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Rotationally symmetric conformal Kähler, Einstein-Maxwell metrics

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ABSTRACT. In this paper, we show that there are non-trivial complete rotationally symmetric conformal Kähler, Einstein metrics on $\mathbb{B}^d$ and $\mathbb{C}^d$, and there are non-trivial complete rotationally symmetric conformal Kähler, Einstein-Maxwell metrics on $\mathbb{B}^{d*}$ and $\mathbb{C}^{d*}$.

CONTENTS

1. Introduction

Let M be a complex m-dimensional Kähler manifold endowed with a Kähler metric g with respect to an integrable almost complex structure J. A Hermitian metric $\widetilde{g}$ on (M, J) is called a conformally Kähler, Einstein-Maxwell metric (cKEM metric for short) if it satisfies the following three conditions:

(a) There exists a positive smooth function f on M such that $\widetilde{g} = \frac{1}{f^2} g$.
(b) The Ricci tensor $\mathrm{Ric}_{\widetilde{g}}$ of the metric $\widetilde{g}$ satisfies $\mathrm{Ric}_{\widetilde{g}}(J\cdot, J\cdot) = \mathrm{Ric}_{\widetilde{g}}(\cdot, \cdot)$.
(c) The scalar curvature $s_{\widetilde{g}}$ of $\widetilde{g}$ is constant.

The Ricci tensors of $\widetilde{g}$ and g are related by (see e.g. [6], 1.161)

$$\mathrm{Ric}_{\widetilde{g}} = \mathrm{Ric}_g + \frac{2(m-1)}{f} D^2 f - \frac{1}{f^2}(f\Delta_g f + (2m-1)|\mathrm{d}f|_g^2)g,$$

where D denotes the Levi-Civita connection of g, $|\cdot|_g$ is the tensor norm induced by g, and $\Delta_g := \delta\mathrm{d} + \mathrm{d}\delta$ is the Hodge Laplacian with respect to g.

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Let s_g be the Riemann scalar curvature of g, thus the scalar curvature $s_{\widetilde{g}}$ of $\widetilde{g}$ is given by

$$\begin{aligned} s_{\widetilde{g}} &= f^2 s_g - 2(2m-1) f \Delta_g f - 2m(2m-1)|\mathrm{d}f|_g^2 \\ &= f^2 s_g + \frac{2(2m-1)}{m-1} f^{m+1} \Delta_g f^{-m+1}. \end{aligned}$$

Note that $\mathrm{Ric}_{\widetilde{g}}(J\cdot, J\cdot) = \mathrm{Ric}_{\widetilde{g}}(\cdot,\cdot)$ if and only if $D^2 f(J\cdot,\cdot) = \sqrt{-1}(\partial\overline{\partial} f)(\cdot,\cdot)$, or if and only if

$$\overline{\partial}(\mathrm{grad}^{1,0} f) = \overline{\partial}(g^{j\bar{q}} f_{\bar{q}} \partial_j) = 0.$$

Setting f to be constant yields constant scalar curvature Kähler metric (cscK metric for short), so cKEM metrics with nonconstant f are referred to as non-trivial. For the case of compact, so far, not many examples of non-trivial cKEM metrics are known. Page in [18] and Chen-LeBrun-Weber in [7] have shown that the one-point-blow-up and the two-point-blow-up of $\mathbb{CP}^2$ admit the conformally Kähler Einstein metrics, respectively. The conformally Kähler Einstein metrics also constructed by Apostolov-Calderbank-Gauduchon in [1, 2] on 4-orbifolds and by Bérard Bergery [5] on $\mathbb{P}^1$-bundles over Fano Kähler-Einstein manifolds. Non-Einstein cKEM examples are given by LeBrun [16, 17] on $\mathbb{CP}^1 \times \mathbb{CP}^1$ and the one-point-blow-up of $\mathbb{CP}^2$, by Koca-Tønnesen-Friedman [12] on ruled surfaces of higher genus, and by Futaki-Ono [9] on $\mathbb{CP}^1 \times M$ where M is a compact constant scalar curvature Kähler manifold of arbitrary dimension. For more research on cKEM metrics, please refer to Apostolov-Maschler [3], Apostolov-Maschler-Tonnesen-Friedman [4], Futaki-Ono [10] and Lahdili [13, 15, 14].

In this paper, we study the existence of cKEM metrics on non-compact manifolds. Specifically, we study the existence of cKEM metrics on rotationally invariant domains. Let

$$\begin{aligned} &\mathbb{B}^d := \{z = (z_1, \ldots, z_d) \in \mathbb{C}^d : \|z\|^2 = \sum_{j=1}^d |z_j|^2 < 1\}, \\ &\mathbb{B}^{d*} := \mathbb{B}^d \setminus \{0\},\ \mathbb{C}^{d*} := \mathbb{C}^d \setminus \{0\}. \end{aligned}$$

The following Theorem 1.1, Theorem 1.2 and Theorem 1.3 are the main results of this paper.

Theorem 1.1. *For $d \geq 2$ let g_F be a Kähler metric on a domain $\Omega = \mathbb{B}^d$ or $\mathbb{C}^d$ associated with the Kähler form $\omega_F = \sqrt{-1}\partial\overline{\partial}\Phi_F$, namely $g_F(X,Y) = \omega_F(X, JY)$ for all $X, Y \in T_z\Omega$ and each $z \in \Omega$, where $\Phi_F(z,\overline{z}) = F(t)$ and $t = \ln\|z\|^2$.*

For a given $a < 0$, set $F(-\infty) = 0$, $x = F'(t)$, $\varphi(x) = F''(t)$, $f = ax+1 > 0$, $\widetilde{g} = \frac{1}{f^2} g_F$ and

$$\psi(d,u) = \sum_{k=0}^{d-2} (k+1) \frac{\Gamma(d-1)\Gamma(d+1)}{\Gamma(d+1+k)\Gamma(d-1-k)} u^k.$$

(i) $\widetilde{g}$ *is a complete Ricci flat metric on* $\mathbb{C}^d$*, that is* $\mathrm{Ric}_{\widetilde{g}} = 0$*, if and only if*

$$\varphi(x) = x(1+ax)^2\psi(d, ax),\ x \in [0, -\frac{1}{a}).$$

(ii) *For a given* $\lambda < 0$*,* $\widetilde{g}$ *is a complete Einstein metric on* $\mathbb{B}^d$*, that is* $\mathrm{Ric}_{\widetilde{g}} = \lambda\widetilde{g}$*, if and only if*

$$\varphi(x) = x(1+ax)^2\psi(d, ax) - \frac{\lambda}{d+1}x^2\psi(d+1, ax),\ x \in [0, -\frac{1}{a}).$$

In particular, if

$$\begin{cases} a = -1, \\ \lambda = -(4d-2), \\ F(t) = e^t = \|z\|^2, \\ f(t) = 1 - e^t = 1 - \|z\|^2, \end{cases}$$

then $\widetilde{g}$ *is an Einstein metric on* $\mathbb{B}^d$*.*

(iii) *The complete rotationally symmetric cKEM metric* $\widetilde{g}$ *on* $\mathbb{B}^d$ *or* $\mathbb{C}^d$ *must be an Einstein metric.*

Theorem 1.2. *For* $d \geq 2$ *let* g_F *be a Kähler metric on a domain* $\Omega = \mathbb{B}^{d*}$ *or* $\mathbb{C}^{d*}$ *associated with the Kähler form* $\omega_F = \sqrt{-1}\partial\overline{\partial}\Phi_F$*, namely* $g_F(X,Y) = \omega_F(X, JY)$ *for* $\forall X, Y \in T_z\Omega$ *and each* $z \in \Omega$*, where* $\Phi_F(z, \overline{z}) = F(t)$ *and* $t = \ln\|z\|^2$*.*

For a given $a < 0$*, set* $x = F'(t)$*,* $\varphi(x) = F''(t)$*,* $f = ax + 1 > 0$ *and* $\widetilde{g} = \frac{1}{f^2}g_F$*.*

For a constant $\beta \leq 0$ *and any* $x_0 \in (0, -\frac{1}{a})$*, let*

$$\varphi(x) = \frac{(1+ax)^{2d-1}}{x^{d-1}}\int_{x_0}^{x}(x-u)\frac{u^{d-2}\left(d(d-1)(1+au)^2 - \beta u\right)}{(1+au)^{2d+1}}\mathrm{d}u \tag{1}$$

for $x \in (x_0, -\frac{1}{a})$*.*

(i) *If* $\beta = 0$ *and* $\varphi(x)$ *is defined on* $(x_0, -\frac{1}{a})$ *by* (1)*, then* $\widetilde{g}$ *is a complete cKEM metric on* $\mathbb{C}^{d*}$ *with zero scalar curvature.*

(ii) *If* $\beta < 0$ *and* $\varphi(x)$ *is defined* $(x_0, -\frac{1}{a})$ *by* (1)*, then* $\widetilde{g}$ *is a complete cKEM metric on* $\mathbb{B}^{d*}$ *with negative scalar curvature* 2β*.*

Theorem 1.3. *For* $d \geq 2$ *let* g_F *be a Kähler metric on a rotationally invariant domain* $\Omega \subset \mathbb{C}^d$ *associated with the Kähler form* $\omega_F = \sqrt{-1}\partial\overline{\partial}\Phi_F$*, namely* $g_F(X,Y) = \omega_F(X, JY)$ *for* $\forall X, Y \in T_z\Omega$ *and each* $z \in \Omega$*, where* $\Phi_F(z, \overline{z}) = F(t)$ *and* $t = \ln\|z\|^2$*.*

Let $x = F'(t)$*,* $\varphi(x) = F''(t)$*,* $f = x$ *and* $\widetilde{g} = \frac{1}{f^2}g_F$*.*

For a constant $\beta \leq 0$ and any $x_0 > 0$, let

$$\begin{aligned}\varphi(x) &= x^d \int_{x_0}^{x} (x-u)\frac{d(d-1)u-\beta}{u^{d+2}}\mathrm{d}u && (2)\\ &= x - \frac{\beta}{d(d+1)} + \frac{\beta - d^2 x_0}{d x_0^d} x^d + \frac{(d^2-1)x_0 - \beta}{(d+1)x_0^{d+1}} x^{d+1}. && (3)\end{aligned}$$

(i) *If $\beta = 0$ and $\varphi(x)$ is defined on $(0, x_0)$ by (3), then $\widetilde{g}$ is a complete cKEM metric on $\mathbb{C}^{d*}$ with zero scalar curvature.*

(ii) *If $\beta < 0$ and $\varphi(x)$ is defined on $(0, x_0)$ by (3), then $\widetilde{g}$ is a complete cKEM metric on*

$$\Omega := \{z \in \mathbb{C}^d : \|z\|^2 > 1\}$$

with negative scalar curvature 2β.

