, the A-convolution $*_A$ is commutative, so we are done. $\square$

Two special cases for structured A-functions were proven in [BO19]. Specifically, for any structured B, $(c(B^1), *_B)$ and $(c(B^1), *_{B^1})$ are both Abelian groups. The case of $B = B^1 = \Theta$ constitutes Remark 7 of [CH93].

3. Perfect A-functions

Lemma 3.1. *If $A \in C(A)$ and A is split, then $*_A$ is associative.*

Proof. Suppose $d \in A(m)$ and $m \in A(n)$. Since A is transitive, $d \in A(n)$, and since A is symmetric, $\frac{m}{d} \in A(m)$. Since $\frac{n}{m} \in A\left(\frac{n}{m}\right)$, $m \in A(n)$, and A is split, this means $A\left(\frac{n}{d}\right) = A\left(\frac{m}{d}\right) \cdot A\left(\frac{n}{m}\right)$, so $\frac{m}{d} \in A\left(\frac{n}{d}\right)$. Thus, the conditions for associativity of $*_A$, as proven in [Nar63], are satisfied.

Conversely, suppose $\frac{m}{d} \in A\left(\frac{n}{d}\right)$ and $d \in A(n)$. Since A is split, $\frac{m}{d} \in A\left(\frac{n}{d}\right)$ and $d \in A(m)$. Furthermore, since $d \in A(d)$, $A(d) \cdot A\left(\frac{n}{d}\right) = A(n)$ and $\frac{m}{d} \in A\left(\frac{n}{d}\right)$, so $m \in A(n)$. $\qquad\square$

Definition 3.2. *If A is such that $A \in C(A)$ and $A^1 \supseteq A$, then we call A **perfect**.*

3.1. Conditions and examples for perfect A-functions.

Definition 3.3. *For any A-function A, we call an integer $n > 1$ A-**primitive** if $A(n) = \{1, n\}$. We denote the set of all A-primitive positive integers by $\mathbb{P}_A$. For any $n \in \mathbb{N}$, we denote the set of all A-primitive elements of $A(n)$ by $P_A(n)$.*

We demonstrate the importance of the A-primitive notion with the following theorem, which gives an alternative definition of perfect A-functions.

Theorem 3.4. *An A-function A is perfect if and only if for all $n \in \mathbb{N}$,*

$$A(n) = \prod_{q \in P_A(n)} \{1, q\}, \tag{3.1}$$

and for all $d \in A(n)$, $P_A(d)$ is the unique subset of $P_A(n)$, the product of whose elements is equal d, where the product of sets denotes the set of all possible pointwise products and the empty product of sets is equal to $\{1\}$.

Proof. Let $A \in \mathbb{A}$ be such that the given conditions hold, and m and n be such that $m \in A(mn)$. Then, (3.1) can be decomposed as

$$A(mn) = \left(\prod_{q \in P_A(m)} \{1, q\}\right)\left(\prod_{q \in P_A(n)} \{1, q\}\right) = A(m) \cdot A(n),$$

so $A \in C(A)$. Also, since $mn \in A(mn)$, (3.1) requires $P_A(m)$ and $P_A(n)$ to be disjoint. Therefore, from the product formula, $A(m) \cap A(n) = \{1\}$, so $A^1 \supseteq A$ and A is perfect.

We prove the other implication by induction on n. For $n = 1$, this is trivial. Let $n \geqslant 2$ and suppose the statement is true for all lesser values of n. If $A(n) = \{1, n\}$, then this is trivial. Otherwise, let $d \in A(n)$ with $d \neq 1, n$. Since $A \in C(A)$,

$$A(n) = A(d) \cdot A\left(\frac{n}{d}\right) = \left(\prod_{q \in P_A(d)} \{1, q\}\right)\left(\prod_{q \in P_A\left(\frac{n}{d}\right)} \{1, q\}\right).$$

Since $A^1 \supseteq A$, $A(d) \cap A\left(\frac{n}{d}\right) = \{1\}$, which implies that $P_A(d) \cap P_A\left(\frac{n}{d}\right) = \varnothing$, so this product simplifies to (3.1). The fact that $P_A(d) \subseteq P_A(n)$ is evident from this.

