

Gysin restriction of topological and Hodge-theoretic characteristic classes for singular spaces

Markus Banagl

ABSTRACT. We establish formulae that show how the topological characteristic L -classes of Goresky and MacPherson, as well as the Hodge-theoretic Hirzebruch type characteristic classes defined by Brasselet, Schürmann and Yokura transform under Gysin restrictions associated to normally nonsingular embeddings of singular spaces. We find that both types of classes transform in the same manner. These results suggest a method of normally nonsingular expansions for computing the above characteristic classes. We illustrate this method by computing Goresky-MacPherson L -classes of some singular Schubert varieties.

CONTENTS

1. Introduction	1273
2. The L -class of a pseudomanifold	1278
3. Behavior of the L -class under normally nonsingular inclusions	1278
4. Application: The L -class of singular Schubert varieties	1301
5. Hodge-theoretic characteristic classes	1305
6. Hodge classes and normally nonsingular inclusions	1311
References	1333

1. Introduction

We establish Verdier-Riemann-Roch type formulae that describe the behavior of both the topological L -classes of Goresky and MacPherson ([31]) and the Hodge-theoretic Hirzebruch-type characteristic classes IT_{1*} defined by Brasselet, Schürmann and Yokura ([15]) under Gysin restrictions associated to normally nonsingular embeddings of singular spaces. In doing

Received February 19, 2020.

2010 *Mathematics Subject Classification*. 57R20, 55N33, 57N80, 32S60, 32S20, 55R12.

Key words and phrases. Characteristic classes, stratified spaces, intersection homology, Gysin homomorphism.

This work is supported by Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC-2181/1 - 390900948 (the Heidelberg STRUCTURES Excellence Cluster).

so, we have a twofold goal in mind: We hope that the formulae will contribute to illuminating the relationship between the Goresky-MacPherson L -class and the class IT_{1*} . In [15, Remark 5.4], these are conjectured to be equal for pure-dimensional compact complex algebraic varieties; this is also suggested by Cappell and Shaneson in [17], [65]. For compact complex algebraic varieties that are rational homology manifolds, the conjecture has been established by de Bobadilla and Pallares in [13]. Second, these formulae can serve as the basis of concrete recursive methods for computing these characteristic classes for specific complex projective algebraic varieties. We illustrate this approach by computing Goresky-MacPherson L -classes for some singular Schubert varieties. The Chern-Schwartz-MacPherson classes of Schubert varieties were computed by Aluffi and Mihalcea in [2]. A generalized Verdier-type Riemann-Roch theorem for Chern-Schwartz-MacPherson classes has been obtained by Schürmann ([63]).

In [34], Hirzebruch introduced cohomological L -classes L^* for smooth manifolds as certain polynomials with rational coefficients in the tangential Pontrjagin classes. In view of the signature theorem, Thom described the Poincaré duals of these L -classes by organizing the signatures of submanifolds with trivial normal bundle into a homology class, using global transversality and bordism invariance of the signature. For oriented compact polyhedral pseudomanifolds X , stratified without strata of odd codimension, Goresky and MacPherson employed their intersection homology to obtain a bordism invariant signature and thus, using Thom's method, L -classes $L_i(X) \in H_i(X; \mathbb{Q})$ in (ordinary) homology, [31]. Irreducible complex projective algebraic varieties are examples of such X . It turned out later that one can move well beyond spaces with only strata of even codimension: The work of Siegel yields L -classes for Witt spaces ([67]). These are spaces for which the middle-perversity middle-degree intersection homology of even-dimensional links vanishes. A general sheaf-theoretic treatment of L -classes for arbitrary pseudomanifolds has been given in [3] and [4], where a local obstruction theory in terms of Lagrangian structures along strata of odd codimension is described.

Due on the one hand to their close relation to the normal invariant map on the set of homotopy smoothings of a Poincaré complex, and on the other hand to their remarkable invariance under homeomorphisms, discovered by Novikov, Hirzebruch's L -classes have come to occupy a central role in high-dimensional manifold classification theory. A particularly striking illustration is a classical result of Browder and Novikov, which can readily be deduced from the smooth surgery exact sequence: a closed, smooth, simply connected manifold of even dimension at least 5 is determined, up to finite ambiguity, by its homotopy type and its L -classes. By work of Cappell and Weinberger ([23], [72]), the Goresky-MacPherson L -class can be assigned a similar role in the global classification of singular spaces, and it is still a topological invariant. But much less is known about the transformational

laws that govern its behavior, and this is reflected in the sparsity of concrete calculations that have been carried out. In [43], Maxim and Schürmann calculated the classes T_{1*} and IT_{1*} for arbitrary toric varieties, and showed that $T_{1*} = IT_{1*} = L_*$ for projective simplicial toric varieties. (Such varieties are orbifolds, and hence rational homology manifolds.)

The triviality of normal bundles, required by the construction of Goresky-MacPherson-Thom, is frequently not very natural in projective algebraic geometry. For example, it often prevents one in practice to build recursive L -class calculations involving transverse sections of singular projective varieties. From this point of view, one should thus seek to incorporate nontrivial normal geometry into the singular L -class picture, and this is what we do in the first half of the present paper. An oriented normally nonsingular inclusion $g : Y \hookrightarrow X$ of real codimension c has a *Gysin homomorphism* $g^! : H_*(X; \mathbb{Q}) \rightarrow H_{*-c}(Y; \mathbb{Q})$ on ordinary homology. Our Theorem 3.18 asserts:

Theorem. *Let $g : Y \hookrightarrow X$ be a normally nonsingular inclusion of closed oriented even-dimensional piecewise-linear Witt pseudomanifolds (for example projective complex algebraic varieties). Let ν be the topological normal bundle of g . Then*

$$g^!L_*(X) = L^*(\nu) \cap L_*(Y).$$

Since singular spaces do not possess a tangent bundle, one cannot use naturality and the Whitney product formula to deduce this as in the case of a smooth manifold. Our guiding philosophy is the following: drop down to ordinary homology as late as possible from more elevated theories such as \mathbb{L}^\bullet -homology, or better yet, bordism. Then, on bordism it is possible to see the relation geometrically, using, in particular, geometric descriptions of cobordism due to Buoncrisiano-Rourke-Sanderson in terms of mock bundles. In implementing this philosophy, we use Ranicki’s symmetric algebraic L -theory, Siegel’s Witt bordism, natural transformations from bordism to L -theory as introduced recently in joint work with Laures and McClure ([8]), and various unblocked and blocked bundle theories and Thom spectra, notably the aforementioned theory of mock bundles [16].

In the course of carrying out this program, we prove that the Witt-bordism Gysin map sends the Witt-bordism fundamental class of X to the Witt-bordism fundamental class of Y , $g^![X]_{\text{Witt}} = [Y]_{\text{Witt}}$ (Theorem 3.15). Using this, we prove that the \mathbb{L}^\bullet -homology Gysin restriction sends the $\mathbb{L}^\bullet(\mathbb{Q})$ -homology fundamental class of X to the $\mathbb{L}^\bullet(\mathbb{Q})$ -homology fundamental class of Y , $g^![X]_{\mathbb{L}} = [Y]_{\mathbb{L}}$ (Theorem 3.17). Finally, one arrives at the above theorem on L -classes essentially by localizing at zero.

In Section 4, we apply the above Gysin Theorem in computing L -classes of some singular Schubert varieties. The examples we consider are sufficiently singular so as not to satisfy global Poincaré duality for ordinary homology

with rational coefficients. It seems that L -classes of singular Schubert varieties have not been computed before. The Chern-Schwartz-MacPherson classes of Schubert varieties were computed by Aluffi and Mihalcea in [2].

If ξ is a complex vector bundle over a base space B with Chern roots a_i , Hirzebruch had also defined a generalized Todd class $T_y^*(\xi) \in H^*(B) \otimes \mathbb{Q}[y]$, whose specialization to $y = 1$ is the L -class, $T_1^* = L^*$. Let X be a possibly singular complex algebraic variety of pure dimension, let $MHM(X)$ denote the abelian category of Morihiko Saito's algebraic mixed Hodge modules on X and $K_0(MHM(X))$ the associated Grothendieck group. A motivic Hirzebruch class transformation

$$MHT_{y*} : K_0(MHM(X)) \rightarrow H_{2*}^{\text{BM}}(X) \otimes \mathbb{Q}[y^{\pm 1}, (1+y)^{-1}]$$

to Borel-Moore homology has been defined by Brasselet, Schürmann and Yokura in [15], based on insights of Totaro [69]. Applying this to the mixed Hodge object \mathbb{Q}_X^H , one gets a homological characteristic class $T_{y*}(X) = MHT_{y*}[\mathbb{Q}_X^H]$ such that for X smooth and $y = 1$, $T_{1*}(X) = L_*(X)$. For this reason, $T_{1*}(X)$ has been called the Hodge L -class of X . However, examples of singular curves show that generally $T_{1*}(X) \neq L_*(X)$. This suggests applying MHT_{y*} to the intersection Hodge module IC_X^H , which yields an intersection generalized Todd class $IT_{y*}(X) = MHT_{y*}[IC_X^H[-\dim_{\mathbb{C}} X]]$. If X is an algebraic rational homology manifold, then $\mathbb{Q}_X^H[\dim_{\mathbb{C}} X] \cong IC_X^H$, so $IT_{y*}(X) = T_{y*}(X)$.

In the second half of the present paper, we prove that IT_{1*} transforms under Gysin restrictions associated to suitably normally nonsingular closed algebraic regular embeddings in the same manner as the Goresky-MacPherson L -class in the above Theorem. In the algebraic setting, one uses the algebraic normal bundle of a regular embedding. Since such a bundle need not generally reflect the complex topology near the subvariety, the Gysin result requires a tightness assumption (Definition 6.1), which holds automatically in transverse situations. We introduce a condition called *upward normal nonsingularity* (Definition 6.5), which requires for a tight regular embedding that the exceptional divisor in the blow-up relevant to deformation to the normal cone be normally nonsingular. This holds in suitably transverse situations and is related to the clean blow-ups of Cheeger, Goresky and MacPherson. Our Algebraic Gysin Theorem 6.30 is:

Theorem. *Let X, Y be pure-dimensional compact complex algebraic varieties and let $g : Y \hookrightarrow X$ be an upwardly normally nonsingular embedding. Let $N = N_Y X$ be the algebraic normal bundle of g and let ν denote the topological normal bundle of the topologically normally nonsingular inclusion underlying g . Then*

$$g^! IT_{1*}(X) = L^*(N) \cap IT_{1*}(Y) = L^*(\nu) \cap IT_{1*}(Y).$$

Viewed in conjunction, our Gysin theorems may be interpreted as further evidence towards a conjectural equality $IT_{1*} = L_*$ ([15, Remark 5.4])

for pure-dimensional compact complex algebraic varieties. Sections 2 – 4 deal with the topological L -class of piecewise-linear (PL) pseudomanifolds, whereas the remaining Sections 5 and 6 are concerned with the Hodge-theoretic class IT_{1*} . These two parts can be read independently.

The behavior of the L -class for singular spaces under Gysin transfers associated to finite degree covers is already completely understood. In [6], we showed that for a closed oriented Whitney stratified pseudomanifold X admitting Lagrangian structures along strata of odd codimension (e.g. X Witt) and $p : X' \rightarrow X$ an orientation preserving topological covering map of finite degree, the L -class of X transfers to the L -class of the cover, i.e.

$$p_!L_*(X) = L_*(X'),$$

where $p_! : H_*(X; \mathbb{Q}) \rightarrow H_*(X'; \mathbb{Q})$ is the transfer induced by p . This enabled us, for example, to establish the above conjecture for normal connected complex projective 3-folds X that have at worst canonical singularities, trivial canonical divisor, and $\dim H^1(X; \mathcal{O}_X) > 0$. (Note that such varieties are rational homology manifolds.) In the complex algebraic setting, results concerning the multiplicativity of the χ_y -genus (which in the smooth compact context corresponds to the signature for $y = 1$) under finite covers were obtained by A. Libgober and L. Maxim in [40, Lemma 2.3]. J. Schürmann discusses going up-and-down techniques for the behavior of the motivic Chern class transformation MHC_y under étale morphisms in [62, Cor. 5.11, Cor. 5.12]. Let $\sigma(X)$ denote the signature of a compact Witt space X . If X is a complex projective algebraic variety, then by Saito’s intersection cohomology Hodge index theorem ([56], [42, Section 3.6]), $IT_{1,0}(X) = \sigma(X) = L_0(X)$, that is, the conjecture is known to hold in degree 0. Furthermore, Cappell, Maxim, Schürmann and Shaneson [21, Cor. 1.2] have shown that the conjecture holds for orbit spaces $X = Y/G$, where Y is a projective G -manifold and G a finite group of algebraic automorphisms. The conjecture holds for simplicial projective toric varieties [43, Corollary 1.2(iii)] and for certain complex hypersurfaces with isolated singularities [20, Theorem 4.3].