Remark 1.4. The complete cKEM metrics in Theorems 1.2 and 1.3 are not Einstein metrics.

Remark 1.5. The complete cKEM metric $\widetilde{g}$ with zero scalar curvature defined in Theorem 1.3 is not proportional to the complete cKEM metric $\widetilde{g}$ with zero scalar curvature defined in Theorem 1.2.

The paper is organized as follows. In Section 2, by the momentum profiles $\varphi(x)$ (refer to Hwang-Singer [11]) of rotationally symmetric Kähler metrics g_F, we derive ordinary differential equations for rotationally symmetric cKEM metrics. In Section 3, in order to discuss the completeness of metrics, we give the geodesics of radial conformally Kähler metrics. In Section 4, Section 5 and Section 6, by using the conclusions of Section 2 and Section 3, we obtain proofs for Theorem 1.1, Theorem 1.2 and Theorem 1.3, respectively.

2. Radial conformally Kähler, Einstein-Maxwell metrics

For convenience, we need Lemma 2.1 and Remark 2.2 of [8], namely:

Lemma 2.1. *Let*

$$T \equiv (T_{i\bar{j}}) = \frac{\partial^2 \Phi_F}{\partial z^t \partial \bar{z}},$$

where $\Phi_F(z, \bar{z}) = F(t), t = \ln r^2, r = \|z\|, z \in \mathbb{C}^d$. Then

$$T = \frac{F'}{r^2} I_d + \frac{F'' - F'}{r^4} \bar{z}^t z, \tag{4}$$

$$\det T = \frac{(F')^{d-1} F''}{r^{2d}}, \tag{5}$$

and

$$T^{-1} = \frac{r^2}{F'} I_d + (\frac{1}{F''} - \frac{1}{F'})\bar{z}^t z, \tag{6}$$

where $z = (z_1, \ldots, z_d)$, $\bar{z} = (\overline{z_1}, \ldots, \overline{z_d})$, and z^t denotes transpose of z.

Remark 2.2. From (4), we obtain that $\Phi_F(z,\overline{z}) = F(t)$ with $t = \ln \|z\|^2$ is a Kähler potential on the domain $\Omega \setminus \{0\}$ if and only if $F'(t) > 0$ and $F''(t) > 0$ for $d > 1$, or $F''(t) > 0$ for $d = 1$.

Lemma 2.3. *Under the situation of Lemma 2.1, if a Hermitian matrix T is positive definite, f is a function in t and*

$$\overline{\partial}\left(\frac{\partial f}{\partial \overline{z}} T^{-1}\right) = 0,$$

then there exist constants a, b such that $f = aF'(t) + b$.

Proof. Let $x = F'(t)$ and $\varphi(x) = F''(t)$. By Lemma 2.1, we have

$$\frac{\partial f}{\partial \overline{z}} = \frac{\mathrm{d} f}{\mathrm{d} x}\varphi(x)\frac{z}{r^2},$$

and

$$\frac{\partial f}{\partial \overline{z}} T^{-1} = \frac{\mathrm{d} f}{\mathrm{d} x} z.$$

Thus

$$\overline{\partial}\left(\frac{\partial f}{\partial \overline{z}} T^{-1}\right) = \overline{\partial}\left(\frac{\mathrm{d} f}{\mathrm{d} x}\right) z = 0,$$

that is

$$\overline{\partial}\left(\frac{\mathrm{d} f}{\mathrm{d} x}\right) = 0.$$

Since

$$\overline{\partial}\left(\frac{\mathrm{d} f}{\mathrm{d} x}\right) = \frac{\mathrm{d}^2 f}{\mathrm{d} x^2}\varphi(x)\overline{\partial} t$$

and $\varphi(x) > 0$, so

$$\frac{\mathrm{d}^2 f}{\mathrm{d} x^2} = 0.$$

Then there exist constants a, b such that $f = ax + b$. □

By Lemma 2.3, if g_F is the Kähler metric associated with the Kähler form $\omega_F = \sqrt{-1}\partial\overline{\partial}F(t)$ with $t = \log \|z\|^2$ on a rotationally invariant domain $\Omega \subset \mathbb{C}^d$, and $\widetilde{g} = \frac{1}{f^2} g_F$ is a rotationally symmetric cKEM metric on Ω, then f must be forms $f = aF'(t) + b$.

Theorem 2.4. *Let g_F be a Kähler metric on a rotationally invariant domain $\Omega \subset \mathbb{C}^d$ associated with the Kähler form $\omega_F = \sqrt{-1}\partial\overline{\partial}\Phi_F$, namely $g_F(X,Y) = \omega_F(X, JY)$, where $\Phi_F(z,\overline{z}) = F(t)$, $t = \ln r^2, r^2 = \|z\|^2$.*

Set $x = F'(t)$, $\varphi(x) = F''(t)$, $f = ax + b = aF'(t) + b > 0$ and $\widetilde{g} = \frac{1}{f^2} g_F$.

(i) *The scalar curvature $s_{\widetilde{g}}$ is constant for $\widetilde{g}$ if and only if*

$$\left(\frac{x^{d-1}}{f^{2d-1}}\varphi\right)''(x) = \frac{x^{d-1}}{f(x)^{2d-1}}\left(\frac{d(d-1)}{x} - \frac{k_{\widetilde{g}}}{f(x)^2}\right), \tag{7}$$

where $k_{\widetilde{g}} = \frac{1}{2} s_{\widetilde{g}}$ is a constant.

(ii) $\widetilde{g}$ *is an Einstein metric, that is* $\mathrm{Ric}_{\widetilde{g}} = \lambda\widetilde{g}$*, if and only if*

$$\left(\frac{\varphi}{f}\right)''(x) + \frac{\lambda}{f(x)^3} = 0 \tag{8}$$

for $d = 1$ *and*

$$\varphi'(x) + \left(\frac{d-1}{x} - \frac{a}{ax-b} - \frac{(2d-1)a}{ax+b}\right)\varphi(x) + \left(d + \frac{4abd-\lambda}{2a(ax-b)} - \frac{\lambda}{2a(ax+b)}\right) = 0. \tag{9}$$

for $d > 1$*, where* λ *is a constant.*

Proof. Let $k_{\widetilde{g}} = \frac{1}{2}s_{\widetilde{g}}$ and $k_{g_F} = \frac{1}{2}s_{g_F}$. Then

$$\begin{aligned} k_{\widetilde{g}} &= f^2 k_{g_F} + 2(2d-1)f\triangle_{g_F} f - d(2d-1)|\mathrm{d}f|^2_{g_F} \\ &= f^2 k_{g_F} - \frac{2(2d-1)}{d-1} f^{d+1}\triangle_{g_F} f^{-d+1}, \end{aligned}$$

where

$$\triangle_{g_F} = \mathrm{Tr}\left(T^{-1}\tfrac{\partial^2}{\partial z^t \partial \bar{z}}\right),\ k_{g_F} = -\mathrm{Tr}\left(T^{-1}\tfrac{\partial^2 \log\det T}{\partial z^t \partial \bar{z}}\right),$$

for the expression of T, see Lemma 2.1.

(i) Since

$$\frac{\partial f^{1-d}}{\partial z_i} = (1-d)f^{-d}\frac{\mathrm{d}f}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\partial t}{\partial z_i} = a(1-d)f^{-d}\varphi\frac{\overline{z_i}}{r^2}$$

and

$$\frac{\partial^2 f^{1-d}}{\partial z_i \partial \overline{z_j}} = a(1-d)\varphi\left((f^{-d}\varphi)'_x\frac{z_j\overline{z_i}}{r^4} + f^{-d}\frac{\delta_{ij}r^2 - z_j\overline{z_i}}{r^4}\right),$$

by (6), it follows that

$$\begin{aligned} \triangle_{g_F} f^{1-d} &= \mathrm{Tr}\left(T^{-1}\frac{\partial^2 f^{1-d}}{\partial z^t \partial \overline{z}}\right) \\ &= a(1-d)\frac{\varphi}{r^4}\mathrm{Tr}\left\{\left(\frac{r^2}{x}I_d + \frac{x-\varphi}{x\varphi}\overline{z}^t z\right)\left(f^{-d}r^2 I_d + ((f^{-d}\varphi)'_x - f^{-d})\overline{z}^t z\right)\right\} \\ &= a(1-d)\frac{f^{-d}\varphi}{r^2}\mathrm{Tr}\left\{\frac{r^2}{x}I_d + \left(\frac{\varphi'}{\varphi} - \frac{ad}{f} - \frac{1}{x}\right)\overline{z}^t z\right\} \\ &= a(1-d)f^{-d}\varphi\left(\frac{\varphi'}{\varphi} - \frac{ad}{f} + \frac{d-1}{x}\right). \end{aligned}$$

Using (2.11) of [8], we obtain

$$\begin{aligned} k_{\widetilde{g}} &= f^2 k_{g_F} - \frac{2(2d-1)}{d-1} f^{d+1} \triangle_{g_F} f^{-d+1} \\ &= f^2 k_{g_F} + 2a(2d-1)f\left\{\varphi' + \left(\frac{d-1}{x} - \frac{ad}{f}\right)\varphi\right\} \\ &= f^2\left\{\frac{d(d-1)}{x} - \varphi'' - \frac{2(d-1)}{x}\varphi' - \frac{(d-1)(d-2)}{x^2}\varphi\right\} \\ &\quad + 2a(2d-1)f\left\{\varphi' + \left(\frac{d-1}{x} - \frac{ad}{f}\right)\varphi\right\}, \end{aligned}$$

that is,

$$\begin{aligned} &\varphi'' + \left(\frac{2(d-1)}{x} - \frac{2a(2d-1)}{f}\right)\varphi' \\ &\quad + \left\{\frac{(d-1)(d-2)}{x^2} - \frac{2a(2d-1)}{f}\left(\frac{d-1}{x} - \frac{ad}{f}\right)\right\}\varphi + \frac{k_{\widetilde{g}}}{f^2} = \frac{d(d-1)}{x}, \end{aligned}$$

or

$$\left(\frac{x^{d-1}}{f^{2d-1}}\varphi\right)'' = \frac{x^{d-1}}{f^{2d-1}}\left(\frac{d(d-1)}{x} - \frac{k_{\widetilde{g}}}{f^2}\right).$$