To prove that $P_A(d)$ is the unique subset of $P_A(n)$ whose product of elements is n, suppose $Q \subseteq P_A(n)$ is such that the product of elements in Q

is d. Let

$$c = d \left(\prod_{q \in P_A(d) \cap Q} q \right)^{-1} = \prod_{q \in P_A(d) \smallsetminus Q} q = \prod_{q \in Q \smallsetminus P_A(d)} q.$$

By the inductive hypothesis, since $P_A(d) \smallsetminus Q \subseteq P_A(d)$, we have $c \in A(d)$. Similarly, $Q \smallsetminus P_A(d) \subseteq P_A(n) \smallsetminus P_A(d) = P_A\left(\frac{n}{d}\right)$, so $c \in A\left(\frac{n}{d}\right)$. However, $A(d) \cap A\left(\frac{n}{d}\right) = \{1\}$, so $c = 1$ and hence $P_A(d) = Q$. □

Corollary 3.5. *If $f \in c(A)$, then f is uniquely determined by its values at A-primitive elements.*

Proof. Since each n may be uniquely factored into distinct A-primitive elements q, we see that

$$f(n) = \prod_{q \in P_A(n)} f(q). \tag{3.2}$$

□

We give examples of some perfect A-functions appearing in the literature.

Example 3.6. *Let B be a structured A-function and $\chi_B(q)$ be the character of any B-prime q as defined in Section 1.2. Then, B^1 is perfect, and $\mathbb{P}_{B^1}$ is the set of prime powers of the form q^a, where q is a B-prime and $1 \leqslant a < \chi_B(q)$.*

Example 3.7. *Let $Z \in \mathbb{A}$ be the A-function defined by $Z(n) = \{1, n\}$. Then $Z \in C(Z)$ and since $Z \subseteq D^1$, $Z^1 \supseteq D^2 \supseteq D^1 \supseteq Z$ by the alternating property of the iterate. Hence $Z \subseteq Z^1$, so Z is perfect. Furthermore, $P_Z = \mathbb{N} \smallsetminus \{1\}$ and $c(Z)$ is the set of all arithmetical functions f such that $f(1) = 1$. Note that Z is not multiplicative.*

Example 3.8. *Consider D^3, the multiplicative A-function corresponding to the 3-ary divisibility relation of Cohen [Coh90]. At a prime power p^a, with $a \neq 3$ or 6, $D^3(p^a) = \{1, p^a\}$. If $a = 3$, then $D^3(p^3) = \{1, p, p^2, p^3\}$ and if $a = 6$, then $D^3(p^6) = \{1, p^2, p^4, p^6\}$. By Theorem 2 of [BGSVB18], $(D^3)^1 = D^4 \supseteq D^3$.*

To see that $D^3 \in C(D^3)$, observe that $p \in D^3(p^3)$ and $D^3(p) \cdot D^3(p^2) = D^3(p^3)$. Furthermore, $p^2 \in D^3(p^6) = D^3(p^2) \cdot D^3(p^4)$, and all other conditions to check are trivially satisfied. Hence, D^3 is perfect.

Given that D^1 and D^3 are perfect, one might wonder whether any of the other k-ary divisibility relations are perfect. Theorem 2 of [BGSVB18] indicates that $D^{2k+1} \subseteq D^{2k+2}$ for all k; however, the question remains if $D^{2k+1} \in C(D^{2k+1})$ for $k > 1$. We pose this as an open question. A necessary condition for this is that the $*_{D^{2k+1}}$ convolution is associative for some $k > 1$. Whether this weaker condition is the case is a question posed by Haukkanen [Hau00] in his work on the k-ary convolutions.