Following the overall strategy introduced in the present paper, formulae describing the behavior of the Goresky-MacPherson L -class and of IT_{1*} under transfer homomorphisms associated to fiber bundles with nonsingular positive dimensional fiber can also be obtained, but deserve treatment in a separate paper.

Convention. Ordinary homology and cohomology of spaces will be taken with coefficients in \mathbb{Q} , unless noted otherwise.

Acknowledgements. We express our gratitude to Jörg Schürmann, whose thoughtful comments on an earlier version of this paper lead to numerous improvements, and to Laurentiu Maxim for providing helpful information on certain aspects of mixed Hodge modules.

2. The L -class of a pseudomanifold

If ξ is a real vector bundle over a topological space B , let

$$L^*(\xi) = L^0(\xi) + L^1(\xi) + L^2(\xi) + \cdots \in H^{4*}(B; \mathbb{Q}), \quad L^0(\xi) = 1,$$

denote its cohomological Hirzebruch L -class with $L^i(\xi) \in H^{4i}(B; \mathbb{Q})$. For a closed oriented smooth manifold M of real dimension n ,

$$L_*(M^n) = L_n(M) + L_{n-4}(M) + L_{n-8}(M) + \cdots$$

denotes the Poincaré dual of the Hirzebruch L -class $L^*(M) = L^*(TM)$ of the tangent bundle $\xi = TM$ of M . Thus

$$L_i(M) \in H_i(M; \mathbb{Q}), \quad L_{n-4i}(M) = L^i(M) \cap [M].$$

We have

$$L_n(M) = L^0(M) \cap [M] = 1 \cap [M] = [M],$$

and if M has real dimension $n = 4k$, then

$$\epsilon_* L_0(M) = \epsilon_*(L^k(M) \cap [M]) = \sigma(M),$$

where $\sigma(M)$ denotes the signature of M .

Let X be a compact oriented piecewise-linear (PL) pseudomanifold of dimension n . Such a pseudomanifold can be equipped with a choice of PL stratification, and there is a PL-intrinsic such stratification. Siegel called X a *Witt space* if the middle degree, lower middle perversity rational intersection homology of even-dimensional links of strata vanishes, [67]. This condition turns out to be independent of the choice of PL stratification, [32, Section 2.4]. A pure-dimensional complex algebraic variety can be Whitney stratified, and thus PL stratified, without strata of odd dimension and is thus a Witt space. Compact Witt spaces X have homological L -classes

$$L_i(X) \in H_i(X; \mathbb{Q}) \cong \text{Hom}(H^i(X; \mathbb{Q}), \mathbb{Q}),$$

on which a cohomology class $\xi \in H^i(X; \mathbb{Q})$, stably represented as $\xi = f^*(u)$, $f: X \rightarrow S^i$ transverse with regular value $p \in S^i$, evaluates to $\langle \xi, L_i(X) \rangle = \sigma(f^{-1}(p))$, where $u \in H^i(S^i)$ is the appropriate generator and σ denotes the signature. Note that the transverse preimage $f^{-1}(p)$ is again a Witt space. Using L^2 -forms on the top stratum with respect to conical Riemannian metrics, Cheeger gave a local formula for $L_*(X)$ in terms of eta-invariants of links, [25]. Again $\epsilon_* L_0(X) = \sigma(X)$ and if $X = M$ is a smooth manifold, then $L_i(X)$ agrees with the above Poincaré duals $L_i(M)$ of Hirzebruch's class.

3. Behavior of the L -class under normally nonsingular inclusions

Let $g: Y \hookrightarrow X$ be an inclusion of compact oriented stratified pseudomanifolds. If the inclusion is normally nonsingular with trivial normal bundle, then, by the very definition of the L -class, there is a clear relationship between the L -classes of X and Y . In the projective algebraic situation, the triviality assumption on the normal bundle is not very natural, and

it becomes important to understand the relationship of these characteristic classes for arbitrary normal bundles. This will be accomplished in the present section by establishing a precise formula (one of the main results of this paper) involving the Gysin transfer associated to the normally non-singular embedding g . The formula is motivated by the special case of a smooth embedding of manifolds, where it is easily established (see below).

Definition 3.1. A *topological stratification* of a topological space X is a filtration

$$X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_1 \supseteq X_0 \supseteq X_{-1} = \emptyset$$

by closed subsets X_i such that the difference sets $X_i - X_{i-1}$ are topological manifolds of pure dimension i , unless empty. The connected components X_α of these difference sets are called the *strata*. We will often write stratifications as $\mathcal{X} = \{X_\alpha\}$.

The following definition is due to Siebenmann [66]; see also Schürmann [61, Def. 4.2.1, p. 232].

Definition 3.2. A topological stratification $\{X_i\}$ of X is called *locally cone-like* if for all $x \in X_i - X_{i-1}$ there is an open neighborhood U of x in X , a compact topological space L with filtration

$$L = L_{n-i-1} \supseteq L_{n-i-2} \supseteq \cdots \supseteq L_0 \supseteq L_{-1} = \emptyset,$$

and a filtration preserving homeomorphism $U \cong \mathbb{R}^i \times \text{cone}^\circ(L)$, where $\text{cone}^\circ(L)$ denotes the open cone on L .

Locally cone-like topological stratifications are frequently also called *cs-stratifications*. We understand an *algebraic stratification* of a complex algebraic variety X to be a locally cone-like topological stratification $\{X_{2i}\}$ of X such that all subspaces X_{2i} are closed algebraic subsets of X . Complex algebraic Whitney stratifications are algebraic stratifications in this sense.

Definition 3.3. Let X be a topological space with locally cone-like topological stratification $\mathcal{X} = \{X_\alpha\}$ and let Y be any topological space. An embedding $g : Y \hookrightarrow X$ is called *normally nonsingular* (with respect to \mathcal{X}), if

- (1) $\mathcal{Y} := \{Y_\alpha := X_\alpha \cap Y\}$ is a locally cone-like topological stratification of Y ,
- (2) there exists a topological vector bundle $\pi : E \rightarrow Y$ and
- (3) there exists a (topological) embedding $j : E \rightarrow X$ such that
 - (a) $j(E)$ is open in X ,
 - (b) $j|_Y = g$, and
 - (c) the homeomorphism $j : E \xrightarrow{\cong} j(E)$ is stratum preserving, where the open set $j(E)$ is endowed with the stratification $\{X_\alpha \cap j(E)\}$ and E is endowed with the stratification $\mathcal{E} = \{\pi^{-1}Y_\alpha\}$.

Note that the above stratification \mathcal{E} of the total space E is automatically topologically locally cone-like.

Definition 3.4. If X and Y are complex algebraic varieties and $g : Y \hookrightarrow X$ a closed algebraic embedding whose underlying topological embedding $g(\mathbb{C})$ in the complex topology is normally nonsingular, then we will call g and $g(\mathbb{C})$ *compatibly stratifiable* if there exists an algebraic stratification \mathcal{X} of X such that $g(\mathbb{C})$ is normally nonsingular with respect to \mathcal{X} and the induced stratification \mathcal{Y} is an algebraic stratification of Y .

An oriented normally nonsingular inclusion $g : Y \hookrightarrow X$ of real codimension c has a *Gysin map*

$$g^! : H_*(X; \mathbb{Q}) \longrightarrow H_{*-c}(Y; \mathbb{Q})$$

on ordinary singular homology, given as follows: Let $u \in H^c(E, E_0; \mathbb{Q})$ denote the Thom class in ordinary cohomology of the rank c vector bundle $\pi : E \rightarrow Y$, where $E_0 \subset E$ denotes the complement of the zero section in E . Then $g^!$ is the composition

$$H_k(X; \mathbb{Q}) \rightarrow H_k(X, X - Y; \mathbb{Q}) \xrightarrow[e_*]{e^*} H_k(E, E_0; \mathbb{Q})$$

$$\xrightarrow[\cong]{u \cap -} H_{k-c}(E; \mathbb{Q}) \xrightarrow[\cong]{\pi_*} H_{k-c}(Y; \mathbb{Q}),$$

where we use the embedding $j : E \rightarrow X$ in defining the excision isomorphism e_* . For classes $x \in H^p(X; \mathbb{Q})$, $y \in H_*(X; \mathbb{Q})$, the formula

$$g^!(x \cap y) = g^*x \cap g^!y \tag{3.1}$$

holds, provided either p or the real codimension c is even ([7, Lemma 5, p. 613], [12, Ch. V, §6.2]). In the special case of a smooth embedding $g : N \hookrightarrow M$ of closed oriented even-dimensional smooth manifolds, the Gysin transfer maps the fundamental class $[M]$ of M to the fundamental class $[N]$ of N . Thus in this case, using naturality and the Whitney sum formula, and with ν the normal bundle $E \rightarrow N$ of N in M ,

$$g^!L_*(M) = g^*L^*(TM) \cap g^![M] = L^*(g^*TM) \cap [N] = L^*(\nu) \cap L_*(N).$$

(All involved classes lie in even degrees and hence no signs enter.) In this section, we shall show that this relation continues to hold for normally nonsingular inclusions of singular spaces. Note that when the normal bundle is trivial, the formula becomes $g^!L_*(M) = L_*(N)$, as it should be. In fact, for trivial normal bundle, the relation $g^!L_*(X) = L_*(Y)$ was already pointed out by Cappell and Shaneson [18] in the singular context, even for general Verdier self-dual complexes of sheaves.

Complex algebraic pure-dimensional varieties are Witt spaces in the sense of Siegel [67]. Bordism of Witt spaces, denoted by $\Omega_*^{\text{Witt}}(-)$, is a generalized homology theory represented by a spectrum MWITT. For a (real) codimension c normally nonsingular inclusion $g : Y^{n-c} \hookrightarrow X^n$ of (compact, oriented)

Witt spaces, we will define a Gysin map

$$g^! : \Omega_k^{\text{Witt}}(X) \longrightarrow \Omega_{k-c}^{\text{Witt}}(Y),$$

and we shall prove that it sends the Witt-orientation of X , represented by the identity map, to the Witt orientation of Y . This will then be applied in proving the analogous statement for the $\mathbb{L}^\bullet(\mathbb{Q})$ -homology orientations, using the machinery of Banagl-Laures-McClure [8]. We write $\mathbb{L}^\bullet = \mathbb{L}^\bullet(\mathbb{Z}) = \mathbb{L}^\bullet\langle 0 \rangle(\mathbb{Z})$ for Ranicki’s connected symmetric algebraic L -spectrum with homotopy groups $\pi_n(\mathbb{L}^\bullet) = L^n(\mathbb{Z})$, the symmetric L -groups of the ring of integers. Localization $\mathbb{Z} \rightarrow \mathbb{Q}$ induces a map $\mathbb{L}^\bullet(\mathbb{Z}) \rightarrow \mathbb{L}^\bullet(\mathbb{Q})$ and $\pi_n(\mathbb{L}^\bullet(\mathbb{Q})) = L^n(\mathbb{Q})$ with

$$L^n(\mathbb{Q}) \cong \begin{cases} \mathbb{Z} \oplus (\mathbb{Z}/2)^\infty \oplus (\mathbb{Z}/4)^\infty, & n \equiv 0(4) \\ 0, & n \not\equiv 0(4). \end{cases}$$

As far as cobordism is concerned, the idea is to employ the framework of Buoncrisiano, Rourke and Sanderson [16], which provides a geometric description of cobordism in terms of mock bundles, as well as geometric descriptions of Thom classes in cobordism, and cap products between cobordism and bordism.

3.1. Thom classes in cobordism. Our approach requires uniform notions of Thom spaces and Thom classes in cobordism for various types of bundle theories and cobordism theories. This will now be set up.