(ii) Using

$$\mathrm{Ric}_{\widetilde{g}} = \mathrm{Ric}_{g_F} + \frac{2(d-1)}{f}D^2 f - \frac{1}{f^2}(-2f\triangle_{g_F} f + (2d-1)|\mathrm{d}f|^2_{g_F})g_F,$$

we get

$$\mathrm{Ric}_{g_F} + \frac{2(d-1)}{f}D^2 f = \frac{1}{f^2}(\lambda - 2f\triangle_{g_F} f + (2d-1)|\mathrm{d}f|^2_{g_F})g_F,$$

namely,

$$\begin{aligned} &\frac{2(d-1)}{f}\sqrt{-1}\partial\overline{\partial} f \\ &\quad = \frac{1}{f^2}(\lambda - 2f\triangle_{g_F} f + (2d-1)|\mathrm{d}f|^2_{g_F})\sqrt{-1}\partial\overline{\partial}\Phi_F + \sqrt{-1}\partial\overline{\partial}\log\det T \end{aligned}$$

or

$$\begin{aligned} \frac{2(d-1)}{f}\frac{\partial^2 f}{\partial z^t \partial \overline{z}} &= \frac{1}{f^2}(\lambda - 2f\triangle_{g_F} f \\ &\qquad + (2d-1)|\mathrm{d}f|^2_{g_F})\frac{\partial^2 \Phi_F}{\partial z^t \partial \overline{z}} + \frac{\partial^2 \log\det T}{\partial z^t \partial \overline{z}}. \quad (10) \end{aligned}$$

Let

$$h(t) := \log\det T(t) = (d-1)\log F'(t) + \log F''(t) - dt.$$

Note that

$$\frac{\partial t}{\partial z^t}\frac{\partial t}{\partial \overline{z}} = \frac{\overline{z}^t z}{r^4},$$
$$\frac{\partial^2 t}{\partial z^t \partial \overline{z}} = \frac{1}{r^2} I_d - \frac{\overline{z}^t z}{r^4},$$
$$h'(t) = (d-1)\frac{F''(t)}{F'(t)} + \frac{F'''(t)}{F''(t)} - d = (d-1)\frac{\varphi(x)}{x} + \varphi'(x) - d,$$

and

$$h''(t) = \left\{(d-1)\left(\frac{\varphi(x)}{x}\right)' + \varphi''(x)\right\}\varphi(x),$$

so

$$\begin{aligned}
\frac{\partial^2 \log\det T}{\partial z^t \partial \overline{z}} &= \frac{h'}{r^2} I_d + \frac{h''-h'}{r^4}\overline{z}^t z,\\
\frac{\partial f}{\partial z^t} &= f'\varphi\frac{\overline{z}^t}{r^2},\\
\frac{\partial f}{\partial \overline{z}} &= f'\varphi\frac{z}{r^2},\\
\frac{\partial^2 f}{\partial z^t \partial \overline{z}} &= (f'\varphi)'\varphi\frac{\overline{z}^t z}{r^4} + f'\varphi\left(\frac{I_d}{r^2} - \frac{\overline{z}^t z}{r^4}\right)\\
&= \frac{f'\varphi}{r^2} I_d + \frac{((f'\varphi)' - f')\varphi}{r^4}\overline{z}^t z\\
&= \frac{a\varphi}{r^2} I_d + \frac{a(\varphi'-1)\varphi}{r^4}\overline{z}^t z,\\
|\mathrm{d}f|^2_{g_F} &= 2\mathrm{Tr}\left(T^{-1}\frac{\partial f}{\partial z^t}\frac{\partial f}{\partial \overline{z}}\right)\\
&= 2\frac{(f'\varphi)^2}{r^4}\mathrm{Tr}\left\{\left(\frac{r^2}{F'} I_d + (\frac{1}{F''} - \frac{1}{F'})\overline{z}^t z\right)\overline{z}^t z\right\}\\
&= 2(f')^2\varphi = 2a^2\varphi
\end{aligned}$$

and

$$\begin{aligned}
\triangle_{g_F} f &= \mathrm{Tr}\left(T^{-1}\frac{\partial^2 f}{\partial z^t \partial \overline{z}}\right)\\
&= \mathrm{Tr}\left\{\left(\frac{r^2}{x} I_d + (\frac{1}{\varphi} - \frac{1}{x})\overline{z}^t z\right)\left(\frac{a\varphi}{r^2} I_d + \frac{a(\varphi'-1)\varphi}{r^4}\overline{z}^t z\right)\right\}\\
&= \frac{a\varphi}{r^2}\mathrm{Tr}\left\{\frac{r^2}{x} I_d + \left(\frac{1}{\varphi} - \frac{1}{x}\right)\overline{z}^t z\right\}\\
&\quad + \frac{a(\varphi'-1)\varphi}{r^4}\mathrm{Tr}\left\{\frac{r^2}{x}\overline{z}^t z + \left(\frac{1}{\varphi} - \frac{1}{x}\right)\overline{z}^t z\overline{z}^t z\right\}\\
&= a\left(\varphi' + (d-1)\frac{\varphi}{x}\right).
\end{aligned}$$

From (10), we get

$$\frac{A}{r^2}I_d+\frac{B}{r^4}\overline{z}^t z=0, \tag{11}$$

where

$$\begin{aligned}
A &= x\frac{\lambda-2f\triangle_{g_F}f+(2d-1)|\mathrm{d}f|^2_{g_F}}{f^2}+h'-2a(d-1)\frac{\varphi}{f}\\
&= x\left(\frac{\lambda}{f^2}+\frac{-2a\left(\varphi'+(d-1)\frac{\varphi}{x}\right)}{f}+\frac{2(2d-1)a^2\varphi}{f^2}\right)\\
&\quad +(d-1)\frac{\varphi}{x}+\varphi'-d-2a(d-1)\frac{\varphi}{f}\\
&= (1-\frac{2ax}{f})\varphi'+\left(\frac{d-1}{x}-\frac{4a(d-1)}{f}+\frac{2(2d-1)a^2x}{f^2}\right)\varphi\\
&\quad +\frac{\lambda x}{f^2}-d\\
&= x\left\{\left(\frac{1}{x}-\frac{2a}{f}\right)\varphi'+\left(\frac{d-1}{x^2}-\frac{4a(d-1)}{xf}+\frac{2(2d-1)a^2}{f^2}\right)\varphi\right.\\
&\quad \left.+\frac{\lambda}{f^2}-\frac{d}{x}\right\},
\end{aligned} \tag{12}$$

and

$$B=(\varphi-x)\frac{\lambda-2f\triangle_{g_F}f+(2d-1)|\mathrm{d}f|^2_{g_F}}{f^2} +h''-h'-2a(d-1)\frac{(\varphi'-1)\varphi}{f}. \tag{13}$$

Then,

$$\begin{aligned}
A+B &= \varphi\frac{\lambda-2f\triangle_{g_F}f+(2d-1)|\mathrm{d}f|^2_{g_F}}{f^2}+h''-2a(d-1)\frac{\varphi'\varphi}{f}\\
&= \varphi\left\{\frac{\lambda}{f^2}+\frac{-2a\left(\varphi'+(d-1)\frac{\varphi}{x}\right)}{f}+\frac{2(2d-1)a^2\varphi}{f^2}\right.\\
&\quad \left.+(d-1)\left(\frac{\varphi}{x}\right)'+\varphi''-2a(d-1)\frac{\varphi'}{f}\right\}\\
&= \varphi\left\{\varphi''+\left(\frac{d-1}{x}-\frac{2ad}{f}\right)\varphi'\right.\\
&\quad \left.+\left(\frac{2(2d-1)a^2}{f^2}-\frac{2a(d-1)}{xf}-\frac{d-1}{x^2}\right)\varphi+\frac{\lambda}{f^2}\right\}.
\end{aligned}$$

Using

$$\det\left(\mu I_d-\frac{A}{r^2}I_d-\frac{B}{r^4}\overline{z}^t z\right)=\left(\mu-\frac{A}{r^2}\right)^{d-1}\left(\mu-\frac{A+B}{r^2}\right),$$

we have that (11) is equivalent to $A+B=0$ for $d=1$ and $A=0$, $A+B=0$ for $d>1$.

Note that if

$$\left(\frac{1}{x}-\frac{2a}{f}\right)\varphi'+\left(\frac{d-1}{x^2}-\frac{4a(d-1)}{xf}+\frac{2(2d-1)a^2}{f^2}\right)\varphi+\frac{\lambda}{f^2}-\frac{d}{x}=0 \quad (14)$$

and

$$\varphi''(x)+\left(\frac{d-1}{x}-\frac{2ad}{f}\right)\varphi' \\ +\left(\frac{2(2d-1)a^2}{f^2}-\frac{2a(d-1)}{xf}-\frac{d-1}{x^2}\right)\varphi+\frac{\lambda}{f^2}=0, \quad (15)$$

then (14) $\times (d-1)+$ (15) gives

$$\varphi''(x)+2\left(\frac{d-1}{x}-\frac{(2d-1)a}{f}\right)\varphi'+\left(\frac{2d(2d-1)a^2}{f^2}-\frac{2a(d-1)(2d-1)}{xf}\right. \\ \left.+\frac{(d-1)(d-2)}{x^2}\right)\varphi+\frac{\lambda d}{f^2}-\frac{d(d-1)}{x}=0.$$

In fact, this is equation (7), where $k_{\widetilde{g}}=d\lambda$.

Now we prove that if equation (14) holds, then equation (15) must be true. Let

$$\begin{aligned} Q_1 : &= \frac{\frac{d-1}{x^2}-\frac{4a(d-1)}{xf}+\frac{2(2d-1)a^2}{f^2}}{\frac{1}{x}-\frac{2a}{f}}=\frac{d-1}{x}-\frac{a}{ax-b}-\frac{(2d-1)a}{ax+b}, \\ Q_2 : &= \frac{\frac{\lambda}{f^2}-\frac{d}{x}}{\frac{1}{x}-\frac{2a}{f}}=d+\frac{4abd-\lambda}{2a(ax-b)}-\frac{\lambda}{2a(ax+b)}, \end{aligned}$$

and

$$P_1:=\frac{d-1}{x}-\frac{2ad}{f},\ P_2:=\frac{2(2d-1)a^2}{f^2}-\frac{2a(d-1)}{xf}-\frac{d-1}{x^2},\ P_3:=\frac{\lambda}{f^2},$$

and

$$C:=\varphi'+Q_1\varphi+Q_2,\ D:=\varphi''+P_1\varphi'+P_2\varphi+P_3.$$

Then,

$$\frac{\partial C}{\partial x}=\varphi''+Q_1\varphi'+\frac{\partial Q_1}{\partial x}\varphi+\frac{\partial Q_2}{\partial x},$$

where

$$\begin{aligned} \frac{\partial Q_1}{\partial x} &= \frac{(2d-1)a^2}{(ax+b)^2}+\frac{a^2}{(ax-b)^2}-\frac{d-1}{x^2}, \\ \frac{\partial Q_2}{\partial x} &= -\frac{4abd-\lambda}{2(ax-b)^2}+\frac{\lambda}{2(ax+b)^2}. \end{aligned}$$

Thus,

$$\frac{\partial C}{\partial x} - D = (Q_1 - P_1)\left(\varphi' + \frac{\frac{\partial Q_1}{\partial x} - P_2}{Q_1 - P_1}\varphi + \frac{\frac{\partial Q_2}{\partial x} - P_3}{Q_1 - P_1}\right),$$

where

$$Q_1 - P_1 = -\frac{2ab}{a^2x^2 - b^2},\ \frac{\frac{\partial Q_1}{\partial x} - P_2}{Q_1 - P_1} = Q_1,\ \frac{\frac{\partial Q_2}{\partial x} - P_3}{Q_1 - P_1} = Q_2.$$

This shows that

$$D = \frac{\partial C}{\partial x} - (Q_1 - P_1)C.$$

Therefore, if $C = 0$, then $D = 0$.

