

# Product Hardy spaces associated with para-accretive functions and $Tb$ theorem

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ABSTRACT. We introduce the Hardy spaces associated with para-accretive functions in product domains and demonstrate their atomic decompositions as well as duality. Then, we establish the endpoint version of product  $Tb$  theorem with respect to our Hardy spaces and the dual spaces.

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## 1. Introduction

Calderón and Zygmund [2] introduced a class of convolution singular integral operators that generalize the Hilbert transform and Riesz transforms. The  $L^2$ -boundedness of such convolution operators follows from the Plancherel theorem. For non-convolution singular integral operators, David and Journé [5] gave a general criterion for the  $L^2$ -boundedness that is the remarkable  $T1$  theorem. Unfortunately, the  $T1$  theorem cannot be applied

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to the Cauchy integral on a Lipschitz curve defined by

$$C(f)(x) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{f(y)}{(x-y) + i(a(x) - a(y))} dy,$$

where the function  $a(x)$  satisfies the Lipschitz condition. Meyer first observed that  $C(b) = 0$  provided  $b(x) = 1 + ia'(x)$ . Therefore, if the function 1 in the  $T1$  theorem is allowed to be replaced by an accretive function  $b$  which is a bounded complex-valued function satisfying  $\text{Re } b(x) \geq \delta > 0$  almost everywhere, then this result would imply the  $L^2$ -boundedness of the Cauchy integrals on all Lipschitz curves. Replacing the function 1 by an accretive function  $b$ , McIntosh and Meyer [25] obtained a  $Tb$  theorem. Finally, David, Journé and Semmes [6] proved a new  $Tb$  theorem by replacing the function 1 by the so-called para-accretive functions  $b$ . To extend the  $Tb$  theorem to Hardy spaces, Han, Lee and Lin [13] introduced a new class of Hardy spaces associated to a para-accretive function  $b$ , denoted by  $H^p_b$ , which was given by those distributions such that their Littlewood-Paley  $g$ -functions associated to  $b$  belong to  $L^p$ , and showed the  $H^p_b$ -boundedness of singular integer operators.

By taking the space  $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$  along with two-parameter family of dilations  $(x, y) \mapsto (\delta_1 x, \delta_2 y), x \in \mathbb{R}^{n_1}, y \in \mathbb{R}^{n_2}, \delta_i > 0, i = 1, 2$ , instead of the classical one-parameter dilation, R. Fefferman and Stein [10] studied the product convolution singular integral operators which satisfy analogous conditions enjoyed by the double Hilbert transform defined on  $\mathbb{R} \times \mathbb{R}$ . Journé [22] generalized the product convolution singular integral operators to the product non-convolution singular integral operators and introduced a class of singular integral operators which coincides with the product convolution singular integral operators on product spaces. Moreover, Journé [22] proved the product  $T1$  theorem. Suppose  $b(x_1, x_2) = b_1(x_1)b_2(x_2)$ , where  $b_1$  and  $b_2$  are para-accretive functions on  $\mathbb{R}^{n_1}$  and  $\mathbb{R}^{n_2}$ , respectively. A *generalized singular integral operator* is a continuous linear operator  $T$  from  $bC^{\eta}_0(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  into  $(bC^{\eta}_0(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}))'$  for all  $\eta > 0$  if the kernel of  $T$  is a singular integral kernel and, for  $f_1, g_1 \in C^{\eta}_0(\mathbb{R}^{n_1})$  with  $\text{supp}(f_1) \cap \text{supp}(g_1) = \emptyset$  and  $f_2, g_2 \in C^{\eta}_0(\mathbb{R}^{n_2})$  with  $\text{supp}(f_2) \cap \text{supp}(g_2) = \emptyset$ ,

$$\begin{aligned} & \langle M_b T M_b f_1 \otimes f_2, g_1 \otimes g_2 \rangle \\ &= \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} b_2(x_2)b_1(x_1)g_1(x_1)g_2(x_2)K(x_1, x_2, y_1, y_2) \\ & \quad \times b_2(y_2)b_1(y_1)f_1(y_1)f_2(y_2)dx_1dx_2dy_1dy_2, \end{aligned}$$

where  $M_b$  denotes the multiplication operator by  $b$ ; that is,  $M_b f(x) = b(x)f(x)$ . Han, Lee and Lin [15] obtained the following product  $Tb$  theorem.

**Theorem A** ([15]). *Suppose that  $b_1$  and  $b_2$  are para-accretive functions on  $\mathbb{R}^{n_1}$  and  $\mathbb{R}^{n_2}$ , respectively,  $b(x_1, x_2) = b_1(x_1)b_2(x_2)$ , and  $T$  is a generalized*

singular integral operator. Then  $T$  and  $\tilde{T}$  are bounded on  $L^2(\mathbb{R}^{n_1+n_2})$  if and only if  $Tb, {}^tTb, \tilde{T}b, {}^t\tilde{T}b \in BMO(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  and  $M_bTM_b \in WBP$ .

Here  ${}^tTb, \tilde{T}b$  and  ${}^t\tilde{T}b$  denote the three types of adjoint and partial adjoint operators of  $T$ . For simplicity we do not repeat all the definitions here, for the detail of the definition we refer to [15]. We also note that a similar result of the product  $Tb$  theorem was obtained by Ou [28] as well.

In 2013, Han, Li and Lu [18] developed a satisfactory theory of multiparameter Hardy spaces in the framework of spaces of homogeneous type under only the doubling condition and some regularity assumption of the underlying metric spaces. Later Han, Li and Ward [19] established the theory of multiparameter Hardy spaces on space of homogeneous type in the sense of Coifman and Weiss, without assuming any extra conditions. Such a metric space of homogeneous type includes the model case of Carnot-Carathéodory spaces intrinsic to a family of vector fields satisfying Hörmander's condition of finite rank. Recently, Hart [21] presented a bilinear  $Tb$  theorem for singular operators, and proved the product Lebesgue space bounds for bilinear Riesz transform defined on Lipschitz curve as an application of his  $Tb$  theorem.

Consider the Cauchy integrals on product domain  $\mathbb{R} \times \mathbb{R}$

$$C_{\text{pro}}f(x_1, x_2) = \text{p.v.} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(y_1, y_2)}{((x_1 - y_1) + i(a_1(x_1) - a_1(y_1)))} \\ \times \frac{1}{((x_2 - y_2) + i(a_2(x_2) - a_2(y_2)))} dy_1 dy_2,$$

which is a particular testing example operator for product  $Tb$  theorem. It is well-known that  $C_{\text{pro}}$  is not bounded on Chang-Fefferman product Hardy space  $H^1(\mathbb{R} \times \mathbb{R})$ .

Thus, a natural question arises. What is the right version of product Hardy spaces associated to the para-accretive  $b_1(x) = 1 + ia'_1(x)$  and  $b_2(x) = 1 + ia'_2(x)$  such that the operator  $C_{\text{pro}}$  is bounded on these product Hardy spaces? In this paper we focus on the following

**Question 1:** Can one develop the product Hardy spaces  $H^p_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  associated to para-accretive functions  $b_1, b_2$  for  $p_0 < p < \infty, p_0 < 1$ ?

**Question 2:** Motivated by the duality results between product Hardy space  $H^1$  and product BMO on  $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$  of Chang and R. Fefferman, can we establish the duality theory for  $H^p_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}), p_0 < p \leq 1$ ?

**Question 3:** What is the analogous endpoint version of product  $Tb$  theorem for singular integral operators  $T$  when  $p = 1$  and  $p = \infty$  or more generally for  $p_0 < p \leq 1$ ?

These questions will be answered affirmatively. We will employ a unified approach to answer these questions. This approach is achieved by the following steps:

1. We first introduce new Banach spaces which are spaces of product test functions and distributions in our framework. These spaces on the one-parameter space  $\mathbb{R}^n$  associated with para-accretive function were introduced in [12], and on the spaces of test functions and distributions on product spaces of homogeneous type in the sense of Coifman and Weiss only satisfying the doubling condition and the regularity assumption were introduced in [18]. In this paper, we introduce spaces of test functions and distributions on product spaces associated with para-accretive functions.

2. We then establish discrete Calderón's identity on such product test function spaces. The classical Calderón's identity was first used by Calderón in [1]. Such an identity is a very powerful tool, in particular, in the theory of wavelet analysis. See [26] for more details. Using Coifman's decomposition of the identity operator, David, Journé and Semmes [6] provided a Calderón-type identity which is a key tool to prove the  $T1$  theorems on space of homogeneous type and the  $Tb$  theorem on  $\mathbb{R}^n$ . The continuous and discrete versions of Calderón's identities associated with para-accretive function were developed in [12], [13], [14] and [24]. In this paper, we provide discrete Calderón's identity on the product spaces associated with para-accretive functions. This identity will be the main tool for us to establish the whole product theory.

3. We next demonstrate the Plancherel-Pôlya-type (or sup-inf) inequality in the multiparameter setting. The classical Plancherel-Pôlya inequality says that the  $L^p$  norm of  $f$  whose Fourier transform has compact support is equivalent to the  $\ell^p$  norm of the restrictions of  $f$  at appropriate lattices. This kind of inequality was first proved in [13] on one-parameter space associated with para-accretive function. In this paper we prove such inequalities on product spaces associated with para-accretive functions. As an immediate consequence of the Plancherel-Pôlya inequality, the product Hardy space  $H^p_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  is well defined.

4. We then consider the atomic decomposition theory on product domains. The atomic decomposition for one variable space  $H^p_b$  is given by [13] and [14]. For product Hardy space, Chang and R. Fefferman provided the atomic decomposition of  $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ . Using the atomic decomposition, Han, Lee and Lin [15] proved the  $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  boundedness of Journé's product singular integrals. Lee [23] also obtained the theory in weighted product Hardy space. In this paper, we obtain the atomic decomposition in  $H^p_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \cap L^2(\mathbb{R}^{n_1+n_2})$ .

5. We develop the dual spaces of  $H^p_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ . Coifman and Meyer [27] introduced a new Hardy space  $H^1_b$  that can be defined by all functions  $f$  such that  $bf \in H^1$ . Similarly,  $BMO_b$ , the dual of  $H^1_b$ , is defined as follows: a function  $f \in BMO_b$  if and only if  $f = bg$ , where  $g \in BMO$ . For  $p < 1$ , the dual space of  $H^p_b$  is defined similarly via the dual space of the classical Hardy spaces  $H^p$ . Han, Li and Lu [17] established and studied the product type Carleson measure space  $CMO^p$  and proved that it is the dual of the product

$H^p$  for  $0 < p_0 < p \leq 1$  for some  $p_0$  on product spaces of homogeneous type. In this paper, we introduce the Carleson measure space  $CMO_{b_1 b_2}^p$  associated with the para-accretive functions  $b_1$  and  $b_2$ , and prove that the dual of the product Hardy space  $H_{b_1 b_2}^p$  studied in this paper can be characterized by  $CMO_{b_1 b_2}^p$ . In particular,  $CMO_{b_1 b_2}^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ , the dual of  $H_{b_1 b_2}^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ . For the definition of para-accretive functions, we refer to Section 2.

6. We finally establish endpoint version of  $Tb$  theorem for singular integral operators  $T$  on product spaces (see Definition 5.1). In this paper we apply vector-valued singular integral, Calderón's identity, Littlewood-Paley theory and the almost orthogonality together with Fefferman's rectangle atomic decomposition and Journé's covering lemma to show that Journé's product singular integrals are bounded on  $H_{b_1 b_2}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  and  $BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  under some conditions. To be more specific, we have the following

**Theorem 1.1.** *Suppose that  $b_i$  are para-accretive functions on  $\mathbb{R}^{n_i}$  for  $i = 1, 2$  and  $T$  is a singular integral operator in Journé's class with regularity exponent  $\varepsilon$  (see Definition 5.1). Let  $\max_{i=1,2} \left\{ \frac{n_i}{n_i + \varepsilon} \right\} < p \leq 1$ .*

- (1)  $T_1^*(b_1) = T_2^*(b_2) = 0$  if and only if  $T$  is bounded from  $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  to  $H_{b_1 b_2}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ .
- (2)  $T_1^*(1) = T_2^*(1) = 0$  if and only if  $TM_{b_1 b_2}$  is bounded from  $H_{b_1 b_2}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  to  $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ .
- (3)  $T_1^*(b_1) = T_2^*(b_2) = 0$  if and only if  $TM_{b_1 b_2}$  is bounded from  $H_{b_1 b_2}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  to  $H_{b_1 b_2}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ .

For the details of definition of  $T_1^*(1)$ ,  $T_2^*(1)$ ,  $T_1^*(b_1)$  and  $T_2^*(b_2)$ , we refer to Section 5.

By the duality of  $H^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  with  $BMO(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  and the duality of  $H_{b_1 b_2}^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  with  $BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ , we also have

**Theorem 1.2.** *Suppose that  $b_i$  are para-accretive functions on  $\mathbb{R}^{n_i}$  for  $i = 1, 2$  and  $T$  is a singular integral operator in Journé's class with regularity exponent  $\varepsilon$ .*

- (1)  $T_1(b_1) = T_2(b_2) = 0$  if and only if  $T$  admits a bounded extension from  $BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  to  $BMO(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ .
- (2)  $T_1(1) = T_2(1) = 0$  if and only if  $M_{b_1 b_2} T$  admits a bounded extension from the space  $BMO(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  to  $BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ .
- (3)  $T_1(b_1) = T_2(b_2) = 0$  if and only if  $M_{b_1 b_2} T$  admits a bounded extension from the space  $BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  to  $BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ .

For the details of definition of  $T_1(1)$ ,  $T_2(1)$ ,  $T_1(b_1)$  and  $T_2(b_2)$ , we refer to Section 5.

This paper is organized as follows. In Section 2, we develop the Littlewood-Paley-Stein theory on  $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ . To do this, we first recall some basic definitions, notion and known results established in the one parameter case, and then introduce the spaces of test functions and distributions

on  $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ . We prove discrete Calderón’s identity and the Plancherel-Pôlya inequalities (Theorems 2.8 and 2.13). We introduce the Littlewood-Paley-Stein square function and define the Hardy space  $H^p_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ . Some important properties of  $H^p_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  are also proved in this section. In Section 3, we establish a new atomic decomposition for  $H^p_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \cap L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  (Theorem 3.1). As an application, we show that the Hardy space  $H^p_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \cap L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  can be defined by  $b_1 b_2 f \in H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \cap L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  (Corollary 3.2). Section 4 deals with the dual of  $H^p_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ . We introduce the generalized Carleson measure space  $CMO^p_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  and show that  $CMO^p_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  is the dual space of  $H^p_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  (Theorem 4.1). Finally, we show the boundedness and endpoint estimates of Journé’s product singular integrals (Theorems 1.1 and 1.2).

## 2. The Littlewood-Paley-Stein theory on $H^p_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$

**2.1. The Littlewood-Paley-Stein theory on  $H^p_b(\mathbb{R}^n)$ .** We begin by recalling the definition of para-accretive functions (see for example [6, 12]).

**Definition 2.1.** A bounded complex-valued function  $b$  defined on  $\mathbb{R}^n$  is said to be para-accretive if there exist constants  $C, \gamma > 0$  such that for each cube  $Q \subset \mathbb{R}^n$ , there is a  $Q' \subset Q$  with  $\gamma|Q| \leq |Q'|$  and satisfies

$$\frac{1}{|Q|} \left| \int_{Q'} b(x) dx \right| \geq C.$$

By the Lebesgue differentiation theorem, it is easy to show that  $|b(x)| \geq C > 0$  almost everywhere.

The following class of “test functions” associated to a para-accretive function was introduced in [12].

**Definition 2.2.** Fix two exponents  $0 < \beta \leq 1$  and  $\gamma > 0$ . Let  $b$  be a para-accretive function. A function  $f$  defined on  $\mathbb{R}^n$  is said to be a *test function of type  $(\beta, \gamma)$  centered at  $x_0 \in \mathbb{R}^n$  with width  $d > 0$*  if  $f$  satisfies

$$|f(x)| \leq C \frac{d^\gamma}{(d + |x - x_0|)^{n+\gamma}}, \tag{1}$$

$$|f(x) - f(x')| \leq C \left( \frac{|x - x'|}{d + |x - x_0|} \right)^\beta \frac{d^\gamma}{(d + |x - x_0|)^{n+\gamma}} \tag{2}$$

for  $|x - x'| \leq \frac{d+|x-x_0|}{2}$ , and

$$\int_{\mathbb{R}^n} f(x)b(x) dx = 0.$$

We write  $\mathcal{M}^{(\beta, \gamma)}(x_0, d)$  for the collection of all test functions of type  $(\beta, \gamma)$  centered at  $x_0 \in \mathbb{R}^n$  with width  $d > 0$ . For  $f \in \mathcal{M}^{(\beta, \gamma)}(x_0, d)$ , the norm of  $f$

in  $\mathcal{M}^{(\beta,\gamma)}(x_0, d)$  is defined by

$$\|f\|_{\mathcal{M}^{(\beta,\gamma)}(x_0, d)} = \inf \{C : (2.1) \text{ and } (2.2) \text{ hold}\}.$$

We denote  $\mathcal{M}^{(\beta,\gamma)}(0, 1)$  simply by  $\mathcal{M}^{(\beta,\gamma)}$ . Then  $\mathcal{M}^{(\beta,\gamma)}$  is a Banach space under the norm  $\|f\|_{\mathcal{M}^{(\beta,\gamma)}}$ . It is easy to check that for any  $x_0 \in \mathbb{R}^n$  and  $d > 0$ ,  $\mathcal{M}^{(\beta,\gamma)}(x_0, d) = \mathcal{M}^{(\beta,\gamma)}$  with equivalent norms. As usual, we write

$$b\mathcal{M}^{(\beta,\gamma)} = \{bg \mid g \in \mathcal{M}^{(\beta,\gamma)}\}.$$

If  $f \in b\mathcal{M}^{(\beta,\gamma)}$  and  $f = bg$  for  $g \in \mathcal{M}^{(\beta,\gamma)}$ , then the norm of  $f$  is defined by  $\|f\|_{b\mathcal{M}^{(\beta,\gamma)}} = \|g\|_{\mathcal{M}^{(\beta,\gamma)}}$ . The dual space  $(b\mathcal{M}^{(\beta,\gamma)})'$  consists of all linear functionals  $\mathcal{L}$  from  $b\mathcal{M}^{(\beta,\gamma)}$  to  $\mathbb{C}$  satisfying

$$|\mathcal{L}(f)| \leq C\|f\|_{b\mathcal{M}^{(\beta,\gamma)}} \quad \text{for all } f \in b\mathcal{M}^{(\beta,\gamma)}.$$

We also need the definition of an approximation to the identity associated to a para-accretive function  $b$ .

**Definition 2.3** ([12]). Let  $b$  be a para-accretive function. A sequence of operators  $\{S_k\}_{k \in \mathbb{Z}}$  is called an *approximation to the identity associated to  $b$*  if  $S_k(x, y)$ , the kernels of  $S_k$ , are functions from  $\mathbb{R}^n \times \mathbb{R}^n$  into  $\mathbb{C}$  and there exist a constant  $C$  and some  $0 < \varepsilon \leq 1$  such that, for all  $k \in \mathbb{Z}$  and all  $x, x', y$  and  $y' \in \mathbb{R}^n$ ,

- (i)  $|S_k(x, y)| = 0$  if  $|x - y| \geq C2^{-k}$  and  $|S_k(x, y)| \leq C2^{kn}$ ,
- (ii)  $|S_k(x, y) - S_k(x', y)| \leq C2^{k(1+\varepsilon)}|x - x'|^\varepsilon$ ,
- (iii)  $|S_k(x, y) - S_k(x, y')| \leq C2^{k(n+\varepsilon)}|y - y'|^\varepsilon$ ,
- (iv)  $||S_k(x, y) - S_k(x, y')| - |S_k(x', y) - S_k(x', y')|| \leq C2^{2k(n+\varepsilon)}|x - x'|^\varepsilon|y - y'|^\varepsilon$ ,
- (v)  $\int_{\mathbb{R}^n} S_k(x, y)b(y)dy = 1$  for all  $k \in \mathbb{Z}$  and  $x \in \mathbb{R}^n$ ,
- (vi)  $\int_{\mathbb{R}^n} S_k(x, y)b(x)dx = 1$  for all  $k \in \mathbb{Z}$  and  $y \in \mathbb{R}^n$ .

*Remark 2.1.* Let  $D_k(x, y) := S_k(x, y) - S_{k-1}(x, y)$ . It is clear that  $D_k(x, \cdot) \in \mathcal{M}^{(\varepsilon,\varepsilon)}(x, 2^{-k})$  and hence  $D_k(x, \cdot) \in \mathcal{M}^{(\varepsilon,\varepsilon)}$ . Similarly,  $D_k(\cdot, y) \in \mathcal{M}^{(\varepsilon,\varepsilon)}$ . By definition, it is clear to see that  $bD_k(x, \cdot) \in b\dot{\mathcal{M}}^{(\beta,\gamma)}$  for  $0 < \beta, \gamma < \varepsilon$ .

We now recall the definition of  $H_b^p$ .

**Definition 2.4** ([13]). Suppose that  $\{S_k\}_{k \in \mathbb{Z}}$  is an approximation to the identity associated to a para-accretive function  $b$  with regularity exponent  $\varepsilon$ . Set  $D_k = S_k - S_{k-1}$ . For  $0 < \beta, \gamma < \varepsilon$ , denote by  $\dot{\mathcal{M}}^{(\beta,\gamma)}$  the closure of  $\mathcal{M}^{(\varepsilon,\varepsilon)}$  with respect to the norm  $\|\cdot\|_{\mathcal{M}^{(\beta,\gamma)}}$ . The Hardy space  $H_b^p, \frac{n}{n+\varepsilon} < p \leq 1$ , is the collection of  $f \in (b\dot{\mathcal{M}}^{(\beta,\gamma)})'$  satisfying

$$\|f\|_{H_b^p} := \|g(f)\|_p < \infty,$$

where  $g(f)$ , the Littlewood-Paley  $g$ -function of  $f$ , is defined by

$$g(f)(x) := \left\{ \sum_k |D_k(bf)(x)|^2 \right\}^{1/2}.$$

We would like to point out that one of main results in [13] is the following Plancherel-Pôlya type inequality.