The term *fibration* will always mean Hurewicz fibration. A *sectioned fibration* is a pair (ξ, s) , where ξ is a fibration $p : E \rightarrow B$, $s : B \rightarrow E$ is a section of p , and the inclusion of the image of s in E is a fiberwise cofibration over B . Let $(S^n, *)$ be a pointed n -sphere. An $(S^n, *)$ -fibration is a sectioned S^n -fibration (ξ, s) such that $(p^{-1}(b), s(b))$ is pointed homotopy equivalent to $(S^n, *)$ for every $b \in B$. Such $(S^n, *)$ -fibrations are classified by maps into a classifying space BF_n . In particular, over BF_n , there is a universal $(S^n, *)$ -fibration γ_n^F .

Definition 3.5. The *Thom space* of an $(S^n, *)$ -fibration $\alpha = (\xi, s)$ is defined to be

$$\text{Th}(\alpha) := E/s(B).$$

(See Rudyak [55].) Let $(\xi, s), (\xi', s')$ be $(S^n, *)$ -fibrations with ξ, ξ' given by $p : E \rightarrow B, p' : E' \rightarrow B'$, respectively. A *morphism of $(S^n, *)$ -fibrations* $\phi : (\xi, s) \rightarrow (\xi', s')$ is a pair $\phi = (g, f)$, where $f : B \rightarrow B'$ and $g : E \rightarrow E'$ are maps such that $p' \circ g = f \circ p$,

$$g| : (p^{-1}(b), s(b)) \longrightarrow (p'^{-1}(f(b)), s'(f(b)))$$

is a pointed homotopy equivalence for all $b \in B$, and ϕ respects the sections, i.e. $g \circ s = s' \circ f$. The composition of two morphisms of $(S^n, *)$ -fibrations is again an $(S^n, *)$ -fibration and the identity is a morphism of

$(S^n, *)$ -fibrations. Thus $(S^n, *)$ -fibrations form a category. A morphism $\phi : \alpha = (\xi, s) \rightarrow (\xi', s') = \alpha'$ of $(S^n, *)$ -fibrations induces a map

$$\mathrm{Th}(\phi) : \mathrm{Th}(\alpha) = E/s(B) \longrightarrow E'/s'(B') = \mathrm{Th}(\alpha').$$

In this way, $\mathrm{Th}(\cdot)$ becomes a functor on the category of $(S^n, *)$ -fibrations. Let $\theta = \theta_1^F$ denote the trivial (product) $(S^1, *)$ -fibration over a point. Then, using fiberwise homotopy smash product \wedge^h , $\gamma_n^F \wedge^h \theta$ is an $(S^{n+1}, *)$ -fibration over BF_n , and hence has a classifying morphism $\phi_n : \gamma_n^F \wedge^h \theta \rightarrow \gamma_{n+1}^F$ of $(S^{n+1}, *)$ -fibrations. This yields in particular maps $f_n : \mathrm{BF}_n \rightarrow \mathrm{BF}_{n+1}$ and we denote the stable classifying space by BF . In addition to BF_n , the following classifying spaces will be relevant:

- BSO_n , classifying oriented real vector bundles,
- BSPL_n , classifying oriented PL $(\mathbb{R}^n, 0)$ -bundles
(and oriented PL microbundles),
- BSTOP_n , classifying oriented topological $(\mathbb{R}^n, 0)$ -bundles
(and oriented topological microbundles),
- $\mathrm{BS}\widetilde{\mathrm{PL}}_n$, classifying oriented PL closed disc block bundles,
- BG_n , classifying spherical fibrations with fiber S^{n-1} .

The unoriented versions of these spaces will be denoted by omitting the ‘S’. For the theory of block bundles, due to Rourke and Sanderson, we ask the reader to consult [51], [52], [53], and [54]; the definition of a block bundle will be briefly reviewed further below. There is a homotopy commutative diagram

$$\begin{array}{ccccc} \mathrm{BSO}_n & \xrightarrow{LR} & \mathrm{BSPL}_n & \xrightarrow{\text{forget}} & \mathrm{BSTOP}_n \\ & & \downarrow & & \downarrow \\ & & \mathrm{BS}\widetilde{\mathrm{PL}}_n & \longrightarrow & \mathrm{BG}_n \longrightarrow \mathrm{BF}_n, \end{array}$$

whose philosophy here is that we can flush Thom space issues down to the level of BF_n . Thus, a vector bundle has an underlying microbundle, [46, p. 55, Example (2)]. The leftmost horizontal arrow is due to Lashof and Rothenberg [38], who showed that O_n -vector bundles can be triangulated. The left vertical arrow is due to Rourke and Sanderson: A PL microbundle gives rise to a unique equivalence class of PL block bundles, [51]. A PL block bundle determines a unique spherical fibration with fiber S^{n-1} , [52, Cor. 5.9, p. 23]. (Also cf. Casson [24].) Of course, given an (oriented) topological $(\mathbb{R}^n, 0)$ -bundle, one can delete the zero-section to obtain an S^{n-1} -fibration, which describes the composition $\mathrm{BSTOP}_n \rightarrow \mathrm{BG}_n$. Consider $S^0 = \{-1, +1\}$ as the trivial S^0 -bundle θ_0 over a point. Given an S^{n-1} -fibration ξ , there is a canonical $(S^n, *)$ -fibration ξ^\bullet associated to it, namely $\xi^\bullet := \xi * \theta_0$ (fiberwise unreduced suspension). Note that the fiberwise unreduced suspension ξ^\bullet can be given a canonical section, by consistently taking north poles (say). This describes the map $\mathrm{BG}_n \rightarrow \mathrm{BF}_n$.

To fix notation, let ξ be a rank n oriented vector bundle over the polyhedron $X = |K|$ of a finite simplicial complex K . Then ξ has a classifying map $\xi : X \rightarrow \text{BSO}_n$. (We denote classifying maps and the bundle they classify by the same letter.) Composing with the Lashof-Rothenberg map LR , we get a classifying map

$$\xi_{\text{PL}} : X \rightarrow \text{BSPL}_n,$$

which determines an underlying oriented PL $(\mathbb{R}^n, 0)$ -bundle (or PL microbundle) over X . We compose further with the map $\text{BSPL}_n \rightarrow \text{BS}\widehat{\text{SPL}}_n$ and get a classifying map

$$\xi_{\text{PLB}} : X \rightarrow \text{BS}\widehat{\text{SPL}}_n,$$

which determines an underlying oriented PL block bundle ξ_{PLB} over X . On the other hand, we may compose ξ_{PL} with the forget map to obtain a classifying map

$$\xi_{\text{TOP}} : X \rightarrow \text{BSTOP}_n,$$

which determines an underlying oriented topological $(\mathbb{R}^n, 0)$ -bundle (or topological microbundle) ξ_{TOP} over X . Composing with the map $\text{BSTOP}_n \rightarrow \text{BG}_n$, we receive a classifying map

$$\xi_{\text{G}} : X \rightarrow \text{BG}_n,$$

which determines an underlying S^{n-1} -fibration ξ_{G} over X , which in turn has an underlying $(S^n, *)$ -fibration $\xi^\bullet = \xi_{\text{G}}^\bullet$.

Definition 3.6. Let ξ be a real vector bundle, or PL/topological $(\mathbb{R}^n, 0)$ -bundle, or PL closed disc block bundle, or S^{n-1} -fibration. Then the *Thom space* $\text{Th}(\xi)$ of ξ is defined to be the Thom space of its underlying $(S^n, *)$ -fibration,

$$\text{Th}(\xi) := \text{Th}(\xi^\bullet).$$

In particular for an oriented vector bundle ξ ,

$$\text{Th}(\xi) = \text{Th}(\xi_{\text{PL}}) = \text{Th}(\xi_{\text{PLB}}) = \text{Th}(\xi_{\text{TOP}}) = \text{Th}(\xi^\bullet).$$

Uniform constructions of Thom spectra can be given via the notion of Thom spectrum of a map f . Let X be a CW complex and $f : X \rightarrow \text{BF}$ a continuous map. The Thom spaces $\text{Th}(f_n^* \gamma_n^F)$ of the pullbacks under $f_n : X_n \rightarrow \text{BF}_n$ of the universal $(S^n, *)$ -fibrations form a spectrum $\text{Th}(f)$, whose structure maps are induced on Thom spaces by the morphisms $f_n^* \gamma_n^F \oplus \theta \rightarrow f_{n+1}^* \gamma_{n+1}^F$. Here, f_n is the restriction of f to an increasing and exhaustive CW-filtration $\{X_n\}$ of X such that $f(X_n) \subset \text{BF}_n$. The spectrum $\text{Th}(f)$ is called the *Thom spectrum of the map f* . This construction applies to the map $f : \text{BSTOP} \rightarrow \text{BF}$, filtered by $f_n : \text{BSTOP}_n \rightarrow \text{BF}_n$, and yields the Thom spectrum $\text{MSTOP} = \text{Th}(\text{BSTOP} \rightarrow \text{BF})$. Note that $f_n^* \gamma_n^F$ has classifying map $f_n : \text{BSTOP}_n \rightarrow \text{BF}_n$, but so does the underlying $(S^n, *)$ -fibration $(\gamma_n^{\text{STOP}})^\bullet$ of the universal oriented topological $(\mathbb{R}^n, 0)$ -bundle γ_n^{STOP} over BSTOP_n . Hence $f_n^* \gamma_n^F$ and $(\gamma_n^{\text{STOP}})^\bullet$ are equivalent $(S^n, *)$ -fibrations

and so have homotopy equivalent Thom spaces. Similarly, we obtain Thom spectra $\text{MSPL} = \text{Th}(\text{BSPL} \rightarrow \text{BF})$ and $\text{MSO} = \text{Th}(\text{BSO} \rightarrow \text{BF})$. These spectra $\text{MSO}, \text{MSPL}, \text{MSTOP}$ are commutative ring spectra, Rudyak [55, Cor. IV.5.22, p. 261].

Let $\Omega_n^{\text{STOP}}(-), \Omega_n^{\text{SPL}}(-),$ and $\Omega_n^{\text{SO}}(-)$ denote bordism of oriented topological, or PL, or smooth manifolds. The Pontrjagin-Thom theorem provides natural isomorphisms

$$\Omega_n^{\text{STOP}}(X) \cong \text{MSTOP}_n(X), \Omega_n^{\text{SPL}}(X) \cong \text{MSPL}_n(X), \Omega_n^{\text{SO}}(X) \cong \text{MSO}_n(X).$$

(In the TOP case, this requires Kirby-Siebenmann topological transversality in high dimensions, and the work of Freedman and Quinn in dimension 4.)

We shall next construct maps between Thom spectra, using the following general principle: Let X', X be CW complexes with CW filtrations $\{X'_n\}, \{X_n\}$, respectively. Let $g : X' \rightarrow X$ be a map with $g(X'_n) \subset X_n$. Let $f : X \rightarrow \text{BF}$ be a map as above so that $\text{Th}(f)$ is defined. Then composition gives a map $f' = fg : X' \rightarrow \text{BF}$ such that the Thom spectrum $\text{Th}(f')$ is defined as well. The map g induces a map of spectra

$$\text{Th}(f') \longrightarrow \text{Th}(f).$$

Applying this principle to $g : \text{BSPL} = X' \rightarrow X = \text{BSTOP}$, with $f : X = \text{BSTOP} \rightarrow \text{BF}$ as in the above definition of MSTOP , yields a map of spectra

$$\phi_F : \text{MSPL} = \text{Th}(f') \longrightarrow \text{Th}(f) = \text{MSTOP}.$$

Similarly, we get $\phi_{LR} : \text{MSO} \rightarrow \text{MSPL}$ using the Lashof-Rothenberg map.