Finally, by (15) we have (8), and by (14) we get (9). □

3. The geodesics of radial conformally Kähler metrics

In this section, we give the geodesics of radial conformally Kähler metrics. To this end, we first give the following Lemma 3.1.

Lemma 3.1. *Let D be the Chern connection for the m-dimensional Kähler manifold (M, g), f be a positive real smooth function on M, and $\widetilde{D}$ be the Levi-Civita connection for a Riemannian metric $\widetilde{g} = \frac{1}{f^2}g$ on M. For locally holomorphic coordinates $z = (z_1, \ldots, z_m)$ of M, let $g_{i\bar{j}} = g(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j})$ and*

$$D\begin{pmatrix} \frac{\partial}{\partial z^t} \\ \frac{\partial}{\partial \bar{z}^t} \end{pmatrix} = \Omega_{\mathbb{C}} \otimes \begin{pmatrix} \frac{\partial}{\partial z^t} \\ \frac{\partial}{\partial \bar{z}^t} \end{pmatrix}, \quad \widetilde{D}\begin{pmatrix} \frac{\partial}{\partial z^t} \\ \frac{\partial}{\partial \bar{z}^t} \end{pmatrix} = \widetilde{\Omega}_{\mathbb{C}} \otimes \begin{pmatrix} \frac{\partial}{\partial z^t} \\ \frac{\partial}{\partial \bar{z}^t} \end{pmatrix},$$

where

$$T = (g_{i\bar{j}}), \quad \Omega_{\mathbb{C}} = \begin{pmatrix} (\partial\, T)T^{-1} & 0 \\ 0 & (\overline{\partial}\, \overline{T})\overline{T}^{-1} \end{pmatrix}.$$

Then

$$\begin{aligned} \widetilde{\Omega}_{\mathbb{C}} \;=\; & \begin{pmatrix} (\partial\, T)T^{-1} & 0 \\ 0 & (\overline{\partial}\, \overline{T})\overline{T}^{-1} \end{pmatrix} - \frac{\mathrm{d}f}{f}I_{2m} - \frac{1}{f}\begin{pmatrix} \frac{\partial f}{\partial z^t} \\ \frac{\partial f}{\partial \bar{z}^t} \end{pmatrix}\begin{pmatrix} \mathrm{d}z & \mathrm{d}\bar{z} \end{pmatrix} \\ & + \frac{1}{f}\begin{pmatrix} 0 & T \\ \overline{T} & 0 \end{pmatrix}\begin{pmatrix} \mathrm{d}z^t \\ \mathrm{d}\bar{z}^t \end{pmatrix}\begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial \bar{z}} \end{pmatrix}\begin{pmatrix} 0 & \overline{T}^{-1} \\ T^{-1} & 0 \end{pmatrix}, \end{aligned}$$

where I_{2m} is the identity matrix of order $2m$.

Proof. Let

$$z_i = x_i + \sqrt{-1}y_i, e_i \in \left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_m}\right\},$$

$$e_i^* \in \{\mathrm{d}x_1, \ldots, \mathrm{d}x_m, \mathrm{d}y_1, \ldots, \mathrm{d}y_m\},$$

$$g_{ij} = g(e_i, e_j), \widetilde{g}_{ij} = \frac{1}{f^2}g_{ij}, g_{\mathbb{R}} = (g_{ij}), (g^{ij}) = (g_{ij})^{-1}, (\widetilde{g}^{ij}) = (\widetilde{g}_{ij})^{-1}$$

and

$$\mathbf{\Gamma}_{ij}^k = \frac{1}{2} g^{kl}(e_i g_{jl} + e_j g_{il} - e_l g_{ij}),\ \widetilde{\mathbf{\Gamma}}_{ij}^k = \frac{1}{2} \widetilde{g}^{kl}(e_i \widetilde{g}_{jl} + e_j \widetilde{g}_{il} - e_l \widetilde{g}_{ij}),$$

where $\mathbf{\Gamma}_{ij}^k$ and $\widetilde{\mathbf{\Gamma}}_{ij}^k$ are the Christoffel symbols of the Riemann metrics g and $\widetilde{g}$, respectively. Then

$$\widetilde{\mathbf{\Gamma}}_{ij}^k = \mathbf{\Gamma}_{ij}^k - \frac{f_i}{f}\delta_j^k - \frac{f_j}{f}\delta_i^k + \frac{f_l}{f} g^{kl} g_{ij},$$

where

$$f_i = e_i f,\ \delta_i^j = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

This implies that

$$\widetilde{D} \begin{pmatrix} \frac{\partial}{\partial x^t} \\ \\ \frac{\partial}{\partial y^t} \end{pmatrix} = \widetilde{\Omega}_{\mathbb{R}} \otimes \begin{pmatrix} \frac{\partial}{\partial x^t} \\ \\ \frac{\partial}{\partial y^t} \end{pmatrix},$$

where

$$\begin{aligned} \widetilde{\Omega}_{\mathbb{R}} \quad = \quad & \Omega_{\mathbb{R}} - \frac{\mathrm{d}f}{f} I_{2m} - \frac{1}{f} \begin{pmatrix} \frac{\partial f}{\partial x^t} \\ \\ \frac{\partial f}{\partial y^t} \end{pmatrix} \begin{pmatrix} \mathrm{d}x & \mathrm{d}y \end{pmatrix} \\ & + \frac{1}{f} g_{\mathbb{R}} \begin{pmatrix} \mathrm{d}x^t \\ \\ \mathrm{d}y^t \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{pmatrix} g_{\mathbb{R}^{-1}}, \end{aligned}$$

$$\Omega_{\mathbb{R}} = (\Omega_i^j),\ \Omega_i^j = \Gamma_{ik}^j e_k^*,\ \mathrm{d}x = (\mathrm{d}x_1, \ldots, \mathrm{d}x_m),\ \mathrm{d}x^t = (\mathrm{d}x)^t$$

and

$$\frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x^m}\right),\ \frac{\partial f}{\partial x^t} = \left(\frac{\partial f}{\partial x}\right)^t.$$

Since

$$\begin{pmatrix} \frac{\partial}{\partial x^t} \\ \\ \frac{\partial}{\partial y^t} \end{pmatrix} = P \begin{pmatrix} \frac{\partial}{\partial z^t} \\ \\ \frac{\partial}{\partial \overline{z}^t} \end{pmatrix}, \quad \begin{pmatrix} \mathrm{d}x & \mathrm{d}y \end{pmatrix} = \begin{pmatrix} \mathrm{d}z & \mathrm{d}\overline{z} \end{pmatrix} P^{-1},$$

where

$$P = \begin{pmatrix} I_m & I_m \\ \\ \sqrt{-1} I_m & -\sqrt{-1} I_m \end{pmatrix}.$$

It follows that

$$\widetilde{\Omega}_{\mathbb{R}} = \Omega_{\mathbb{R}} - \frac{\mathrm{d}f}{f} I_{2m} - \frac{1}{f} P \begin{pmatrix} \frac{\partial f}{\partial z^t} \\ \\ \frac{\partial f}{\partial \overline{z}^t} \end{pmatrix} \begin{pmatrix} \mathrm{d}z & \mathrm{d}\overline{z} \end{pmatrix} P^{-1} + \frac{1}{f} g_{\mathbb{R}} (P^t)^{-1} \begin{pmatrix} \mathrm{d}z^t \\ \mathrm{d}\overline{z}^t \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial \overline{z}} \end{pmatrix} P^t g_{\mathbb{R}}^{-1}$$

and $\widetilde{\Omega}_{\mathbb{C}} = P^{-1} \widetilde{\Omega}_{\mathbb{R}} P$.

By

$$g_{\mathbb{R}} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \ T = \frac{1}{2}(A + \sqrt{-1}B),$$

and

$$\begin{aligned} & P^{-1} g_{\mathbb{R}} (P^{-1})^t \\ = & \tfrac{1}{4} \begin{pmatrix} I_m & -\sqrt{-1} I_m \\ I_m & \sqrt{-1} I_m \end{pmatrix} \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \begin{pmatrix} I_m & I_m \\ -\sqrt{-1} I_m & \sqrt{-1} I_m \end{pmatrix} \\ = & \begin{pmatrix} 0 & \frac{1}{2}(A + \sqrt{-1}B) \\ \frac{1}{2}(A - \sqrt{-1}B) & 0 \end{pmatrix} \\ = & \begin{pmatrix} 0 & T \\ \overline{T} & 0 \end{pmatrix}, \end{aligned}$$

we get

$$\begin{aligned} \widetilde{\Omega}_{\mathbb{C}} &= P^{-1} \widetilde{\Omega}_{\mathbb{R}} P \\ &= P^{-1} \Omega_{\mathbb{R}} P - \tfrac{\mathrm{d}f}{f} I_{2m} - \tfrac{1}{f} \begin{pmatrix} \frac{\partial f}{\partial z^t} \\ \\ \frac{\partial f}{\partial \overline{z}^t} \end{pmatrix} \begin{pmatrix} \mathrm{d}z & \mathrm{d}\overline{z} \end{pmatrix} \\ &\quad + \tfrac{1}{f} P^{-1} g_{\mathbb{R}} (P^t)^{-1} \begin{pmatrix} \mathrm{d}z^t \\ \mathrm{d}\overline{z}^t \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial \overline{z}} \end{pmatrix} P^t g_{\mathbb{R}}^{-1} P \\ &= \Omega_{\mathbb{C}} - \tfrac{\mathrm{d}f}{f} I_{2m} - \tfrac{1}{f} \begin{pmatrix} \frac{\partial f}{\partial z^t} \\ \\ \frac{\partial f}{\partial \overline{z}^t} \end{pmatrix} \begin{pmatrix} \mathrm{d}z & \mathrm{d}\overline{z} \end{pmatrix} \\ &\quad + \tfrac{1}{f} \begin{pmatrix} 0 & T \\ \overline{T} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{d}z^t \\ \mathrm{d}\overline{z}^t \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial \overline{z}} \end{pmatrix} \begin{pmatrix} 0 & \overline{T}^{-1} \\ T^{-1} & 0 \end{pmatrix}. \end{aligned}$$

$\square$

Lemma 3.2. *Let g_F be a Kähler metric on a rotationally invariant domain $\Omega \subset \mathbb{C}^d$ associated with the Kähler form $\omega_F = \sqrt{-1}\partial\overline{\partial}\Phi_F$, namely*

$g_F(X,Y)=\omega_F(X,JY)$, where $\Phi_F(z,\overline{z})=F(t)$, $t=\ln r^2, r^2=\|z\|^2$. Let $f=aF'(t)+b>0$, $\widetilde{g}=\frac{1}{f^2}g_F$, and

$$\gamma(\tau)=\tau z_0,\ \|z_0\|^2=1.$$

Then γ is a geodesic of $(\Omega,\widetilde{g})$.