Example 3.9. *Let $F = \{1, 2, 3, 5, 8, 13, \cdots\}$ be the set of Fibonacci numbers. By Zeckendorf's Theorem [Zec72], for all $a \in \mathbb{N}$, there is a unique subset $F_a \subseteq F$ containing non-consecutive Fibonacci numbers whose sum of elements is a. Therefore, we can define a multiplicative A-function such that for any prime power p^a, $A(p^a) = \{p^b : F_b \subseteq F_a\}$. Then A is perfect with $\mathbb{P}_A = \{p^r : p \in \mathbb{P}, r \in F\}$.*

3.2. Rings of class-A arithmetical functions. The below theorem generalizes the following oft-unstated result: That the set of multiplicative arithmetical functions forms a commutative *ring* with unity under the unitary convolution.

Theorem 3.10. *Let A be an A-function. Then, $(c(A), *_A, \cdot)$ forms a commutative ring with unity, where $\cdot$ is the usual pointwise multiplication of functions, if and only if A is perfect.*

Proof. Suppose A is perfect. We show that A is split as well. Suppose that $m \in A(mn)$, $a \in A(m)$, and $b \in A(n)$, where a and b are both not 1. Since $A^1 \supseteq A$, $A(m) \cap A(n) = \{1\}$, which implies that $P_A(m) \cap P_A(n) = \varnothing$. By Theorem 3.4, $P_A(a) \subseteq P_A(m)$ and $P_A(b) \subseteq P_A(n)$, so $P_A(a) \cap P_A(b) = \varnothing$ and $P_A(ab) = P_A(a) \cup P_A(b)$. Therefore, $P_A(a) \subseteq P_A(ab)$, so $a \in A(ab)$. Therefore, A is split. By Theorem 2.9, $*_A$ preserves $c(A)$, and by Lemma 3.1, $*_A$ is associative. Since A is symmetric, $*_A$ is also commutative. Hence, $c(A)$ forms an Abelian group under $*_A$.

To see that this group may be endowed with a ring structure, observe that the product of two arithmetical functions in $c(A)$ is still in $c(A)$. Finally, for any $f, g, h \in c(A)$,

$$(f \cdot (g *_A h))(n) = ((g *_A h) \cdot f)(n)$$

$$= f(n) \sum_{d \in A(n)} g(d) h\left(\frac{n}{d}\right)$$

$$= \sum_{d \in A(n)} f(d) g(d) f\left(\frac{n}{d}\right) h\left(\frac{n}{d}\right)$$

$$= ((f \cdot g) *_A (f \cdot h))(n),$$

from which we may conclude the distributive law holds. The unity of this ring is the function $u(n) = 1$.

We now show that the assumptions $A \in C(A)$ and $A^1 \supseteq A$ are necessary. By Theorem 2.9, in order for $*_A$ to preserve $c(A)$, $A \in C(A)$ and $\tau_A \in c(A)$ are necessary. However, $\tau_A \in c(A)$ means that for every $m \in A(mn)$, $\tau_A(m)\tau_A(n) = \tau_A(mn)$. Since $A(mn) = A(m) \cdot A(n)$, this requires a bijection between $A(mn)$ and $A(m) \cdot A(n)$, so $A(m) \cap A(n) = \{1\}$. Hence, $A^1 \supseteq A$. $\square$

Corollary 3.11. *If A is perfect and μ_A is the inverse of u under $*_A$, then for all n, $\mu_A(n) \in \{-1, 1\}$.*

Proof. Since $(c(A), *_A)$ is a group, $\mu_A \in c(A)$, so it is determined entirely by its value on A-primitive elements by Corollary 3.5. However, $\mu_A(q) = -1$ for all $q \in \mathbb{P}_A$. Hence, $\mu_A(n) = (-1)^{|P_A(n)|}$. $\qquad\square$

Corollary 3.12. *If B is a structured A-function, then $(c(B^1), *_{B^1}, \cdot)$ is a commutative ring.*