We turn to uniform constructions of Thom classes in cobordism theory. First, say, for topological bundles: Let ξ be an oriented topological $(\mathbb{R}^n, 0)$ -bundle. Then ξ is classified by a map $t : X \rightarrow \text{BSTOP}_n$ and has an underlying $(S^n, *)$ -fibration ξ^\bullet with classifying map the composition

$$X \xrightarrow{t} \text{BSTOP}_n \xrightarrow{f_n} \text{BF}_n.$$

Let ζ_n^T be the $(S^n, *)$ -fibration such that $\text{MSTOP}_n = \text{Th}(\zeta_n^T)$, i.e. $\zeta_n^T = f_n^* \gamma_n^F$. (This is nothing but $(\gamma_n^{\text{STOP}})^\bullet$.) Then

$$t^* \zeta_n^T = t^* f_n^* \gamma_n^F = \xi^\bullet,$$

with corresponding morphism $\psi : \xi^\bullet \rightarrow \zeta_n^T$ of $(S^n, *)$ -fibrations. This morphism induces on Thom spaces a map

$$\text{Th}(\psi) : \text{Th}(\xi^\bullet) \longrightarrow \text{Th}(\zeta_n^T) = \text{MSTOP}_n.$$

By Definition 3.6, $\text{Th}(\xi^\bullet) = \text{Th}(\xi)$. So we may write $\text{Th}(\psi)$ as

$$\text{Th}(\psi) : \text{Th}(\xi) \longrightarrow \text{Th}(\zeta_n^T) = \text{MSTOP}_n.$$

Suspension and composition with the structure maps of MSTOP gives a map of spectra

$$\Sigma^\infty \text{Th}(\xi) \longrightarrow \Sigma^n \text{MSTOP}.$$

Here $\Sigma^\infty Y$ denotes the suspension spectrum of a space Y , and $\Sigma^n E$ of a spectrum E is the spectrum with $(\Sigma^n E)_k = E_{n+k}$. The map of spectra determines a homotopy class

$$u_{\text{STOP}}(\xi) \in [\Sigma^\infty \text{Th}(\xi), \Sigma^n \text{MSTOP}] = \widetilde{\text{MSTOP}}^n(\text{Th}(\xi)).$$

This class $u_{\text{STOP}}(\xi)$ is called the *Thom class* of ξ in oriented topological cobordism and is indeed an MSTOP-orientation of ξ^\bullet in the sense of Dold.

We proceed in a similar way to construct the Thom class of a PL bundle: Let ξ be an oriented PL $(\mathbb{R}^n, 0)$ -bundle over a compact polyhedron X . Then ξ is classified by a map $h : X \rightarrow \text{BSPL}_n$. Forgetting the PL structure, we have an underlying topological $(\mathbb{R}^n, 0)$ -bundle ξ_{TOP} , classified by the composition

$$X \xrightarrow{h} \text{BSPL}_n \xrightarrow{g_n} \text{BSTOP}_n.$$

This topological bundle in turn has an underlying $(S^n, *)$ -fibration $(\xi_{\text{TOP}})^\bullet$ with classifying map the composition

$$X \xrightarrow{h} \text{BSPL}_n \xrightarrow{g_n} \text{BSTOP}_n \xrightarrow{f_n} \text{BF}_n.$$

Of course ξ itself has an underlying $(S^n, *)$ -fibration ξ^\bullet and $\xi^\bullet = (\xi_{\text{TOP}})^\bullet$. Let ζ_n^P be the $(S^n, *)$ -fibration such that $\text{MSPL}_n = \text{Th}(\zeta_n^P)$, i.e. $\zeta_n^P = (f_n g_n)^* \gamma_n^F$. (This is nothing but $(\gamma_n^{\text{SPL}})^\bullet$.) Then

$$h^* \zeta_n^P = h^* g_n^* f_n^* \gamma_n^F = \xi^\bullet,$$

with corresponding morphism $\phi : \xi^\bullet \rightarrow \zeta_n^P$ of $(S^n, *)$ -fibrations. This morphism induces on Thom spaces a map

$$\text{Th}(\phi) : \text{Th}(\xi^\bullet) \longrightarrow \text{Th}(\zeta_n^P) = \text{MSPL}_n.$$

By Definition 3.6, $\text{Th}(\xi^\bullet) = \text{Th}(\xi)$. So we may write $\text{Th}(\phi)$ as

$$\text{Th}(\phi) : \text{Th}(\xi) \longrightarrow \text{Th}(\zeta_n^P) = \text{MSPL}_n.$$

We arrive thus at a map of spectra

$$\Sigma^\infty \text{Th}(\xi) \longrightarrow \Sigma^n \text{MSPL},$$

which determines a homotopy class

$$u_{\text{SPL}}(\xi) \in [\Sigma^\infty \text{Th}(\xi), \Sigma^n \text{MSPL}] = \widetilde{\text{MSPL}}^n(\text{Th}(\xi)).$$

This class $u_{\text{SPL}}(\xi)$ is called the *Thom class* of ξ in oriented PL cobordism. As in the topological case, one verifies that this is an MSPL-orientation of ξ^\bullet . Earlier, we had constructed a map of Thom spectra $\phi_F : \text{MSPL} \rightarrow \text{MSTOP}$. Recall that the underlying topological bundle ξ_{TOP} of a PL bundle ξ_{PL} and ξ_{PL} itself have the same Thom space,

$$\text{Th}(\xi_{\text{PL}}) = \text{Th}(\xi^\bullet) = \text{Th}(\xi_{\text{TOP}}).$$

Lemma 3.7. *Let ξ_{PL} be an oriented PL $(\mathbb{R}^n, 0)$ -bundle. On cobordism groups, the induced map*

$$\phi_F : \widetilde{\text{MSPL}}^n(\text{Th}(\xi_{\text{PL}})) \longrightarrow \widetilde{\text{MSTOP}}^n(\text{Th}(\xi_{\text{TOP}}))$$

maps the Thom class of ξ_{PL} to the Thom class of the underlying topological $(\mathbb{R}^n, 0)$ -bundle ξ_{TOP} ,

$$\phi_F(u_{\text{SPL}}(\xi_{\text{PL}})) = u_{\text{STOP}}(\xi_{\text{TOP}}).$$

The proof is a standard verification.

The cobordism Thom class of an oriented real vector bundle ξ can be similarly fit into this picture. If n is the rank of ξ , then ξ has a *Thom class*

$$u_{\text{SO}}(\xi) \in [\Sigma^\infty \text{Th}(\xi), \Sigma^n \text{MSO}] = \widetilde{\text{MSO}}^n(\text{Th}(\xi))$$

in smooth oriented cobordism. Recall that we had earlier described a map $\phi_{LR} : \text{MSO} \rightarrow \text{MSPL}$ based on the Lashof-Rothenberg map. The following compatibility result is again standard (and readily verified).

Lemma 3.8. *Let ξ be a rank n oriented vector bundle over a compact polyhedron X . On cobordism groups, the induced map*

$$\phi_{LR} : \widetilde{\text{MSO}}^n(\text{Th}(\xi)) \longrightarrow \widetilde{\text{MSPL}}^n(\text{Th}(\xi_{\text{PL}}))$$

maps the Thom class of ξ to the Thom class of the underlying oriented PL $(\mathbb{R}^n, 0)$ -bundle ξ_{PL} ,

$$\phi_{LR}(u_{\text{SO}}(\xi)) = u_{\text{SPL}}(\xi_{\text{PL}}).$$

3.2. Ranicki's Thom class in \mathbb{L}^\bullet -cohomology. We review Ranicki's definition of a Thom class for topological $(\mathbb{R}^n, 0)$ -bundles (or microbundles) in symmetric \mathbb{L} -cohomology. He constructs a map

$$\sigma^* : \text{MSTOP} \longrightarrow \mathbb{L}^\bullet,$$

see [50, p. 290]. Let X be the polyhedron of a finite simplicial complex and $\xi : X \rightarrow \text{BSTOP}_n$ a topological $(\mathbb{R}^n, 0)$ bundle (or microbundle) over X . Then, following [50, pp. 290, 291], ξ has a canonical \mathbb{L}^\bullet -cohomology orientation

$$u_{\mathbb{L}}(\xi) \in \widetilde{\mathbb{L}^\bullet}^n(\text{Th}(\xi)),$$

which we shall also refer to as the \mathbb{L}^\bullet -cohomology Thom class of ξ , defined by

$$u_{\mathbb{L}}(\xi) := \sigma^*(u_{\text{STOP}}(\xi)).$$

The morphism of spectra $\mathbb{L}^\bullet(\mathbb{Z}) \rightarrow \mathbb{L}^\bullet(\mathbb{Q})$ coming from localization induces a homomorphism

$$\widetilde{\mathbb{L}^\bullet}^n(\text{Th}(\xi)) \longrightarrow \widetilde{\mathbb{L}^\bullet}(\mathbb{Q})^n(\text{Th}(\xi)).$$

We denote the image of $u_{\mathbb{L}}(\xi)$ under this map again by the same symbol $u_{\mathbb{L}}(\xi) \in \widetilde{\mathbb{L}^\bullet}(\mathbb{Q})^n(\text{Th}(\xi))$.

3.3. Geometric description of the MSO-Thom class. Buoncristiano, Rourke and Sanderson [16] give a geometric description of MSPL-cobordism and use it to obtain in particular a geometric description of the Thom class $u_{\text{SPL}}(\xi)$, which we reviewed homotopy theoretically in Section 3.1. The geometric cocycles are given by (oriented) mock bundles, whose definition we recall here. The polyhedron of a ball complex K is denoted by $|K|$.

Definition 3.9. Let K be a finite ball complex and q an integer (possibly negative). A q -mock bundle ξ^q/K with base K and total space $E(\xi)$ consists of a PL map $p : E(\xi) \rightarrow |K|$ such that, for each $\sigma \in K$, $p^{-1}(\sigma)$ is a compact PL manifold of dimension $q + \dim(\sigma)$, with boundary $p^{-1}(\partial\sigma)$. The preimage $\xi(\sigma) := p^{-1}(\sigma)$ is called the *block* over σ .

The empty set is regarded as a manifold of any dimension; thus $\xi(\sigma)$ may be empty for some cells $\sigma \in K$. Note that if σ^0 is a 0-dimensional cell of K , then $\partial\sigma^0 = \emptyset$ and thus $p^{-1}(\partial\sigma) = \emptyset$. Hence the blocks over 0-dimensional cells are *closed* manifolds. For our purposes, we need *oriented* mock bundles, which are defined using incidence numbers of cells and blocks: Suppose that $(M^n, \partial M)$ is an oriented PL manifold and $(N^{n-1}, \partial N)$ is an oriented PL manifold with $N \subset \partial M$. Then an incidence number $\epsilon(N, M) = \pm 1$ is defined by comparing the orientation of N with that induced on N from M (the induced orientation of ∂M is defined by taking the inward normal last); $\epsilon(N, M) = +1$ if these orientation agree and -1 if they disagree. An *oriented cell complex* K is a cell complex in which each cell is oriented. We then have the incidence number $\epsilon(\tau, \sigma)$ defined for faces $\tau^{n-1} < \sigma^n \in K$.

Definition 3.10. An *oriented mock bundle* is a mock bundle ξ/K over an oriented (finite) ball complex K in which every block is oriented (i.e. is an oriented PL manifold) such that for each $\tau^{n-1} < \sigma^n \in K$, $\epsilon(\xi(\tau), \xi(\sigma)) = \epsilon(\tau, \sigma)$.

The following auxiliary result is an analog of [16, Lemma 1.2, p. 21]:

Lemma 3.11. *Let (K, K_0) be a (finite) ball complex pair such that $|K|$ is an n -dimensional (compact) Witt space with (possibly empty) boundary $\partial|K| = |K_0|$. Orient K in such a way that the sum of oriented n -balls is a cycle rel boundary. (This is possible since $|K|$, being a Witt space, is an oriented pseudomanifold-with-boundary.) Let ξ/K be an oriented q -mock bundle over K . Then the total space $E(\xi)$ is an $(n + q)$ -dimensional (compact) Witt space with boundary $p^{-1}(\partial|K|)$.*

Proof. One merely has to modify the proof of [16, Lemma 1.2] for the Witt context, see also the proof of the IP-ad theorem [8, Theorem 4.4]. First, choose a structuring of K as a *structured cone complex* in the sense of McCrory [45] by choosing points $\hat{\sigma}$ in the interior of σ for every cell $\sigma \in K$. The associated first derived subdivision \hat{K} is a simplicial complex and induces a concept of dual cells $D(\sigma)$ for cells $\sigma \in K$. Let $X = |K|$ be the underlying polyhedron of K . Polyhedra have intrinsic PL stratifications,

[1]. In particular, points in X have intrinsic links L with respect to this stratification. The simplicial link of $\hat{\sigma}$ in \hat{K} is a suspension of the intrinsic link L at $\hat{\sigma}$. Then the polyhedron of the dual complex of σ can be written in terms of the intrinsic link L as

$$|D(\sigma)| \cong D^{j-k} \times \text{cone}(L),$$

where D^{j-k} denotes a closed disc of dimension $j - k$.