**Proposition 2.5** ([13]). *Suppose that  $\{S_k\}_{k \in \mathbb{Z}}$  and  $\{R_k\}_{k \in \mathbb{Z}}$  are approximations to the identity associated to  $b$  with regularity exponent  $\varepsilon$ , and  $\frac{n}{n+\varepsilon} < p < \infty$ . Set  $D_k = S_k - S_{k-1}$  and  $E_k = R_k - R_{k-1}$ . Then, for  $f \in (b\mathcal{M}^{(\beta, \gamma)})'$ ,*

$$\begin{aligned} & \left\| \left\{ \sum_k \sum_{Q_k} \left( \sup_{z \in Q_k} |E_k(bf)(z)| \right)^2 \chi_{Q_k} \right\}^{1/2} \right\|_p \\ & \approx \left\| \left\{ \sum_k \sum_{Q_k} \left( \inf_{z \in Q_k} |D_k(bf)(z)| \right)^2 \chi_{Q_k} \right\}^{1/2} \right\|_p, \end{aligned}$$

where  $Q_k$  are dyadic cubes with side length  $\ell(Q_k) = 2^{-k-N}$  for some fixed positive large integer  $N$  and  $\chi_{Q_k}$  are characteristic functions of cubes  $Q_k$ .

We have the continuous and discrete versions of the Calderón reproducing formula.

**Proposition 2.6** ([12], [14] and [24]). *Let  $b$  be a para-accretive function and  $\{S_k\}$  be an approximation to the identity associated to  $b$  with regularity exponent  $\varepsilon$ . Set  $D_k = S_k - S_{k-1}$ . Then there exists a family of operators  $\{\tilde{D}_k\}$  such that, for  $f \in \mathcal{M}^{(\varepsilon, \varepsilon)}$ ,*

$$f(x) = \sum_k D_k M_b \tilde{D}_k (bf)(x)$$

and

$$f(x) = \sum_k \sum_{Q_k} D_k(x, x_{Q_k}) \tilde{D}_k(bf)(x_{Q_k}) \int_{Q_k} b(y) dy,$$

where  $\sum_{Q_k}$  runs over all dyadic cubes  $Q_k$ 's with side length  $\ell(Q_k) = 2^{-k-N}$  for some fixed positive large integer  $N$ ,  $x_{Q_k}$  is any fixed point in  $Q_k$ , and the series converges in  $\mathcal{M}^{(\beta, \gamma)}$  for  $0 < \beta, \gamma < \varepsilon$  and in  $L^q, 1 < q < \infty$ . Moreover,  $\tilde{D}_k(x, y)$ , the kernels of  $\tilde{D}_k$ , satisfy the following estimates: for  $0 < \varepsilon' < \varepsilon$ , there exists a constant  $C > 0$  such that

$$|\tilde{D}_k(x, y)| \leq C \frac{2^{-k\varepsilon'}}{(2^{-k} + |x - y|)^{n+\varepsilon'}},$$

$$|\tilde{D}_k(x, y) - \tilde{D}_k(x, y')| \leq C \left( \frac{|y - y'|}{(2^{-k} + |x - y|)} \right)^{\varepsilon'} \frac{2^{-k\varepsilon'}}{(2^{-k} + |x - y|)^{n+\varepsilon'}}$$



for  $|y - y'| \leq (2^{-k} + |x - y|)/2$ ,

$$\int_{\mathbb{R}^n} \tilde{D}_k(x, y)b(y) dy = 0 \quad \text{for all } k \in \mathbb{Z} \text{ and } x \in \mathbb{R}^n,$$

$$\int_{\mathbb{R}^n} \tilde{D}_k(x, y)b(x) dx = 0 \quad \text{for all } k \in \mathbb{Z} \text{ and } y \in \mathbb{R}^n.$$

**2.2. Test functions and distributions associated to para-accretive functions.** We first introduce spaces of test functions and distributions on  $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$  associated to para-accretive functions.

**Definition 2.7.** Fix four exponents  $0 < \beta_1, \beta_2 \leq 1$  and  $\gamma_1, \gamma_2 > 0$ . Let  $b_1$  and  $b_2$  be para-accretive functions on  $\mathbb{R}^{n_1}$  and  $\mathbb{R}^{n_2}$ , respectively. A function  $f$  defined on  $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$  is said to be a *test function of type*  $(\beta_1, \beta_2, \gamma_1, \gamma_2)$  centered at  $(x_0, y_0) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$  with width  $d_1, d_2 > 0$  if  $f$  satisfies

- (i)  $\|f(\cdot, y)\|_{\mathcal{M}^{(\beta_1, \gamma_1)}(x_0, d_1)} \leq C \frac{d_2^{\gamma_2}}{(d_2 + |y - y_0|)^{n_2 + \gamma_2}}$ ;
- (ii)  $\|f(\cdot, y) - f(\cdot, y')\|_{\mathcal{M}^{(\beta_1, \gamma_1)}(x_0, d_1)} \leq C \left(\frac{|y - y'|}{d_2 + |y - y_0|}\right)^{\beta_2} \frac{d_2^{\gamma_2}}{(d_2 + |y - y_0|)^{n_2 + \gamma_2}}$  for  $|y - y'| \leq \frac{d_2 + |y - y_0|}{2}$ ;
- (iii) properties (i) and (ii) also hold with  $x$  and  $y$  interchanged;
- (iv)  $\int_{\mathbb{R}^{n_1}} f(x, y)b_1(x)dx = \int_{\mathbb{R}^{n_2}} f(x, y)b_2(y)dy = 0$  for all  $x \in \mathbb{R}^{n_1}$  and  $y \in \mathbb{R}^{n_2}$ .

If  $f$  is a test function of type  $(\beta_1, \beta_2, \gamma_1, \gamma_2)$  centered at  $(x_0, y_0) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$  with width  $d_1, d_2 > 0$ , we write

$$f \in \mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}(x_0, y_0, d_1, d_2),$$

and the norm of  $f$  is defined by

$$\|f\|_{\mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}(x_0, y_0, d_1, d_2)} = \inf \{C : \text{(i), (ii) and (iii) hold}\}.$$

Similarly, we denote by  $\mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}$  the class of  $\mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}(0, 0, 1, 1)$ . We can check that  $\mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)} = \mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}(x_0, y_0, d_1, d_2)$  with equivalent norms for all  $(x, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ . Let  $\dot{\mathcal{M}}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}$  denote the completion of the space  $\mathcal{M}^{(\varepsilon_1, \varepsilon_2, \varepsilon_1, \varepsilon_2)}$  in  $\mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}$  with  $0 < \beta_i, \gamma_i < \varepsilon_i, i = 1, 2$ . For  $f \in \dot{\mathcal{M}}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}$ , we define  $\|f\|_{\dot{\mathcal{M}}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}} = \|f\|_{\mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}}$ . As usual, we write

$$b_1 b_2 \mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)} = \{b_1 b_2 g \mid g \in \mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}\}.$$

If  $f \in b_1 b_2 \mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}$  and  $f = b_1 b_2 g$  for  $g \in \mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}$ , then the norm of  $f$  is defined by

$$\|f\|_{b_1 b_2 \mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}} = \|g\|_{\mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}}.$$

The dual space  $(b_1 b_2 \mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)})'$  consists of all linear functionals  $\mathcal{L}$  from  $b_1 b_2 \mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}$  to  $\mathbb{C}$  satisfying

$$|\mathcal{L}(f)| \leq C \|f\|_{b_1 b_2 \mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}} \quad \text{for all } f \in b_1 b_2 \mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}.$$

**2.3. Discrete Calderón identity on test function spaces.** Through the paper, we always denote  $b_{iQ_i}$  by

$$\frac{1}{|Q_i|} \int_{Q_i} b_i(x_i) dx_i, \quad i = 1, 2.$$

**Theorem 2.8.** *Let  $D_{k_i}$  and  $\tilde{D}_{k_i}$  be the operators given in Proposition 2.6 on  $\mathbb{R}^{n_i}$ ,  $i = 1, 2$ . Then*

$$\begin{aligned} f(x_1, x_2) &= \sum_{k_1} \sum_{k_2} \sum_{Q_{k_1}} \sum_{Q_{k_2}} |Q_{k_1}| |Q_{k_2}| D_{k_1}(x_1, x_{Q_{k_1}}) D_{k_2}(x_2, y_{Q_{k_2}}) \\ &\quad \times \tilde{D}_{k_1} M_{b_1} \tilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}}) b_{1Q_{k_1}} b_{2Q_{k_2}}, \end{aligned}$$

where the series converges in  $\mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}$  for  $0 < \beta_i, \gamma_i < \varepsilon_i$ ,  $i = 1, 2$  and in  $L^p$ ,  $1 < p < \infty$ .

**Proof.** The proof of this theorem is based on the method of iteration and some known estimates on  $\mathbb{R}^n$ . We first show the  $L^p$ ,  $1 < p < \infty$ , convergence. Denote

$$\begin{aligned} g(x_1, x_2) &= f(x_1, x_2) - \sum_{|k_1| \leq L_1} \sum_{|k_2| \leq L_2} \sum_{Q_{k_1}} \sum_{Q_{k_2}} |Q_{k_1}| |Q_{k_2}| D_{k_1}(x_1, x_{Q_{k_1}}) \\ &\quad \times D_{k_2}(x_2, y_{Q_{k_2}}) \tilde{D}_{k_1} M_{b_1} \tilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}}) b_{1Q_{k_1}} b_{2Q_{k_2}} \\ &=: g_1(x_1, x_2) + g_2(x_1, x_2), \end{aligned}$$

where

$$\begin{aligned} g_1(x_1, x_2) &= \sum_{|k_2| \leq L_2} \sum_{Q_{k_2}} |Q_{k_2}| D_{k_2}(x_2, y_{Q_{k_2}}) \tilde{D}_{k_2} M_{b_2} f(x_1, y_{Q_{k_2}}) b_{2Q_{k_2}} \\ &\quad - \sum_{|k_1| \leq L_1} \sum_{Q_{k_1}} |Q_{k_1}| D_{k_1}(x_1, x_{Q_{k_1}}) b_{1Q_{k_1}} \tilde{D}_{k_1} M_{b_1} \\ &\quad \times \left( \sum_{|k_2| \leq L_2} \sum_{Q_{k_2}} |Q_{k_2}| D_{k_2}(x_2, y_{Q_{k_2}}) \tilde{D}_{k_2} M_{b_2} f(\cdot, y_{Q_{k_2}}) b_{2Q_{k_2}} \right) (x_{Q_{k_1}}) \end{aligned}$$

and

$$\begin{aligned} g_2(x_1, x_2) &= f(x_1, x_2) - \sum_{|k_2| \leq L_2} \sum_{Q_{k_2}} |Q_{k_2}| D_{k_2}(x_2, y_{Q_{k_2}}) \tilde{D}_{k_2} M_{b_2} f(x_1, y_{Q_{k_2}}) b_{2Q_{k_2}}. \end{aligned}$$

We now need the following two estimates on  $\mathbb{R}^n$ , which was proved in [13] (the original idea and version is due to [12], see also [7] for a systematical introduction): there exists a constant  $C$  such that for  $f \in L^p(\mathbb{R}^n)$ ,  $1 < p < \infty$ , and any integers  $L$ ,

$$\left\| \sum_{|k| \leq L} \sum_{Q_k} |Q_k| D_k(x_2, y_{Q_k}) \tilde{D}_k(bf)(y_{Q_k}) b_{Q_k} \right\|_p \leq C \|f\|_p \tag{3}$$

and

$$\begin{aligned} & \left\| \sum_{|k| \leq L} \sum_{Q_k} |Q_k| D_k(x_2, y_{Q_k}) \tilde{D}_k(bf)(y_{Q_k}) b_{Q_k} - f \right\|_p \\ & \leq C \left\| \left\{ \sum_{|k| \geq L} \sum_{Q_k} |D_k(f)|^2 \right\}^{1/2} \right\|_p. \end{aligned} \tag{4}$$

Using (3) first and then (4) yields

$$\|g_1\|_{L^p} \leq C \left\| \left\{ \sum_{|k_1| \geq L_1} \sum_{|k_2| \leq L_2} |D_{k_1} D_{k_2}(f)|^2 \right\}^{1/2} \right\|_p \rightarrow 0 \quad \text{as } L_1 \rightarrow \infty.$$

Similarly,

$$\|g_2\|_{L^p} \rightarrow 0 \quad \text{as } L_2 \rightarrow \infty.$$

Hence, the  $L^p$  convergence follows.

To see the convergence in the space of test functions, we need the following estimates on  $\mathbb{R}^n$  which, again, was proved in [13] with the original idea and version in [12], see also [7]): for  $f \in \mathcal{M}^{(\beta, \gamma)}$  and any integers  $L$ ,

$$\left\| \sum_{|k| \leq L} \sum_{Q_k} |Q_k| D_k(x_2, y_{Q_k}) \tilde{D}_k(M_b f)(y_{Q_k}) b_{Q_k} \right\|_{\mathcal{M}^{(\beta, \gamma)}} \leq C \|f\|_{\mathcal{M}^{(\beta, \gamma)}} \tag{5}$$

and

$$\begin{aligned} & \left\| \sum_{|k| \leq L} \sum_{Q_k} |Q_k| D_k(x_2, y_{Q_k}) \tilde{D}_k(M_b f)(y_{Q_k}) b_{Q_k} - f \right\|_{\mathcal{M}^{(\beta', \gamma')}} \\ & \leq C 2^{-L\delta} \|f\|_{\mathcal{M}^{(\beta, \gamma)}}, \end{aligned} \tag{6}$$

where  $C$  is a constant,  $0 < \beta' < \beta$  and  $0 < \gamma' < \gamma$ .

We observe that if  $f \in \mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}$ , then  $\|f(\cdot, x_2)\|_{\mathcal{M}^{(\beta_1, \gamma_1)}}$ , as a function of the variable  $x_2$ , is in  $\mathcal{M}^{(\beta_2, \gamma_2)}$  and

$$\| \|f(\cdot, \cdot)\|_{\mathcal{M}^{(\beta_1, \gamma_1)}} \|_{\mathcal{M}^{(\beta_2, \gamma_2)}} \leq \|f\|_{\mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}}.$$

Similarly,

$$\| \|f(\cdot, \cdot)\|_{\mathcal{M}^{(\beta_2, \gamma_2)}} \|_{\mathcal{M}^{(\beta_1, \gamma_1)}} \leq \|f\|_{\mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}}.$$

Therefore, we obtain

$$\|g_1(\cdot, x_2)\|_{\mathcal{M}^{(\beta'_1, \gamma'_1)}}$$

$$\begin{aligned} &\leq C2^{-L_1\delta} \left\| \sum_{|k_2| \leq L_2} \sum_{Q_{k_2}} |Q_{k_2}| D_{k_2}(x_2, y_{Q_{k_2}}) \tilde{D}_{k_2} M_{b_2} f(x_1, y_{Q_{k_2}}) b_{2Q_{k_2}} \right\|_{\mathcal{M}^{(\beta_1, \gamma_1)}} \\ &\leq C2^{-L_1\delta} \left\| \|f(\cdot, \cdot)\|_{\mathcal{M}^{(\beta_2, \gamma_2)}} \frac{d_2^{\gamma_2}}{(d_2 + |x_2 - y_0|)^{n_2 + \gamma_2}} \right\|_{\mathcal{M}^{(\beta_1, \gamma_1)}} \\ &\leq C2^{-L_1\delta} \|f(\cdot, \cdot)\|_{\mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}} \frac{d_2^{\gamma_2}}{(d_2 + |x_2 - y_0|)^{n_2 + \gamma_2}} \end{aligned}$$

and, similarly,

$$\|g_2(\cdot, x_2)\|_{\mathcal{M}^{(\beta'_1, \gamma'_1)}} \leq C2^{-L_2\delta} \|f\|_{\mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}} \frac{d_2^{\gamma_2}}{(d_2 + |x_2 - y_0|)^{n_2 + \gamma_2}}.$$

Noting that  $g(x_1, x_2) - g(x_1, x'_2) = [g_1(x_1, x_2) - g_1(x_1, x'_2)] + [g_2(x_1, x_2) - g_2(x_1, x'_2)]$  and repeating the same estimates imply

$$\begin{aligned} &\|g(\cdot, x_2) - g(\cdot, x'_2)\|_{\mathcal{M}^{(\beta'_1, \gamma'_1)}} \\ &\leq C(2^{-L_1\delta} + 2^{-L_2\delta}) \|f\|_{\mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}} \left(\frac{|x_2 - x'_2|}{d_2 + |x_2 - y_0|}\right)^{\beta_2} \frac{d_2^{\gamma_2}}{(d_2 + |x_2 - y_0|)^{n_2 + \gamma_2}}, \end{aligned}$$

where  $|x_2 - x'_2| \leq \frac{d_2 + |x_2 - y_0|}{2}$ . The same proof can be carried out to the estimates if we interchange the roles of  $x$  and  $y$ . Hence,

$$\|g\|_{\dot{\mathcal{M}}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}} \leq C(2^{-L_1\delta} + 2^{-L_2\delta}) \|f\|_{\mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}}, \tag{7}$$

which yields the convergence in  $\dot{\mathcal{M}}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}$ . □

Using the same argument, we also have continuous Calderón identity on test function spaces as follows.

**Theorem 2.9.** *Let  $D_{k_i}$  and  $\tilde{D}_{k_i}$  be given in Proposition 2.6 on  $\mathbb{R}^{n_i}$ ,  $i = 1, 2$ . Then*

$$f = \sum_{k_1} \sum_{k_2} D_{k_1} M_{b_1} D_{k_2} M_{b_2} \tilde{D}_{k_1} M_{b_1} \tilde{D}_{k_2} M_{b_2} f,$$

where the series converges in  $\dot{\mathcal{M}}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}$  for  $0 < \beta_i, \gamma_i < \varepsilon_i$ ,  $i = 1, 2$  and in  $L^q$ ,  $1 < q < \infty$ .

**Theorem 2.10.** *Let  $D_{k_i}$  and  $\tilde{D}_{k_i}$  be the operators given in Proposition 2.6 on  $\mathbb{R}^{n_i}$ ,  $i = 1, 2$ . Then for  $f \in b_1 b_2 \dot{\mathcal{M}}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}$*

$$\begin{aligned} f(x_1, x_2) &= \sum_{k_1} \sum_{k_2} \sum_{Q_{k_1}} \sum_{Q_{k_2}} |Q_{k_1}| |Q_{k_2}| b_1(x_1) D_{k_1}(x_1, x_{Q_{k_1}}) b_2(x_2) D_{k_2}(x_2, y_{Q_{k_2}}) \\ &\quad \times \tilde{D}_{k_1} \tilde{D}_{k_2} f(x_{Q_{k_1}}, y_{Q_{k_2}}) b_{1Q_{k_1}} b_{2Q_{k_2}}, \end{aligned}$$

where the series converges in  $b_1 b_2 \dot{\mathcal{M}}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}$  for  $0 < \beta_i, \gamma_i < \varepsilon_i$ ,  $i = 1, 2$  and in  $L^p$ ,  $1 < p < \infty$ .

**Proof.** Let  $f \in b_1 b_2 \dot{\mathcal{M}}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}$ . Then there exists a  $g \in \dot{\mathcal{M}}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}$  such that  $f = b_1 b_2 g$ . Since  $\|f\|_{b_1 b_2 \mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}} = \|g\|_{\mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}}$  and Theorem 2.8, the proof is completed.  $\square$

By the duality, we have the following theorem.

**Theorem 2.11.** *Let  $D_{k_i}$  and  $\tilde{D}_{k_i}$  be the operators given in Proposition 2.6 on  $\mathbb{R}^{n_i}$ ,  $i = 1, 2$ . Then for  $f \in (b_1 b_2 \mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)})'$  with  $0 < \beta_i, \gamma_i < \varepsilon_i$ ,  $i = 1, 2$*

$$f(x_1, x_2) = \sum_{k_1} \sum_{k_2} \sum_{Q_{k_1}} \sum_{Q_{k_2}} |Q_{k_1}| |Q_{k_2}| \tilde{D}_{k_1}^*(x_1, x_{Q_{k_1}}) \tilde{D}_{k_2}^*(x_2, y_{Q_{k_2}}) \times D_{k_1}^* D_{k_2}^* f(x_{Q_{k_1}}, y_{Q_{k_2}}) b_{1Q_{k_1}} b_{2Q_{k_2}},$$

where the series converges in the sense of distribution, where  $D_{k_i}^*$ ,  $i = 1, 2$  denote the adjoint operator of  $D_{k_i}$ .

By [12] and [14], we also have the following continuous Calderón reproducing formula. For  $f \in \mathcal{M}^{(\varepsilon, \varepsilon)}$ ,

$$f(x) = \sum_k \tilde{D}_k M_b D_k (bf)(x),$$

where  $\tilde{D}_k(x, y)$ , the kernels of  $\tilde{D}_k$  satisfy the same conditions of  $\tilde{D}_k(y, x)$ . By the above argument, we also have

**Theorem 2.12.** *Let  $D_{k_i}$  and  $\tilde{D}_{k_i}$  be the operators given in Proposition 2.6 on  $\mathbb{R}^{n_i}$ ,  $i = 1, 2$ . Then for  $f \in (b_1 b_2 \mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)})'$  with  $0 < \beta_i, \gamma_i < \varepsilon_i$ ,  $i = 1, 2$*

$$f(x_1, x_2) = \sum_{k_1} \sum_{k_2} \sum_{Q_{k_1}} \sum_{Q_{k_2}} |Q_{k_1}| |Q_{k_2}| D_{k_1}^*(x_1, x_{Q_{k_1}}) D_{k_2}^*(x_2, y_{Q_{k_2}}) \times \tilde{D}_{k_1} \tilde{D}_{k_2} f(x_{Q_{k_1}}, y_{Q_{k_2}}) b_{1Q_{k_1}} b_{2Q_{k_2}},$$

where the series converges in the sense of distribution.

**2.4. Plancherel-Pôlya inequality on test function spaces.** Using the discrete Calderón identity we prove the following Plancherel-Pôlya inequality on product domains associated with para-accretive functions.