Proof. Since $c(B^1, *_{B^1})$ is an Abelian group, $B^1 \in C(B^1)$. By Corollary 1.14, $B^1 \subseteq B$, so by Lemma 1.6, $B^2 \supseteq B^1$. Thus B^1 is perfect. $\qquad\square$

The following result demonstrates that these special rings are all isomorphic to each other:

Proposition 3.13. *Suppose $(c(A), *_A, \cdot)$ forms a commutative ring with unity as in Theorem 3.10. Then $(c(A), *_A, \cdot)$ is isomorphic to $(\mathcal{A}, +, \cdot)$, the set of all arithmetical functions under usual addition and multiplication operations.*

Proof. Let $\mathbb{P}_A$ be the set of A-primitive elements. Note that for any perfect A, $\mathbb{P}_A$ is infinite since it includes every prime number. Let $\psi : \mathbb{P}_A \to \mathbb{N}$ be any bijection and $\Phi : \mathcal{A} \to c(A)$ be given by

$$\Phi(f)(n) = \prod_{q \in P_A(n)} f(\psi(q)), \tag{3.3}$$

with $\Phi(f)(1) = 1$.

First we demonstrate this is a homomorphism of rings. Note that by Corollary 3.5, the image on any $f \in \mathcal{A}$ will be class-A, since we are defining $\Phi(f)$ on A-primitive elements. If $f, g \in \mathcal{A}$ then,

$$\Phi(f + g)(n) = \prod_{q \in P_A(n)} (f(\psi(q)) + g(\psi(q))).$$

By distributing this product,

$$\Phi(f + g)(n) = \sum_{d \in A(n)} \left(\prod_{q \in P_A(d)} f(\psi(q)) \right) \left(\prod_{q \in P_A\left(\frac{n}{d}\right)} g(\psi(q)) \right)$$

$$= \sum_{d \in A(n)} \Phi(f)(d) \Phi(g)\left(\frac{n}{d}\right) = (\Phi(f) *_A \Phi(g))(n).$$

For products,

$$\Phi(f \cdot g)(n) = \prod_{q \in P_A(n)} (f \cdot g)(\psi(n))$$

$$= \left(\prod_{q \in P_A(n)} f(\psi(n)) \right) \left(\prod_{q \in P_A(n)} g(\psi(n)) \right)$$

$$= \Phi(f)(n) \Phi(g)(n).$$

Finally, $\Phi(u)(n) = 1$ for all n, so unity is preserved and Φ is a homomorphism of rings. Since the identity in $c(A)$ under $*_A$ is ι, it is easy to see that

$\Phi(f) = \iota$ if and only if f is the zero function, so Φ is injective. To see that Φ is surjective, note that since $f \in c(A)$ is determined entirely by its value at A-primitive numbers, we may create any $f \in c(A)$ by letting $g \in \mathcal{A}$ be such that $f(q) = g(\psi(q))$ for all A-primitive q. Hence, Φ is a bijection. $\square$

By the above result, all rings of the form $(c(A), *_A, \cdot)$ for perfect A are isomorphic to each other. In the next section, we show that a large number of perfect A-functions may be obtained from a given perfect A-function through the application of special structures.

3.3. Perfect A-functions and structures.

Definition 3.14. *Let γ be a structure of size m. We say γ is **perfect** if it is the upper left $m \times m$ principal minor of the Cohen Triangle (at some prime p) of a perfect A-function A.*

Theorem 3.15. *Suppose $A \in \mathbb{A}$ is perfect and multiplicative, and γ is a perfect structure. Then A^γ is perfect.*