Now let $(X, \partial X) = (|K|, |K_0|)$ be an n -dimensional PL pseudomanifold-with-boundary, where (K, K_0) is a ball complex pair with K structured as described above so that dual blocks of balls are defined. Assume that X is Witt and ξ is an oriented q -mock bundle over K . By the arguments in the proof of [8, Lemma 4.6], the total space $E(\xi)$ is a PL pseudomanifold with collared boundary $p^{-1}(|K_0|)$. (Those arguments do indeed cover the present case, since they only require that the base $(|K|, |K_0|)$, as well as the blocks over cells of that base, be PL pseudomanifolds with collared boundary — IP or Witt conditions are irrelevant for this argument.)

An orientation of $E(\xi)$ is induced by the given orientation data as follows: Triangulate $E(\xi)$ so that each block $\xi(\sigma)$ is a subcomplex. Let s be a top dimensional simplex of $E(\xi)$ in this triangulation. Then there is a unique block $\xi(\sigma_s)$ that contains s . This block is an oriented PL manifold with boundary (since ξ is oriented as a mock bundle), and this orientation induces an orientation of s . Note that $\sigma_s \in K$ is n -dimensional. The sum of all n -dimensional oriented cells in K is a cycle rel $|K_0|$, since $|K|$ is a Witt space, and thus in particular oriented. Then the preservation of incidence numbers between base cells and blocks implies that the sum of all s is a cycle rel $p^{-1}(|K_0|)$. Hence $E(\xi)$ is oriented as a pseudomanifold-with-boundary.

It remains to be shown that $E(\xi) - \partial E(\xi)$ satisfies the Witt condition. Let $x \in E(\xi) - \partial E(\xi)$ be a point in the interior of the total space. There is a unique $\sigma \in K$ for which x is in the interior of the block $\xi(\sigma)$. Note that then $p(x)$ lies in the interior of σ . Let $d = \dim \sigma$.

By the arguments used to prove [16, Lemma II.1.2] and [39, Prop. 6.6], there exists (inductively, using collars) a compact neighborhood N of x in $E(\xi)$, a compact neighborhood $V \cong D^{q+d}$ of x in the $(q + d)$ -dimensional manifold $\xi(\sigma)$, and a PL homeomorphism

$$N \cong V \times |D(\sigma)|.$$

Since

$$N \cong V \times |D(\sigma)| \cong D^{q+d} \times D^{j-k} \times \text{cone}(L) \cong D^{q+d+j-k} \times \text{cone}(L),$$

by a PL homeomorphism which sends x to $(0, c)$, where $c \in \text{cone}(L)$ denotes the cone vertex, we conclude that the intrinsic link at x in $E(\xi)$ is the intrinsic link L of σ at $p(x)$ in $|K|$. If this link has even dimension $2k$, then $IH_k^{\overline{m}}(L; \mathbb{Q}) = 0$ since $|K|$ is a Witt space. But then this condition is also satisfied for the intrinsic link at x in $E(\xi)$. Hence $E(\xi) - \partial E(\xi)$ is Witt. \square

If $|K|$ is a compact Witt space with boundary $\partial|K| = |K_0|$ for a subcomplex $K_0 \subset K$, and ξ is an oriented mock bundle over K which is empty over K_0 , then by Lemma 3.11,

$$\partial E(\xi) = p^{-1}(\partial|K|) = p^{-1}(|K_0|) = \emptyset,$$

i.e. $E(\xi)$ is a *closed* Witt space.

Let $L \subset K$ be a subcomplex. Oriented mock bundles ξ_0 and ξ_1 over K , both empty over L , are *cobordant*, if there is an oriented mock bundle η over $K \times I$, empty over $L \times I$, such that $\eta|_{K \times 0} \cong \xi_0$, $\eta|_{K \times 1} \cong \xi_1$. This is an equivalence relation and we set

$$\Omega_{\text{SPL}}^q(K, L) := \{[\xi^q/K] : \xi|_L = \emptyset\},$$

where $[\xi^q/K]$ denotes the cobordism class of the oriented q -mock bundle ξ^q/K over K . Then the duality theorem [16, Thm. II.3.3] asserts that $\Omega_{\text{SPL}}^{-*}(-)$ is Spanier-Whitehead dual to oriented PL bordism $\Omega_{*}^{\text{SPL}}(-) \cong \text{MSPL}_{*}(-)$; see also [16, Remark 3, top of p. 32]. But so is $\text{MSPL}^{*}(-)$. Hence Spanier-Whitehead duality provides an isomorphism

$$\beta : \Omega_{\text{SPL}}^{-q}(K, L) \cong \text{MSPL}^q(K, L) \tag{3.2}$$

for compact $|K|, |L|$, which is natural with respect to inclusions $(K', L') \subset (K, L)$. This is the geometric description of oriented PL cobordism that we will use. We shall now give an explicit description of the isomorphism β in (3.2). We write $X = |K|$ and $Y = |L|$ for the associated polyhedra, and assume them to be compact. Embed X into some sphere S^N so that we have inclusions $Y \subset X \subset S^N$. We write X^c, Y^c for the complements of X, Y in the sphere. We can regard X^c and Y^c also as compact polyhedra by removing the interior of derived neighborhoods of X and Y . Then, according to [16, Duality Theorem II.3.3], there is a natural isomorphism

$$\phi : \Omega_{\text{SPL}}^{-q}(X, Y) \xrightarrow{\cong} \Omega_{N-q}^{\text{SPL}}(Y^c, X^c).$$

The Thom-Pontrjagin construction gives a natural isomorphism

$$\tau : \Omega_{N-q}^{\text{SPL}}(Y^c, X^c) \xrightarrow{\cong} \text{MSPL}_{N-q}(Y^c, X^c),$$

and Alexander duality provides an isomorphism

$$\alpha : \text{MSPL}_{N-q}(Y^c, X^c) \xrightarrow{\cong} \text{MSPL}^q(X, Y),$$

which is natural with respect to inclusions. On the technical level, we work with $\alpha := (-1)^N \gamma_t$, where γ_t is Switzer's Alexander duality map [68, Thm. 14.11, p. 313]. This choice of sign guarantees that for the n -ball,

$$\alpha : \text{MSPL}_0(\overset{\circ}{D}^n) = \text{MSPL}_0(Y^c, X^c) \rightarrow \text{MSPL}^n(D^n, \partial D^n) = \text{MSPL}^n(X, Y)$$

sends the unit $1 \in \text{MSPL}_0(\text{pt}) = \pi_0(\text{MSPL}) = \text{MSPL}^0(\text{pt})$ to the element

$$\sigma^n \in \widetilde{\text{MSPL}}^n(S^n) = \text{MSPL}^n(D^n, \partial D^n),$$

obtained by suspending the unit n times. Then β in (3.2) is the composition

$$\Omega_{\text{SPL}}^{-q}(X, Y) \xrightarrow{\phi} \Omega_{N-q}^{\text{SPL}}(Y^c, X^c) \xrightarrow{\tau} \text{MSPL}_{N-q}(Y^c, X^c) \xrightarrow{\alpha} \text{MSPL}^q(X, Y).$$

Let us describe ϕ in more detail, following [16]: Let $N(X), N(Y)$ be derived neighborhoods of X, Y in S^N . Note that $N(X)$ and $N(Y)$ are manifolds with boundaries $\partial N(X), \partial N(Y)$. With $j : (X, Y) \hookrightarrow (NX, NY)$ the inclusion, pullback (restriction) of mock bundles defines a map

$$j^* : \Omega_{\text{SPL}}^{-q}(NX, NY) \longrightarrow \Omega_{\text{SPL}}^{-q}(X, Y),$$

which is an isomorphism. Amalgamation defines a map

$$\text{amal} : \Omega_{\text{SPL}}^{-q}(NX, NY) \longrightarrow \Omega_{n+q}^{\text{SPL}}(NX - NY, \partial NX - \partial NY),$$

which works as follows: Given a mock bundle over NX , the amalgamation, i.e. the union of all its blocks, i.e. the total space, is a manifold, since the blocks are manifolds and the base NX is a manifold as well (this is [16, Lemma 1.2, p. 21]). The projection gives a map of the amalgamation to NX . Furthermore, the boundary of the amalgamation is the material lying over ∂NX . Moreover, if the mock bundle is empty over NY , then the boundary of the amalgamation will not map to ∂NY . Thus we have a map as claimed. Finally the inclusion

$$j : (NX - NY, \partial NX - \partial NY) \hookrightarrow (Y^c, X^c)$$

induces a map

$$j_* : \Omega_{N-q}^{\text{SPL}}(NX - NY, \partial NX - \partial NY) \longrightarrow \Omega_{N-q}^{\text{SPL}}(Y^c, X^c).$$

Then ϕ is the composition

$$\begin{array}{ccc} \Omega_{\text{SPL}}^{-q}(X, Y) & \xleftarrow[\cong]{j^*} & \Omega_{\text{SPL}}^{-q}(NX, NY) \\ & \searrow \phi & \downarrow \text{amal} \\ & & \Omega_{N-q}^{\text{SPL}}(NX - NY, \partial NX - \partial NY) \\ & & \downarrow j_* \\ & & \Omega_{N-q}^{\text{SPL}}(Y^c, X^c). \end{array}$$

The following example illustrates the behavior of ϕ and will be used later.

Example 3.12. We consider the n -ball $X = D^n$ and its boundary sphere $Y = \partial D^n$. Take $N = n$ and embed D^n into $S^N = S^n$ as the upper hemisphere so that ∂D^n is embedded as the equatorial sphere. Then NY is a closed band containing the equator and NX is the union of this band with the upper hemisphere. The complement X^c is the open lower hemisphere and the complement Y^c is the disjoint union of open upper and lower hemisphere. Note that D^n may be regarded as the total space of a trivial block bundle ([52]) over a point. A block bundle always has a zero section, which

for the trivial block bundle over a point is the inclusion $i : \{0\} \hookrightarrow D^n$, where D^n is triangulated so that its center 0 is a vertex. (Then i is a simplicial inclusion.) The BRS-Thom class $u_{BRS}(\xi)$ of a block bundle ξ is explained further below, in (3.3). For $\xi = \epsilon^n$, the trivial n -block bundle over a point, it is given by

$$u_{BRS}(\epsilon^n) = [\{0\} \xrightarrow{i} D^n] \in \Omega_{\text{SPL}}^{-n}(D^n, \partial D^n).$$

Here, we interpret the inclusion $\{0\} \hookrightarrow D^n$ as the projection of a $(-n)$ -mock bundle over D^n with block $\{0\}$ over the cell D^n and empty blocks over all boundary cells of the polyhedron D^n . We shall compute the image under ϕ of this element $u_{BRS}(\epsilon^n)$. The center 0 includes into NX , so we have $\{0\} \hookrightarrow NX$. Again, we interpret this inclusion as a mock bundle over NX , so it defines an element $[\{0\} \hookrightarrow NX] \in \Omega_{\text{SPL}}^{-n}(NX, NY)$, as the blocks over the equatorial band NY are all empty. Induced mock bundles are given by pulling back under simplicial maps. If j is the (simplicial) inclusion $j : X \hookrightarrow NX$, then the pullback of the mock bundle $\{0\} \hookrightarrow NX$ is given by $j^*[\{0\} \hookrightarrow NX] = [\{0\} \hookrightarrow D^n]$. To compute the amalgamation of the mock bundle $\{0\} \hookrightarrow NX$ over NX , we observe that its total space consists of only one block (namely $\{0\}$), so there is nothing to amalgamate. Thus

$$\text{amal}[\{0\} \hookrightarrow NX] = [\{0\} \hookrightarrow NX - NY] \in \Omega_0^{\text{SPL}}(NX - NY, \partial NX - \partial NY).$$

(Note that $\{0\} \notin NY$.) Now the boundary ∂NY of the band NY consists of two disjoint circles, one in the upper hemisphere, the other in the lower hemisphere. The circle in the lower hemisphere is ∂NX . Therefore, $\partial NX - \partial NY = \emptyset$. In particular,

$$\Omega_0^{\text{SPL}}(NX - NY, \partial NX - \partial NY) = \Omega_0^{\text{SPL}}(NX - NY).$$