Proof. Let D be the Chern connection for the Kähler manifold (Ω,g), $\widetilde{D}$ be the Levi-Civita connection for the Riemannian manifold $(\Omega,\widetilde{g})$.

Set

$$D\begin{pmatrix}\frac{\partial}{\partial z^t}\\ \\ \frac{\partial}{\partial \overline{z}^t}\end{pmatrix}=\Omega_{\mathbb{C}}\otimes\begin{pmatrix}\frac{\partial}{\partial z^t}\\ \\ \frac{\partial}{\partial \overline{z}^t}\end{pmatrix},\ \widetilde{D}\begin{pmatrix}\frac{\partial}{\partial z^t}\\ \\ \frac{\partial}{\partial \overline{z}^t}\end{pmatrix}=\widetilde{\Omega}_{\mathbb{C}}\otimes\begin{pmatrix}\frac{\partial}{\partial z^t}\\ \\ \frac{\partial}{\partial \overline{z}^t}\end{pmatrix},$$

where

$$\Omega_{\mathbb{C}}=\begin{pmatrix}(\partial\, T)T^{-1} & 0\\ 0 & (\overline{\partial}\,\overline{T})\overline{T^{-1}}\end{pmatrix},\quad \partial:=\mathrm{d}z\frac{\partial}{\partial z^t},\quad \overline{\partial}:=\mathrm{d}\overline{z}\frac{\partial}{\partial \overline{z}^t},$$

and

$$\begin{aligned}\widetilde{\Omega}_{\mathbb{C}} &= \begin{pmatrix}(\partial\, T)T^{-1} & 0\\ 0 & (\overline{\partial}\,\overline{T})\overline{T^{-1}}\end{pmatrix}\\ &\quad -\frac{\mathrm{d}f}{f}I_{2d}-\frac{1}{f}\begin{pmatrix}\frac{\partial f}{\partial z^t}\\ \frac{\partial f}{\partial \overline{z}^t}\end{pmatrix}\begin{pmatrix}\mathrm{d}z & \mathrm{d}\overline{z}\end{pmatrix}\\ &\quad +\frac{1}{f}\begin{pmatrix}0 & T\\ \overline{T} & 0\end{pmatrix}\begin{pmatrix}\mathrm{d}z^t\\ \mathrm{d}\overline{z}^t\end{pmatrix}\begin{pmatrix}\frac{\partial f}{\partial z} & \frac{\partial f}{\partial \overline{z}}\end{pmatrix}\begin{pmatrix}0 & \overline{T^{-1}}\\ T^{-1} & 0\end{pmatrix},\end{aligned}\tag{16}$$

for the definition of T, refer to Lemma 2.1.

For convenience, let

$$\widetilde{\Omega}_1:=\frac{\mathrm{d}f}{f}I_{2d},\ \widetilde{\Omega}_2:=\frac{1}{f}\begin{pmatrix}\frac{\partial f}{\partial z^t}\\ \\ \frac{\partial f}{\partial \overline{z}^t}\end{pmatrix}\begin{pmatrix}\mathrm{d}z & \mathrm{d}\overline{z}\end{pmatrix}$$

and

$$\widetilde{\Omega}_3:=\frac{1}{f}\begin{pmatrix}0 & T\\ \overline{T} & 0\end{pmatrix}\begin{pmatrix}\mathrm{d}z^t\\ \mathrm{d}\overline{z}^t\end{pmatrix}\begin{pmatrix}\frac{\partial f}{\partial z} & \frac{\partial f}{\partial \overline{z}}\end{pmatrix}\begin{pmatrix}0 & \overline{T^{-1}}\\ T^{-1} & 0\end{pmatrix}.$$

Then

$$\widetilde{\Omega}_{\mathbb{C}}=\Omega_{\mathbb{C}}-\widetilde{\Omega}_1-\widetilde{\Omega}_2+\widetilde{\Omega}_3.$$

Using

$$\gamma'(\tau)=z_0\frac{\partial}{\partial z^t}+\overline{z_0}\frac{\partial}{\partial \overline{z}^t},$$

we have

$$\begin{aligned}\|\gamma'(\tau)\|^2_{\widetilde{g}} &= \left(z_0\frac{\partial}{\partial z^t}+\overline{z_0}\frac{\partial}{\partial \overline{z}^t}, z_0\frac{\partial}{\partial z^t}+\overline{z_0}\frac{\partial}{\partial \overline{z}^t}\right)_{\widetilde{g}}\\ &= \frac{2}{f^2}z_0T\overline{z_0}^t\Big|_{z=\tau z_0}=\frac{2}{\tau^2f^2}zT\overline{z}^t\Big|_{z=\tau z_0}=\frac{2F''(t)}{\tau^2 f^2(t)}.\end{aligned}$$

Let

$$\begin{aligned} e(\tau): &= \frac{\gamma'(\tau)}{\|\gamma'(\tau)\|_{\widetilde{g}}} = \frac{\tau f(t)}{\sqrt{2F''(t)}}\left(z_0\frac{\partial}{\partial z^t}+\overline{z_0}\frac{\partial}{\partial \overline{z}^t}\right) \\ &= \frac{\tau f(t)}{\sqrt{2F''(t)}}\begin{pmatrix} z_0 & \overline{z_0} \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial z^t} \\ \frac{\partial}{\partial \overline{z}^t} \end{pmatrix}. \end{aligned}$$

Then

$$\begin{aligned} \widetilde{D}_{\gamma'}e\Big|_{z=\tau z_0} &= \frac{\tau f(t)}{\sqrt{2F''(t)}}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\log(\frac{\tau f(t)}{\sqrt{2F''(t)}})\begin{pmatrix} z_0 & \overline{z_0} \end{pmatrix}\right. \\ &\quad \left. + \begin{pmatrix} z_0 & \overline{z_0} \end{pmatrix}\langle\gamma', \widetilde{\Omega}_{\mathbb{C}}\rangle\Big|_{z=\tau z_0}\right)\begin{pmatrix} \frac{\partial}{\partial z^t} \\ \frac{\partial}{\partial \overline{z}^t} \end{pmatrix}, \end{aligned}$$

where $t = \log\|z\|^2\big|_{z=\tau z_0} = 2\log\tau$, and the symbol $\langle\cdot,\cdot\rangle$ indicates the dual pairing between differential forms and tangent vectors.

By $t = 2\log\tau$, we get

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\log(\frac{\tau f(t)}{\sqrt{2F''(t)}}) = \frac{1}{\tau}\left(1+\frac{2f'(t)}{f(t)}-\frac{F'''(t)}{F''(t)}\right).$$

Since

$$T^{-1}\big|_{z=\tau z_0} = \tau^2\left(\frac{1}{F'(t)}I_d + (\frac{1}{F''(t)}-\frac{1}{F'(t)})\overline{z_0}^t z_0\right)$$

and

$$\begin{aligned} \partial T &= \left(\frac{F'''(t)-F''(t)}{r^4}\partial t - \frac{2(F''(t)-F'(t))}{r^6}\mathrm{d}z\overline{z}^t\right)\overline{z}^t z \\ &\quad + \frac{F''(t)-F'(t)}{r^4}\overline{z}^t\mathrm{d}z + \left(\frac{F''(t)}{r^2}\partial t - \frac{F'(t)}{r^4}\mathrm{d}z\overline{z}^t\right)I_d \\ &= \frac{F'''(t)-3F''(t)+2F'(t)}{r^6}\mathrm{d}z\overline{z}^t\overline{z}^t z + \frac{F''(t)-F'(t)}{r^4}\overline{z}^t\mathrm{d}z \\ &\quad + \frac{F''(t)-F'(t)}{r^4}\mathrm{d}z\overline{z}^t I_d, \end{aligned}$$

it follows that

$$\begin{aligned} \langle\gamma', \partial T\rangle|_{z=\tau z_0} &= \frac{F'''(t)-3F''(t)+2F'(t)}{\tau^6}z_0\overline{\tau z_0}^t\overline{\tau z_0}^t\tau z_0 \\ &\quad + \frac{F''(t)-F'(t)}{\tau^4}\overline{\tau z_0}^t z_0 + \frac{F''(t)-F'(t)}{\tau^4}z_0\overline{\tau z_0}^t I_d \\ &= \frac{F''(t)-F'(t)}{\tau^3}I_d + \frac{F'''(t)-2F''(t)+F'(t)}{\tau^3}\overline{z_0}^t z_0 \end{aligned}$$

and

$$\begin{aligned}\langle\gamma',(\partial T)T^{-1}\rangle|_{z=\tau z_0} &= \left(\frac{F''(t)-F'(t)}{\tau}I_d+\frac{F'''(t)-2F''(t)+F'(t)}{\tau}\overline{z_0}^t z_0\right)\\ &\quad\times\left(\frac{1}{F'(t)}I_d+(\frac{1}{F''(t)}-\frac{1}{F'(t)})\overline{z_0}^t z_0\right)\\ &= \frac{1}{\tau}\left(\frac{F''(t)-F'(t)}{F'(t)}I_d+\frac{F'(t)F'''(t)-F''(t)F''(t)}{F'(t)F''(t)}\overline{z_0}^t z_0\right).\end{aligned}$$

Similarly,

$$\langle\gamma',(\overline{\partial}\,\overline{T})\,\overline{T}^{-1}\rangle|_{z=\tau z_0}=\frac{1}{\tau}\left(\frac{F''(t)-F'(t)}{F'(t)}I_d+\frac{F'(t)F'''(t)-F''(t)F''(t)}{F'(t)F''(t)}z_0^t\overline{z_0}\right).$$