**Theorem 2.13.** *Let  $\{S_{k_i}\}_{k_i \in \mathbb{Z}}$  and  $\{R_{k_i}\}_{k_i \in \mathbb{Z}}$  be two approximations to the identity associated to  $b_i$  with regularity exponent  $\varepsilon$  on  $\mathbb{R}^{n_i}$  and  $\frac{n_i}{n_i + \varepsilon} < p < \infty$ ,  $i = 1, 2$ . Set  $D_{k_i} = S_{k_i} - S_{k_i-1}$  and  $E_{k_i} = R_{k_i} - R_{k_i-1}$ . Then, for  $f \in (b_1 b_2 \mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)})'$ ,*

$$\left\| \left\{ \sum_{k_1} \sum_{k_2} \sum_{Q_{k_1}} \sum_{Q_{k_2}} \left( \sup_{z_1 \in Q_{k_1}} \sup_{z_2 \in Q_{k_2}} |E_{k_1} M_{b_1} E_{k_2} M_{b_2}(f)(z_1, z_2)| \right)^2 \chi_{Q_{k_1}} \chi_{Q_{k_2}} \right\}^{\frac{1}{2}} \right\|_p$$

$$\approx \left\| \left\{ \sum_{k_1} \sum_{k_2} \sum_{Q_{k_1}} \sum_{Q_{k_2}} \left( \inf_{z_1 \in Q_{k_1}} \inf_{z_2 \in Q_{k_2}} |D_{k_1} M_{b_1} D_{k_2} M_{b_2}(f)(z_1, z_2)| \right)^2 \chi_{Q_{k_1}} \chi_{Q_{k_2}} \right\}^{\frac{1}{2}} \right\|_p,$$

where  $Q_{k_i}$  are dyadic cubes with side length  $\ell(Q_{k_i}) = 2^{-k_i - N}$  for some fixed positive large integer  $N$  and  $\chi_{Q_{k_i}}$  are characteristic functions of cubes  $Q_{k_i}$ ,  $i = 1, 2$ .

**Proof.** Given  $f \in (b_1 b_2 \mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)})'$ , Theorem 2.12 shows

$$\langle f, g \rangle = \left\langle \sum_{k_1} \sum_{k_2} \sum_{Q_{k_1}} \sum_{Q_{k_2}} |Q_{k_1}| |Q_{k_2}| D_{k_1}(\cdot, x_{Q_{k_1}}) D_{k_2}(\cdot, y_{Q_{k_2}}) \right. \\ \left. \times \tilde{D}_{k_1} M_{b_1} \tilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}}) b_{1Q_{k_1}} b_{2Q_{k_2}}, g \right\rangle,$$

where  $g \in b_1 b_2 \mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}$ . Hence, for any  $j_1, j_2 \in \mathbb{Z}$ , we have

$$E_{j_1} M_{b_1} E_{j_2} M_{b_2}(f)(x_1, x_2) \\ = \sum_{k_1} \sum_{k_2} \sum_{Q_{k_1}} \sum_{Q_{k_2}} |Q_{k_1}| |Q_{k_2}| E_{j_1} M_{b_1} E_{j_2} M_{b_2} D_{k_1}(x_1, x_{Q_{k_1}}) D_{k_2}(x_2, y_{Q_{k_2}}) \\ \times \tilde{D}_{k_1} M_{b_1} \tilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}}) b_{1Q_{k_1}} b_{2Q_{k_2}}.$$

For  $0 < \varepsilon' < \varepsilon$  and  $x_1 \in Q_{j_1}$ , the almost orthogonality estimate (see [20, Lemma 4.3] and [24, p. 10]) gives

$$|E_{j_1} M_{b_1} D_{k_1}(x_1, x_{Q_{k_1}})| \leq C 2^{-|j_1 - k_1| \varepsilon'} \frac{2^{-(j_1 \wedge k_1) \varepsilon'}}{(2^{-(j_1 \wedge k_1)} + |x_1 - x_{Q_{k_1}}|)^{n_1 + \varepsilon'}}.$$

Therefore, for every  $z_1 \in Q_{j_1}$  and  $z_2 \in Q_{j_2}$ , we have

$$|E_{j_1} M_{b_1} E_{j_2} M_{b_2}(f)(z_1, z_2)| \\ \leq C \sum_{k_1} \sum_{k_2} \sum_{Q_{k_1}} \sum_{Q_{k_2}} |Q_{k_1}| |Q_{k_2}| 2^{-|j_1 - k_1| \varepsilon'} \frac{2^{-(j_1 \wedge k_1) \varepsilon'}}{(2^{-(j_1 \wedge k_1)} + |z_1 - x_{Q_{k_1}}|)^{n_1 + \varepsilon'}} \\ \times 2^{-|j_2 - k_2| \varepsilon'} \frac{2^{-(j_2 \wedge k_2) \varepsilon'}}{(2^{-(j_2 \wedge k_2)} + |z_2 - y_{Q_{k_2}}|)^{n_2 + \varepsilon'}} |\tilde{D}_{k_1} M_{b_1} \tilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}})|,$$

where  $\varepsilon' < \varepsilon$ . Thus,

$$\sup_{z_1 \in Q_{j_1}} \sup_{z_2 \in Q_{j_2}} |E_{j_1} M_{b_1} E_{j_2} M_{b_2}(f)(z_1, z_2)| \chi_{Q_{j_1}}(x_1) \chi_{Q_{j_2}}(x_2) \\ \leq C \sum_{k_1} \sum_{k_2} \sum_{Q_{k_1}} \sum_{Q_{k_2}} |Q_{k_1}| |Q_{k_2}| 2^{-|j_1 - k_1| \varepsilon'} \frac{2^{-(j_1 \wedge k_1) \varepsilon'}}{(2^{-(j_1 \wedge k_1)} + |x_1 - x_{Q_{k_1}}|)^{n_1 + \varepsilon'}} \\ \times 2^{-|j_2 - k_2| \varepsilon'} \frac{2^{-(j_2 \wedge k_2) \varepsilon'}}{(2^{-(j_2 \wedge k_2)} + |x_2 - y_{Q_{k_2}}|)^{n_2 + \varepsilon'}} |$$

$$\times \tilde{D}_{k_1} M_{b_1} \tilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}}) |\chi_{Q_{j_1}}(x_1) \chi_{Q_{j_2}}(x_2)|.$$

By an estimate in [10, pp. 147-148],

$$\begin{aligned} & \sum_{Q_{k_1}} \frac{2^{-(j_1 \wedge k_1) \varepsilon'}}{(2^{-(j_1 \wedge k_1)} + |x_1 - x_{Q_{k_1}}|)^{n_1 + \varepsilon'}} |\tilde{D}_{k_1} M_{b_1} \tilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}})| \\ & \leq C 2^{(k_1 \wedge j_1) n_1} 2^{[k_1 - (k_1 \wedge j_1)] n_1 / r} \\ & \quad \times \left\{ M_1 \left( \sum_{Q_{k_1}} |\tilde{D}_{k_1} M_{b_1} \tilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}})| \chi_{Q_{k_1}} \right)^r \right\}^{1/r} (x_1), \end{aligned}$$

where  $M_1$  is the Hardy-Littlewood maximal function on  $\mathbb{R}^{n_1}$  and  $\max_{i=1,2} \frac{n_i}{n_i + \varepsilon'} < r < p$ . Therefore,

$$\begin{aligned} & \sup_{z_1 \in Q_{j_1}} \sup_{z_2 \in Q_{j_2}} |E_{j_1} M_{b_1} E_{j_2} M_{b_2}(f)(z_1, z_2)| \chi_{Q_{j_1}}(x_1) \chi_{Q_{j_2}}(x_2) \\ & \leq C \sum_{k_1} \sum_{k_2} |Q_{k_1}| |Q_{k_2}| 2^{-|j_1 - k_1| \varepsilon'} 2^{(k_1 \wedge j_1) n_1} 2^{[k_1 - (k_1 \wedge j_1)] n_1 / r} \\ & \quad \times 2^{-|j_2 - k_2| \varepsilon'} 2^{(k_2 \wedge j_2) n_2} 2^{[k_2 - (k_2 \wedge j_2)] n_2 / r} \\ & \quad \times \left\{ M_2 \left( \sum_{Q_{k_1}} \left\{ M_1 \left( \sum_{Q_{k_1}} |\tilde{D}_{k_1} M_{b_1} \tilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}})| \chi_{Q_{k_1}} \right)^r \right\} (x_1) \right. \right. \\ & \quad \left. \left. \times \chi_{Q_{k_2}} \right) \right\}^{1/r} (x_2) \chi_{Q_{j_1}}(x_1) \chi_{Q_{j_2}}(x_2), \end{aligned}$$

where  $M_2$  is the Hardy-Littlewood maximal function on  $\mathbb{R}^{n_2}$ . By Cauchy-Schwartz inequality,

$$\begin{aligned} & \sup_{z_1 \in Q_{j_1}} \sup_{z_2 \in Q_{j_2}} |E_{j_1} M_{b_1} E_{j_2} M_{b_2}(f)(z_1, z_2)|^2 \chi_{Q_{j_1}}(x_1) \chi_{Q_{j_2}}(x_2) \\ & \leq C \sum_{k_1} \sum_{k_2} 2^{-k_1 n_1} 2^{-|j_1 - k_1| \varepsilon'} 2^{(k_1 \wedge j_1) n_1} 2^{[k_1 - (k_1 \wedge j_1)] n_1 / r} \\ & \quad \times 2^{-k_2 n_2} 2^{-|j_2 - k_2| \varepsilon'} 2^{(k_2 \wedge j_2) n_2} 2^{[k_2 - (k_2 \wedge j_2)] n_2 / r} \\ & \quad \times \left\{ M_2 \left( \sum_{Q_{k_1}} \left\{ M_1 \left( \sum_{Q_{k_1}} |\tilde{D}_{k_1} M_{b_1} \tilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}})| \chi_{Q_{k_1}} \right)^r \right\} (x_1) \right. \right. \\ & \quad \left. \left. \times \chi_{Q_{k_2}} \right) \right\}^{2/r} (x_2) \chi_{Q_{j_1}}(x_1) \chi_{Q_{j_2}}(x_2) \end{aligned}$$

since

$$\sup_{j_i} \sum_{k_i} 2^{-k_i n_i} 2^{-|j_i - k_i| \varepsilon'} 2^{(k_i \wedge j_i) n_i} 2^{[k_i - (k_i \wedge j_i)] n_i / r} < \infty, \quad i = 1, 2.$$

This yields

$$\begin{aligned} & \left\{ \sum_{j_1} \sum_{j_2} \sum_{Q_{j_1}} \sum_{Q_{j_2}} \sup_{z_1 \in Q_{j_1}} \sup_{z_2 \in Q_{j_2}} |E_{j_1} M_{b_1} E_{j_2} M_{b_2}(f)(z_1, z_2)|^2 \chi_{Q_{j_1}}(x_1) \chi_{Q_{j_2}}(x_2) \right\}^{1/2} \\ & \leq C \left\{ \sum_{j_1} \sum_{j_2} \sum_{k_1} \sum_{k_2} 2^{-k_1 n_1} 2^{-|j_1 - k_1| \varepsilon'} 2^{(k_1 \wedge j_1) n_1} 2^{[k_1 - (k_1 \wedge j_1)] n_1 / r} \right. \\ & \quad \times 2^{-k_2 n_2} 2^{-|j_2 - k_2| \varepsilon'} 2^{(k_2 \wedge j_2) n_2} 2^{[k_2 - (k_2 \wedge j_2)] n_2 / r} \\ & \quad \times \left. \left\{ M_2 \left( \sum_{Q_{k_1}} \left\{ M_1 \left( \sum_{Q_{k_1}} |\tilde{D}_{k_1} M_{b_1} \tilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}})| \chi_{Q_{k_1}} \right)^r \right\} (x_1) \right. \right. \right. \\ & \quad \left. \left. \left. \times \chi_{Q_{k_2}} \right) \right\}^{2/r} (x_2) \right\}^{1/2} \\ & \leq C \left\{ \sum_{k_1, k_2} \left\{ M_2 \left( \sum_{Q_{k_1}} \left\{ M_1 \left( \sum_{Q_{k_1}} |\tilde{D}_{k_1} M_{b_1} \tilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}})| \chi_{Q_{k_1}} \right)^r \right\} (x_1) \right. \right. \right. \\ & \quad \left. \left. \left. \times \chi_{Q_{k_2}} \right) \right\}^{2/r} (x_2) \right\}^{1/2}. \end{aligned}$$

Since  $x_{Q_{k_1}}$  and  $y_{Q_{k_2}}$  are any fixed point in  $Q_{k_1}$  and  $Q_{k_2}$ , respectively, by the Fefferman-Stein vector-valued maximal function inequality twice with  $r < p$ , the proof is completed.  $\square$

**2.5. The Hardy spaces on product domains.**

**Definition 2.14.** Let  $\{S_{k_i}\}_{k_i \in \mathbb{Z}}$  be approximations to the identity associated to  $b_i$  with regularity exponent  $\varepsilon$  on  $\mathbb{R}^{n_i}$  and  $\frac{n_i}{n_i + \varepsilon} < p < \infty$ ,  $i = 1, 2$ . Set  $D_{k_i} = S_{k_i} - S_{k_i - 1}$ . For  $0 < \beta_i, \gamma_i < \varepsilon$ ,  $f \in (b_1 b_2 \mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)})'$ , the Littlewood-Paley  $\mathcal{G}$ -function of  $f$  is defined by

$$\mathcal{G}(f)(x_1, x_2) := \left\{ \sum_{k_1} \sum_{k_2} |D_{k_1} M_{b_1} D_{k_2} M_{b_2}(f)(x_1, x_2)|^2 \right\}^{1/2}.$$

We point out that due the Remark 2.1, this discrete Littlewood-Paley square function is well-defined. Strictly speaking we shall denote it by  $\mathcal{G}_{b_1 b_2}(f)(x_1, x_2)$ , since it is associated with the functions  $b_1$  and  $b_2$ . However, for the sake of simplicity, we drop the subscript  $b_1 b_2$ .

Applying the result of one parameter and using the iteration as given in [10], we immediately obtain

**Theorem 2.15.** *If  $f \in L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ ,  $1 < p < \infty$ , then  $\|\mathcal{G}(f)\|_p \approx \|f\|_p$ .*

We point out that the following discrete Littlewood-Paley function is more convenient for the study of the Hardy space  $H_{b_1 b_2}^p$  when  $p \leq 1$ .



**Definition 2.16.** Let  $\{S_{k_i}\}_{k_i \in \mathbb{Z}}$  be approximations to the identity associated to  $b_i$  with regularity exponent  $\varepsilon$  on  $\mathbb{R}^{n_i}$  and  $\frac{n_i}{n_i + \varepsilon} < p < \infty, i = 1, 2$ . Set  $D_{k_i} = S_{k_i} - S_{k_i - 1}$ . For  $0 < \beta_i, \gamma_i < \varepsilon, f \in (b_1 b_2 \mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)})'$ , the discrete Littlewood-Paley  $\mathcal{G}_d$ -function of  $f$  is defined by

$$\mathcal{G}_d(f)(x_1, x_2) := \left\{ \sum_{k_1} \sum_{k_2} \sum_{Q_{k_1}} \sum_{Q_{k_2}} |D_{k_1} M_{b_1} D_{k_2} M_{b_2}(f)(x_1, x_2)|^2 \chi_{Q_{k_1}}(x_1) \chi_{Q_{k_2}}(x_2) \right\}^{1/2}.$$

By the Plancherel-Pôlya inequality, the  $L^p$  norm of these two kinds of Littlewood-Paley functions are equivalent.

**Theorem 2.17.** Let  $0 < \beta_i, \gamma_i < \varepsilon$  and  $\frac{n_i}{n_i + \varepsilon} < p < \infty, i = 1, 2$ . If  $f \in (b_1 b_2 \mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)})'$ , then  $\|\mathcal{G}_d(f)\|_p \approx \|\mathcal{G}(f)\|_p$ .

We now define the Hardy spaces as follows.

**Definition 2.18.** Suppose that  $\{S_{k_i}\}_{k_i \in \mathbb{Z}}$  are approximations to the identity associated to  $b_i$  with regularity exponent  $\varepsilon$  on  $\mathbb{R}^{n_i}$  and  $\frac{n_i}{n_i + \varepsilon} < p \leq 1, i = 1, 2$ . Set  $D_{k_i} = S_{k_i} - S_{k_i - 1}$ . For  $0 < \beta_i, \gamma_i < \varepsilon$ , the Hardy space  $H_{b_1 b_2}^p$  is the collection of  $f \in (b_1 b_2 \mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)})'$  satisfying

$$\|f\|_{H_{b_1 b_2}^p} := \|\mathcal{G}_d(f)\|_p < \infty.$$

*Remark 2.2.* Let  $f \in H_{b_1 b_2}^p \subseteq (b_1 b_2 \mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)})'$  and denote by

$$f_L(x_1, x_2) = \sum_{|k_1| \leq L} \sum_{|k_2| \leq L} \sum_{Q_{k_1}} \sum_{Q_{k_2}} |Q_{k_1}| |Q_{k_2}| |D_{k_1}^*(x_1, x_{Q_{k_1}}) D_{k_2}^*(x_2, y_{Q_{k_2}})| \times \tilde{D}_{k_1} \tilde{D}_{k_2} f(x_{Q_{k_1}}, y_{Q_{k_2}}) b_{1Q_{k_1}} b_{2Q_{k_2}}.$$

Using Theorem 2.12 and applying the same argument for the proof of Plancherel-Pôlya inequality, we obtain that

$$\|\mathcal{G}_d(f - f_L)\|_p \leq C \left\| \left\{ \sum_{|k_1| > L} \sum_{|k_2| > L} \sum_{Q_{k_1}} \sum_{Q_{k_2}} |\tilde{D}_{k_1} M_{b_1} \tilde{D}_{k_2} M_{b_2}(f)(x_1, x_2)|^2 \times \chi_{Q_{k_1}}(x_1) \chi_{Q_{k_2}}(x_2) \right\}^{1/2} \right\|_p.$$

Since  $\|\mathcal{G}_d(f)\|_p$  and

$$\left\| \left\{ \sum_{k_1} \sum_{k_2} \sum_{Q_{k_1}} \sum_{Q_{k_2}} |\tilde{D}_{k_1} M_{b_1} \tilde{D}_{k_2} M_{b_2}(f)(x_1, x_2)|^2 \chi_{Q_{k_1}}(x_1) \chi_{Q_{k_2}}(x_2) \right\}^{1/2} \right\|_p$$

are equivalent (see [14, p. 75] for one variable) and  $\|\mathcal{G}_d(f)\|_p < \infty$ , we have  $\|\mathcal{G}_d(f - f_L)\|_p$  tend to zero as  $L \rightarrow \infty$ . It is clear that

$$D_{k_1}(\cdot, x_{Q_{k_1}}) D_{k_2}(\cdot, x_{Q_{k_1}}) \in \mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}$$

and hence  $f_L \in \mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}$ . We obtain that  $\mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)}$  is dense in  $H_{b_1 b_2}^p$ . Since  $\mathcal{M}^{(\beta_1, \beta_2, \gamma_1, \gamma_2)} \subseteq H_{b_1 b_2}^p \cap L^2$ , the subset  $H_{b_1 b_2}^p \cap L^2$  is dense in  $H_{b_1 b_2}^p$ .

### 3. Atomic decomposition for $H_{b_1 b_2}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$

Let  $\max\{\frac{n_1}{n_1 + \varepsilon_1}, \frac{n_2}{n_2 + \varepsilon_2}\} < p \leq 1$ . A  $(p, 2)$ -atoms of  $H_{b_1 b_2}^p$  is a function  $a(x_1, x_2)$  defined on  $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$  whose support is contained in some open set  $\Omega$  of finite measure such that

- (1)  $\|a\|_{L^2} \leq |\Omega|^{1/2-1/p}$ ,
- (2)  $a$  can be further decomposed into  $p$  elementary particles  $a_R$  as follows:
  - (i)  $a = \sum_{R \in \mathcal{M}(\Omega)} a_R$ , where  $\mathcal{M}(\Omega)$  denotes the class of all maximal dyadic subrectangles of  $\Omega$  and  $a_R$  is supported in  $5R$  where  $R \subset \Omega$  (say  $R = I \times J$ ),
  - (ii)  $\int_I a_R(x_1, \tilde{x}_2) b_1(x_1) dx_1 = \int_J a_R(\tilde{x}_1, x_2) b_2(x_2) dx_2 = 0$  for each  $\tilde{x}_1 \in I, \tilde{x}_2 \in J$ ,
  - (iii)  $\sum_{R \in \mathcal{M}(\Omega)} \|a_R\|_{L^2}^2 \leq |\Omega|^{1-2/p}$ .

We establish a new atomic decomposition for  $H_{b_1 b_2}^p \cap L^2$  as follows.