Since Y^c is the disjoint union of two open discs, and X^c is the lower one of these discs, we have by excision

$$\Omega_0^{\text{SPL}}(Y^c, X^c) = \Omega_0^{\text{SPL}}(D^{2n}) = \Omega_0^{\text{SPL}}(\{0\}),$$

where D^{2n} is the upper open disc, i.e. the one containing the point 0. Under this identification,

$$j_*[\{0\} \hookrightarrow NX - NY] = [\{0\} \xrightarrow{\text{id}} \{0\}] \in \Omega_0^{\text{SPL}}(\text{pt}).$$

We have shown that

$$\phi(u_{BRS}(\epsilon^n)) = [\{0\} \hookrightarrow Y^c] \in \Omega_0^{\text{SPL}}(Y^c, X^c).$$

Let I denote the unit interval. Recall that a PL (closed disc) q -block bundle ξ^q/K consists of a PL total space $E(\xi)$ and a ball complex K covering a polyhedron $|K|$ such that $|K| \subset E(\xi)$, for each n -ball σ in K , there is a (closed) PL $(n + q)$ -ball $\beta(\sigma) \subset E(\xi)$ (called the *block* over σ) and a PL homeomorphism of pairs

$$(\beta(\sigma), \sigma) \cong (I^{n+q}, I^n),$$

$E(\sigma)$ is the union of all blocks $\beta(\sigma)$, $\sigma \in K$, the interiors of the blocks are disjoint, and if L is the complex covering the polyhedron $\sigma_1 \cap \sigma_2$, then $\beta(\sigma_1) \cap \beta(\sigma_2)$ is the union of the blocks over cells in L . So a block bundle need not have a projection from the total space to the base, but it always has a canonical zero section $i : K \hookrightarrow E(\xi)$. The trivial q -block bundle has total space $E(\xi) = |K| \times I^q$ and blocks $\beta(\sigma) = \sigma \times I^q$ for each $\sigma \in K$. The cube I^q has boundary $\Sigma^{q-1} = \partial I^q$ and a block preserving PL homeomorphism $\Delta^n \times I^q \xrightarrow{\cong} \Delta^n \times I^q$ restricts to a block preserving PL homeomorphism $\Delta^n \times \Sigma^{q-1} \xrightarrow{\cong} \Delta^n \times \Sigma^{q-1}$, where Δ^n is the standard n -simplex. Hence there is a homomorphism $\widetilde{\text{PL}}_n(I) \rightarrow \widetilde{\text{PL}}_n(\Sigma)$ of semi-simplicial groups given by restriction, [53, p. 436]. On classifying spaces, this map induces $\text{B}\widetilde{\text{PL}}_n \rightarrow \text{B}\widetilde{\text{PL}}_n(\Sigma)$. Thus a closed disc block bundle ξ has a well-defined *sphere block bundle* $\dot{\xi}$, see [52, §5, p. 19f], whose total space \dot{E} is a PL subspace $\dot{E} \subset E(\xi)$ of the total space of ξ .

Now let $\xi : |K| \rightarrow \text{BSPL}_n$ be an oriented PL closed disc block bundle of rank n over a finite complex K . Then ξ has a Thom class as follows (cf. [16, p. 26]): Let $i : K \rightarrow E = E(\xi)$ be the zero section. Endow E with the ball complex structure given by taking the blocks $\beta(\sigma)$ of the bundle ξ as balls, together with the balls of a suitable ball complex structure on the total space \dot{E} of the sphere block bundle $\dot{\xi}$. Then $i : K \rightarrow E$ is the projection of an oriented $(-n)$ -mock bundle, and thus determines an element

$$u_{BRS}(\xi) := [i] \in \Omega_{\text{SPL}}^{-n}(E, \dot{E}), \tag{3.3}$$

which we shall call the *BRS-Thom class* of ξ . Note that if σ is a cell in \dot{E} , then $i^{-1}(\sigma) = \sigma \cap |K| = \emptyset$, so $[i]$ defines indeed a class rel \dot{E} . The BRS-Thom class is natural, [16, p. 27].

Let $\xi : |K| \rightarrow \text{BSPL}_n$ be an oriented PL $(\mathbb{R}^n, 0)$ -bundle. This bundle has a Thom class

$$u_{\text{SPL}}(\xi) \in \widetilde{\text{MSPL}}^n(\text{Th}(\xi)),$$

as discussed in Section 3.1. Composing with the map $\text{BSPL}_n \rightarrow \text{B}\widetilde{\text{SPL}}_n$, we get a map $\xi_{\text{PLB}} : |K| \rightarrow \text{B}\widetilde{\text{SPL}}_n$, which is the classifying map of the underlying oriented PL block bundle ξ_{PLB} of ξ .

Lemma 3.13. *For the trivial oriented PL $(\mathbb{R}^n, 0)$ -bundle ϵ^n over a point, the isomorphism (3.2),*

$$\beta : \Omega_{\text{SPL}}^{-n}(D^n, \partial D^n) \cong \text{MSPL}^n(D^n, \partial D^n),$$

maps the BRS-Thom class $u_{BRS}(\epsilon_{\text{PLB}}^n)$ to the Thom class $u_{\text{SPL}}(\epsilon^n)$.

Proof. The isomorphism β is the composition

$$\Omega_{\text{SPL}}^{-n}(D^n, \partial D^n) \xrightarrow{\phi} \Omega_0^{\text{SPL}}(Y^c, X^c) \xrightarrow{\tau} \text{MSPL}_0(Y^c, X^c) \xrightarrow{\alpha} \text{MSPL}^n(D^n, \partial D^n).$$

Note that the underlying PL block bundle ϵ^n_{PLB} of ϵ^n is the trivial block bundle over a point. Thus, by Example 3.12,

$$\phi(u_{\text{BRS}}(\epsilon^n_{\text{PLB}})) = [\{0\} \hookrightarrow Y^c] \in \Omega_0^{\text{SPL}}(Y^c, X^c)$$

and under the identification $\Omega_0^{\text{SPL}}(Y^c, X^c) = \Omega_0^{\text{SPL}}(D^{on}) = \Omega_0^{\text{SPL}}(\{0\})$, we have

$$\phi(u_{\text{BRS}}(\epsilon^n_{\text{PLB}})) = [\{0\} \xrightarrow{\text{id}} \{0\}] \in \Omega_0^{\text{SPL}}(\{0\}) = \Omega_0^{\text{SPL}}(\text{pt}).$$

The Thom-Pontrjagin construction τ sends the class $[\text{id}_{\{0\}}]$ to the unit $1 \in \text{MSPL}_0(\text{pt})$. Finally, the Alexander duality map α sends the unit $1 \in \text{MSPL}_0(\text{pt})$ to $\sigma^n \in \text{MSPL}^n(D^n, \partial D^n)$. So

$$\beta(u_{\text{BRS}}(\epsilon^n_{\text{PLB}})) = \alpha\tau\phi(u_{\text{BRS}}(\epsilon^n_{\text{PLB}})) = \alpha\tau[\text{id}_{\{0\}}] = \alpha(1) = \sigma^n.$$

Directly from the construction of u_{SPL} one sees that $u_{\text{SPL}}(\epsilon^n) = \sigma^n$ as well. \square

Lemma 3.14. *Let $\xi : |K| \rightarrow \text{BSPL}_n$ be an oriented PL $(\mathbb{R}^n, 0)$ -bundle, $|K|$ compact. Under the isomorphism β in (3.2), the BRS-Thom class $u_{\text{BRS}}(\xi_{\text{PLB}})$ of the underlying oriented PL block bundle gets mapped to the Thom class $u_{\text{SPL}}(\xi)$.*

Proof. We write $X = |K|$ for the compact polyhedron of K . Let $x \in X$ be a point. The bundle ξ has a projection $p : E \rightarrow X$ and we can speak of the fiber $E_x = p^{-1}(x) \cong \mathbb{R}^n$ over x . Let $E_0 \subset E$ be the complement of the zero section and let $E_{0x} = E_x \cap E_0 \cong \mathbb{R}^n - \{0\}$. Let E' denote the total space of the block bundle ξ_{PLB} , and \dot{E}' the total space of the sphere block bundle of ξ_{PLB} . We may identify $\text{MSPL}_{\text{SPL}}^n(E', \dot{E}') \cong \text{MSPL}_{\text{SPL}}^n(E, E_0)$, since E'/\dot{E}' and $\text{Th}(\xi_{\text{PLB}}) = \text{Th}(\xi^\bullet) = \text{Th}(\xi)$ are naturally homotopy equivalent. Let $\xi_{\text{PLB}}|_{\{x\}}$ denote the restriction of ξ_{PLB} to $\{x\}$, where we subdivide K so that x becomes a vertex, if necessary. Let E'_x denote the total space of $\xi_{\text{PLB}}|_{\{x\}}$, and \dot{E}'_x the total space of the sphere block bundle of $\xi_{\text{PLB}}|_{\{x\}}$. The inclusions

$$(E'_x, \dot{E}'_x) \hookrightarrow (E', \dot{E}'), (E_x, E_{0x}) \hookrightarrow (E, E_0)$$

will be denoted by j_x . By naturality of the isomorphism β with respect to inclusions of pairs, the diagram

$$\begin{CD} \Omega_{\text{SPL}}^{-n}(E', \dot{E}') @>{\cong}_{\beta}>> \text{MSPL}^n(E', \dot{E}') @= \text{MSPL}^n(E, E_0) \\ @V{j_x^*}VV @VV{j_x^*}V @VV{j_x^*}V \\ \Omega_{\text{SPL}}^{-n}(E'_x, \dot{E}'_x) @>{\cong}_{\beta}>> \text{MSPL}^n(E'_x, \dot{E}'_x) @= \text{MSPL}^n(E_x, E_{0x}) \end{CD}$$

commutes. As X is compact, it has finitely many path components X_1, \dots, X_m . For every $i = 1, \dots, m$, choose a point $x_i \in X_i$. We shall compute the fiber restrictions of our two classes to these points. Let $x \in \{x_1, \dots, x_m\}$. Directly from the construction of u_{SPL} , we have $j_x^*u_{\text{SPL}}(\xi) = \sigma^n \in \widehat{\text{MSPL}}^n(S^n) \cong \widehat{\text{MSPL}}^0(S^0)$. In particular, $u_{\text{SPL}}(\xi)$ is an orientation

for ξ (and ξ^\bullet) in Dold’s sense. For $\beta(u_{BRS}(\xi_{PLB}))$ we have, using the above commutative diagram, the naturality of both the BRS-Thom class and u_{SPL} , and Lemma 3.13,

$$\begin{aligned} j_x^*(\beta(u_{BRS}(\xi_{PLB}))) &= \beta(j_x^*u_{BRS}(\xi_{PLB})) = \beta(u_{BRS}(\xi_{PLB}|_{\{x\}})) \\ &= u_{SPL}(\xi|_{\{x\}}) = j_x^*u_{SPL}(\xi). \end{aligned}$$

This shows that $\beta(u_{BRS}(\xi_{PLB}))$ is also an orientation for ξ . Since MSPL is a connected spectrum, an orientation u in $MSPL^n(E, E_0)$ for ξ is uniquely determined by $j_x^*(u)$, $x \in \{x_1, \dots, x_m\}$ ([68, 14.8, p. 311]). The above calculation shows that the MSPL-orientations $u_{SPL}(\xi)$ and $\beta(u_{BRS}(\xi_{PLB}))$ have the same restrictions under the j_x^* and thus $u_{SPL}(\xi) = \beta(u_{BRS}(\xi_{PLB}))$. □

3.4. Witt bordism and cap products. Recall that we had the map $\phi_{LR} : MSO \rightarrow MSPL$ due to Lashof-Rothenberg. Let MWITT be the spectrum representing Witt-bordism $\Omega_*^{Witt}(-)$, considered explicitly first in [28]. Curran proves in [28, Thm. 3.6, p. 117] that MWITT is an MSO-module spectrum. It is even an MSPL-module spectrum because the product of a Witt space and an oriented PL manifold is again a Witt space. (Further remarks on the structure of MWITT will be made in Section 3.7 below.) Thus there is a cap product

$$\cap : MSPL^c(X, A) \otimes MWITT_n(X, A) \longrightarrow MWITT_{n-c}(X).$$

By Buoncristiano-Rourke-Sanderson, a geometric description of this cap product is given as follows: One uses the isomorphism (3.2) to think of the cap product as a product

$$\cap : \Omega_{SPL}^{-c}(K, L) \otimes \Omega_n^{Witt}(|K|, |L|) \longrightarrow \Omega_{n-c}^{Witt}(|K|)$$

for finite ball complexes K with subcomplex $L \subset K$. Let us first discuss the absolute case $L = \emptyset$, and then return to the relative one. If $f : Z \rightarrow |K|$ is a continuous map from an n -dimensional closed Witt space Z to $|K|$, and ξ^q is a q -mock bundle over K (with $q = -c$), then one defines (cf. [16, p. 29])

$$[\xi^q/K] \cap [f : Z \rightarrow |K|] := [h : E(f^*\xi) \rightarrow |K|] \in \Omega_{n-c}^{Witt}(|K|),$$

where h is the diagonal arrow in the cartesian diagram

$$\begin{array}{ccc} E(f^*\xi) & \longrightarrow & E(\xi) \\ \downarrow & \searrow h & \downarrow p \\ Z & \xrightarrow{f'} & K. \end{array}$$

Here, we subdivide simplicially, homotope f to a simplicial map f' , and use the fact ([16, II.2, p. 23f]) that mock bundles admit pullbacks under simplicial maps. By Lemma 3.11, $E(f^*\xi)$ is a closed Witt space. For the relative case, we observe that if $(Z, \partial Z)$ is a compact Witt space with boundary, $f : (Z, \partial Z) \rightarrow (|K|, |L|)$ maps the boundary into $|L|$, and $\xi|_L = \emptyset$, then

$f^*\xi|_{\partial Z} = \emptyset$ and so $\partial E(f^*\xi) = \emptyset$, i.e. the Witt space $E(f^*\xi)$ is *closed*. Hence it defines an absolute bordism class.