So

$$\begin{aligned}\begin{pmatrix} z_0 & \overline{z_0}\end{pmatrix}\langle\gamma',\Omega_{\mathbb{C}}\rangle\Big|_{z=\tau z_0} &= \begin{pmatrix} z_0 & \overline{z_0}\end{pmatrix}\begin{pmatrix} A & 0\\ 0 & B\end{pmatrix}\\ &= \frac{F'''(t)-F''(t)}{\tau F''(t)}\begin{pmatrix} z_0 & \overline{z_0}\end{pmatrix},\end{aligned}$$

where

$$\begin{aligned}A &= \frac{1}{\tau}\left(\frac{F''-F'}{F'}(t)I_d+\frac{F'F'''-F''F''}{F'F''}(t)\overline{z_0}^t z_0\right),\\ B &= \frac{1}{\tau}\left(\frac{F''-F'}{F'}(t)I_d+\frac{F'F'''-F''F''}{F'F''}(t)z_0^t\overline{z_0}\right).\end{aligned}$$

By direct calculation, we obtain

$$\begin{aligned}\left\langle\gamma',\frac{\mathrm{d}f}{f}\right\rangle\Big|_{z=\tau z_0} &= \frac{1}{f(t)}\frac{\mathrm{d}f}{\mathrm{d}\tau}\Big|_{z=\tau z_0}=\frac{2f'(t)}{\tau f(t)},\\ \begin{pmatrix} z_0 & \overline{z_0}\end{pmatrix}\langle\gamma',\widetilde{\Omega}_1\rangle\Big|_{z=\tau z_0} &= \frac{2f'(t)}{\tau f(t)}\begin{pmatrix} z_0 & \overline{z_0}\end{pmatrix},\end{aligned}$$

and

$$\begin{aligned}\begin{pmatrix} z_0 & \overline{z_0}\end{pmatrix}\langle\gamma',\widetilde{\Omega}_2\rangle\Big|_{z=\tau z_0} &= \frac{1}{f(t)}\begin{pmatrix} z_0 & \overline{z_0}\end{pmatrix}\begin{pmatrix}\frac{\partial f}{\partial z^t}\\ \frac{\partial f}{\partial \overline{z}^t}\end{pmatrix}\begin{pmatrix}\frac{dz}{d\tau} & \frac{d\overline{z}}{d\tau}\end{pmatrix}\Big|_{z=\tau z_0}\\ &= \frac{f'(t)}{\tau f(t)}\begin{pmatrix} z_0 & \overline{z_0}\end{pmatrix}\begin{pmatrix}\overline{z_0}^t\\ z_0^t\end{pmatrix}\begin{pmatrix} z_0 & \overline{z_0}\end{pmatrix}\\ &= \frac{2f'(t)}{\tau f(t)}\begin{pmatrix} z_0 & \overline{z_0}\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}
&\begin{pmatrix} z_0 & \overline{z_0} \end{pmatrix} \langle \gamma', \widetilde{\Omega}_3 \rangle \Big|_{z=\tau z_0} \\
= \ & \frac{1}{f(t)} \begin{pmatrix} z_0 & \overline{z_0} \end{pmatrix} \begin{pmatrix} 0 & T \\ \overline{T} & 0 \end{pmatrix} \begin{pmatrix} z_0^t \\ \overline{z_0}^t \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial \overline{z}} \end{pmatrix} \begin{pmatrix} 0 & \overline{T^{-1}} \\ T^{-1} & 0 \end{pmatrix} \Big|_{z=\tau z_0} \\
= \ & \frac{2f'(t)F''(t)}{\tau^3 f(t)} \begin{pmatrix} \overline{z_0} & z_0 \end{pmatrix} \begin{pmatrix} 0 & \overline{T^{-1}} \\ T^{-1} & 0 \end{pmatrix} \Big|_{z=\tau z_0} \\
= \ & \frac{2f'(t)}{\tau f(t)} \begin{pmatrix} z_0 & \overline{z_0} \end{pmatrix}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\begin{pmatrix} z_0 & \overline{z_0} \end{pmatrix} \langle \gamma', \widetilde{\Omega}_{\mathbb{C}} \rangle \Big|_{z=\tau z_0} &= \begin{pmatrix} z_0 & \overline{z_0} \end{pmatrix} \langle \gamma', \Omega_{\mathbb{C}} - \widetilde{\Omega}_1 - \widetilde{\Omega}_2 + \widetilde{\Omega}_3 \rangle \Big|_{z=\tau z_0} \\
&= \frac{1}{\tau} \left(\frac{F'''(t) - F''(t)}{F''(t)} - \frac{2f'(t)}{f(t)} \right) \begin{pmatrix} z_0 & \overline{z_0} \end{pmatrix}.
\end{aligned}$$

Finally, we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \log\Big(\frac{\tau f(t)}{\sqrt{2F''(t)}}\Big) \Big|_{z=\tau z_0} \begin{pmatrix} z_0 & \overline{z_0} \end{pmatrix} + \begin{pmatrix} z_0 & \overline{z_0} \end{pmatrix} \langle \gamma', \widetilde{\Omega}_{\mathbb{C}} \rangle \Big|_{z=\tau z_0} = 0,$$

namely,

$$\widetilde{D}_{\gamma'} \frac{\gamma'}{\|\gamma'\|_{\widetilde{g}}} \Big|_{z=\tau z_0} = 0,$$

which implies that γ is a geodesic of $(\Omega, \widetilde{g})$. $\square$

4. Proof of Theorem 1.1

4.1. Proof of part (i) and part (ii) of Theorem 1.1.

Proof of part (i) **and part** (ii) **of Theorem 1.1 .** Since $F'(t) > 0$ and $F''(t) > 0$ for $t > -\infty$, using $F(-\infty) = 0$ it follows that

$$\lim_{t \to -\infty} x(t) = \lim_{t \to -\infty} F'(t) = 0.$$

Because the metric g_F is smooth at $z = 0$, so $\varphi(0) = 0$ and $\varphi'(0) = 1$. By (9), we get

$$\begin{cases} \varphi'(x) + \left(\frac{d-1}{x} - \frac{a}{ax-1} - \frac{(2d-1)a}{ax+1} \right) \varphi(x) \\ \\ + \left(d + \frac{4ad-\lambda}{2a(ax-1)} - \frac{\lambda}{2a(ax+1)} \right) = 0, \\ \\ \varphi(0) = 0. \end{cases} \tag{17}$$

(i) For $\lambda = 0$, let $\varphi(x) = x(ax+1)^2\psi(d,u)$ with $u = ax$, then $\psi(d,0) = 1$. According to (17), we get

$$\begin{cases} \left\{(d-2)u^2 - 2(d-2)u + d\right\}\psi(d,u) - d \\ +u(1-u^2)\frac{\mathrm{d}\psi}{\mathrm{d}u}(d,u) = 0, \\ \psi(d,0) = 1, \end{cases} \tag{18}$$

which implies

$$\psi(d,-1) = \frac{d}{4d-6},\ \psi(d,1) = \frac{d}{2}. \tag{19}$$

Solving the equation (18), we have

$$\psi(d,u) = \sum_{k=0}^{d-2}(k+1)\frac{\Gamma(d-1)\Gamma(d+1)}{\Gamma(d+1+k)\Gamma(d-1-k)}u^k \tag{20}$$

$$= \frac{d(1-u)(1+u)^{2d-3}}{u^d}\int_0^u \frac{v^{d-1}}{(1-v)^2(1+v)^{2d-2}}\mathrm{d}v. \tag{21}$$

Thus,

$$\psi(d,-u) = \frac{d(1+u)(1-u)^{2d-3}}{u^d}\int_0^u \frac{v^{d-1}}{(1+v)^2(1-v)^{2d-2}}\mathrm{d}v. \tag{22}$$

Hence,

$$\varphi(x) = x(1+ax)^2\psi(d,ax) > 0,\ x \in (0, -\frac{1}{a}).$$

For a given $x_0 \in (0, -\frac{1}{a})$, let

$$t = \int_{x_0}^{x(t)} \frac{\mathrm{d}v}{\varphi(v)},\ F(t) = \int_0^{x(t)} \frac{v\mathrm{d}v}{\varphi(v)}.$$

Then

$$\lim_{x\to 0^+} t = -\infty,\ \lim_{x\to(-\frac{1}{a})^-} t = +\infty,$$

which implies that the g_F is defined on $\mathbb{C}^d$.

For any $z_0 \in \mathbb{C}^d$ with $\|z_0\|^2 = 1$, from Lemma 3.2, ray

$$C : z(\tau) = \tau z_0,\ \tau \in [0, +\infty)$$

is a geodesic with respect to the metric $\widetilde{g}$. Using

$$z'(\tau) = (z_0\frac{\partial}{\partial z^t} + \overline{z_0}\frac{\partial}{\partial \overline{z}^t})|_{z=\tau z_0}$$

and (4), the square norm of the tangent vector of C at τz_o with respect to the metric $\widetilde{g}$ is

$$\begin{aligned}
|z'|^2_{\widetilde{g}}(\tau) &= \left(z_0\frac{\partial}{\partial z^t}+\overline{z_0}\frac{\partial}{\partial \overline{z}^t}, z_0\frac{\partial}{\partial z^t}+\overline{z_0}\frac{\partial}{\partial \overline{z}^t}\right)_{\widetilde{g}}\Big|_{z=\tau z_0}\\
&= 2\left(z_0\frac{\partial}{\partial z^t}, \overline{z_0}\frac{\partial}{\partial \overline{z}^t}\right)_{\widetilde{g}}\Big|_{z=\tau z_0}\\
&= \frac{2}{(1+aF'(t))^2}\times z_0\left(\frac{F'(t)}{r^2}I_d+\frac{F''(t)-F'(t)}{r^4}\overline{\tau z_0}^t\tau z_o\right)\overline{z_0}^t\\
&= \frac{2}{(1+aF'(t))^2}\left(\frac{F'(t)}{r^2}+\frac{F''(t)-F'(t)}{r^4}\tau^2\right)\\
&= \frac{2F''(t)}{(1+aF'(t))^2\tau^2}=\frac{2e^{-t}F''(t)}{(1+aF'(t))^2},
\end{aligned}$$

where $r^2=\|\tau z_0\|^2=\tau^2=e^t$. So the length of C is

$$\begin{aligned}
l &= \int_0^{+\infty}|z'|_{\widetilde{g}}(\tau)\mathrm{d}\tau=\frac{\sqrt{2}}{2}\int_{-\infty}^{+\infty}\frac{\sqrt{F''(t)}}{1+aF'(t)}\mathrm{d}t\\
&= \frac{\sqrt{2}}{2}\int_0^{-\frac{1}{a}}\frac{1}{(1+ax)\sqrt{\varphi(x)}}\mathrm{d}x=+\infty.
\end{aligned}$$

This shows that the metric $\widetilde{g}$ is complete on $\mathbb{C}^d$.