**Theorem 3.1.** *Let  $b_i$  be para-accretive functions on  $\mathbb{R}^{n_i}, i = 1, 2$  and  $\max\{\frac{n_1}{n_1 + \varepsilon_1}, \frac{n_2}{n_2 + \varepsilon_2}\} < p \leq 1$ . For  $f \in H_{b_1 b_2}^p \cap L^2$ , there exist a sequence of  $(p, 2)$ -atoms  $\{a_i\}$  of  $H_{b_1 b_2}^p$  and a sequence of scales  $\{\lambda_i\}$  such that  $f = \sum \lambda_i a_i$  and  $\sum |\lambda_i|^p \leq C \|f\|_{H_{b_1 b_2}^p}^p$ . Moreover, the series converges in both  $H_{b_1 b_2}^p$  and  $L^2$  norms.*

**Proof.** For  $k \in \mathbb{Z}$ , let

$$\Omega_k = \{x \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \mathcal{G}_d(f)(x_1, x_2) > 2^k\}$$

and

$$\mathcal{R}_k = \left\{ \text{dyadic rectangle } R : |R \cap \Omega_k| \geq \frac{1}{2}|R| \text{ and } |R \cap \Omega_{k+1}| < \frac{1}{2}|R| \right\}.$$

By Calderón reproducing formula (Theorem 2.8),

$$\begin{aligned} f(x_1, x_2) &= \sum_{k_1} \sum_{k_2} \sum_{Q_{k_1}} \sum_{Q_{k_2}} |Q_{k_1}| |Q_{k_2}| D_{k_1}(x_1, x_{Q_{k_1}}) D_{k_2}(x_2, y_{Q_{k_2}}) \\ &\quad \times \tilde{D}_{k_1} M_{b_1} \tilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}}) b_{1Q_{k_1}} b_{2Q_{k_2}} \\ &= \sum_{k \in \mathbb{Z}} \sum_{R_{k_1 k_2} \in \mathcal{R}_k} |R_{k_1 k_2}| D_{k_1}(x_1, x_{Q_{k_1}}) D_{k_2}(x_2, y_{Q_{k_2}}) \\ &\quad \times \tilde{D}_{k_1} M_{b_1} \tilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}}) b_{1Q_{k_1}} b_{2Q_{k_2}}, \end{aligned}$$

where  $R_{k_1 k_2} = Q_{k_1} \times Q_{k_2}$ . We rewrite the above decomposition as

$$f(x_1, x_2) = \sum_k \lambda_k a_k(x_1, x_2), \tag{8}$$

where

$$a_k(x_1, x_2) = \frac{1}{\lambda_k} \sum_{R_{k_1 k_2} \in \mathcal{R}_k} |R_{k_1 k_2}| D_{k_1}(x_1, x_{Q_{k_1}}) D_{k_2}(x_2, y_{Q_{k_2}}) \\ \times \tilde{D}_{k_1} M_{b_1} \tilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}}) b_{1Q_{k_1}} b_{2Q_{k_2}},$$

and

$$\lambda_k = C \left\| \left\{ \sum_{R_{k_1 k_2} \in \mathcal{R}_k} |\tilde{D}_{k_1} M_{b_1} \tilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}})|^2 \chi_{R_{k_1 k_2}} \right\}^{1/2} \right\|_2 \|\tilde{\Omega}_k\|^{1/p-1/2}$$

Let  $\tilde{\Omega}_k = \{(x_1, x_2) : M_s(\chi_{\Omega_k})(x_1, x_2) > \frac{1}{100}\}$ . Then, for each  $R \in \mathcal{R}_k$ , there exists a maximal dyadic subrectangle  $\tilde{R} \in \mathcal{M}(\tilde{\Omega}_k)$  such that  $R \subset \tilde{R}$ . For each  $S \in \mathcal{M}(\tilde{\Omega}_k)$ , set

$$a_S(x_1, x_2) = \frac{1}{\lambda_k} \sum_{\widetilde{R_{k_1 k_2}}=S} |R_{k_1 k_2}| D_{k_1}(x_1, x_{Q_{k_1}}) D_{k_2}(x_2, y_{Q_{k_2}}) \\ \times \tilde{D}_{k_1} M_{b_1} \tilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}}) b_{1Q_{k_1}} b_{2Q_{k_2}}.$$

Then  $a_k(x_1, x_2) = \sum_{S \in \mathcal{M}(\tilde{\Omega}_k)} a_S(x_1, x_2)$ . Since  $R_{k_1 k_2} \in \mathcal{R}_k$  and  $5R_{k_1 k_2} \subset \tilde{\Omega}_k$ , we have

$$\bigcup_{R_{k_1 k_2} \in \mathcal{R}_k} 5R_{k_1 k_2} \subseteq \tilde{\Omega}_k.$$

This implies that  $a_k$  is supported in an open set  $\tilde{\Omega}_k$  and  $a_S$  is supported on  $5S$ . The vanishing moment conditions of  $a_S$  follow from the assumption of  $D_{k_1}$  and  $D_{k_2}$ . To verify the size conditions of atom, the duality and the discrete Littlewood-Paley square function estimates on  $L^2$  show

$$\left\| \sum_{R_{k_1 k_2} \in \mathcal{R}_k} |R_{k_1 k_2}| D_{k_1}(x_1, x_{Q_{k_1}}) D_{k_2}(x_2, y_{Q_{k_2}}) \right. \\ \left. \times \tilde{D}_{k_1} M_{b_1} \tilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}}) b_{1Q_{k_1}} b_{2Q_{k_2}} \right\|_2 \\ = \sup_{\|g\|_2 \leq 1} \sum_{R_{k_1 k_2} \in \mathcal{R}_k} |R_{k_1 k_2}| D_{k_1} D_{k_2} g(x_{Q_{k_1}}, y_{Q_{k_2}}) \\ \times \tilde{D}_{k_1} M_{b_1} \tilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}}) b_{1Q_{k_1}} b_{2Q_{k_2}} \\ \leq C \sup_{\|g\|_2 \leq 1} \left\| \left\{ \sum_{R_{k_1 k_2} \in \mathcal{R}_k} |\tilde{D}_{k_1} M_{b_1} \tilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}})|^2 \chi_{R_{k_1 k_2}} \right\}^{1/2} \right\|_2$$

$$\begin{aligned} & \times \left\| \left\{ \sum_{R_{k_1 k_2} \in \mathcal{R}_k} |D_{k_1} D_{k_2} g(x_{Q_{k_1}}, y_{Q_{k_2}})|^2 \chi_{R_{k_1 k_2}} \right\}^{1/2} \right\|_2 \\ & \leq C \left\| \left\{ \sum_{R_{k_1 k_2} \in \mathcal{R}_k} |\tilde{D}_{k_1} M_{b_1} \tilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}})|^2 \chi_{R_{k_1 k_2}} \right\}^{1/2} \right\|_2. \end{aligned}$$

This yields

$$\|a_k\|_2 \leq |\tilde{\Omega}_k|^{1/2-1/p}.$$

Following the same proof, we have

$$\sum_{S \in \mathcal{M}(\tilde{\Omega}_k)} \|a_S\|^2 \leq |\tilde{\Omega}_k|^{1-2/p}.$$

Note that  $|\tilde{\Omega}_k| \leq C|\Omega_k|$  due to the maximal theorem. Since  $(x_1, x_2) \in R_{k_1 k_2} \in \mathcal{R}_k$  implies  $M_s(\chi_{R_{k_1 k_2} \cap \tilde{\Omega}_k \setminus \Omega_{k+1}})(x_1, x_2) > \frac{1}{2}$ , we have  $\chi_{R_{k_1 k_2}} \leq 2M_s(\chi_{R_{k_1 k_2} \cap \tilde{\Omega}_k \setminus \Omega_{k+1}})$  and then  $\chi_{R_{k_1 k_2}} \leq 4M_s^2(\chi_{R_{k_1 k_2} \cap \tilde{\Omega}_k \setminus \Omega_{k+1}})$ . Thus, by the Fefferman-Stein vector valued inequality,

$$\begin{aligned} & \left\| \left\{ \sum_{R_{k_1 k_2} \in \mathcal{R}_k} |\tilde{D}_{k_1} M_{b_1} \tilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}})|^2 \chi_{R_{k_1 k_2}} \right\}^{1/2} \right\|_2^2 \\ & \leq C \iint \sum_{R_{k_1 k_2} \in \mathcal{R}_k} |\tilde{D}_{k_1} M_{b_1} \tilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}})| \\ & \quad \times M_s(\chi_{R_{k_1 k_2} \cap \tilde{\Omega}_k \setminus \Omega_{k+1}})(x_1, x_2)|^2 dx_1 dx_2 \\ & \leq C \iint_{\tilde{\Omega}_k \setminus \Omega_{k+1} + \mathcal{R}_{k_1 k_2} \in \mathcal{R}_k} \sum |\tilde{D}_{k_1} M_{b_1} \tilde{D}_{k_2} M_{b_2} f(x_{Q_{k_1}}, y_{Q_{k_2}})| \chi_{R_{k_1 k_2}}(x_1, x_2)|^2 dx_1 dx_2 \\ & \leq C 2^{2k} |\tilde{\Omega}_k|. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_k |\lambda_k|^p & \leq C \sum_k 2^{kp} |\tilde{\Omega}_k|^{p/2} |\tilde{\Omega}_k|^{1-p/2} \\ & = C \sum_k 2^{kp} |\tilde{\Omega}_k| \\ & \leq C \|\mathcal{G}_d\|_p^p \leq C \|f\|_{H_{b_1 b_2}^p}^p. \end{aligned}$$

This ends the proof of Theorem 3.1. □

As an application of Theorem 3.1, we have

**Corollary 3.2.** *Let  $b_i$  be para-accretive functions on  $\mathbb{R}^{n_i}$ ,  $i = 1, 2$ , and*

$$\max \left\{ \frac{n_1}{n_1 + \varepsilon_1}, \frac{n_2}{n_2 + \varepsilon_2} \right\} < p \leq 1.$$

Then  $f \in H^p_{b_1 b_2} \cap L^2$  if and only if  $b_1 b_2 f \in H^p \cap L^2$ , and the norm of  $f$  on  $H^p_{b_1 b_2}$  and the norm of  $b_1 b_2 f$  on  $H^p$  are equivalent.

The definition of product  $H^p$  is the same with  $H^p_{b_1 b_2}$  where  $b_1 = b_2 = 1$  and the classical product atom is the same with  $(p, 2)$ -atoms of  $H^p_{b_1 b_2}$  for  $b_1 = b_2 = 1$ .

#### 4. The dual spaces of $H^p_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$

We now consider the dual spaces of  $H^p_{b_1 b_2}$ . For one variable, the dual of  $H^1_b$  is  $BMO_b = \{bf : f \in BMO\}$  given by Meyer and Coifman [27]. As a consequence of Corollary 3.2,

the proof of dual space of  $H^p_{b_1 b_2}$  is just a copy of the classical product Hardy space  $H^p$ . More precisely, let  $CMO^p_{b_1 b_2} = \{f = b_1 b_2 h : h \in CMO^p \cap L^2\}$  with  $\|f\|_{CMO^p_{b_1 b_2}} = \|h\|_{CMO^p}$ , where the product type Carleson measure space  $CMO^p$  was given in [17, pages 647 & 653] and [18, pages 340–341]. Denote

$$CMO^p_{b_1 b_2} = \left\{ f : \text{there is a sequence } \{f_m\} \text{ in } CMO^p_{b_1 b_2} \right. \\ \left. \text{such that } \langle f, g \rangle = \lim_{m \rightarrow \infty} \langle f_m, g \rangle \text{ for all } g \in H^p_{b_1 b_2} \cap L^2 \right\}$$

and  $\|f\|_{CMO^p_{b_1 b_2}} = \lim_{m \rightarrow \infty} \|f_m\|_{CMO^p_{b_1 b_2}}$ .

**Theorem 4.1.** *Suppose that  $b_i$  are para-accretive functions on  $\mathbb{R}^{n_i}$  for  $i = 1, 2$ . Let  $\max_{i=1,2} \left\{ \frac{n_i}{n_i + \varepsilon_i/2} \right\} < p \leq 1$ . The dual space of  $H^p_{b_1 b_2}$  is  $CMO^p_{b_1 b_2}$  in the following sense.*

- (a) *For each  $g \in CMO^p_{b_1 b_2}$ , the linear functional  $\ell_g : f \mapsto \langle f, g \rangle$ , defined initially on  $H^p_{b_1 b_2} \cap L^2$ , has a continuous extension to  $H^p_{b_1 b_2}$  and  $\|\ell_g\| \leq C\|g\|_{CMO^p_{b_1 b_2}}$ .*
- (b) *Conversely, every continuous linear functional  $\ell$  on  $H^p_{b_1 b_2}$  can be realized as  $\ell = \ell_g$ , defined initially on  $H^p_{b_1 b_2} \cap L^2$ , for some  $g \in CMO^p_{b_1 b_2}$  and  $\|g\|_{CMO^p_{b_1 b_2}} \leq C\|\ell\|$ .*

*In particular, when  $p = 1$ , we obtain  $BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) = \{b_1 b_2 g : g \in BMO(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})\}$ . We refer the reader to [17, pages 647 & 653] or [18, pages 340–341] for the definition and details about  $BMO(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ .*

**Proof.** For each  $g \in CMO^p_{b_1 b_2}$ , there exists a sequence  $\{g_m\}$  in  $CMO^p_{b_1 b_2}$  such that  $\langle f, g \rangle = \lim_{m \rightarrow \infty} \langle f, g_m \rangle$  for  $f \in H^p_{b_1 b_2} \cap L^2$ . By the definition,  $g_m = b_1 b_2 h_m$  with  $h_m \in CMO^p \cap L^2$ , and  $\|g_m\|_{CMO^p_{b_1 b_2}} = \|h_m\|_{CMO^p}$ . For  $f \in H^p_{b_1 b_2} \cap L^2$ , Corollary 3.2 gives  $b_1 b_2 f \in H^p \cap L^2$  and  $\|f\|_{H^p_{b_1 b_2}} \approx \|b_1 b_2 f\|_{H^p}$ .

The dual of  $H^p$  (cf. [17]) yields

$$\begin{aligned} |\ell_g(f)| &= |\langle f, g \rangle| = \left| \lim_{m \rightarrow \infty} \langle f, b_1 b_2 h_m \rangle \right| = \left| \lim_{m \rightarrow \infty} \langle b_1 b_2 f, h_m \rangle \right| \\ &\leq \lim_{m \rightarrow \infty} \|b_1 b_2 f\|_{H^p} \|h_m\|_{CMO^p} \\ &\leq C \|f\|_{H^p_{b_1 b_2}} \|g\|_{CMO^p_{b_1 b_2}}. \end{aligned}$$

Since  $H^p_{b_1 b_2} \cap L^2$  is dense in  $H^p_{b_1 b_2}$ , the map  $\ell_g$  can be extended to a continuous linear functional on  $H^p_{b_1 b_2}$  satisfying  $\|\ell_g\| \leq C \|g\|_{CMO^p_{b_1 b_2}}$ .

Conversely, let  $\ell \in (H^p_{b_1 b_2})'$  and define  $\ell_1$  by  $\ell_1(b_1 b_2 f) = \ell(f)$  for  $f \in H^p_{b_1 b_2} \cap L^2$ . It follows from Corollary 3.2 that  $\ell_1$  is a linear functional on  $H^p \cap L^2$ . By the duality argument between  $H^p$  and  $CMO^p$ , there exists  $h \in CMO^p$  such that

$$\begin{aligned} \ell(f) &= \ell_1(b_1 b_2 f) = \langle b_1 b_2 f, h \rangle \\ &= \lim_{m \rightarrow \infty} \langle b_1 b_2 f, h_m \rangle = \lim_{m \rightarrow \infty} \langle f, b_1 b_2 h_m \rangle \quad \text{for } f \in H^p_{b_1 b_2} \cap L^2, \end{aligned}$$

where  $h_m \in CMO^p \cap L^2$  and  $\|b_1 b_2 h_m\|_{CMO^p_{b_1 b_2}} = \|h_m\|_{CMO^p}$ . Let  $\langle f, g \rangle = \lim_{m \rightarrow \infty} \langle f, b_1 b_2 h_m \rangle$ . Then  $\|g\|_{CMO^p_{b_1 b_2}} = \lim_{m \rightarrow \infty} \|b_1 b_2 h_m\|_{CMO^p_{b_1 b_2}} \leq C \|\ell_1\| \leq C \|\ell\|$ .  $\square$

### 5. The boundedness and endpoint estimates of singular integral operators on product spaces

We start with recalling the definition of a Calderón-Zygmund kernel. A continuous complex-valued function  $K(x, y)$  defined on  $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x, y) : x = y\}$  is called a *Calderón-Zygmund kernel* if there exist constant  $C > 0$  and a regularity exponent  $\varepsilon \in (0, 1]$  such that

- (i)  $|K(x, y)| \leq C|x - y|^{-n}$
- (ii)  $|K(x, y) - K(x', y)| \leq C|x - x'|^\varepsilon|x - y|^{-n-\varepsilon}$  if  $|x - x'| \leq |x - y|/2$
- (iii)  $|K(x, y) - K(x, y')| \leq C|y - y'|^\varepsilon|x - y|^{-n-\varepsilon}$  if  $|y - y'| \leq |x - y|/2$ .

The smallest such constant  $C$  is denoted by  $|K|_{CZ}$ .

We say that an operator  $T$  is a *Calderón-Zygmund operator* if  $T$  is a continuous linear operator from  $C^\infty_0(\mathbb{R}^n)$  into its dual associated with a Calderón-Zygmund kernel  $K(x, y)$  given by

$$\langle Tf, g \rangle = \iint g(x)K(x, y)f(y)dydx$$

for all test functions  $f$  and  $g$  with disjoint supports and  $T$  is bounded on  $L^2(\mathbb{R}^n)$ . If  $T$  is a Calderón-Zygmund operator associated with a kernel  $K$ , its Calderón-Zygmund operator norm is defined by  $\|T\|_{CZ} = \|T\|_{L^2 \rightarrow L^2} + |K|_{CZ}$ . Of course, in general, one cannot conclude that a singular integral operator  $T$  is bounded on  $L^2(\mathbb{R}^n)$  because Plancherel's theorem doesn't work for non-convolution operators. However, if one assumes that  $T$  is bounded

on  $L^2(\mathbb{R}^n)$ , then the  $L^p, 1 < p < \infty$ , boundedness follows from Calderón-Zygmund’s real variable method. The characterization of the  $L^2(\mathbb{R}^n)$  boundedness of non-convolution singular integral operators was finally proved by the remarkable  $T1$  theorem by David and Journé [5], in which they gave a general criterion for the  $L^2$ -boundedness of singular integral operators. Let  $T$  be a singular integral operator defined for functions on  $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$  by

$$Tf(x_1, x_2) = \iint_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} K(x_1, x_2, y_1, y_2) f(y_1, y_2) dy_1 dy_2,$$

for  $(x_1, x_2)$  is outside the support of  $f$ . For each  $x_1, y_1 \in \mathbb{R}^{n_1}$ , set  $\tilde{K}^1(x_1, y_1)$  to be the singular integral operator acting on functions on  $\mathbb{R}^{n_2}$  with the kernel  $\tilde{K}^1(x_1, y_1)(x_2, y_2) = K(x_1, x_2, y_1, y_2)$  and, similarly,  $\tilde{K}^2(x_2, y_2)(x_1, y_1) = K(x_1, x_2, y_1, y_2)$ .

**Definition 5.1.** A singular integral operator  $T$  is said to be in *Journé’s class* if the associated kernel  $K(x_1, x_2, y_1, y_2)$  satisfies the following conditions. There exist constants  $C > 0$  and  $\varepsilon \in (0, 1]$  such that

- (A<sub>1</sub>)  $T$  is bounded on  $L^2(\mathbb{R}^{n_1+n_2})$ ,
- (A<sub>2</sub>)  $\|\tilde{K}^1(x_1, y_1)\|_{CZ} \leq C|x_1 - y_1|^{-n_1}$ ,  
 $\|\tilde{K}^1(x_1, y_1) - \tilde{K}^1(x_1, y'_1)\|_{CZ} \leq C|y_1 - y'_1|^\varepsilon|x_1 - y_1|^{-(n_1+\varepsilon)}$   
for  $|y_1 - y'_1| \leq |x_1 - y_1|/2$ ,  
 $\|\tilde{K}^1(x_1, y_1) - \tilde{K}^1(x'_1, y_1)\|_{CZ} \leq C|x_1 - x'_1|^\varepsilon|x_1 - y_1|^{-(n_1+\varepsilon)}$   
for  $|x_1 - x'_1| \leq |x_1 - y_1|/2$ ,
- (A<sub>3</sub>)  $\|\tilde{K}^2(x_2, y_2)\|_{CZ} \leq C|x_2 - y_2|^{-n_2}$  ,  
 $\|\tilde{K}^2(x_2, y_2) - \tilde{K}^2(x_2, y'_2)\|_{CZ} \leq C|y_2 - y'_2|^\varepsilon|x_2 - y_2|^{-(n_2+\varepsilon)}$   
for  $|y_2 - y'_2| \leq |x_2 - y_2|/2$ ,  
 $\|\tilde{K}^2(x_2, y_2) - \tilde{K}^2(x'_2, y_2)\|_{CZ} \leq C|x_2 - x'_2|^\varepsilon|x_2 - y_2|^{-(n_2+\varepsilon)}$   
for  $|x_2 - x'_2| \leq |x_2 - y_2|/2$ .

Let  $L^2_{0,0}(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n): f \text{ has compact support with } \int f(x)dx = 0\}$ . Suppose that  $T$  is a singular integral in Journé’s class. By [8, p. 840],  $T$  is bounded from  $H^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  to  $L^1(\mathbb{R}^{n_1+n_2})$ . Note that if  $\varphi^1 \in L^2_{0,0}(\mathbb{R}^{n_1})$  and  $\varphi^2 \in L^2_{0,0}(\mathbb{R}^{n_2})$ , then  $\varphi^1(y_1)\varphi^2(y_2) \in H^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ . Thus,  $T(\varphi^1\varphi^2)(x_1, x_2) \in L^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ . This implies that  $T(\varphi^1\varphi^2)(x_1, x_2)$ , as a function of  $x_1$ , is an integrable function on  $\mathbb{R}^{n_1}$ . Similarly,  $T(\varphi^1\varphi^2)(x_1, x_2)$ , as a function of  $x_2$ , is an integrable function on  $\mathbb{R}^{n_2}$ . Now we say that  $T_1^*(1) = 0$  if

$$\int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} K(x_1, x_2, y_1, y_2) \varphi^1(y_1) \varphi^2(y_2) dy_1 dy_2 dx_1 = 0$$

for all  $\varphi^1 \in L^2_{0,0}(\mathbb{R}^{n_1})$ ,  $\varphi^2 \in L^2_{0,0}(\mathbb{R}^{n_2})$ , and  $x_2 \in \mathbb{R}^{n_2}$ . Similarly,  $T_2^*(1) = 0$  if

$$\int_{\mathbb{R}^{n_2}} \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} K(x_1, x_2, y_1, y_2) \varphi^1(y_1) \varphi^2(y_2) dy_1 dy_2 dx_2 = 0$$

for all  $\varphi^1 \in L_{0,0}^2(\mathbb{R}^{n_1})$ ,  $\varphi^2 \in L_{0,0}^2(\mathbb{R}^{n_2})$ , and  $x_1 \in \mathbb{R}^{n_1}$ .

We also say that  $T_1^*(b_1) = 0$  if

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} K(x_1, x_2, y_1, y_2) b_1(x_1) \varphi^1(y_1) \varphi^2(y_2) dy_1 dy_2 dx_1 = 0$$

for all  $\varphi^1 \in L_{0,0}^2(\mathbb{R}^{n_1})$ ,  $\varphi^2 \in L_{0,0}^2(\mathbb{R}^{n_2})$ , and  $x_2 \in \mathbb{R}^{n_2}$ . Similarly,  $T_1^*(b_2) = 0$  if

$$\int_{\mathbb{R}^{n_2}} \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} K(x_1, x_2, y_1, y_2) b_2(x_2) \varphi^1(y_1) \varphi^2(y_2) dy_1 dy_2 dx_2 = 0$$

for all  $\varphi^1 \in L_{0,0}^2(\mathbb{R}^{n_1})$ ,  $\varphi^2 \in L_{0,0}^2(\mathbb{R}^{n_2})$ , and  $x_1 \in \mathbb{R}^{n_1}$ .

We now prove the main result of this article.