3.5. The Gysin map on Witt bordism. For a (real) codimension c normally nonsingular inclusion $g : Y^{n-c} \hookrightarrow X^n$ of closed oriented PL pseudomanifolds, we define a Gysin map

$$g^! : \Omega_k^{\text{Witt}}(X) \longrightarrow \Omega_{k-c}^{\text{Witt}}(Y),$$

and we shall prove that it sends the Witt orientation of X to the Witt orientation of Y , if X and Y are Witt spaces. This will then be applied in proving the analogous statement for the \mathbb{L}^\bullet -homology orientations.

Let ν be the normal bundle of the embedding g . By definition of normal nonsingularity, ν is a vector bundle over Y , and it is canonically oriented since X and Y are oriented. Thus ν is classified by a continuous map $\nu : Y \rightarrow \text{BSO}_c$. As explained in Section 3.1, ν determines an oriented PL $(\mathbb{R}^c, 0)$ -bundle ν_{PL} , an oriented PL (closed disc) block bundle ν_{PLB} , and an oriented topological $(\mathbb{R}^c, 0)$ -bundle ν_{TOP} . Let $E = E(\nu_{\text{PLB}})$ denote the total space of the PL block bundle ν_{PLB} . Then E is a compact PL pseudomanifold with boundary $\partial E = \dot{E} = \dot{E}(\nu_{\text{PLB}})$. (This uses that Y is closed.) The Thom space $\text{Th}(\nu)$ of ν is homotopy equivalent to the PL space $\text{Th}'(\nu_{\text{PLB}}) := E \cup_{\dot{E}} \text{cone } \dot{E}$. The standard map $j : X \rightarrow \text{Th}'(\nu_{\text{PLB}})$ is the identity on an open tubular neighborhood of Y in X and sends points farther away from Y to the cone point $\infty \in \text{Th}'(\nu_{\text{PLB}})$. As in Ranicki [49, p. 186], this map extends to a map $j : X_+ \rightarrow \text{Th}'(\nu_{\text{PLB}})$ by sending the additional disjoint point to ∞ . By Lashof-Rothenberg triangulation, we can and will assume that j is simplicial. This map induces a homomorphism

$$j_* : \Omega_k^{\text{Witt}}(X) = \Omega_k^{\text{Witt}}(X_+, \text{pt}) \longrightarrow \Omega_k^{\text{Witt}}(\text{Th}'(\nu_{\text{PLB}}), \infty) \cong \Omega_k^{\text{Witt}}(E, \dot{E}).$$

Recall from Section 3.4 that we had a cap product

$$\cap : \Omega_{\text{SPL}}^{-c}(E, \dot{E}) \otimes \Omega_k^{\text{Witt}}(E, \dot{E}) \longrightarrow \Omega_{k-c}^{\text{Witt}}(E),$$

which we had described geometrically. Capping with the BRS-Thom class

$$u_{\text{BRS}}(\nu_{\text{PLB}}) \in \Omega_{\text{SPL}}^{-c}(E, \dot{E}),$$

we get a map

$$u_{\text{BRS}}(\nu_{\text{PLB}}) \cap - : \Omega_k^{\text{Witt}}(E, \dot{E}) \longrightarrow \Omega_{k-c}^{\text{Witt}}(E).$$

Composing this with the above map j_* , we get the *Witt-bordism Gysin map*

$$g^! := (u_{\text{BRS}}(\nu_{\text{PLB}}) \cap -) \circ j_* :$$

$$\Omega_k^{\text{Witt}}(X) \longrightarrow \Omega_k^{\text{Witt}}(E, \dot{E}) \longrightarrow \Omega_{k-c}^{\text{Witt}}(E) \cong \Omega_{k-c}^{\text{Witt}}(Y),$$

where the last isomorphism is the inverse of the isomorphism induced by the zero section. A closed n -dimensional Witt space X^n has a canonical *Witt-bordism fundamental class*

$$[X]_{\text{Witt}} := [\text{id} : X \rightarrow X] \in \Omega_n^{\text{Witt}}(X).$$

Theorem 3.15. *The Witt-bordism Gysin map $g^!$ of a (real) codimension c normally nonsingular inclusion $g : Y^{n-c} \hookrightarrow X^n$ of closed (oriented) Witt spaces sends the Witt-bordism fundamental class of X to the Witt-bordism fundamental class of Y :*

$$g^![X]_{\text{Witt}} = [Y]_{\text{Witt}}.$$

Proof. The image of $[\text{id} : X \rightarrow X]$ under j_* is

$$[j : X \rightarrow \text{Th}'(\nu_{\text{PLB}})] \in \Omega_n^{\text{Witt}}(\text{Th}'(\nu_{\text{PLB}}), \infty) \cong \Omega_n^{\text{Witt}}(E, \dot{E}).$$

The BRS-Thom class of ν_{PLB} is given by the class $[i : Y \rightarrow E]$ of the zero-section. Under the identification $\Omega_{\text{SPL}}^{-c}(E, \dot{E}) \cong \Omega_{\text{SPL}}^{-c}(\text{Th}'(\nu_{\text{PLB}}), \infty)$, it is represented by composing i with the inclusion $E \rightarrow \text{Th}'(\nu_{\text{PLB}})$. We call the resulting map again $i : Y \rightarrow \text{Th}'(\nu_{\text{PLB}})$; it is a $(-c)$ -mock bundle projection, where $\text{Th}'(\nu_{\text{PLB}})$ is equipped with a ball complex structure which contains ∞ as a zero dimensional ball. Since Y does not touch ∞ , this mock bundle is empty over the ball ∞ . The cap product

$$[i : Y \rightarrow \text{Th}'(\nu_{\text{PLB}})] \cap [j : X \rightarrow \text{Th}'(\nu_{\text{PLB}})],$$

is given by $[h]$, where h is the diagonal arrow in the cartesian diagram

$$\begin{array}{ccc} E(j^*(i)) & \longrightarrow & Y \\ \downarrow & \searrow h & \downarrow i \\ X & \xrightarrow{j} & \text{Th}'(\nu_{\text{PLB}}). \end{array}$$

The pullback $E(j^*(i))$ is just Y and the above diagram is

$$\begin{array}{ccc} Y & \xrightarrow{\text{id}} & Y \\ \downarrow g & \searrow h & \downarrow i \\ X & \xrightarrow{j} & \text{Th}'(\nu_{\text{PLB}}). \end{array}$$

(Recall that j is the identity in a tubular neighborhood of Y ; the points of X that are mapped under j to the zero section are precisely the points of Y .) So

$$[i : Y \rightarrow \text{Th}'(\nu_{\text{PLB}})] \cap [j : X \rightarrow \text{Th}'(\nu_{\text{PLB}})] = [h] = [i].$$

Now under the isomorphism

$$i_* : \Omega_{n-c}^{\text{Witt}}(Y) \xrightarrow{\cong} \Omega_{n-c}^{\text{Witt}}(E),$$

the Witt-bordism fundamental class $[\text{id} : Y \rightarrow Y]$ is sent to $[i]$. □

3.6. The Gysin map on \mathbb{L}^\bullet -homology. We continue in the context of Section 3.5. Thus $g : Y^{n-c} \hookrightarrow X^n$ is a normally nonsingular inclusion of closed Witt spaces with normal vector bundle ν . The canonical map $j : X_+ \rightarrow \text{Th}(\nu)$ induces a homomorphism

$$j_* : \mathbb{L}^\bullet(\mathbb{Q})_*(X) \longrightarrow \widetilde{\mathbb{L}}^\bullet(\mathbb{Q})_*(\text{Th}(\nu)).$$

As discussed in Section 3.2, the oriented topological $(\mathbb{R}^c, 0)$ -bundle ν_{TOP} determined by ν has an \mathbb{L}^\bullet -cohomology Thom class

$$u_{\mathbb{L}}(\nu_{\text{TOP}}) \in \widetilde{\mathbb{L}}^\bullet(\mathbb{Q})^c(\text{Th}(\nu_{\text{TOP}})),$$

defined by

$$u_{\mathbb{L}}(\nu_{\text{TOP}}) = \epsilon_{\mathbb{Q}} \sigma^*(u_{\text{STOP}}(\nu_{\text{TOP}})),$$

where $\epsilon_{\mathbb{Q}}$ is induced by $\mathbb{L}^\bullet(\mathbb{Z}) \rightarrow \mathbb{L}^\bullet(\mathbb{Q})$. Capping with this class, we receive a map

$$u_{\mathbb{L}}(\nu_{\text{TOP}}) \cap - : \widetilde{\mathbb{L}}^\bullet(\mathbb{Q})_k(\text{Th}(\nu)) \longrightarrow \mathbb{L}^\bullet(\mathbb{Q})_{k-c}(Y).$$

Composing this with the above map j_* on $\mathbb{L}^\bullet(\mathbb{Q})$ -homology, we get the \mathbb{L}^\bullet -homology Gysin map

$$g^! := (u_{\mathbb{L}}(\nu_{\text{TOP}}) \cap -) \circ j_* : \mathbb{L}^\bullet(\mathbb{Q})_k(X) \longrightarrow \mathbb{L}^\bullet(\mathbb{Q})_k(\text{Th}(\nu)) \longrightarrow \mathbb{L}^\bullet(\mathbb{Q})_{k-c}(Y).$$

(Of course this map can be defined over \mathbb{L}^\bullet , but we only need it over $\mathbb{L}^\bullet(\mathbb{Q})$.)