Proof. We prove that the conditions for $(A^\gamma)^1 \supseteq A^\gamma$ and $A \in C(A)$ are satisfied on the prime powers. Since A is multiplicative, this is sufficient. Let A be multiplicative and A and γ be perfect, with γ of size m. Then observe that $p^{mb+s} \in A^\gamma(p^{ma+r})$ if and only if $s \in \gamma(r)$ and $p^b \in A(p^a)$, but these conditions imply that $s \in \gamma^1(r)$ and $p^b \in A^1(p^a)$, so $p^{mb+s} \in (A^1)^{\gamma^1}(p^{ma+r})$, and hence $A^\gamma \subseteq (A^1)^{\gamma^1}$. However, by Lemma 1.15, $(A^\gamma)^1 \supseteq (A^1)^{\gamma^1}$, so $A^\gamma \subseteq (A^1)^{\gamma^1}$.

We now demonstrate that $A^\gamma \in C(A^\gamma)$ on the prime powers. Suppose $p^{mb+s} \in A(p^{ma+r})$ and consider $A^\gamma(p^{mb+s}) \cdot A^\gamma(p^{m(a-b)+r-s})$. Let $c = a - b$ and $t = r - s$. Since $s < r$, we need not be concerned with t being negative. Then, $A^\gamma(p^{mb+s}) \cdot A^\gamma(p^{mc+t}) = \{p^{mw+y} \,|\, p^w \in A(p^b), y \in \gamma(s)\} \cdot \{p^{mx+z} \,|\, p^x \in A(p^c), z \in \gamma(t)\} = \{p^{m(w+x)+y+z} \,|\, p^w \in A(p^b), p^x \in A(p^c), y \in \gamma(s), z \in \gamma(t)\}$. Each $p^{m(w+x)+y+z}$ in this set is some element $p^{mv+u} \in A^\gamma(p^{ma+r})$, where $v = w + x$ and $u = y + z$. Furthermore, each such prime power may be obtained in this way, so $A^\gamma \in C(A^\gamma)$, and thus A^γ is perfect. $\square$

Corollary 3.16. *Suppose $A \supsetneq \Theta$. Then, $(c(A), *_A)$ is not a group.*

Proof. Using Lemma 1.6, $A^1 \subseteq \Theta^1 = \Theta \subsetneq A$, contradicting the second premise of Theorem 2.9. $\square$

4. Extensions to the convolutions of Davison

In [Dav66], Davison introduced the so-called K**-convolution** of arithmetical functions f and g as follows:

Definition 4.1. *Let K be a complex-valued function of two positive integer inputs, the second of which is a divisor of the first. Let f and g be*

*arithmetical functions. Then, the K-**convolution** of f and g is given by*

$$(f *_K g)(n) := \sum_{d \mid n} K(n,d) f(d) g\left(\frac{n}{d}\right). \tag{4.1}$$

We call any such K a K-**function**. It is clear that any Narkiewicz A-convolution A is given by a Davison K-convolution K_A where

$$K_A(n,d) := \begin{cases} 1 & d \in A(n) \\ 0 & d \notin A(n) \end{cases} \tag{4.2}$$

Definition 4.2. *If K is a K-function, we define the **support** of K to be the A-function A_K such that $d \in A_K(n)$ if and only if $K(n,d) \neq 0$.*

See [Hau89] for a more recent treatment of Davison K-convolutions, including a natural generalization of multiplicativity in the context of K-convolutions.

One might wonder if, given that A-functions are in a sense special cases of K-functions, generalizations of our previous theorems might be obtained. In fact, a generalization of Theorem 2.9 may be naturally obtained. In particular, we may discuss the conditions under which a given K-convolution preserves class-B arithmetical functions for a specific $B \in \mathbb{A}$.