**Proof of the “if part” of Theorem 1.1.** We prove (1) only since the proof of (2) and (3) are similar. We use the skill in the proof of [16, Theorem 1]. We define the Hilbert space  $\mathcal{H}$  by

$$\mathcal{H} = \left\{ \{a_{k,j}\}_{k,j \in \mathbb{Z}} : \|\{a_{k,j}\}\|_{\mathcal{H}} = \left( \sum_{k,j \in \mathbb{Z}} |a_{k,j}|^2 \right)^{1/2} < \infty \right\}.$$

For simplicity, we use  $D_{k_1, k_2} M_b$  and  $\tilde{D}_{k_1, k_2} M_b$  to express  $D_{k_1} M_{b_1} D_{k_2} M_{b_2}$  and  $\tilde{D}_{k_1} M_{b_1} \tilde{D}_{k_2} M_{b_2}$ , respectively. We also denote  $\int dv$  by  $\int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} dv_1 dv_2$  and similarly for other variables. Set  $T_{k_1, k_2}(f) = D_{k_1, k_2} M_b T(f)$ . For  $f \in L^2(\mathbb{R}^{n_1+n_2}) \cap H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ , by the continuous Calderón identity,

$$T_{k_1, k_2}(f)(x_1, x_2) = D_{k_1, k_2} M_b T \left( \sum_{j_1} \sum_{j_2} M_b D_{j_1, j_2} M_b \tilde{D}_{j_1, j_2}(f)(x_1, x_2) \right).$$

Hence, the kernel  $T_{k_1, k_2}(x_1, x_2, y_1, y_2)$  of  $T_{k_1, k_2}$  is given by

$$\begin{aligned} & T_{k_1, k_2}(x_1, x_2, y_1, y_2) \\ &= \sum_{j_1} \sum_{j_2} \iint D_{k_1}(x_1, u_1) b_1(u_1) D_{k_2}(x_2, u_2) b_2(u_2) K(u_1, u_2, v_1, v_2) \\ & \quad \times M_b D_{j_1, j_2} M_b \tilde{D}_{j_1}(v_1, y_1) \tilde{D}_{j_2}(v_2, y_2) dudv. \end{aligned}$$

By the definition of  $H_{b_1 b_2}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ , the  $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) - H_{b_1 b_2}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  boundedness of  $T$  is equivalent to the  $H^p - L_{\mathcal{H}}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  boundedness of the  $\mathcal{H}$ -valued operator  $\mathcal{L}$  which maps  $f$  into  $\{T_{k_1, k_2}(f)\}_{k_1, k_2 \in \mathbb{Z}}$ . Note that the  $L^2(\mathbb{R}^{n_1+n_2})$  boundedness of  $T$  and the product Littlewood-Paley estimate imply that  $\mathcal{L}$  is bounded from  $L^2(\mathbb{R}^{n_1+n_2})$  to  $L_{\mathcal{H}}^2(\mathbb{R}^{n_1+n_2})$ . Let  $\varepsilon$  be the regularity exponent satisfying  $(A_2)$  and  $(A_3)$ . We will prove that  $\{T_{k_1, k_2}(x_1, x_2, y_1, y_2)\}_{k_1, k_2 \in \mathbb{Z}}$  satisfies the following estimates:

$$(B_1) \quad \|\{T_{k_1, k_2}(x_1, x_2, y_1, y_2)\}\|_{\mathcal{H}} \leq C|x_1 - y_1|^{-n_1} |x_2 - y_2|^{-n_2},$$

$$(B_2) \quad \text{for } \varepsilon' < \varepsilon,$$



$$\begin{aligned}
 & \text{(i)} \quad \left\| \{T_{k_1, k_2}(x_1, x_2, y_1, y_2) - T_{k_1, k_2}(x_1, x_2, y'_1, y_2)\} \right\|_{\mathcal{H}} \\
 & \leq C \frac{|y_1 - y'_1|^{\varepsilon'}}{|x_1 - y_1|^{n_1 + \varepsilon'}} |x_2 - y_2|^{-n_2} \quad \text{if } |y_1 - y'_1| \leq |x_1 - y_1|/2, \\
 & \text{(ii)} \quad \left\| \{T_{k_1, k_2}(x_1, x_2, y_1, y_2) - T_{k_1, k_2}(x_1, x_2, y_1, y'_2)\} \right\|_{\mathcal{H}} \\
 & \leq C \frac{|y_2 - y'_2|^{\varepsilon'}}{|x_2 - y_2|^{n_2 + \varepsilon'}} |x_1 - y_1|^{-n_1} \quad \text{if } |y_2 - y'_2| \leq |x_2 - y_2|/2,
 \end{aligned}$$

(B<sub>3</sub>) for  $\varepsilon' < \varepsilon$ ,

$$\begin{aligned}
 & \left\| \{ [T_{k_1, k_2}(x_1, x_2, y_1, y_2) - T_{k_1, k_2}(x_1, x_2, y'_1, y_2)] \right. \\
 & \quad \left. - [T_{k_1, k_2}(x_1, x_2, y_1, y'_2) - T_{k_1, k_2}(x_1, x_2, y'_1, y'_2)] \right\|_{\mathcal{H}} \\
 & \leq C \frac{|y_1 - y'_1|^{\varepsilon'} |y_2 - y'_2|^{\varepsilon'}}{|x_1 - y_1|^{n_1 + \varepsilon'} |x_2 - y_2|^{n_2 + \varepsilon'}} \\
 & \quad \text{if } |y_1 - y'_1| \leq |x_1 - y_1|/2 \text{ and } |y_2 - y'_2| \leq |x_2 - y_2|/2.
 \end{aligned}$$

To this end, according to the almost orthogonal estimates, we decompose the kernel  $T_{k_1, k_2}(x_1, x_2, y_1, y_2)$  as follows.

$$\begin{aligned}
 & T_{k_1, k_2}(x_1, x_2, y_1, y_2) \\
 & = \sum_{j_1} \sum_{j_2} \iint D_{k_1}(x_1, u_1) b_1(u_1) D_{k_2}(x_2, u_2) b_2(u_2) K(u_1, u_2, v_1, v_2) \\
 & \quad \times M_b D_{j_1, j_2} M_b \tilde{D}_{j_1}(v_1, y_1) \tilde{D}_{j_2}(v_2, y_2) dudv \\
 & = \left( \sum_{j_1 \geq k_1} \sum_{j_2 \geq k_2} + \sum_{j_1 \geq k_1} \sum_{j_2 < k_2} + \sum_{j_1 < k_1} \sum_{j_2 \geq k_2} + \sum_{j_1 < k_1} \sum_{j_2 < k_2} \right) \iint D_{k_1}(x_1, u_1) b_1(u_1) \\
 & \quad \times D_{k_2}(x_2, u_2) b_2(u_2) K(u_1, u_2, v_1, v_2) M_b D_{j_1, j_2} M_b \tilde{D}_{j_1}(v_1, y_1) \tilde{D}_{j_2}(v_2, y_2) dudv \\
 & := T_{k_1, k_2}^1(x_1, x_2, y_1, y_2) + T_{k_1, k_2}^2(x_1, x_2, y_1, y_2) \\
 & \quad + T_{k_1, k_2}^3(x_1, x_2, y_1, y_2) + T_{k_1, k_2}^4(x_1, x_2, y_1, y_2).
 \end{aligned}$$

The estimates of (B<sub>1</sub>) – (B<sub>3</sub>) for  $\{T_{k_1, k_2}(x_1, x_2, y_1, y_2)\}_{k_1, k_2 \in \mathbb{Z}}$  will follow easily by the following lemma.

**Lemma 5.2.** *For  $1 \leq j \leq 4$  and  $k_1, k_2 \in \mathbb{Z}$ , there exists a constant  $C$  such that*

(D<sub>1</sub>) for  $\varepsilon' < \varepsilon$ ,

$$|T_{k_1, k_2}^j(x_1, x_2, y_1, y_2)| \leq C \frac{2^{-k_1 \varepsilon'}}{(2^{-k_1} + |x_1 - y_1|)^{n_1 + \varepsilon'}} \frac{2^{-k_2 \varepsilon'}}{(2^{-k_2} + |x_2 - y_2|)^{n_2 + \varepsilon'}},$$

(D<sub>2</sub>) for  $\varepsilon'' < \varepsilon'$ ,

$$\begin{aligned}
 & \text{(i)} \quad |T_{k_1, k_2}^j(x_1, x_2, y_1, y_2) - T_{k_1, k_2}^j(x_1, x_2, y'_1, y_2)| \\
 & \leq C \left( \frac{|y_1 - y'_1|}{2^{-k_1}} \right)^{\varepsilon''} \frac{2^{-k_1 \varepsilon'}}{(2^{-k_1} + |x_1 - y_1|)^{n_1 + \varepsilon'}} \frac{2^{-k_2 \varepsilon'}}{(2^{-k_2} + |x_2 - y_2|)^{n_2 + \varepsilon'}} \\
 & \quad \text{if } |y_1 - y'_1| \leq 2^{-k_1 - 1},
 \end{aligned}$$

$$(ii) \quad |T_{k_1, k_2}^j(x_1, x_2, y_1, y_2) - T_{k_1, k_2}^j(x_1, x_2, y_1, y_2')| \\ \leq C \left( \frac{|y_2 - y_2'|}{2^{-k_2}} \right)^{\varepsilon''} \frac{2^{-k_1 \varepsilon'}}{(2^{-k_1} + |x_1 - y_1|)^{n_1 + \varepsilon'}} \frac{2^{-k_2 \varepsilon'}}{(2^{-k_2} + |x_2 - y_2|)^{n_2 + \varepsilon'}} \\ \text{if } |y_2 - y_2'| \leq 2^{-k_2 - 1},$$

(D<sub>3</sub>) for  $\varepsilon'' < \varepsilon'$ ,

$$\| \{ [T_{k_1, k_2}(x_1, x_2, y_1, y_2) - T_{k_1, k_2}(x_1, x_2, y_1', y_2)] \\ - [T_{k_1, k_2}(x_1, x_2, y_1, y_2') - T_{k_1, k_2}(x_1, x_2, y_1', y_2')] \} \|_{\mathcal{H}} \\ \leq C \left( \frac{|y_1 - y_1'|}{2^{-k_1}} \right)^{\varepsilon''} \left( \frac{|y_2 - y_2'|}{2^{-k_2}} \right)^{\varepsilon''} \frac{2^{-k_1 \varepsilon'}}{(2^{-k_1} + |x_1 - y_1|)^{n_1 + \varepsilon'}} \\ \frac{2^{-k_2 \varepsilon'}}{(2^{-k_2} + |x_2 - y_2|)^{n_2 + \varepsilon'}} \\ \text{if } |y_1 - y_1'| \leq 2^{-k_1 - 1} \text{ and } |y_2 - y_2'| \leq 2^{-k_2 - 1}.$$

**Proof.** We will use the iteration method which reduces the product case to the classical case. We first check that  $T_{k_1, k_2}^1(x_1, x_2, y_1, y_2)$  satisfies the estimates (D<sub>1</sub>) – (D<sub>3</sub>). For fixed  $k_1, x_1$  and  $y_1$ , set

$$\mathcal{K}_2(u_2, v_2) \\ = \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_1}} D_{k_1}(x_1, u_1) b_1(u_1) K(u_1, u_2, v_1, v_2) b_1(v_1) D_{j_1} M_{b_1} \tilde{D}_{j_1}(v_1, y_1) du_1 dv_1.$$

Since  $T_1^*(b_1) = 0$ , we have

$$\mathcal{K}_2(u_2, v_2) = \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_1}} (D_{k_1}(x_1, u_1) - D_{k_1}(x_1, y_1)) b_1(u_1) K(u_1, u_2, v_1, v_2) \\ \times b_1(v_1) D_{j_1} M_{b_1} \tilde{D}_{j_1}(v_1, y_1) du_1 dv_1.$$

By [15, Theorem 3.1] with  $j_1 \geq k_1$ , the operator  $S$  associated with the kernel  $\mathcal{K}_2(u_2, v_2)$  is a Calderón-Zygmund operator and satisfies, for  $\varepsilon' < \varepsilon$ ,

$$\|S\|_{CZ} \leq C 2^{-(j_1 - k_1)\varepsilon'} \frac{2^{-k_1 \varepsilon'}}{(2^{-k_1} + |x_1 - y_1|)^{n_1 + \varepsilon'}}. \tag{9}$$

Note that the condition  $T_2^*(b_2) = 0$  implies  $S^*(b_2) = 0$ . Therefore, first writing

$$T_{k_1, k_2}^1(x_1, x_2, y_1, y_2) = \sum_{j_1 \geq k_1} \sum_{j_2 \geq k_2} \int_{\mathbb{R}^{n_2} \times \mathbb{R}^{n_2}} D_{k_2}(x_2, u_2) b_2(u_2) \mathcal{K}_2(u_2, v_2) \\ \times b_2(v_2) D_{j_2} M_{b_2} \tilde{D}_{j_2}(v_2, y_2) du_2 dv_2$$

and then applying the almost orthogonal estimate for  $\mathcal{K}_2(u_2, v_2)$  with the norm estimate in (9) imply

$$|T_{k_1, k_2}^1(x_1, x_2, y_1, y_2)| \leq C \frac{2^{-k_1 \varepsilon'}}{(2^{-k_1} + |x_1 - y_1|)^{n_1 + \varepsilon'}} \frac{2^{-k_2 \varepsilon'}}{(2^{-k_2} + |x_2 - y_2|)^{n_2 + \varepsilon'}}.$$

This shows that  $T_{k_1, k_2}^1(x_1, x_2, y_1, y_2)$  satisfies the estimate  $(D_1)$ . To check  $(D_2)$ (i), we write

$$\begin{aligned} & T_{k_1, k_2}^1(x_1, x_2, y_1, y_2) - T_{k_1, k_2}^1(x_1, x_2, y'_1, y_2) \\ &= \sum_{j_1 \geq k_1} \sum_{j_2 \geq k_2} \int_{\mathbb{R}^{n_2} \times \mathbb{R}^{n_2}} D_{k_2}(x_2, u_2) b_2(u_2) \mathcal{K}_{2,2}(u_2, v_2) \\ & \quad \times b_2(v_2) D_{j_2} M_{b_2} \tilde{D}_{j_2}(v_2, y_2) du_2 dv_2, \end{aligned}$$

where for fixed  $k_1, x_1, y_1, y'_1$ ,

$$\begin{aligned} \mathcal{K}_{2,2}(u_2, v_2) &= \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_1}} D_{k_1}(x_1, u_1) b_1(u_1) K(u_1, u_2, v_1, v_2) \\ & \quad \times b_1(v_1) D_{j_1}(v_1, z_1) b_1(z_1) (\tilde{D}_{j_1}(z_1, y_1) - \tilde{D}_{j_1}(z_1, y'_1)) du_1 dv_1. \end{aligned}$$

By the similar argument of the proof in [15, Theorem 3.1] (but simpler since  $T$  is bounded on  $L^2$ ), we obtain that  $\mathcal{K}_{2,2}(u_2, v_2)$  is a Calderón-Zygmund kernel and

$$|\mathcal{K}_{2,2}|_{CZ} \leq C \left( \frac{|y_1 - y'_1|}{2^{-k_1}} \right)^{\varepsilon''} \frac{2^{-(j_1 - k_1)\varepsilon'} 2^{-k_1\varepsilon'}}{(2^{-k_1} + |x_1 - y_1|)^{n_1 + \varepsilon'}} \quad \text{for } |y_1 - y'_1| \leq 2^{-k_1 - 1}. \tag{10}$$

The almost orthogonal estimate together with the estimate of (10) yields

$$\begin{aligned} & \left| \sum_{j_1 \geq k_1} \sum_{j_2 \geq k_2} \int_{\mathbb{R}^{n_2} \times \mathbb{R}^{n_2}} D_{k_2}(x_2, u_2) b_2(u_2) \mathcal{K}_{2,2}(u_2, v_2) \right. \\ & \quad \left. \times b_2(v_2) D_{j_2} M_{b_2} \tilde{D}_{j_2}(v_2, y_2) du_2 dv_2 \right| \\ & \leq \left( \frac{|y_1 - y'_1|}{2^{-k_1}} \right)^{\varepsilon''} \frac{2^{-k_1\varepsilon'}}{(2^{-k_1} + |x_1 - y_1|)^{n_1 + \varepsilon'}} \frac{2^{-k_2\varepsilon'}}{(2^{-k_2} + |x_2 - y_2|)^{n_2 + \varepsilon'}} \end{aligned}$$

for  $|y_1 - y'_1| \leq 2^{-k_1 - 1}$  and hence  $(D_2)$ (i) follows. The proof of  $(D_2)$ (ii) is the same. To prove  $(D_3)$  for the kernel  $T_{k_1, k_2}^1(x_1, x_2, y_1, y_2)$ , we write

$$\begin{aligned} & [T_{k_1, k_2}^1(x_1, x_2, y_1, y_2) - T_{k_1, k_2}^1(x_1, x_2, y'_1, y_2)] \\ & \quad - [T_{k_1, k_2}^1(x_1, x_2, y_1, y'_2) - T_{k_1, k_2}^1(x_1, x_2, y'_1, y'_2)] \\ &= \sum_{j_1 \geq k_1} \sum_{j_2 \geq k_2} \iiint D_{k_1}(x_1, u_1) b_1(u_1) D_{k_2}(x_2, u_2) b_2(u_2) K(u_1, u_2, v_1, v_2) \\ & \quad \times b_1(v_1) D_{j_1}(v_1, z_1) b_1(z_1) (\tilde{D}_{j_1}(z_1, y_1) - \tilde{D}_{j_1}(z_1, y'_1)) \\ & \quad \times b_2(v_2) D_{j_2}(v_2, z_2) b_2(z_2) (\tilde{D}_{j_2}(z_2, y_2) - \tilde{D}_{j_2}(z_2, y'_2)) dudvdz \\ &= \sum_{j_1 \geq k_1} \sum_{j_2 \geq k_2} \int_{\mathbb{R}^{n_2}} \int_{\mathbb{R}^{n_2}} \int_{\mathbb{R}^{n_2}} D_{k_2}(x_2, u_2) b_2(u_2) \mathcal{K}_{2,2}(u_2, v_2) \\ & \quad \times b_2(v_2) D_{j_2}(v_2, z_2) b_2(z_2) (\tilde{D}_{j_2}(z_2, y_2) - \tilde{D}_{j_2}(z_2, y'_2)) du_2 dv_2 dz_2. \end{aligned}$$

By the almost orthogonal estimate together with the estimate of (10), we obtain that the kernel  $T_{k_1, k_2}^1(x_1, x_2, y_1, y_2)$  satisfies  $(D_3)$ .

Since the proofs for  $T_{k_1, k_2}^j(x_1, x_2, y_1, y_2), j = 2, 3$  are similar, we estimate the kernel  $T_{k_1, k_2}^2(x_1, x_2, y_1, y_2)$  only. We rewrite

$$\begin{aligned} & T_{k_1, k_2}^2(x_1, x_2, y_1, y_2) \\ &= \sum_{j_1 \geq k_1} \sum_{j_2 < k_2} \iint D_{k_1}(x_1, u_1) b_1(u_1) D_{k_2}(x_2, u_2) b_2(u_2) K(u_1, u_2, v_1, v_2) \\ &\quad \times M_b D_{j_1, j_2} M_b \tilde{D}_{j_1}(v_1, y_1) \tilde{D}_{j_2}(v_2, y_2) dudv \\ &= \sum_{j_1 \geq k_1} \sum_{j_2 < k_2} \iint \int_{\mathbb{R}^{n_2}} D_{k_1}(x_1, u_1) b_1(u_1) D_{k_2}(x_2, u_2) b_2(u_2) K(u_1, u_2, v_1, v_2) \\ &\quad \times b_1(v_1) D_{j_1} M_{b_1} \tilde{D}_{j_1}(v_1, y_1) b_2(v_2) D_{j_2}(v_2, z_2) b_2(z_2) \tilde{D}_{j_2}(z_2, y_2) dudvdz_2 \\ &= \sum_{j_1 \geq k_1} \sum_{j_2 < k_2} \iint \int_{\mathbb{R}^{n_2}} D_{k_1}(x_1, u_1) b_1(u_1) D_{k_2}(x_2, u_2) b_2(u_2) K(u_1, u_2, v_1, v_2) \\ &\quad \times b_1(v_1) D_{j_1} M_{b_1} \tilde{D}_{j_1}(v_1, y_1) b_2(v_2) (D_{j_2}(v_2, z_2) - D_{j_2}(x_2, z_2)) \\ &\quad \times b_2(z_2) \tilde{D}_{j_2}(z_2, y_2) dudvdz_2 \\ &+ \sum_{j_1 \geq k_1} \sum_{j_2 < k_2} \iint \int_{\mathbb{R}^{n_2}} D_{k_1}(x_1, u_1) b_1(u_1) D_{k_2}(x_2, u_2) b_2(u_2) K(u_1, u_2, v_1, v_2) \\ &\quad \times b_1(v_1) D_{j_1} M_{b_1} \tilde{D}_{j_1}(v_1, y_1) b_2(v_2) D_{j_2}(x_2, z_2) b_2(z_2) \tilde{D}_{j_2}(z_2, y_2) dudvdz_2 \\ &:= T_{k_1, k_2}^{2,1}(x_1, x_2, y_1, y_2) + T_{k_1, k_2}^{2,2}(x_1, x_2, y_1, y_2). \end{aligned}$$

The proof for  $T_{k_1, k_2}^{2,1}(x_1, x_2, y_1, y_2)$  is similar to  $T_{k_1, k_2}^1(x_1, x_2, y_1, y_2)$ , so we leave details to the reader. Write

$$P_{k_2}(x_2, y_2) = \sum_{j_2 < k_2} \int_{\mathbb{R}^{n_2}} D_{j_2}(x_2, z_2) b_2(z_2) \tilde{D}_{j_2}(z_2, y_2) dz_2.$$

Note that  $T_1^*(b_1) = 0$ . We have

$$\begin{aligned} & T_{k_1, k_2}^{2,2}(x_1, x_2, y_1, y_2) \\ &= \sum_{j_1 \geq k_1} \iint (D_{k_1}(x_1, u_1) - D_{k_1}(x_1, v_1)) b_1(u_1) D_{k_2}(x_2, u_2) b_2(u_2) \\ &\quad \times K(u_1, u_2, v_1, v_2) b_1(v_1) D_{j_1} M_{b_1} \tilde{D}_{j_1}(v_1, y_1) b_2(v_2) P_{k_2}(x_2, y_2) dudv. \end{aligned}$$