(ii) For $\lambda<0$, let $\varphi(x)=x+\mu(\lambda)x^2h(u)$ with $u=ax$ and

$$\mu(\lambda)=\frac{(4d-2)a-\lambda}{d+1}.$$

If $\lambda=(4d-2)a$, then $\varphi(x)=x$ is a solution of the equation (17).

If $\lambda\neq(4d-2)a$, from (17), we get

$$\begin{cases} u(1-u^2)h'(u)+\left\{(d-1)u^2-2(d-1)u+d+1\right\}h(u)\\ -(d+1)=0,\\ h(0)=1.\end{cases}\tag{23}$$

Comparing (23) with (18), we obtain $h(u)=\psi(d+1,u)$, so

$$\begin{aligned}
\varphi(x) &= x+\mu(0)x^2h(u)-\frac{\lambda}{d+1}x^2h(u)\\
&= x(1+ax)^2\psi(d,ax)-\frac{\lambda}{d+1}x^2\psi(d+1,ax)>0.
\end{aligned}$$

for $x\in(0,-\frac{1}{a}]$.

Let

$$t=\int_{-\frac{1}{a}}^{x(t)}\frac{\mathrm{d}v}{\varphi(v)},\qquad F(t)=\int_0^{x(t)}\frac{v\mathrm{d}v}{\varphi(v)}.$$

Then

$$\lim_{x\to 0^+} t = -\infty, \qquad \lim_{x\to(-\frac{1}{a})^-} t = 0,$$

which implies that the g_F is defined on $\mathbb{B}^d$.

For any $z_0 \in \mathbb{C}^d$ with $\|z_0\|^2 = 1$, let

$$C : z(\tau) = \tau z_0,\ \tau \in [0, 1)\,.$$

Then the length of the geodesic C:

$$\begin{aligned} l &= \int_0^1 |z'|_{\widetilde{g}}(\tau)\mathrm{d}\tau = \frac{\sqrt{2}}{2}\int_{-\infty}^0 \frac{\sqrt{F''(t)}}{1+aF'(t)}\mathrm{d}t \\ &= \frac{\sqrt{2}}{2}\int_0^{-\frac{1}{a}} \frac{1}{(1+ax)\sqrt{\varphi(x)}}\mathrm{d}x = +\infty. \end{aligned}$$

This proves that the metric $\widetilde{g}$ is complete.

Finally, if $a = -1$ and $\lambda = (4d-2)a = -(4d-2)$, then $\varphi(x) = x$, thus

$$F(t) = e^t = \|z\|^2 \text{ and } f(t) = 1 + aF'(t) = 1 - e^t = 1 - \|z\|^2.$$

□

The proof of (20) is given below.

Proof of (20). For $\alpha > 0$, let

$$\begin{cases} u(1-u^2)\frac{\mathrm{d}\psi}{\mathrm{d}u}(u) + \left\{(\alpha-2)u^2 - 2(\alpha-2)u + \alpha\right\}\psi(u) - \alpha = 0, \\ \psi(0) = 1. \end{cases} \tag{24}$$

and

$$\psi(u) = \sum_{k=0}^{+\infty} c_k u^k.$$

Then

$$(u - u^3)\frac{\mathrm{d}\psi}{\mathrm{d}u} = c_1 u + 2c_2 u^2 + \sum_{k=3}^{+\infty}\left(kc_k - (k-2)c_{k-2}\right)u^k,$$

and

$$\begin{aligned} &\left\{(\alpha-2)u^2 - 2(\alpha-2)u + \alpha\right\}\psi(u) \\ &= \alpha c_0 + \{\alpha c_1 - 2(\alpha-2)c_0\}\,u + \{(\alpha-2)c_0 - 2(\alpha-2)c_1 + \alpha c_2\}\,u^2 \\ &+ \sum_{k=3}^{+\infty}\{(\alpha-2)c_{k-2} - 2(\alpha-2)c_{k-1} + \alpha c_k\}\,u^k, \end{aligned}$$

and

$$\sum_{k=3}^{+\infty}\left\{(\alpha+k)c_k-2(\alpha-2)c_{k-1}+(\alpha-k)c_{k-2}\right\}u^k+\alpha(c_0-1)$$
$$+\left\{(\alpha+1)c_1-2(\alpha-2)c_0\right\}u$$
$$+\left\{(\alpha-2)c_0-2(\alpha-2)c_1+(\alpha+2)c_2\right\}u^2=0.$$

Thus,

$$\begin{cases} \alpha(c_0-1)=0, \\ (\alpha+1)c_1-2(\alpha-2)c_0=0, \\ (\alpha+2)c_2-2(\alpha-2)c_1+(\alpha-2)c_0=0, \\ (\alpha+k)c_k-2(\alpha-2)c_{k-1}+(\alpha-k)c_{k-2}=0, \quad k\geq 3. \end{cases} \tag{25}$$

Solving (25), we have

$$\begin{cases} c_0=1, \\ c_1=2\frac{\alpha-2}{\alpha+1}, \\ c_2=3\frac{(\alpha-2)(\alpha-3)}{(\alpha+1)(\alpha+2)}, \\ c_k=(k+1)\frac{(\alpha-2)\cdots(\alpha-k-1)}{(\alpha+1)\cdots(\alpha+k)}, \quad k\geq 3, \end{cases} \tag{26}$$

that is,

$$\psi(u)=\sum_{k=0}^{+\infty}(k+1)\frac{\Gamma(\alpha-1)\Gamma(\alpha+1)}{\Gamma(\alpha+1+k)\Gamma(\alpha-1-k)}u^k. \tag{27}$$

$\square$

Remark 4.1. From (24), we have

$$\psi'(u) = \frac{(\alpha-2)u^2-2(\alpha-2)u+\alpha}{u^3-u}\psi(u)-\frac{\alpha}{u^3-u},$$
$$\psi'(0) = \frac{2(\alpha-2)}{\alpha+1},$$

and

$$\begin{aligned}\psi''(u) &= \frac{\partial}{\partial u}\left(\frac{(\alpha-2)u^2-2(\alpha-2)u+\alpha}{u^3-u}\right)\psi(u)\\&\quad-\frac{\partial}{\partial u}\left(\frac{\alpha}{u^3-u}\right)+\frac{(\alpha-2)u^2-2(\alpha-2)u+\alpha}{u^3-u}\psi'(u)\\&=\frac{(\alpha-2)(\alpha-3)u^3-3(\alpha-2)(\alpha-3)u^2+(3\alpha^2-9\alpha)u-\alpha(\alpha+1)}{u^2(u-1)(u+1)^2}\psi(u)\\&\quad+\frac{(-\alpha^2+5\alpha)u+\alpha(\alpha+1)}{u^2(u-1)(u+1)^2}.\end{aligned}$$

Thus,

$$\begin{aligned}&\psi''(u)-\frac{2((\alpha-3)u-\alpha)}{u(1+u)}\psi'(u)=\\&\qquad-\frac{(\alpha-2)(\alpha-3)u^2-2\alpha(\alpha-2)u+\alpha^2-\alpha}{u^2(1+u)^2}\psi(u)-\frac{\alpha-\alpha^2}{u^2(1+u)^2}.\end{aligned}$$

Therefore, the equation (24) is equivalent to

$$\begin{cases} u^2(1+u)^2\frac{\mathrm{d}^2\psi}{\mathrm{d}u^2}(u)-2u(1+u)\left\{(\alpha-3)u-\alpha\right\}\frac{\mathrm{d}\psi}{\mathrm{d}u}(u)\\ +\left\{(\alpha-2)(\alpha-3)u^2-2\alpha(\alpha-2)u+\alpha^2-\alpha\right\}\psi(u)\\ +(\alpha-\alpha^2)=0,\\ \psi(0)=1,\ \psi'(0)=\frac{2(\alpha-2)}{\alpha+1}.\end{cases} \tag{28}$$

4.2. Proof of part (iii) of Theorem 1.1.

Proof of part (iii) **of Theorem 1.1.** Let the scalar curvature for $\widetilde{g}$ be equal to 2β.

From the metric g_F is smooth at $z=0$, we get $\varphi(0)=0$ and $\varphi'(0)=1$. By (7), we have

$$\varphi(x)=\frac{(1+ax)^{2d-1}}{x^{d-1}}\int_0^x(x-u)\chi(u)\mathrm{d}u, \tag{29}$$

where

$$\chi(u):=\frac{u^{d-2}\left(d(d-1)(1+au)^2-\beta u\right)}{(1+au)^{2d+1}}. \tag{30}$$

From (29) it follows that φ is a polynomial in x, $\varphi(x)>0$ for $x\in(0,-\frac{1}{a})$,

$$\lim_{x\to0^+}\frac{\varphi(x)}{x}=\lim_{x\to0^+}\frac{\int_0^x\chi(u)\mathrm{d}u}{dx^{d-1}}=1$$

and

$$\lim_{x\to(-\frac{1}{a})^-}\varphi(x)=\frac{-\beta}{2d(2d-1)a^2}\geq0.$$

$(\mathrm{iii}-1)$ If $\beta=0$, then

$$\begin{aligned}\lim_{x\to(-\frac{1}{a})^-}\frac{\varphi(x)}{(1+ax)^2} &= (-a)^{d-1}\lim_{x\to(-\frac{1}{a})^-}\frac{\int_0^x(x-u)\chi(u)\mathrm{d}u}{(1+ax)^{-(2d-3)}}\\ &= (-a)^{d-1}\lim_{x\to(-\frac{1}{a})^-}\frac{\int_0^x\chi(u)\mathrm{d}u}{-(2d-3)a(1+ax)^{-2(d-1)}}\\ &= (-a)^{d-1}\lim_{x\to(-\frac{1}{a})^-}\frac{\chi(x)}{2(d-1)(2d-3)a^2(1+ax)^{-(2d-1)}}\\ &= -\frac{d}{(4d-6)a}>0.\end{aligned}$$

It follows that $\varphi(x)=x(1+ax)^2h(x)$ with $h(x)>0$ for $x\in[0,-\frac{1}{a}]$. Let $u=ax$ and $p(u)=h(x)$. By (7),

$$\begin{cases} u^2(u+1)^2p''(u)-2u(u+1)\left\{(d-3)u-d\right\}p'(u)\\ +\left\{(d-2)(d-3)u^2-2d(d-2)u+d^2-d\right\}p(u)+d-d^2=0,\end{cases}\tag{31}$$

which implies that $p(0)=1$ and

$$\begin{aligned}&u(u+1)^2p''(u)-2(u+1)\left\{(d-3)u-d\right\}p'(u)\\ &+\left\{(d-2)(d-3)u^2-2d(d-2)u+d^2-d\right\}\frac{p(u)-p(0)}{u}\\ &+(d-2)(d-3)u-2d(d-2)=0,\end{aligned}$$

so $p'(0)=\frac{2(d-2)}{d+1}$.