Theorem 4.3. *Let K be a K-function and $B \in \mathbb{A}$. Then, $*_K$ preserves $c(B)$ if and only if the following three conditions are satisfied:*

(1) *$A_K \in C(B)$.*
(2) *If $m \in B(mn)$, $a \in A_K(m)$, and $b \in A_K(n)$, then*

$$K(mn,d) = \sum_{\substack{a \in A_K(m) \\ b \in A_K(n) \\ ab=d}} K(m,a) K(n,b). \tag{4.3}$$

(3) *For every $f \in c(B)$, f is (A_K, B)-split.*

Proof. The sufficiency of these three conditions is obvious and can be proven in the same manner as we prove Theorem 2.9. To see the necessity of conditions 1 and 3, we may use the same trick as before, writing $f_r := f \cdot \mathrm{id}^r$ for $f \in c(B)$ and arriving at a contradiction if either condition fails. Finally, to see that condition 2 is required, consider both sides of the equation $(\mathrm{id}^r *_K u)(mn) = (\mathrm{id}^r *_K u)(m)(\mathrm{id}^r *_K u)(n)$ for $m \in B(mn)$. Since the coefficients in front of each d^r for $d \in B(mn)$ are not dependent on r, we must have equality of coefficients, which is exactly condition 2. $\square$

Corollary 4.4. *If $*_K$ is associative and A_K is reflexive and such that the conditions of Theorem 4.3 are satisfied, then $(c(B), *_K)$ forms a group.*

The case of $B = D^1$ was analyzed in [Dav66] and further in [Fot75], with the latter being in the context of rings of arithmetical functions under pointwise addition and K-convolution.

Corollary 4.5. *Let* $B(mn, m) := \prod_{p \mid n} \binom{\nu_p(mn)}{\nu_p(m)}$ *be the K-function determining the binomial convolution* $*_B$ *of* [Hau96]*. Then,* $(c(D), *_B)$ *forms an Abelian group.*

Proof. Observe that $A_B = D$, and since $D \in C(D)$, condition 2 of Theorem 4.3 is satisfied. Furthermore, every completely multiplicative function is (D, D)-split, so condition 3 is satisfied. Finally, the Chu-Vandermonde identity combined with the multiplicative nature of the formula for $B(mn, n)$ gives us condition 1. It is then a matter of determining that $*_K$ is associative; this was stated without proof in the original paper on K-convolutions by Davison [Dav66], but was verified in [Hau96] by Haukkanen. $\square$

See [TH09] for an extensive discussion of the binomial convolution and its many other properties.

Theorem 2.9 is thus a special case of Theorem 4.3 for $K(mn, m) = 0$ or 1 for all $m, n \in \mathbb{N}$: The second condition implies condition 2 of Theorem 2.9 and, when combined with the first condition, implies condition 1 of Theorem 2.9. The third condition is the same as condition 3 of Theorem 2.9.

We leave the following as an open problem: Under what conditions on $B \in \mathbb{A}$ does there exist a K-function K such that $*_K$ preserves $c(B)$? Clearly, if there exists such a K-function that together with B satisfies the premises of Theorem 4.3, then the question is answered, but the authors have not investigated to a great extent the conditions on specifically B under which such a K may satisfy these premises.

References

[Apo13]   APOSTOL, TOM M. Introduction to analytic number theory. *Springer Science & Business Media*, 2013. 340 pp. ISBN 978-1475755794. 1412

[BGSVB18] BURNETT, JOSEPH VADE; GRAYSON, SAM; SULLIVAN, ZACHARY; VAN NATTA, RICHARD; BANG, LUKE. Arithmetical functions associated with the k-ary divisors of an integer. *Int. J. Math. Math. Sci.* (2018), Art. ID 9349245, 7 pp. MR3816112, Zbl 06904507, doi: 10.1155/2018/9349245. 1407, 1414

[BO19]   BURNETT, JOSEPH VADE; OSTERMAN, OTTO VAUGHN. A generalization of the infinitary divisibility relation: algebraic and analytic properties. *Int. J. Number Theory* **15** (2019), no. 9, 1771–1792. MR4015513, Zbl 07114777, doi: 10.1142/S1793042119500994. 1405, 1406, 1407, 1408, 1409, 1410, 1411, 1412