For fixed  $x_2$ , set

$$\begin{aligned} \mathcal{K}_1(u_1, v_1) &= \int_{\mathbb{R}^{n_2}} \int_{\mathbb{R}^{n_2}} D_{k_2}(x_2, u_2) b_2(u_2) K(u_1, u_2, v_1, v_2) b_2(v_2) du_2 dv_2 \\ &= \int_{\mathbb{R}^{n_2}} \int_{\mathbb{R}^{n_2}} D_{k_2}(x_2, u_2) b_2(u_2) \tilde{K}^1(u_1, v_1)(u_2, v_2) b_2(v_2) du_2 dv_2. \end{aligned}$$

Note that for fixed  $(u_1, v_1)$ ,  $\int_{\mathbb{R}^{n_2}} \tilde{K}^1(u_1, v_1)(u_2, v_2)b_2(v_2)dv_2$ , as a function of the variable  $u_2$ , is a *BMO* function and  $D_{k_2}(x_2, \cdot)b_2$  is a function in  $H^1(\mathbb{R}^{n_2})$  with  $H^1(\mathbb{R}^{n_2})$ -norm uniformly bounded for all  $x_2$  and  $k_2$ . Moreover,

$$\begin{aligned} \left\| \int_{\mathbb{R}^{n_2}} \tilde{K}^1(u_1, v_1)(\cdot, v_2)b_2(v_2)dv_2 \right\|_{BMO(\mathbb{R}^{n_2})} &\leq C \|\tilde{K}^1(u_1, v_1)\|_{CZ} \\ &\leq C|u_1 - v_1|^{-n_1}, \end{aligned}$$

which implies

$$|K_1(u_1, v_1)| \leq C|u_1 - v_1|^{-n_1}.$$

Similarly, for  $|u_1 - u'_1| \leq \frac{1}{2}|u_1 - v_1|$ , we have

$$\begin{aligned} &\left\| \int_{\mathbb{R}^{n_2}} [\tilde{K}^1(u_1, v_1)(\cdot, v_2) - \tilde{K}^1(u'_1, v_1)(\cdot, v_2)]b_2(v_2)dv_2 \right\|_{BMO(\mathbb{R}^{n_2})} \\ &\leq C \|\tilde{K}^1(u_1, v_1) - \tilde{K}^1(u'_1, v_1)\|_{CZ} \\ &\leq C|u_1 - u'_1|^\varepsilon |u_1 - v_1|^{-n_1 - \varepsilon}, \end{aligned}$$

and hence

$$\begin{aligned} &|K_1(u_1, v_1) - K_1(u'_1, v_1)| \\ &= \left| \int_{\mathbb{R}^{n_2}} \int_{\mathbb{R}^{n_2}} D_{k_2}(x_2, u_2)b_2(u_2) \right. \\ &\quad \left. \times [\tilde{K}^1(u_1, v_1)(u_2, v_2) - \tilde{K}^1(u'_1, v_1)(u_2, v_2)]b_2(v_2)du_2dv_2 \right| \\ &\leq C|u_1 - u'_1|^\varepsilon |u_1 - v_1|^{-n_1 - \varepsilon}. \end{aligned}$$

The estimate  $|K_1(u_1, v_1) - K_1(u'_1, v_1)|$  can be obtained by the same approach. Thus,  $K_1(u_1, v_1)$  is a Calderón-Zygmund kernel and  $|K_1|_{CZ} \leq C$ . Note that

$$\begin{aligned} &T_{k_1, k_2}^{2,2}(x_1, x_2, y_1, y_2) \\ &= \sum_{j_1 \geq k_1} \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_1}} (D_{k_1}(x_1, u_1) - D_{k_1}(x_1, v_1))b_1(u_1)K_1(u_1, v_1) \\ &\quad \times b_1(v_1)D_{j_1}M_{b_1}\tilde{D}_{j_1}(v_1, y_1)du_1dv_1P_{k_2}(x_2, y_2). \end{aligned}$$

Note that the kernel  $P_{k_2}(x_2, y_2)$  satisfies the same size and smoothness conditions as a Poisson kernel by using the method in [15, page 133]. Applying the almost orthogonal estimate to  $K_1(u_1, v_1)$  together with the size condition on  $P_{k_2}$ , we get  $(D_1)$  for  $T_{k_1, k_2}^{2,2}(x_1, x_2, y_1, y_2)$ . The estimates of  $(D_2)$  and  $(D_3)$  can be proved by the same way.

Finally, we rewrite

$$\begin{aligned} &T_{k_1, k_2}^4(x_1, x_2, y_1, y_2) \\ &= \sum_{j_1 < k_1} \sum_{j_2 < k_2} \iint D_{k_1}(x_1, u_1)b_1(u_1)D_{k_2}(x_2, u_2)b_2(u_2)K(u_1, u_2, v_1, v_2)b_1(v_1) \end{aligned}$$

$$\begin{aligned}
 & \times D_{j_1, j_2} M_b \tilde{D}_{j_1}(v_1, y_1) b_2(v_2) \tilde{D}_{j_2}(v_2, y_2) dudv \\
 = & \sum_{j_1 < k_1} \sum_{j_2 < k_2} \iiint D_{k_1}(x_1, u_1) b_1(u_1) D_{k_2}(x_2, u_2) b_2(u_2) K(u_1, u_2, v_1, v_2) b_1(v_1) \\
 & \times D_{j_1}(v_1, z_1) b_1(z_1) \tilde{D}_{j_1}(z_1, y_1) b_2(v_2) D_{j_2}(v_2, z_2) b_2(z_2) \tilde{D}_{j_2}(z_2, y_2) dudvdz \\
 = & \sum_{j_1 < k_1} \sum_{j_2 < k_2} \iiint D_{k_1}(x_1, u_1) b_1(u_1) D_{k_2}(x_2, u_2) b_2(u_2) K(u_1, u_2, v_1, v_2) \\
 & \times b_1(v_1) (D_{j_1}(v_1, z_1) - D_{j_1}(x_1, z_1)) b_1(z_1) \tilde{D}_{j_1}(z_1, y_1) \\
 & \times b_2(v_2) (D_{j_2}(v_2, z_2) - D_{j_2}(x_2, z_2)) b_2(z_2) \tilde{D}_{j_2}(z_2, y_2) dudvdz \\
 + & \sum_{j_1 < k_1} \sum_{j_2 < k_2} \iiint D_{k_1}(x_1, u_1) b_1(u_1) D_{k_2}(x_2, u_2) b_2(u_2) K(u_1, u_2, v_1, v_2) \\
 & \times b_1(v_1) D_{j_1}(x_1, z_1) b_1(z_1) \tilde{D}_{j_1}(z_1, y_1) \\
 & \times b_2(v_2) (D_{j_2}(v_2, z_2) - D_{j_2}(x_2, z_2)) b_2(z_2) \tilde{D}_{j_2}(z_2, y_2) dudvdz \\
 + & \sum_{j_1 < k_1} \sum_{j_2 < k_2} \iiint D_{k_1}(x_1, u_1) b_1(u_1) D_{k_2}(x_2, u_2) b_2(u_2) K(u_1, u_2, v_1, v_2) \\
 & \times b_2(v_2) D_{j_2}(x_2, z_2) b_1(v_1) (D_{j_1}(v_1, z_1) - D_{j_1}(x_1, z_1)) \\
 & \times b_1(z_1) \tilde{D}_{j_1}(z_1, y_1) b_2(z_2) \tilde{D}_{j_2}(z_2, y_2) dudvdz \\
 + & \sum_{j_1 < k_1} \sum_{j_2 < k_2} \iiint D_{k_1}(x_1, u_1) b_1(u_1) D_{k_2}(x_2, u_2) b_2(u_2) K(u_1, u_2, v_1, v_2) b_1(v_1) \\
 & \times D_{j_1}(x_1, z_1) b_2(v_2) D_{j_2}(x_2, z_2) b_1(z_1) \tilde{D}_{j_1}(z_1, y_1) b_2(z_2) \tilde{D}_{j_2}(z_2, y_2) dudvdz \\
 = & T_{k_1, k_2}^{4,1}(x_1, x_2, y_1, y_2) + T_{k_1, k_2}^{4,2}(x_1, x_2, y_1, y_2) \\
 & + T_{k_1, k_2}^{4,3}(x_1, x_2, y_1, y_2) + T_{k_1, k_2}^{4,4}(x_1, x_2, y_1, y_2).
 \end{aligned}$$

Since the proof for  $T_{k_1, k_2}^{4,1}(x_1, x_2, y_1, y_2)$  is similar to  $T_{k_1, k_2}^1(x_1, x_2, y_1, y_2)$  and the proofs of  $T_{k_1, k_2}^{4,2}(x_1, x_2, y_1, y_2)$  and  $T_{k_1, k_2}^{4,3}(x_1, x_2, y_1, y_2)$  are similar to  $T_{k_1, k_2}^{2,2}(x_1, x_2, y_1, y_2)$ , we estimate  $T_{k_1, k_2}^{4,4}(x_1, x_2, y_1, y_2)$  only. Let

$$P_{k_i}(x_i, y_i) = \sum_{j_i < k_i} \int_{\mathbb{R}^{n_i}} D_{j_i}(x_i, z_i) b_i(z_i) \tilde{D}_{j_i}(z_i, y_i) dz_i, \quad i = 1, 2.$$

Then

$$\begin{aligned}
 T_{k_1, k_2}^{4,4}(x_1, x_2, y_1, y_2) = & \iint D_{k_1}(x_1, u_1) b_1(u_1) D_{k_2}(x_2, u_2) b_2(u_2) K(u_1, u_2, v_1, v_2) \\
 & \times b_1(v_1) b_2(v_2) dudv P_{k_1}(x_1, y_1) P_{k_2}(x_2, y_2).
 \end{aligned}$$

Note that  $\int K(u_1, u_2, v_1, v_2) b_1(v_1) b_2(v_2) dv_1 dv_2$ , as a function of variables  $u_1$  and  $u_2$ , belongs to  $BMO(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ , and  $D_{k_1}(x_1, u_1) b_1(u_1) D_{k_2}(x_2, u_2) b_2(u_2)$ ,

as function of  $(u_1, u_2)$ , is in  $H^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  with the bounded norm uniformly for all  $k_1, k_2$  and  $x_1, x_2$ . Thus,

$$\iint D_{k_1}(x_1, u_1)b_1(u_1)D_{k_2}(x_2, u_2)b_2(u_2)K(u_1, u_2, v_1, v_2)b_1(v_1)b_2(v_2)dudv$$

is uniformly bounded for all  $k_1, k_2$  and  $x_1, x_2$ . Therefore, estimates  $(D_1) - (D_3)$  for  $T_{k_1, k_2}^{4,4}(x_1, x_2, y_1, y_2)$  are the same as those for  $P_{k_1}(x_1, y_1)P_{k_2}(x_2, y_2)$ . By the similar method in [15, page 133], the kernels  $P_{k_1}$  and  $P_{k_2}$  satisfy the same size and smoothness conditions as a Poisson kernel, and hence  $P_{k_1}(x_1, y_1)P_{k_2}(x_2, y_2)$  satisfies estimates  $(D_1) - (D_3)$ . Thus, the proof of Lemma 2 is completed.  $\square$

Now we demonstrate the regularity of the operator  $T_{k_1, k_2}$  mapping from  $L^2$  into  $L^2_{\mathcal{H}}$ .

**Lemma 5.3.** *Let  $T_{k_1, k_2}$  be defined above and  $\varepsilon$  be the regularity exponent of  $T$ . For  $\varepsilon' < \varepsilon$ ,*

(i) *if  $|y_1 - x_I| \leq |x_1 - x_I|/2$ , then*

$$\begin{aligned} & \left\| \left\{ \int_{\mathbb{R}^{n_2}} [T_{k_1, k_2}(x_1, \cdot, y_1, y_2) - T_{k_1, k_2}(x_1, \cdot, x_I, y_2)] f(y_2) dy_2 \right\} \right\|_{L^2_{\mathcal{H}}(\mathbb{R}^{n_2})} \\ & \leq C \frac{|y_1 - x_I|^{\varepsilon'}}{|x_1 - x_I|^{n_1 + \varepsilon'}} \|f\|_2; \end{aligned}$$

(ii) *if  $|y_2 - y_J| \leq |x_2 - y_J|/2$ , then*

$$\begin{aligned} & \left\| \left\{ \int_{\mathbb{R}^{n_1}} [T_{k_1, k_2}(\cdot, x_2, y_1, y_2) - T_{k_1, k_2}(\cdot, x_2, y_1, y_J)] f(y_1) dy_1 \right\} \right\|_{L^2_{\mathcal{H}}(\mathbb{R}^{n_1})} \\ & \leq C \frac{|y_2 - y_J|^{\varepsilon'}}{|x_2 - y_J|^{n_2 + \varepsilon'}} \|f\|_2. \end{aligned}$$

**Proof.** The proofs of (i) and (ii) are the same, so we show the case (i) only. We will use  $0 < \varepsilon''' < \varepsilon'' < \varepsilon' < \varepsilon$  through the proof. Note that

$$\begin{aligned} & \left\| \left\{ \int_{\mathbb{R}^{n_2}} [T_{k_1, k_2}(x_1, \cdot, y_1, y_2) - T_{k_1, k_2}(x_1, \cdot, x_I, y_2)] f(y_2) dy_2 \right\} \right\|_{L^2_{\mathcal{H}}(\mathbb{R}^{n_2})} \\ & = \int_{\mathbb{R}^{n_2}} \left\| \left\{ \int_{\mathbb{R}^{n_2}} [T_{k_1, k_2}(x_1, x_2, y_1, y_2) - T_{k_1, k_2}(x_1, x_2, x_I, y_2)] f(y_2) dy_2 \right\} \right\|_{\mathcal{H}}^2 dx_2. \end{aligned}$$

We write

$$\begin{aligned} & \int_{\mathbb{R}^{n_2}} [T_{k_1, k_2}(x_1, x_2, y_1, y_2) - T_{k_1, k_2}(x_1, x_2, x_I, y_2)] f(y_2) dy_2 \\ & = \int_{\mathbb{R}^{n_2}} \sum_{j_1} \sum_{j_2} \iint D_{k_1}(x_1, u_1)b_1(u_1)D_{k_2}(x_2, u_2)b_2(u_2)K(u_1, u_2, v_1, v_2) \\ & \quad \times b_1(v_1)(D_{j_1}M_{b_1}\tilde{D}_{j_1}(v_1, y_1) - D_{j_1}M_{b_1}\tilde{D}_{j_1}(v_1, x_I)) \end{aligned}$$

$$\begin{aligned}
 & \times b_2(v_2)D_{j_2}M_{b_2}\tilde{D}_{j_2}(v_2, y_2)f(y_2)dudvdy_2 \\
 = & \sum_{j_1} \sum_{j_2} \iint D_{k_1}(x_1, u_1)b_1(u_1)D_{k_2}(x_2, u_2)b_2(u_2)K(u_1, u_2, v_1, v_2) \\
 & \times b_1(v_1)(D_{j_1}M_{b_1}\tilde{D}_{j_1}(v_1, y_1) - D_{j_1}M_{b_1}\tilde{D}_{j_1}(v_1, x_I)) \\
 & \times b_2(v_2)D_{j_2}M_{b_2}\tilde{D}_{j_2}M_{b_2}f(v_2)dudv \\
 = & \sum_{j_1} \iint D_{k_1}(x_1, u_1)b_1(u_1)D_{k_2}(x_2, u_2)b_2(u_2)K(u_1, u_2, v_1, v_2) \\
 & \times b_1(v_1)(D_{j_1}M_{b_1}\tilde{D}_{j_1}(v_1, y_1) - D_{j_1}M_{b_1}\tilde{D}_{j_1}(v_1, x_I))b_2(v_2)f(v_2)dudv \\
 = & D_{k_2}M_{b_2} \left( \sum_{j_1} \int \int_{\mathbb{R}^{n_1}} D_{k_1}(x_1, u_1)b_1(u_1)K(u_1, \cdot, v_1, v_2) \right. \\
 & \left. \times b_1(v_1)(D_{j_1}M_{b_1}\tilde{D}_{j_1}(v_1, y_1) - D_{j_1}M_{b_1}\tilde{D}_{j_1}(v_1, x_I))b_2(v_2)f(v_2)du_1dv \right) (x_2),
 \end{aligned}$$

where we first write

$$\int_{\mathbb{R}^{n_2}} D_{j_2}M_{b_2}\tilde{D}_{j_2}(v_2, y_2)b_2(y_2)f(y_2)dy_2 = D_{j_2}M_{b_2}\tilde{D}_{j_2}M_{b_2}f(v_2)$$

and then use the Calderón identity  $\sum_{j_2} D_{j_2}M_{b_2}\tilde{D}_{j_2}M_{b_2}f(v_2) = f(v_2)$ . The Littlewood-Paley estimate gives

$$\begin{aligned}
 & \int_{\mathbb{R}^{n_2}} \left\| \left\{ \int_{\mathbb{R}^{n_2}} [T_{k_1, k_2}(x_1, x_2, y_1, y_2) - T_{k_1, k_2}(x_1, x_2, x_I, y_2)] f(y_2)dy_2 \right\} \right\|_{\mathcal{H}}^2 dx_2 \\
 & \leq C \sum_{k_1} \int_{\mathbb{R}^{n_2}} \left| \sum_{j_1} \int \int_{\mathbb{R}^{n_1}} D_{k_1}(x_1, u_1)b_1(u_1)K(u_1, x_2, v_1, v_2) \right. \\
 & \quad \left. \times b_1(y_1)(D_{j_1}M_{b_1}\tilde{D}_{j_1}(v_1, y_1) - D_{j_1}M_{b_1}\tilde{D}_{j_1}(v_1, x_I))b_2(v_2)f(v_2)du_1dv \right|^2 dx_2.
 \end{aligned} \tag{11}$$

Divide the sum  $\sum_{j_1}$  into three parts as follows:

$$\begin{aligned}
 & \int_{\mathbb{R}^{n_2}} \left| \sum_{j_1} \int \int_{\mathbb{R}^{n_1}} D_{k_1}(x_1, u_1)b_1(u_1)K(u_1, x_2, v_1, v_2) \right. \\
 & \quad \left. \times b_1(y_1)(D_{j_1}M_{b_1}\tilde{D}_{j_1}(v_1, y_1) - D_{j_1}M_{b_1}\tilde{D}_{j_1}(v_1, x_I))b_2(v_2)f(v_2)du_1dv \right|^2 dx_2 \\
 & \leq C \int_{\mathbb{R}^{n_2}} \left| \sum_{j_1 \geq k_1} \int_{\mathbb{R}^{n_1}} \int \int_{\mathbb{R}^{n_1}} D_{k_1}(x_1, u_1)b_1(u_1)K(u_1, x_2, v_1, v_2)b_1(v_1) \right. \\
 & \quad \left. \times D_{j_1}(v_1, z_1)b_1(z_1)(\tilde{D}_{j_1}(z_1, y_1) - \tilde{D}_{j_1}(z_1, x_I))b_2(v_2)f(v_2)du_1dvdz_1 \right|^2 dx_2
 \end{aligned}$$



$$\begin{aligned}
& + C \int_{\mathbb{R}^{n_2}} \left| \sum_{j_1 < k_1} \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_1}} D_{k_1}(x_1, u_1) b_1(u_1) K(u_1, x_2, v_1, v_2) b_1(v_1) \right. \\
& \quad \times (D_{j_1}(v_1, z_1) - D_{j_1}(x_1, z_1)) b_1(z_1) (\tilde{D}_{j_1}(z_1, y_1) - \tilde{D}_{j_1}(z_1, x_I)) \\
& \quad \left. \times b_2(v_2) f(v_2) du_1 dv dz_1 \right|^2 dx_2 \\
& + C \int_{\mathbb{R}^{n_2}} \left| \sum_{j_1 < k_1} \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_1}} D_{k_1}(x_1, u_1) b_1(u_1) K(u_1, x_2, v_1, v_2) b_1(v_1) \right. \\
& \quad \times D_{j_1}(x_1, z_1) b_1(z_1) (\tilde{D}_{j_1}(z_1, y_1) - \tilde{D}_{j_1}(z_1, x_I)) b_2(v_2) f(v_2) du_1 dv dz_1 \left. \right|^2 dx_2 \\
& := E + F + G.
\end{aligned}$$

We first consider the item  $G$  and write

$$\begin{aligned}
G & = C \int_{\mathbb{R}^{n_2}} \left| \sum_{j_1 < k_1} \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_1}} D_{k_1}(x_1, u_1) b_1(u_1) K(u_1, x_2, v_1, v_2) b_1(v_1) \right. \\
& \quad \left. \times (D_{j_1} M_{b_1} \tilde{D}_{j_1}(v_1, y_1) - D_{j_1} M_{b_1} \tilde{D}_{j_1}(v_1, x_I)) b_2(v_2) f(v_2) du_1 dv \right|^2 dx_2 \\
& = C |P_{k_1}(v_1, y_1) - P_{k_1}(v_1, x_I)|^2 \int_{\mathbb{R}^{n_2}} \left| \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_1}} D_{k_1}(x_1, u_1) b_1(u_1) \right. \\
& \quad \left. \times K(u_1, x_2, v_1, v_2) b_1(v_1) b_2(v_2) f(v_2) du_1 dv \right|^2 dx_2 \\
& = C \sup_{\|g\|_2 \leq 1} \left( \iint D_{k_1}(x_1, u_1) b_1(u_1) K(u_1, x_2, v_1, v_2) \right. \\
& \quad \left. \times b_1(v_1) b_2(v_2) f(v_2) g(x_2) du_1 dv dx_2 \right)^2 |P_{k_1}(v_1, y_1) - P_{k_1}(v_1, x_I)|^2,
\end{aligned}$$

where  $P_{k_1}(v_1, \cdot) = \sum_{j_1 < k_1} D_{j_1} M_{b_1} \tilde{D}_{j_1}(v_1, \cdot)$ .