Solving (31) with $p(0)=1$ and $p'(0)=\frac{2(d-2)}{d+1}$, and using (28), (24) and (27), we have

$$p(u)=\sum_{k=0}^{d-2}(k+1)\frac{\Gamma(d-1)\Gamma(d+1)}{\Gamma(d+1+k)\Gamma(d-1-k)}u^k.\tag{32}$$

For any $x_0\in(0,-\frac{1}{a})$, let

$$t=\int_{x_0}^{x(t)}\frac{\mathrm{d}u}{\varphi(u)},\ F(t)=\int_0^{x(t)}\frac{u\mathrm{d}u}{\varphi(u)}.$$

We obtain

$$\lim_{x\to0^+}t=-\infty,\quad \lim_{x\to(-\frac{1}{a})^-}t=+\infty,$$

and

$$\int_{x_0}^{-\frac{1}{a}}\frac{\mathrm{d}x}{(1+ax)\sqrt{\varphi(x)}}=+\infty.$$

This means that the metric $\widetilde{g}$ is complete on $\mathbb{C}^d$.

$(\mathrm{iii}-2)$ If $\beta<0$, then $\varphi(x)=xh(x)$ with $h(x)>0$ for $x\in[0,-\frac{1}{a}]$. If $\beta=2d(2d-1)a$, then $\varphi(x)=x$ is a solution of (7).

If $\beta \neq 2d(2d-1)a$, let $u = ax$ and

$$\varphi(x) = x + \frac{2d(2d-1)a - \beta}{d(d+1)} x^2 q(u),$$

by (7), then the polynomial $q(u)$ satisfies the following equation

$$\begin{cases} u^2(u+1)^2 q''(u) - 2u(u+1)\left\{(d-2)u - d - 1\right\} q'(u) \\ + \left\{(d-1)(d-2)u^2 - 2(d^2-1)u + d^2 + d\right\} q(u) - d - d^2 = 0, \end{cases} \tag{33}$$

which implies that

$$q(0) = 1, \ q'(0) = \frac{2(d-1)}{d+2}.$$

Solving (33) with $q(0) = 1$ and $q'(0) = \frac{2(d-1)}{d+2}$, and applying (28), (24) and (27), we have

$$q(u) = \sum_{k=0}^{d-1} (k+1) \frac{\Gamma(d)\Gamma(d+2)}{\Gamma(d+2+k)\Gamma(d-k)} u^k. \tag{34}$$

Let

$$t = \int_{-\frac{1}{a}}^{x(t)} \frac{\mathrm{d}u}{\varphi(u)}, \ F(t) = \int_0^{x(t)} \frac{u \mathrm{d}u}{\varphi(u)}.$$

We obtain

$$\lim_{x \to 0^+} t = -\infty, \quad \lim_{x \to (-\frac{1}{a})^-} t = 0,$$

and

$$\int_0^{-\frac{1}{a}} \frac{\mathrm{d}x}{(1+ax)\sqrt{\varphi(x)}} = +\infty.$$

This shows that the metric $\widetilde{g}$ defined on $\mathbb{B}^d$ is complete.

Combining the above results with parts (i) and (ii) of Theorem 1.1, we conclude that the complete rotationally symmetric cKEM metrics $\widetilde{g}$ on $\mathbb{B}^d$ or $\mathbb{C}^d$ must be an Einstein metric. □

5. Proof of Theorem 1.2

Proof of Theorem 1.2 . Let

$$\chi(u) := \frac{u^{d-2}\left(d(d-1)(1+au)^2 - \beta u\right)}{(1+au)^{2d+1}}.$$

Then

$$\varphi(x) = \frac{(1+ax)^{2d-1}}{x^{d-1}} \int_{x_0}^{x} (x-u)\chi(u) \mathrm{d}u, \tag{35}$$

and $\varphi(x)$ satisfies (7) with $\varphi(x_0) = 0$ and $\varphi'(x_0) = 0$. This shows that the metric $\widetilde{g}$ is a cKEM metric with scalar curvature 2β.

It easy to see that $\varphi(x)>0$ for $x\in(x_0,-\frac{1}{a})$,

$$\begin{aligned}\lim_{x\to x_0^+}\frac{\varphi(x)}{(x-x_0)^2} &= \frac{(1+ax_0)^{2d-1}}{x_0^{d-1}}\lim_{x\to x_0^+}\frac{\int_{x_0}^x\chi(u)\mathrm{d}u}{2(x-x_0)}\\ &= \frac{(1+ax_0)^{2d-1}}{2x_0^{d-1}}\chi(x_0)\\ &= \frac{d(d-1)(1+ax_0)-\beta x_0}{2x_0(1+ax_0)^2}>0,\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to(-\frac{1}{a})^-}\varphi(x) &= (-a)^{d-1}\lim_{x\to(-\frac{1}{a})^-}\frac{\int_{x_0}^x(x-u)\chi(u)\mathrm{d}u}{(1+ax)^{-(2d-1)}}\\ &= (-a)^{d-1}\lim_{x\to(-\frac{1}{a})^-}\frac{\int_{x_0}^x\chi(u)\mathrm{d}u}{-(2d-1)a(1+ax)^{-2d}}\\ &= (-a)^{d-1}\lim_{x\to(-\frac{1}{a})^-}\frac{\chi(x)}{2d(2d-1)a^2(1+ax)^{-2d-1}}\\ &= \frac{-\beta}{2d(2d-1)a^2}\geq 0.\end{aligned}$$

(i) If $\beta=0$, then $\varphi(x)=(x-x_0)^2(1+ax)h(x)$ with $h(x)>0$ for $x\in(x_0,-\frac{1}{a})$. For any $x_1\in(x_0,-\frac{1}{a})$, let

$$t=\int_{x_1}^{x(t)}\frac{\mathrm{d}u}{\varphi(u)},\ F(t)=\int_{x_1}^{x(t)}\frac{u\mathrm{d}u}{\varphi(u)}.$$

We obtain

$$\lim_{x\to x_0^+}t=-\infty,\quad \lim_{x\to(-\frac{1}{a})^-}t=+\infty,$$

and

$$\int_{x_0}^{x_1}\frac{\mathrm{d}x}{(1+ax)\sqrt{\varphi(x)}}=+\infty,\ \int_{x_1}^{-\frac{1}{a}}\frac{\mathrm{d}x}{(1+ax)\sqrt{\varphi(x)}}=+\infty.$$

This indicates that the metric $\widetilde{g}$ is complete on $\mathbb{C}^{d*}$.

(ii) If $\beta<0$, then $\varphi(x)=(x-x_0)^2h(x)$ with $h(x)>0$ for $x\in[x_0,-\frac{1}{a}]$. Let

$$t=\int_{-\frac{1}{a}}^{x(t)}\frac{\mathrm{d}u}{\varphi(u)},\ F(t)=\int_{-\frac{1}{a}}^{x(t)}\frac{u\mathrm{d}u}{\varphi(u)}.$$

We obtain

$$\lim_{x\to x_0^+}t=-\infty,\quad \lim_{x\to(-\frac{1}{a})^-}t=0,$$

and

$$\int_{x_0}^{\frac{1}{2}(x_0-\frac{1}{a})}\frac{\mathrm{d}x}{(1+ax)\sqrt{\varphi(x)}}=+\infty,\ \int_{\frac{1}{2}(x_0-\frac{1}{a})}^{-\frac{1}{a}}\frac{\mathrm{d}x}{(1+ax)\sqrt{\varphi(x)}}=+\infty.$$

This shows that the metric $\widetilde{g}$ is complete on $\mathbb{B}^{d*}$. $\square$

6. Proof of Theorem 1.3

Proof of Theorem 1.3 . Using (2) and (3), we get

$$\begin{aligned}\lim_{x\to x_0^-}\frac{\varphi(x)}{(x-x_0)^2} &= x_0^d\lim_{x\to x_0^-}\frac{\int_{x_0}^{x}\frac{d(d-1)u-\beta}{u^{d+2}}\mathrm{d}u}{2(x-x_0)}\\ &= \frac{x_0^d}{2}\frac{d(d-1)x_0-\beta}{x_0^{d+2}}\\ &= \frac{d(d-1)x_0-\beta}{2x_0^2}>0,\end{aligned}$$

and

$$\lim_{x\to 0^+}\varphi(x)=-\frac{\beta}{d(d+1)}\geq 0.$$

By (2), it easy to see that $\varphi(x)>0$ for $x\in(0,x_0)$, and $\varphi(x)$ satisfies (7) with $\varphi(x_0)=0$ and $\varphi'(x_0)=0$, this shows that the metric $\widetilde{g}$ is a cKEM metric with scalar curvature 2β.

(i) If $\beta=0$, by (3), then $\varphi(x)=x(x-x_0)^2h(x)$ with $h(x)>0$ for $x\in[0,x_0]$. For any $x_1\in(0,x_0)$, let

$$t=\int_{x_1}^{x(t)}\frac{\mathrm{d}u}{\varphi(u)},\ F(t)=\int_{x_1}^{x(t)}\frac{u\mathrm{d}u}{\varphi(u)}.$$

We obtain

$$\lim_{x\to 0^+}t=-\infty,\ \lim_{x\to x_0^-}t=+\infty,$$

and

$$\int_0^{x_1}\frac{\mathrm{d}x}{x\sqrt{\varphi(x)}}=+\infty,\ \int_{x_1}^{x_0}\frac{\mathrm{d}x}{x\sqrt{\varphi(x)}}=+\infty.$$

This indicates that the metric $\widetilde{g}$ is complete on $\mathbb{C}^{d*}$.

(ii) If $\beta<0$, then $\varphi(x)=(x-x_0)^2h(x)$ with $h(x)>0$ for $x\in[0,x_0]$. Let

$$t=\int_{0}^{x(t)}\frac{\mathrm{d}u}{\varphi(u)},\ F(t)=\int_{0}^{x(t)}\frac{u\mathrm{d}u}{\varphi(u)}.$$

We obtain

$$\lim_{x\to 0^+}t=0,\ \lim_{x\to x_0^-}t=+\infty,$$

and

$$\int_0^{\frac{x_0}{2}}\frac{\mathrm{d}x}{x\sqrt{\varphi(x)}}=+\infty,\ \int_{\frac{x_0}{2}}^{x_0}\frac{\mathrm{d}x}{x\sqrt{\varphi(x)}}=+\infty.$$

This shows that the metric $\widetilde{g}$ is complete on $\Omega=\{z\in\mathbb{C}^d:\|z\|>1\}$. $\square$

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