[Coh90]   COHEN, GRAEME L. On an integer's infinitary divisors. *Math. Comp.* **54** (1990), no. 189, 395–411. MR0993927, Zbl 0689.10014, doi: 10.2307/2008701. 1407, 1412, 1414

[CH93]   COHEN, GRAEME L.; HAGIS, PETER, JR. Arithmetic functions associated with the infinitary divisors of an integer. *Internat. J. Math. Math. Sci.* **16** (1993), no. 2, 373–383. MR1210848, Zbl 0774.11006, doi: 10.1155/S0161171293000456. 1407, 1410, 1412

[Dav66]   DAVISON, THOMAS M. K. On arithmetic convolutions. *Canad. Math. Bull.* **9** (1966), 287–296. MR0200226, Zbl 0151.02804, doi: 10.4153/CMB-1966-037-x. 1417, 1418, 1419

[Fot75]   FOTINO, INGRID POPA. Generalized convolution ring of arithmetic functions. *Pacific J. Math.* **61** (1975), no. 1, 103–116. MR0401619, Zbl 0319.10007, doi:10.2140/pjm.1975.61.103. 1418

[Hau89]   HAUKKANEN, PENTTI. On the Davison convolution of arithmetical functions. *Canad. Math. Bull.* **32** (1989), no. 4, 467–473. MR1019413, Zbl 0643.10005, doi:10.4153/CMB-1989-067-4. 1418

[Hau96]   HAUKKANEN, PENTTI. On a binomial convolution of arithmetical functions. *Nieuw Arch. Wisk.* (4) **14** (1996), no. 2, 209–216. MR1402841, Zbl 0859.11005. 1419

[Hau00]   HAUKKANEN, PENTTI. On the k-ary convolution of arithmetical functions. *Fibonacci Quart.* **38** (2000), no. 5, 440–445. MR1796560, Zbl 0962.11006. 1407, 1414

[LS07]   LITSYN, SIMON; SHEVALEV, VLADIMIR. On factorization of integers with restrictions on the exponents. *Integers* **7** (2007), A33, 35 pp. MR2342191, Zbl 1125.11002.

[Nar63]   NARKIEWICZ, WŁADYSŁAW. On a class of arithmetical convolutions. *Colloq. Math.* **10** (1963), 81–94. MR0159778, Zbl 0114.26502, doi:10.4064/cm-10-1-81-94. 1405, 1406, 1412

[SST11]   STEUDING, RAZA; STEUDING, JÖRN; TÓTH LÁSZLÓ. A modified Möbius μ-function. *Rend. Circ. Mat. Palermo* (2) **60** (2011), no. 1–2, 13–21. MR2825685, Zbl 1247.11123, doi:10.1007/s12215-011-0022-x.

[TH09]   TÓTH, LÁSZLÓ; HAUKKANEN, PENTTI. On the binomial convolution of arithmetical functions. *J. Comb. Number Theory* **1** (2009), no. 1, 31–47. MR2681504, Zbl 1242.11008, arXiv:0806.0508. 1419

[Yoc73]   YOCOM, K. L. Totally multiplicative functions in regular convolution rings. *Canad. Math. Bull.* **16** (1973), 119–128. MR0325502, Zbl 0259.10002, doi:10.4153/CMB-1973-023-2. 1410

[Zec72]   ZECKENDORF, EDOUARD. Représentations des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas. *Bull. Soc. Roy. Sci. Liège* **41** (1972), 179–182. MR0308032, Zbl 0252.10011. 1415

(J. V. Burnett) DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF TEXAS AT DALLAS, RICHARDSON, TX 75080, UNITED STATES
Joseph.Burnett@utdallas.edu

(O. V. Osterman) DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF TEXAS AT DALLAS, RICHARDSON, TX 75080, UNITED STATES
Otto.Osterman@utdallas.edu

(R. Lewandowski) DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF TEXAS AT DALLAS, RICHARDSON, TX 75080, UNITED STATES
RWL170000@utdallas.edu