For fixed  $u_1$  and  $v_1$ , set

$$\bar{K}(u_1, v_1) = \int_{\mathbb{R}^{n_2}} \int_{\mathbb{R}^{n_2}} K(u_1, x_2, v_1, v_2) b_2(v_2) f(v_2) g(x_2) dv_2 dx_2.$$

Then the operator associated to the kernel  $\bar{K}(u_1, v_1)$  is a Calderón-Zygmund operator with operator norm  $C\|f\|_2\|g\|_2$ . Since  $\int_{\mathbb{R}^{n_1}} \bar{K}(u_1, v_1) dv_1$  is a *BMO* function for  $u_1$ ,

$$\left| \int_{\mathbb{R}^{n_1}} D_{k_1}(x_1, u_1) b_1(u_1) \int_{\mathbb{R}^{n_1}} \bar{K}(u_1, v_1) b_1(v_1) dv_1 du_1 \right| \leq C\|f\|_2\|g\|_2$$

uniformly for  $x_1$ . Hence, for  $|y_1 - x_I| \leq t/2$ ,

$$\begin{aligned} G &\leq C |P_{k_1}(v_1, y_1) - P_{k_1}(v_1, x_I)|^2 \|f\|_2^2 \\ &\leq C (2^{k_1} |y_1 - x_I|)^{2\varepsilon} \frac{2^{-k_1 2\varepsilon}}{(2^{-k_1} + |x_1 - x_I|)^{2(n_1 + \varepsilon)}} \|f\|_2^2. \end{aligned} \tag{12}$$

To estimate  $E$ , we consider two cases  $\{|x_1 - z_1| > 8c2^{-k_1}\}$  and  $\{|x_1 - z_1| \leq 8c2^{-k_1}\}$ , where the constant  $c$  satisfies item (ii) in the definition of  $\{S_{k_i}\}, i = 1, 2$ . By duality, we get

$$\begin{aligned} E^{1/2} &= C \sup_{\|h\|_2 \leq 1} \int_{\mathbb{R}^{n_2}} h(x_2) \sum_{j_1 \geq k_1} \int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_1}} D_{k_1}(x_1, u_1) b_1(u_1) \\ &\quad \times K(u_1, x_2, v_1, v_2) b_1(v_1) D_{j_1}(v_1, z_1) b_1(z_1) \\ &\quad \times (\tilde{D}_{j_1}(z_1, y_1) - \tilde{D}_{j_1}(z_1, x_I)) b_2(v_2) f(v_2) du_1 dv dz_1 dx_2 \\ &= C \sup_{\|h\|_2 \leq 1} \left( \int_{|x_1 - z_1| > 8c2^{-k_1}} + \int_{|x_1 - z_1| \leq 8c2^{-k_1}} \right) \sum_{j_1 \geq k_1} \iint h(x_2) \\ &\quad \times D_{k_1}(x_1, u_1) b_1(u_1) K(u_1, x_2, v_1, v_2) b_1(v_1) D_{j_1}(v_1, z_1) b_1(z_1) \\ &\quad \times (\tilde{D}_{j_1}(z_1, y_1) - \tilde{D}_{j_1}(z_1, x_I)) b_2(v_2) f(v_2) du_1 dx_2 dv dz_1 \\ &:= E_1 + E_2. \end{aligned}$$

For  $E_1$ , we use the cancellation properties of  $D_{j_1}$  to get

$$\begin{aligned} E_1 &= C \sup_{\|h\|_2 \leq 1} \int_{|x_1 - z_1| > 8c2^{-k_1}} \sum_{j_1 \geq k_1} \iint h(x_2) D_{k_1}(x_1, u_1) b_1(u_1) \\ &\quad \times (K(u_1, x_2, v_1, v_2) - K(u_1, x_2, z_1, v_2)) b_1(v_1) D_{j_1}(v_1, z_1) b_1(z_1) \\ &\quad \times (\tilde{D}_{j_1}(z_1, y_1) - \tilde{D}_{j_1}(z_1, x_I)) b_2(v_2) f(v_2) du_1 dx_2 dv dz_1. \end{aligned}$$

Note that the facts  $|x_1 - z_1| > 8c2^{-k_1}$ ,  $|x_1 - u_1| < c2^{-k_1}$  and  $|v_1 - z_1| < c2^{-j_1} \leq c2^{-k_1}$  easily imply  $|v_1 - z_1| \leq |u_1 - v_1|/2$  and  $|u_1 - v_1| > |x_1 - z_1|/2$ .

We apply (A<sub>2</sub>) to obtain that, for  $|y_1 - x_I| \leq 2^{-k_1-1}$ ,

$$\begin{aligned}
 E_1 &\leq C \sup_{\|h\|_2 \leq 1} \|h\|_2 \|f\|_2 \sum_{j_1 \geq k_1} \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_1}} |D_{k_1}(x_1, u_1)| \frac{|v_1 - z_1|^\varepsilon}{|u_1 - v_1|^{n_1+\varepsilon}} \\
 &\quad \times |D_{j_1}(v_1, z_1)| |\tilde{D}_{j_1}(z_1, y_1) - \tilde{D}_{j_1}(z_1, x_I)| du_1 dv_1 dz_1 \\
 &\leq C \|f\|_2 \sum_{j_1 \geq k_1} \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_1}} 2^{-(j_1-k_1)\varepsilon} |D_{k_1}(x_1, u_1)| \frac{2^{-k_1\varepsilon}}{(2^{-k_1} + |u_1 - v_1|)^{n_1+\varepsilon}} \\
 &\quad \times |D_{j_1}(v_1, z_1)| |\tilde{D}_{j_1}(z_1, y_1) - \tilde{D}_{j_1}(z_1, x_I)| du_1 dv_1 dz_1 \\
 &\leq C \|f\|_2 \sum_{j_1 \geq k_1} \int_{\mathbb{R}^{n_1}} 2^{-(j_1-k_1)(\varepsilon-\varepsilon')} \frac{2^{-k_1\varepsilon}}{(2^{-k_1} + |x_1 - z_1|)^{n_1+\varepsilon}} \left(\frac{|y_1 - x_I|}{2^{-k_1}}\right)^{\varepsilon'} \\
 &\quad \times \left(\frac{2^{-j_1\varepsilon}}{(2^{-j_1} + |z_1 - y_1|)^{n_1+\varepsilon}} + \frac{2^{-j_1\varepsilon}}{(2^{-j_1} + |z_1 - x_I|)^{n_1+\varepsilon}}\right) dz_1 \\
 &\leq C \left(\frac{|y_1 - x_I|}{2^{-k_1}}\right)^{\varepsilon'} \frac{2^{-k_1\varepsilon}}{(2^{-k_1} + |x_1 - x_I|)^{n_1+\varepsilon}} \|f\|_2.
 \end{aligned} \tag{13}$$

By the condition on the support of  $D_{k_1}$ ,

$$\begin{aligned}
 E_2 &= C \sup_{\|h\|_2 \leq 1} \int_{|x_1 - z_1| \leq 8c2^{-k_1}} \sum_{j_1 \geq k_1} \iint_{|u_1 - z_1| \leq 9c2^{-k_1}} h(x_2) D_{k_1}(x_1, u_1) b_1(u_1) \\
 &\quad \times K(u_1, x_2, v_1, v_2) b_1(v_1) D_{j_1}(v_1, z_1) b_1(z_1) \\
 &\quad \times (\tilde{D}_{j_1}(z_1, y_1) - \tilde{D}_{j_1}(z_1, x_I)) b_2(v_2) f(v_2) du_1 dx_2 dv_1 dz_1.
 \end{aligned}$$

Note that  $E_2 = 0$  if  $|z_1 - y_1| > c2^{-k_1}$  and  $|z_1 - x_I| > c2^{-k_1}$ . It implies  $|x_1 - x_I| \leq 10c2^{-k_1}$  provided  $|x_1 - z_1| \leq 8c2^{-k_1}$  and  $|y_1 - x_I| \leq c2^{-k_1-1}$ . This fact will be used later.

Now let  $\eta_0 \in C^\infty(\mathbb{R}^{n_1})$  be 1 on the unit ball and 0 outside the ball  $B(0, 2)$ . Set  $\eta_1 = 1 - \eta_0$ . We use  $T_1^*(b_1) = 0$  to obtain

$$\begin{aligned}
 E_2 &= C \sup_{\|h\|_2 \leq 1} \int_{|x_1 - z_1| \leq 8c2^{-k_1}} \sum_{j_1 \geq k_1} \iint_{|u_1 - z_1| \leq 9c2^{-k_1}} h(x_2) \\
 &\quad \times (D_{k_1}(x_1, u_1) - D_{k_1}(x_1, z_1)) b_1(u_1) K(u_1, x_2, v_1, v_2) b_1(v_1) D_{j_1}(v_1, z_1) \\
 &\quad \times b_1(z_1) (\tilde{D}_{j_1}(z_1, y_1) - \tilde{D}_{j_1}(z_1, x_I)) b_2(v_2) f(v_2) du_1 dx_2 dv_1 dz_1 \\
 &= C \sup_{\|h\|_2 \leq 1} \int_{|x_1 - z_1| \leq 8c2^{-k_1}} \sum_{j_1 \geq k_1} \iint_{|u_1 - z_1| \leq 9c2^{-k_1}} h(x_2) \\
 &\quad \times \left(\eta_0\left(\frac{u_1 - z_1}{2^{-j_1-2}}\right) + \eta_1\left(\frac{u_1 - z_1}{2^{-j_1-2}}\right)\right) (D_{k_1}(x_1, u_1) - D_{k_1}(x_1, z_1)) b_1(u_1) \\
 &\quad \times K(u_1, x_2, v_1, v_2) b_1(v_1) D_{j_1}(v_1, z_1) b_1(z_1) (\tilde{D}_{j_1}(z_1, y_1) - \tilde{D}_{j_1}(z_1, x_I)) \\
 &\quad \times b_2(v_2) f(v_2) du_1 dx_2 dv_1 dz_1
 \end{aligned}$$

$$:= E_{21} + E_{22}.$$

Let

$$f_{j_1, z_1}(u_1) = \eta_0\left(\frac{u_1 - z_1}{2^{-j_1 - 2}}\right)(D_{k_1}(x_1, u_1) - D_{k_1}(x_1, z_1))$$

and

$$g_{j_1, z_1}(v_1) = D_{j_1}(v_1, z_1)b_1(z_1)(\tilde{D}_{j_1}(z_1, y_1) - \tilde{D}_{j_1}(z_1, x_I)).$$

The  $L^2(\mathbb{R}^{n_1+n_2})$  boundedness of  $T$  yields, for  $|y_1 - x_I| \leq 2^{-k_1-1}$ ,

$$\begin{aligned} E_{21} &\leq C \sup_{\|h\|_2 \leq 1} \int \sum_{j_1 \geq k_1} \|h\|_2 \|f_{t', z_1}\|_2 \|g_{t', z_1}\|_2 \|f\|_2 dz_1 \\ &\leq C \sum_{j_1 \geq k_1} \frac{2^{-j_1}}{2^{-k_1(n_1+1)}} 2^{-j_1 n_1/2} \left(\frac{|y_1 - x_I|}{2^{-j_1}}\right)^{\varepsilon'} 2^{j_1 n_1/2} \|f\|_2 \\ &\leq C |y_1 - x_I|^{\varepsilon'} 2^{k_1(n_1+\varepsilon')} \|f\|_2. \end{aligned}$$

To estimate  $E_{22}$ , we use the cancellation property of  $D_{j_1}$  and write

$$\begin{aligned} E_{22} &= C \sup_{\|h\|_2 \leq 1} \int_{|x_1 - z_1| \leq 8c2^{-k_1}} \sum_{j_1 \geq k_1} \iint_{|u_1 - z_1| \leq 9c2^{-k_1}} h(x_2) \eta_1\left(\frac{u_1 - z_1}{2^{-j_1 - 2}}\right) \\ &\quad \times (D_{k_1}(x_1, u_1) - D_{k_1}(x_1, z_1)) b_1(u_1) \\ &\quad \times (K(u_1, x_2, v_1, v_2) - K(u_1, x_2, z_1, v_2)) b_1(v_1) D_{j_1}(v_1, z_1) \\ &\quad \times b_1(z_1) (\tilde{D}_{j_1}(z_1, y_1) - \tilde{D}_{j_1}(z_1, x_I)) b_2(v_2) f(v_2) du_1 dx_2 dv_1 dz_1. \end{aligned}$$

By the conditions on the supports of  $\eta_1$  and  $D_{j_1}$ , we have  $|u_1 - z_1| \geq 4c2^{-j_1}$  and  $|v_1 - z_1| < c2^{-j_1}$ . This gives  $|v_1 - z_1| \leq |u_1 - z_1|/2$ . Applying  $(A_2)$  with the estimate

$$|D_{k_1}(x_1, u_1) - D_{k_1}(x_1, z_1)| \leq C \frac{|u_1 - z_1|}{2^{-k_1(n_1+1)}},$$

we obtain that, for  $|y_1 - x_I| \leq 2^{-k_1-1}$ ,

$$\begin{aligned} E_{22} &\leq C \sum_{j_1 \geq k_1} \int_{\mathbb{R}^{n_1}} \int_{2^{-j_1-2} \leq |u_1 - z_1| \leq 9c2^{-k_1}} \int_{\mathbb{R}^{n_1}} \frac{|u_1 - z_1|}{2^{-k_1(n_1+1)}} \frac{|v_1 - z_1|^\varepsilon}{|u_1 - z_1|^{n_1+\varepsilon}} \|f\|_2 \\ &\quad \times |D_{j_1}(v_1, z_1)| |\tilde{D}_{j_1}(z_1, y_1) - \tilde{D}_{j_1}(z_1, x_I)| dv_1 du_1 dz_1 \\ &\leq C |y_1 - x_I|^{\varepsilon'} 2^{k_1(n_1+\varepsilon')} \|f\|_2. \end{aligned}$$

Thus,  $E_2 \leq C |y_1 - x_I|^{\varepsilon'} 2^{k_1(n_1+\varepsilon')} \|f\|_2$ . By the fact  $|x_1 - x_I| \leq 10c2^{-k_1}$  as mentioned before, we have

$$E_2 \leq C \left(\frac{|y_1 - x_I|}{2^{-k_1}}\right)^{\varepsilon'} \frac{2^{-k_1\varepsilon}}{(2^{-k_1} + |x_1 - x_I|)^{n_1+\varepsilon}} \|f\|_2. \tag{14}$$

The estimate of  $F$  is the same as the estimate of  $E$ . It follows from (12)–(14) that, for  $|y_1 - x_I| \leq 2^{-k_1-1}$ ,

$$\begin{aligned} & \int_{\mathbb{R}^{n_2}} \left| \sum_{j_1} \int_{\mathbb{R}^{n_1}} D_{k_1}(x_1, u_1) b_1(u_1) K(u_1, x_2, v_1, v_2) \right. \\ & \quad \left. \times b_1(y_1) (D_{j_1} M_{b_1} \tilde{D}_{j_1}(v_1, y_1) - D_{j_1} M_{b_1} \tilde{D}_{j_1}(v_1, x_I)) b_2(v_2) f(v_2) du_1 dv \right|^2 dx_2 \\ & \leq C (2^{k_1} |y_1 - x_I|)^{2\varepsilon} \frac{2^{-k_1 2\varepsilon}}{(2^{-k_1} + |x_1 - x_I|)^{2(n_1+\varepsilon)}} \|f\|_2^2. \end{aligned} \tag{15}$$

Inserting (15) into (11), we obtain the desired result (i) of Lemma 5.3.  $\square$

To finish the proof of Theorem 1.1, we need the following general result that follows from Theorem 3.1 and [16, Proposition 4].

**Proposition 5.4.** *Let  $\mathcal{L}$  be a bounded operator from  $L^2(\mathbb{R}^{n_1+n_2})$  to  $L^2_{\mathcal{H}}(\mathbb{R}^{n_1+n_2})$ . Then, for  $0 < p \leq 1$ ,  $\mathcal{L}$  extends to be a bounded operator from  $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  to  $L^p_{\mathcal{H}}(\mathbb{R}^{n_1+n_2})$  if and only if  $\|\mathcal{L}(a)\|_{L^p_{\mathcal{H}}(\mathbb{R}^{n_1+n_2})} \leq C$  for all  $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ -atoms  $a$ , where the constant  $C$  is independent of  $a$ .*

It is known that the  $L^2(\mathbb{R}^{n_1+n_2})$  boundedness of  $T$  and the product Littlewood-Paley estimate imply that  $\mathcal{L}$  is bounded from  $L^2(\mathbb{R}^{n_1+n_2})$  to  $L^2_{\mathcal{H}}(\mathbb{R}^{n_1+n_2})$ . As mentioned before, the  $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) - H^p_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  boundedness of  $T$  is equivalent to the  $H^p - L^p_{\mathcal{H}}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  boundedness of the  $\mathcal{H}$ -valued operator  $\mathcal{L}$  which maps  $f$  into  $\{T_{k_1, k_2}(f)\}_{k_1, k_2 \in \mathbb{Z}}$ . Hence, to show the “if part” of Theorem 1.1, by Proposition 5.4, it suffices to prove

$$\|\{T_{k_1, k_2}(a)\}_{k_1, k_2 \in \mathbb{Z}}\|_{L^p_{\mathcal{H}}(\mathbb{R}^{n_1+n_2})} \leq C \quad \text{for all } H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \text{ atoms } a,$$

where the constant  $C$  is independent of  $a$ .

To do this, we follow R. Fefferman’s idea [9]. Suppose that  $a$  is an  $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  atom supported on an open set  $\Omega \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$  with finite measure. Furthermore,  $a$  can be decomposed as  $a = \sum_{R \in \mathcal{M}(\Omega)} a_R$ , where  $\mathcal{M}(\Omega)$  is the collection of all maximal dyadic subrectangles contained in  $\Omega$ , each  $a_R$  is supported on  $2R = 2I \times 2J$ , the double of  $R = I \times J$ ,  $\int_{2I} a_R(x_1, x_2) dx_1 = 0$  for all  $x_2 \in 2J$ , and  $\int_{2J} a_R(x_1, x_2) dx_2 = 0$  for all  $x_1 \in 2I$ . Here the higher order moments vanishing of  $a_R$  are not needed because we only consider  $\max\{\frac{n_1}{n_1+\varepsilon}, \frac{n_2}{n_2+\varepsilon}\} < p \leq 1$ . Moreover,  $\|a\|_2 \leq |\Omega|^{\frac{1}{2} - \frac{1}{p}}$  and  $\sum_{R \in \mathcal{M}(\Omega)} \|a_R\|_2^2 \leq |\Omega|^{1 - \frac{2}{p}}$ . Let  $\tilde{\Omega} = \{(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : M_s(\chi_{\Omega})(x_1, x_2) > 4^{-n_1-n_2} n_1^{-n_1/2} n_2^{-n_2/2}\}$ , where  $M_s$  is the strong maximal function defined by

$$M_s(f)(x_1, x_2) = \sup_{(x_1, x_2) \in P} \frac{1}{|P|} \int_P |f(y_1, y_2)| dy_1 dy_2,$$

where the supremum is taken over all rectangles  $P$  (a product of a cube in  $\mathbb{R}^{n_1}$  with a cube in  $\mathbb{R}^{n_2}$ ) containing  $(x_1, x_2)$ . It follows from the strong maximal theorem that  $|\tilde{\Omega}| \leq C|\Omega|$ .

We now estimate  $\|\{T_{k_1, k_2}(a)\}_{k_1, k_2 \in \mathbb{Z}}\|_{L_{\mathcal{H}}^p(\mathbb{R}^{n_1+n_2})}$  as follows. Write  $\tilde{\tilde{\Omega}} = \{(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : M_s(\chi_{\tilde{\Omega}})(x_1, x_2) > 4^{-n_1-n_2} n_1^{-n_1/2} n_2^{-n_2/2}\}$  and similarly for  $\tilde{\tilde{\Omega}}$ . Then

$$\begin{aligned} & \int \|\{T_{k_1, k_2}(a)\}(x_1, x_2)\|_{\mathcal{H}}^p dx_1 dx_2 \\ &= \int_{\tilde{\tilde{\Omega}}} \|\{T_{k_1, k_2}(a)\}(x_1, x_2)\|_{\mathcal{H}}^p dx_1 dx_2 + \int_{(\tilde{\tilde{\Omega}})^c} \|\{T_{k_1, k_2}(a)\}(x_1, x_2)\|_{\mathcal{H}}^p dx_1 dx_2. \end{aligned}$$

By Hölder’s inequality, the  $L^2 - L_{\mathcal{H}}^2$  boundedness of  $\mathcal{L}$ , and the size condition of  $a$ ,

$$\begin{aligned} & \int_{\tilde{\tilde{\Omega}}} \|\{T_{k_1, k_2}(a)\}(x_1, x_2)\|_{\mathcal{H}}^p dx_1 dx_2 \\ & \leq \left( \int_{\tilde{\tilde{\Omega}}} \|\{T_{k_1, k_2}(a)\}(x_1, x_2)\|_{\mathcal{H}}^2 dx_1 dx_2 \right)^{\frac{p}{2}} |\tilde{\tilde{\Omega}}|^{1-\frac{p}{2}} \\ & \leq C \|a\|_2^p |\Omega|^{1-\frac{p}{2}} \leq C. \end{aligned}$$

Therefore it remains to deal with

$$\begin{aligned} & \int_{(\tilde{\tilde{\Omega}})^c} \|\{T_{k_1, k_2}(a)\}(x_1, x_2)\|_{\mathcal{H}}^p dx_1 dx_2 \\ & \leq \sum_{R \in \mathcal{M}(\Omega)} \int_{(\tilde{\tilde{\Omega}})^c} \|\{T_{k_1, k_2}(a)\}(x_1, x_2)\|_{\mathcal{H}}^p dx_1 dx_2, \end{aligned}$$

where we use the inequality  $(\alpha + \beta)^p \leq \alpha^p + \beta^p$  for  $p \leq 1$ .

For each  $R = I \times J \in \mathcal{M}(\Omega)$ , we set a larger rectangle  $\tilde{R} = \tilde{I} \times \tilde{J}$  such that  $\tilde{I}$  is the largest dyadic cube containing  $I$  and  $\tilde{I} \times \tilde{J} \subset \tilde{\tilde{\Omega}}$ . Similarly,  $\tilde{\tilde{R}} = \tilde{I} \times \tilde{\tilde{J}}$  where  $\tilde{\tilde{J}}$  is the largest dyadic cube containing  $J$  and  $\tilde{I} \times \tilde{\tilde{J}} \subset \tilde{\tilde{\Omega}}$ . Let  $\mathcal{M}_1(\Omega)$  denote the collection of all dyadic subrectangles  $R \subset \Omega$ ,  $R = I \times J$  that are maximal in the  $x_1$  direction. It is clear that  $R \in \mathcal{M}(\Omega)$  implies  $R \in \mathcal{M}_2(\Omega)$  and  $\tilde{R} \in \mathcal{M}_1(\tilde{\tilde{\Omega}})$ . Define  $\mathcal{M}_2(\Omega)$  similarly. Also note that  $4\sqrt{n_1}\tilde{I} \times 4\sqrt{n_2}\tilde{\tilde{J}} \subset \tilde{\tilde{\Omega}}$ . Then

$$\begin{aligned} & \int_{(\tilde{\tilde{\Omega}})^c} \|\{T_{k_1, k_2}(a_R)\}(x_1, x_2)\|_{\mathcal{H}}^p dx_1 dx_2 \\ & \leq \int_{(4\sqrt{n_1}\tilde{I})^c \times \mathbb{R}^{n_2}} \|\{T_{k_1, k_2}(a_R)\}(x_1, x_2)\|_{\mathcal{H}}^p dx_1 dx_2 \\ & \quad + \int_{\mathbb{R}^{n_1} \times (4\sqrt{n_2}\tilde{\tilde{J}})^c} \|\{T_{k_1, k_2}(a_R)\}(x_1, x_2)\|_{\mathcal{H}}^p dx_1 dx_2 \end{aligned}$$

$$:= U(R) + V(R).$$

We define  $\gamma_1(R) = \gamma_1(R, \Omega) = \frac{\ell(\tilde{I})}{\ell(I)}$  and  $\gamma_2(\tilde{R}) = \gamma_2(\tilde{R}, \tilde{\Omega}) = \frac{\ell(\tilde{J})}{\ell(J)}$ , where  $\ell(I)$  denotes the side length of  $I$ . To estimate  $U(R)$ , we write

$$\begin{aligned} U(R) &= \int_{(4\sqrt{n_1}\tilde{I})^c \times 4\sqrt{n_2}J} \|\{T_{k_1, k_2}(a_R)\}(x_1, x_2)\|_{\mathcal{H}}^p dx_1 dx_2 \\ &\quad + \int_{(4\sqrt{n_1}\tilde{I})^c \times (4\sqrt{n_2}J)^c} \|\{T_{k_1, k_2}(a_R)\}(x_1, x_2)\|_{\mathcal{H}}^p dx_1 dx_2 \\ &:= U_1(R) + U_2(R). \end{aligned}$$

By Hölder’s inequality and Minkowski’s inequality,

$$U_1(R) \leq C|J|^{1-\frac{p}{2}} \int_{(4\sqrt{n_1}\tilde{I})^c} \left( \int_{\mathbb{R}^{n_2}} \|\{T_{k_1, k_2}(a_R)\}(x_1, x_2)\|_{\mathcal{H}}^2 dx_2 \right)^{\frac{p}{2}} dx_1. \tag{16}$$

The cancellation condition of  $a_R$  yields

$$\begin{aligned} &T_{k_1, k_2}(a_R)(x_1, x_2) \\ &= \int T_{k_1, k_2}(x_1, x_2, y_1, y_2) a_R(y_1, y_2) dy_1 dy_2 \\ &= \int [T_{k_1, k_2}(x_1, x_2, y_1, y_2) - T_{t, s}(x_1, x_2, x_I, y_2)] a_R(y_1, y_2) dy_1 dy_2, \end{aligned}$$

where  $x_I$  denotes the center of  $I$ . Now we apply Schwarz’s inequality to get

$$\begin{aligned} &\|\{T_{k_1, k_2}(a_R)\}(x_1, x_2)\|_{\mathcal{H}}^2 \\ &\leq C|I| \int_{2I} \left\| \left\{ \int_{2J} [T_{k_1, k_2}(x_1, x_2, y_1, y_2) \right. \right. \\ &\quad \left. \left. - T_{k_1, k_2}(x_1, x_2, x_I, y_2)] a_R(y_1, y_2) dy_2 \right\} \right\|_{\mathcal{H}}^2 dy_1. \end{aligned}$$

This estimate and Lemma 5.3 imply that, for  $x_1 \in (4\sqrt{n_1}\tilde{I})^c$  and  $y_1 \in 2I$ ,

$$\begin{aligned} &\int_{\mathbb{R}^{n_1}} \|\{T_{k_1, k_2}(a_R)\}(x_1, x_2)\|_{\mathcal{H}}^2 dx_2 \\ &\leq C|I| \int_{2I} \int_{\mathbb{R}^{n_2}} \left\| \left\{ \int_{2J} [T_{k_1, k_2}(x_1, x_2, y_1, y_2) \right. \right. \\ &\quad \left. \left. - T_{k_1, k_2}(x_1, x_2, x_I, y_2)] a_R(y_1, y_2) dy_2 \right\} \right\|_{\mathcal{H}}^2 dx_2 dy_1 \\ &\leq C|I| \left( \frac{\ell(I)^{\varepsilon'}}{|x_1 - x_I|^{n+\varepsilon'}} \right)^2 \|a_R\|_2^2. \end{aligned}$$

Inserting the estimate above into (16) shows

$$U_1(R) \leq C(\gamma_1(R))^{n_1 - (n_1 + \varepsilon')p} |R|^{1-\frac{p}{2}} \|a_R\|_2^p. \tag{17}$$

To estimate  $U_2(R)$ , we use the cancellation conditions of  $a_R$  to write

$$\begin{aligned} & T_{k_1, k_2}(a_R)(x_1, x_2) \\ &= \int \left( T_{k_1, k_2}(x_1, x_2, y_1, y_2) - T_{k_1, k_2}(x_1, x_2, x_I, y_2) \right. \\ &\quad \left. - T_{k_1, k_2}(x_1, x_2, y_1, x_J) + T_{k_1, k_2}(x_1, x_2, x_I, x_J) \right) a_R(y_1, y_2) dy_1 dy_2, \end{aligned}$$

where  $x_J$  is the center of  $J$ . For  $x_1 \in (4\sqrt{n_1}\tilde{I})^c$ ,  $x_2 \in (4\sqrt{n_2}J)^c$ ,  $y_1 \in 2I$ , and  $y_2 \in 2J$ , we have  $|y_1 - x_I| \leq \frac{1}{2}|x_1 - x_I|$  and  $|y_2 - x_J| \leq \frac{1}{2}|x_2 - x_J|$ . Thus, the estimate (B<sub>3</sub>) gives

$$\begin{aligned} & \|\{T_{k_1, k_2}(a_R)\}(x_1, x_2)\|_{\mathcal{H}} \\ & \leq C \left( \int |R| \left( \frac{|y_1 - x_I|^{\varepsilon'}}{|x_1 - x_I|^{n_1 + \varepsilon'}} \frac{|y_2 - x_J|^{\varepsilon'}}{|x_2 - x_J|^{n_2 + \varepsilon'}} |a_R(y_1, y_2)| \right)^2 dy_1 dy_2 \right)^{1/2}. \end{aligned}$$

Hence,

$$U_2(R) \leq C(\gamma_1(R))^{n_1 - (n_1 + \varepsilon')p} |R|^{1 - \frac{p}{2}} \|a_R\|_2^p. \tag{18}$$

Both estimates (17) and (18) give

$$U(R) \leq C(\gamma_1(R))^{n_1 - (n_1 + \varepsilon')p} |R|^{1 - \frac{p}{2}} \|a_R\|_2^p.$$

The estimate for  $V(R)$ , though slightly different from  $U(R)$ , can be handled in much the same manner so that

$$V(R) \leq C(\gamma_2(\tilde{R}))^{n_2 - (n_2 + \varepsilon')p} |R|^{1 - \frac{p}{2}} \|a_R\|_2^p.$$

Summing over  $R$  gives

$$\begin{aligned} & \sum_{R \in \mathcal{M}(\Omega)} \int_{(\tilde{\Omega})^c} \|\{T_{k_1, k_2}(a_R)\}(x_1, x_2)\|_{\mathcal{H}}^p dx_1 dx_2 \\ & \leq C \sum_{R \in \mathcal{M}(\Omega)} (\gamma_1(R))^{n_1 - (n_1 + \varepsilon')p} |R|^{1 - \frac{p}{2}} \|a_R\|_2^p \\ & \quad + C \sum_{R \in \mathcal{M}(\Omega)} (\gamma_2(\tilde{R}))^{n_2 - (n_2 + \varepsilon')p} |R|^{1 - \frac{p}{2}} \|a_R\|_2^p \\ & \leq C \left\{ \left( \sum_{R \in \mathcal{M}_2(\Omega)} |R| (\gamma_1(R))^{-\delta_1} \right)^{1 - \frac{p}{2}} \right. \\ & \quad \left. + \left( \sum_{\tilde{R} \in \mathcal{M}_1(\tilde{\Omega})} |\tilde{R}| (\gamma_2(\tilde{R}))^{-\delta_2} \right)^{1 - \frac{p}{2}} \right\} \left( \sum_{R \in \mathcal{M}(\Omega)} \|a_R\|_2^2 \right)^{\frac{p}{2}}, \end{aligned}$$

where  $\delta_1 = \frac{2[n_1 - (n_1 + \varepsilon')p]}{p-2} > 0$  and  $\delta_2 = \frac{2[n_2 - (n_2 + \varepsilon')p]}{p-2} > 0$ .

To estimate the last part above, we use the following



**Journé’s lemma.** Let  $\mathcal{M}_1(\Omega)$  and  $\mathcal{M}_2(\Omega)$  be defined as above. Then  $\sum_{R \in \mathcal{M}_2(\Omega)} |R|(\gamma_1(R))^{-\delta} \leq C_\delta |\Omega|$  and  $\sum_{R \in \mathcal{M}_1(\Omega)} |R|(\gamma_2(R))^{-\delta} \leq C_\delta |\Omega|$  for any  $\delta > 0$ , where  $C_\delta$  is a constant depending on  $\delta$  only.

Journé’s lemma and the size condition of  $a_R$  imply

$$\sum_{R \in \mathcal{M}(\Omega)} \int_{(\tilde{\Omega})^c} \|\{T_{k_1, k_2}(a_R)\}(x_1, x_2)\|_{\mathcal{H}}^p dx_1 dx_2 \leq C |\Omega|^{1-\frac{p}{2}} |\Omega|^{\frac{p}{2}-1} \leq C.$$

We complete the proof of the “if part” of Theorem 1.1. □

**Proof of Theorem 1.2. “if” part:**

We point out that the “if” part follows directly from duality of  $H^1_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  with  $BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ . We provide the details as follows.

(1) Suppose  $T_1(b_1) = T_2(b_2) = 0$ .

Then for the adjoint operator  $T^*$  of  $T$ , it is clear to see that  $T^*$  satisfies

$$(T^*)_1^*(b_1) = (T^*)_2^*(b_2) = 0.$$

Hence, from (1) in Theorem 1.1 we obtain that  $T^*$  is bounded from  $H^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  to  $H^1_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ .

Then for every  $f \in BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \cap L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ ,  $g \in H^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \cap L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ , we have

$$\begin{aligned} |\langle Tf, g \rangle| &= |\langle f, T^*g \rangle| \leq C \|f\|_{BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \|T^*g\|_{H^1_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \\ &\leq C \|f\|_{BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \|g\|_{H^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \end{aligned}$$

By density argument we can obtain that  $Tf$  is in  $BMO(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ .

(2) Suppose  $T_1(1) = T_2(1) = 0$ .

Then for the adjoint operator  $T^*$  of  $T$ , it is clear to see that  $T^*$  satisfies

$$(T^*)_1^*(1) = (T^*)_2^*(1) = 0.$$

Hence, from (2) in Theorem 1.1 we obtain that  $T^*M_{b_1 b_2}$  is bounded from  $H^1_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  to  $H^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ .

Then for every  $f \in BMO(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \cap L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ ,  $g \in H^1_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \cap L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ , we have

$$\begin{aligned} |\langle M_{b_1 b_2} Tf, g \rangle| &= |\langle f, T^*b_1 b_2 g \rangle| \\ &\leq C \|f\|_{BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \|T^*M_{b_1 b_2}g\|_{H^1_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \\ &\leq C \|f\|_{BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \|g\|_{H^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \end{aligned}$$

By density argument we can obtain that  $M_{b_1 b_2}Tf$  is in  $BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ .

(3) Suppose  $T_1(b_1) = T_2(b_2) = 0$ .

Then for the adjoint operator  $T^*$  of  $T$ , it is clear to see that  $T^*$  satisfies

$$(T^*)_1^*(b_1) = (T^*)_2^*(b_2) = 0.$$

Hence, from (3) in Theorem 1.1 we obtain that  $T^*M_{b_1 b_2}$  is bounded from  $H^1_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  to  $H^1_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ .

Then for every  $f \in BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \cap L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ ,  $g \in H^1_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \cap L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ , we have

$$\begin{aligned} |\langle M_{b_1 b_2} T f, g \rangle| &= |\langle f, T^* b_1 b_2 g \rangle| \\ &\leq C \|f\|_{BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \|T^* M_{b_1 b_2} g\|_{H^1_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \\ &\leq C \|f\|_{BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \|g\|_{H^1_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \end{aligned}$$

By density argument we can obtain that  $M_{b_1 b_2} T f$  is in  $BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ .

**“only if” part:**

We now prove the “only if” part.

(1) Suppose  $T$  admits a bounded extension from  $BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  to  $BMO(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ .

We now consider the function  $f(x_1, x_2) = \chi_1(x_1) f_2(x_2) b_2^{-1}(x_2)$ , where  $\chi_1(x_1) \equiv 1$  on  $\mathbb{R}^{n_1}$ , and  $f_2(x_2) \in C^\infty_0(\mathbb{R}^{n_2})$ . Then it is clear that this  $f(x_1, x_2)$  is in  $L^\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ , and hence it is in  $BMO(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  with

$$\|f\|_{BMO(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} = 0.$$

Then we also have  $g(x_1, x_2) = b_1(x_1) b_2(x_2) f(x_1, x_2)$  is in  $BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  with

$$\|g\|_{BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} = 0.$$

Moreover, since  $Tg$  is in  $BMO(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  with

$$\|Tg\|_{BMO(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \leq \|g\|_{BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} = 0.$$

Hence, we obtain that for all  $\psi_1 \in C^\infty_0(\mathbb{R}^{n_1})$  with  $\int_{\mathbb{R}^{n_1}} \psi_1(x_1) dx_1 = 0$  and all  $\psi_2 \in C^\infty_0(\mathbb{R}^{n_2})$  with  $\int_{\mathbb{R}^{n_2}} \psi_2(x_2) dx_2 = 0$ ,

$$\int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} \psi_1(x_1) \psi_2(x_2) Tg(x_1, x_2) dx_1 dx_2 = 0,$$

since  $\psi_1(x_1) \psi_2(x_2)$  is in  $H^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ . This yields

$$\int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} T^*(\psi_1 \psi_2)(y_1, y_2) b_1(y_1) b_2(y_2) f(y_1, y_2) dy_1 dy_2 = 0,$$

which, together with the definition of  $f$ , gives

$$\int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} T^*(\psi_1 \psi_2)(y_1, y_2) b_1(y_1) f_2(y_2) dy_1 dy_2 = 0,$$

Since  $\psi_1(x_1) \psi_2(x_2)$  is in  $H^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ , we have that  $T^*(\psi_1 \psi_2)$  is in  $L^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \cap L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ . Moreover, since  $f_2(x_2) \in C^\infty_0(\mathbb{R}^{n_2})$  and  $b_2$  is bounded, we see that the set  $\{b_2 f_2 : |b_2| \leq C, f_2 \in C^\infty_0(\mathbb{R}^{n_2})\}$  is dense in  $L^2(\mathbb{R}^{n_2})$ .

We get that

$$\int_{\mathbb{R}^{n_1}} T^*(\psi_1 \psi_2)(y_1, y_2) b_1(y_1) dy_1 = 0,$$

which is

$$\int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} \int_{\mathbb{R}^{n_1}} \psi_1(x_1)\psi_2(x_2)K(x_1, x_2, y_1, y_2)b_1(y_1)dy_1dx_1dx_2 = 0.$$

Based on the definition, this shows that

$$T_1(b_1) = 0.$$

Similarly we can obtain that

$$T_2(b_2) = 0.$$

(2) Suppose  $M_{b_1b_2}T$  admits a bounded extension from  $BMO(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  to  $BMO_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ .

We now consider the function  $f(x_1, x_2) = \chi_1(x_1)f_2(x_2)$ , where  $\chi_1(x_1) \equiv 1$  on  $\mathbb{R}^{n_1}$ , and  $f_2(x_2) \in C_0^\infty(\mathbb{R}^{n_2})$ . Then it is clear that this  $f(x_1, x_2)$  is in  $L^\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ , and hence it is in  $BMO(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  with

$$\|f\|_{BMO(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} = 0.$$

Therefore,  $M_{b_1b_2}Tf$  is in  $BMO_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  with

$$\|M_{b_1b_2}Tf\|_{BMO_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \leq \|f\|_{BMO_{b_1b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} = 0.$$

For all  $\psi_1 \in C_0^\infty(\mathbb{R}^{n_1})$  with  $\int_{\mathbb{R}^{n_1}} \psi_1(x_1)dx_1 = 0$  and all  $\psi_2 \in C_0^\infty(\mathbb{R}^{n_2})$  with  $\int_{\mathbb{R}^{n_2}} \psi_2(x_2)dx_2 = 0$ , we have  $\psi_1(x_1)\psi_2(x_2)$  is in  $H^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \cap L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  and then  $\psi_1(x_1)\psi_2(x_2)b_1^{-1}(x_1)b_2^{-1}(x_2)$  is in  $H_{b_1b_2}^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \cap L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ . Hence

$$\int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} \psi_1(x_1)\psi_2(x_2)b_1^{-1}(x_1)b_2^{-1}(x_2)M_{b_1b_2}Tf(x_1, x_2)dx_1dx_2 = 0,$$

This yields

$$\int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} T^*(\psi_1\psi_2)(y_1, y_2)f(y_1, y_2)dy_1dy_2 = 0,$$

which, together with the definition of  $f$ , gives

$$\int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} T^*(\psi_1\psi_2)(y_1, y_2)f_2(y_2)dy_1dy_2 = 0,$$

Since  $\psi_1(x_1)\psi_2(x_2)$  is in  $H^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ , we have that  $T^*(\psi_1\psi_2)$  is in  $L^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \cap L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ . Moreover, since  $f_2(x_2) \in C_0^\infty(\mathbb{R}^{n_2})$ , we see that the set  $\{f_2 : f_2 \in C_0^\infty(\mathbb{R}^{n_2})\}$  is dense in  $L^2(\mathbb{R}^{n_2})$ . We get that

$$\int_{\mathbb{R}^{n_1}} T^*(\psi_1\psi_2)(y_1, y_2)b_1(y_1)dy_1 = 0,$$

which is

$$\int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} \int_{\mathbb{R}^{n_1}} \psi_1(x_1)\psi_2(x_2)K(x_1, x_2, y_1, y_2)dy_1dx_1dx_2 = 0.$$

Based on the definition, this shows that  $T_1(1) = 0$ . Similarly we can obtain that  $T_2(2) = 0$ .

(3) Suppose  $T$  admits a bounded extension from  $BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  to  $BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ .

We now consider the function  $f(x_1, x_2) = \chi_1(x_1)f_2(x_2)b_2^{-1}(x_2)$ , where  $\chi_1(x_1) \equiv 1$  on  $\mathbb{R}^{n_1}$ , and  $f_2(x_2) \in C_0^\infty(\mathbb{R}^{n_2})$ . Then it is clear that this  $f(x_1, x_2)$  is in  $L^\infty(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ , and hence it is in  $BMO(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  with

$$\|f\|_{BMO(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} = 0.$$

Then we also have  $g(x_1, x_2) = b_1(x_1)b_2(x_2)f(x_1, x_2)$  is in  $BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  with

$$\|g\|_{BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} = 0.$$

Moreover, since  $M_{b_1 b_2}Tg$  is in  $BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  with

$$\|M_{b_1 b_2}Tg\|_{BMO(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} \leq \|g\|_{BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})} = 0.$$

Hence, we obtain that for all  $\psi_1 \in C_0^\infty(\mathbb{R}^{n_1})$  with  $\int_{\mathbb{R}^{n_1}} \psi_1(x_1)dx_1 = 0$  and all  $\psi_2 \in C_0^\infty(\mathbb{R}^{n_2})$  with  $\int_{\mathbb{R}^{n_2}} \psi_2(x_2)dx_2 = 0$ ,

$$\int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} \psi_1(x_1)\psi_2(x_2)b_1^{-1}(x_1)b_2^{-1}(x_2)M_{b_1 b_2}Tg(x_1, x_2)dx_1dx_2 = 0,$$

since  $\psi_1(x_1)\psi_2(x_2)b_1^{-1}(x_1)b_2^{-1}(x_2)$  is in  $H^1_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \cap L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ . This yields

$$\int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} T^*(\psi_1\psi_2)(y_1, y_2)b_1(y_1)b_2(y_2)f(y_1, y_2)dy_1dy_2 = 0,$$

which, together with the definition of  $f$ , gives

$$\int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} T^*(\psi_1\psi_2)(y_1, y_2)b_1(y_1)f_2(y_2)dy_1dy_2 = 0,$$

Since  $\psi_1(x_1)\psi_2(x_2)$  is in  $H^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ , we have that  $T^*(\psi_1\psi_2)$  is in  $L^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \cap L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ . Moreover, since  $f_2(x_2) \in C_0^\infty(\mathbb{R}^{n_2})$  and  $b_2$  is bounded, we see that the set  $\{b_2 f_2 : |b_2| \leq C, f_2 \in C_0^\infty(\mathbb{R}^{n_2})\}$  is dense in  $L^2(\mathbb{R}^{n_2})$ . We get that

$$\int_{\mathbb{R}^{n_1}} T^*(\psi_1\psi_2)(y_1, y_2)b_1(y_1)dy_1 = 0,$$

which is

$$\int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} \int_{\mathbb{R}^{n_1}} \psi_1(x_1)\psi_2(x_2)K(x_1, x_2, y_1, y_2)b_1(y_1)dy_1dx_1dx_2 = 0.$$

Based on the definition, this shows that  $T_1(b_1) = 0$ . Similarly we can obtain that  $T_2(b_2) = 0$ .

The proof of Theorem 1.2 is complete. □

We now finalize the proof of Theorem 1.1.

**Proof of the “only if part” of Theorem 1.1.** (1) Suppose  $T$  is bounded from  $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  to  $H_{b_1 b_2}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ . Then we see that the adjoint  $T^*$  extends to a bounded operator from  $BMO_{b_1 b_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$  to  $BMO(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ . Then apply the “only if” part for (2) of Theorem 1.2, we see that  $T_1^*(b_1) = T_2^*(b_2) = 0$ .

Similarly we can verify the “only if” part of (2) and (3).

The proof of Theorem 1.1 is complete.  $\square$

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