3.7. Relation between Witt- and \mathbb{L}^\bullet -Gysin maps. The spectra $\mathbb{L}^\bullet(\mathbb{Z})$ and $\mathbb{L}^\bullet(\mathbb{Q})$ are ring spectra. The product of two \mathbb{Q} -Witt spaces is again a \mathbb{Q} -Witt space. This implies essentially that MWITT is a ring spectrum; for more details see [8]. There, we constructed a map

$$\tau : \text{MWITT} \longrightarrow \mathbb{L}^\bullet(\mathbb{Q}).$$

(Actually, we even constructed an integral map $\text{MIP} \rightarrow \mathbb{L}^\bullet$, where MIP represents bordism of integral intersection homology Poincaré spaces studied in [33] and [47], but everything works in the same manner for Witt, if one uses the \mathbb{L}^\bullet -spectrum with rational coefficients.) This map is multiplicative, i.e. a ring map, as shown in [8, Section 12]. Using this map τ , a closed Witt space X^n has a canonical $\mathbb{L}^\bullet(\mathbb{Q})$ -homology fundamental class

$$[X]_{\mathbb{L}} \in \mathbb{L}^\bullet(\mathbb{Q})_n(X),$$

which is by definition the image of $[X]_{\text{Witt}}$ under the map

$$\tau_* : \Omega_n^{\text{Witt}}(X) = \text{MWITT}_n(X) \longrightarrow \mathbb{L}^\bullet(\mathbb{Q})_n(X),$$

i.e.

$$[X]_{\mathbb{L}} := \tau_*([X]_{\text{Witt}}).$$

Every oriented PL manifold is a Witt space. Hence there is a map

$$\phi_W : \text{MSPL} \longrightarrow \text{MWITT},$$

which, using the methods of ad-theories and Quinn spectra employed in [8], can be constructed to be multiplicative, i.e. a map of ring spectra. By the construction of τ in [8], the diagram

$$\begin{array}{ccc}
 & \text{MSTOP} \xrightarrow{\sigma^*} & \mathbb{L}^\bullet(\mathbb{Z}) \\
 \nearrow \phi_F & & \downarrow \epsilon_{\mathbb{Q}} \\
 \text{MSPL} & & \\
 \searrow \phi_W & & \\
 & \text{MWITT} \xrightarrow{\tau} & \mathbb{L}^\bullet(\mathbb{Q})
 \end{array} \tag{3.4}$$

homotopy commutes. In the proof of Theorem 3.17 below, we shall use the following standard fact:

Lemma 3.16. *If E is a ring spectrum, F, F' module spectra over E and $\phi : F \rightarrow F'$ an E -module morphism, then the diagram*

$$\begin{array}{ccc}
 E^c(X, A) \otimes F_n(X, A) & \xrightarrow{\cap} & F_{n-c}(X) \\
 \text{id} \otimes \phi_* \downarrow & & \downarrow \phi_* \\
 E^c(X, A) \otimes F'_n(X, A) & \xrightarrow{\cap} & F'_{n-c}(X)
 \end{array}$$

commutes: if $u \in E^c(X, A)$, and $a \in F_n(X, A)$, then

$$\phi_*(u \cap a) = u \cap \phi_*(a).$$

Theorem 3.17. *The \mathbb{L}^\bullet -homology Gysin map $g^!$ of a (real) codimension c normally nonsingular inclusion $g : Y^{n-c} \hookrightarrow X^n$ of closed (oriented) Witt spaces sends the $\mathbb{L}^\bullet(\mathbb{Q})$ -homology fundamental class of X to the $\mathbb{L}^\bullet(\mathbb{Q})$ -homology fundamental class of Y :*

$$g^![X]_{\mathbb{L}} = [Y]_{\mathbb{L}}.$$

Proof. Let ν be the topological normal vector bundle of g . The diagram

$$\begin{array}{ccc}
 \Omega_n^{\text{Witt}}(X) & \xrightarrow{\tau_*} & \mathbb{L}^\bullet(\mathbb{Q})_n(X) \\
 j_* \downarrow & & \downarrow j_* \\
 \Omega_n^{\text{Witt}}(\text{Th}(\nu), \infty) & \xrightarrow{\tau_*} & \mathbb{L}^\bullet(\mathbb{Q})_n(\text{Th}(\nu), \infty)
 \end{array}$$

commutes, since τ_* is a natural transformation of homology theories. We shall prove next that the diagram

$$\begin{array}{ccc}
 \Omega_n^{\text{Witt}}(\text{Th}(\nu), \infty) & \xrightarrow{\tau_*} & \mathbb{L}^\bullet(\mathbb{Q})_n(\text{Th}(\nu), \infty) \\
 u_{\text{BRS}}(\nu_{\text{PLB}}) \cap - \downarrow & & \downarrow u_{\mathbb{L}}(\nu_{\text{TOP}}) \cap - \\
 \Omega_{n-c}^{\text{Witt}}(Y) & \xrightarrow{\tau_*} & \mathbb{L}^\bullet(\mathbb{Q})_{n-c}(Y),
 \end{array} \tag{3.5}$$

commutes as well. Let $a \in \Omega_n^{\text{Witt}}(\text{Th}(\nu), \infty)$ be an element. According to the definition of the \mathbb{L}^\bullet -cohomology Thom class, we have

$$u_{\mathbb{L}}(\nu_{\text{TOP}}) \cap \tau_*(a) = \epsilon_{\mathbb{Q}}\sigma^*(u_{\text{STOP}}(\nu_{\text{TOP}})) \cap \tau_*(a).$$

By Lemma 3.7,

$$\epsilon_{\mathbb{Q}}\sigma^*(u_{\text{STOP}}(\nu_{\text{TOP}})) \cap \tau_*(a) = \epsilon_{\mathbb{Q}}\sigma^*\phi_F(u_{\text{SPL}}(\nu_{\text{PL}})) \cap \tau_*(a).$$

Using diagram (3.4),

$$\epsilon_{\mathbb{Q}}\sigma^*\phi_F(u_{\text{SPL}}(\nu_{\text{PL}})) \cap \tau_*(a) = \tau\phi_W(u_{\text{SPL}}(\nu_{\text{PL}})) \cap \tau_*(a).$$

Here, the symbol \cap denotes the cap-product on \mathbb{L}^\bullet -(co)homology. Using the ring map $\phi_W : \text{MSPL} \rightarrow \text{MWITT}$, the spectrum MWITT becomes an MSPL -module with action map

$$\text{MSPL} \wedge \text{MWITT} \longrightarrow \text{MWITT}$$

given by the composition

$$\text{MSPL} \wedge \text{MWITT} \xrightarrow{\phi_W \wedge \text{id}} \text{MWITT} \wedge \text{MWITT} \longrightarrow \text{MWITT}.$$

Using the ring map $\tau\phi_W : \text{MSPL} \rightarrow \mathbb{L}^\bullet(\mathbb{Q})$, the spectrum $\mathbb{L}^\bullet(\mathbb{Q})$ becomes an MSPL -module with action map

$$\text{MSPL} \wedge \mathbb{L}^\bullet(\mathbb{Q}) \longrightarrow \mathbb{L}^\bullet(\mathbb{Q})$$

given by the composition

$$\text{MSPL} \wedge \mathbb{L}^\bullet(\mathbb{Q}) \xrightarrow{(\tau\phi_W) \wedge \text{id}} \mathbb{L}^\bullet(\mathbb{Q}) \wedge \mathbb{L}^\bullet(\mathbb{Q}) \longrightarrow \mathbb{L}^\bullet(\mathbb{Q}).$$

Hence

$$\tau\phi_W(u_{\text{SPL}}(\nu_{\text{PL}})) \cap \tau_*(a) = u_{\text{SPL}}(\nu_{\text{PL}}) \cap \tau_*(a),$$

where \cap on the left hand side denotes the \mathbb{L}^\bullet -internal cap-product, whereas \cap on the right hand side denotes the cap-product coming from the above structure of $\mathbb{L}^\bullet(\mathbb{Q})$ as an MSPL -module. The homotopy commutative diagram

$$\begin{array}{ccc} \text{MSPL} \wedge \text{MWITT} & \xrightarrow{\text{id} \wedge \tau} & \text{MSPL} \wedge \mathbb{L}^\bullet(\mathbb{Q}) \\ \downarrow \phi_W \wedge \text{id} & & \downarrow (\tau\phi_W) \wedge \text{id} \\ \text{MWITT} \wedge \text{MWITT} & \xrightarrow{\tau \wedge \tau} & \mathbb{L}^\bullet(\mathbb{Q}) \wedge \mathbb{L}^\bullet(\mathbb{Q}) \\ \downarrow & & \downarrow \\ \text{MWITT} & \xrightarrow{\tau} & \mathbb{L}^\bullet(\mathbb{Q}) \end{array}$$

shows that $\tau : \text{MWITT} \rightarrow \mathbb{L}^\bullet(\mathbb{Q})$ is an MSPL-module morphism. Thus by Lemma 3.16,

$$\begin{array}{ccc} \text{MSPL}^c(\text{Th}(\nu), \infty) \otimes \text{MWITT}_n(\text{Th}(\nu), \infty) & \xrightarrow{\cap} & \text{MWITT}_{n-c}(Y) \\ \text{id} \otimes \tau_* \downarrow & & \downarrow \tau_* \\ \text{MSPL}^c(\text{Th}(\nu), \infty) \otimes \mathbb{L}^\bullet(\mathbb{Q})_n(\text{Th}(\nu), \infty) & \xrightarrow{\cap} & \mathbb{L}^\bullet(\mathbb{Q})_{n-c}(Y) \end{array}$$

commutes, so that

$$u_{\text{SPL}}(\nu_{\text{PL}}) \cap \tau_*(a) = \tau_*(u_{\text{SPL}}(\nu_{\text{PL}}) \cap a).$$

By Lemma 3.14, the canonical isomorphism (3.2) identifies the Thom class $u_{\text{SPL}}(\nu_{\text{PL}})$ with the BRS-Thom class $u_{\text{BRS}}(\nu_{\text{PLB}})$. Therefore,

$$\tau_*(u_{\text{SPL}}(\nu_{\text{PL}}) \cap a) = \tau_*(u_{\text{BRS}}(\nu_{\text{PLB}}) \cap a).$$

Altogether then,

$$u_{\mathbb{L}}(\nu_{\text{TOP}}) \cap \tau_*(a) = \tau_*(u_{\text{BRS}}(\nu_{\text{PLB}}) \cap a),$$

which shows that the diagram (3.5) commutes as claimed. We have shown that the diagram

$$\begin{array}{ccc} \Omega_n^{\text{Witt}}(X) & \xrightarrow{\tau_*} & \mathbb{L}^\bullet(\mathbb{Q})_n(X) \\ j_* \downarrow & & \downarrow j_* \\ \Omega_n^{\text{Witt}}(\text{Th}(\nu), \infty) & \xrightarrow{\tau_*} & \mathbb{L}^\bullet(\mathbb{Q})_n(\text{Th}(\nu), \infty) \\ u_{\text{BRS}}(\nu_{\text{PLB}}) \cap - \downarrow & & \downarrow u_{\mathbb{L}}(\nu_{\text{TOP}}) \cap - \\ \Omega_{n-c}^{\text{Witt}}(Y) & \xrightarrow{\tau_*} & \mathbb{L}^\bullet(\mathbb{Q})_{n-c}(Y), \end{array}$$

commutes. Thus the diagram of Gysin maps

$$\begin{array}{ccc} \Omega_n^{\text{Witt}}(X) & \xrightarrow{\tau_*} & \mathbb{L}^\bullet(\mathbb{Q})_n(X) \\ g^! \downarrow & & \downarrow g^! \\ \Omega_{n-c}^{\text{Witt}}(Y) & \xrightarrow{\tau_*} & \mathbb{L}^\bullet(\mathbb{Q})_{n-c}(Y) \end{array}$$

commutes. Using Theorem 3.15, it follows that

$$g^! [X]_{\mathbb{L}} = g^! \tau_* [X]_{\text{Witt}} = \tau_* g^! [X]_{\text{Witt}} = \tau_* [Y]_{\text{Witt}} = [Y]_{\mathbb{L}}.$$

□

Theorem 3.18. *Let $g : Y \hookrightarrow X$ be a normally nonsingular inclusion of closed oriented even-dimensional PL Witt pseudomanifolds. Let ν be the topological normal bundle of g . Then*

$$g^! L_*(X) = L^*(\nu) \cap L_*(Y).$$

Proof. By Theorem 3.17, the \mathbb{L}^\bullet -homology Gysin map $g^!$ of g sends the $\mathbb{L}^\bullet(\mathbb{Q})$ -homology fundamental class of X to the $\mathbb{L}^\bullet(\mathbb{Q})$ -homology fundamental class of Y : $g^![X]_{\mathbb{L}} = [Y]_{\mathbb{L}}$. It remains to analyze what this equation means after we tensor with \mathbb{Q} . By [8, Lemma 11.1],

$$[X]_{\mathbb{L}} \otimes \mathbb{Q} = L_*(X), \quad [Y]_{\mathbb{L}} \otimes \mathbb{Q} = L_*(Y).$$

Furthermore, according to [49, Remark 16.2, p. 176], $u_{\mathbb{L}}(\nu) \otimes \mathbb{Q} = L^*(\nu)^{-1} \cup u_{\mathbb{Q}}$, where $u_{\mathbb{Q}} \in \tilde{H}^c(\text{Th}(\nu); \mathbb{Q})$ is the Thom class of ν in ordinary rational cohomology. (Note that Ranicki omits cupping with $u_{\mathbb{Q}}$ in his notation.) Thus

$$\begin{aligned} L_*(Y) &= [Y]_{\mathbb{L}} \otimes \mathbb{Q} = (g^![X]_{\mathbb{L}}) \otimes \mathbb{Q} = (j_*[X]_{\mathbb{L}} \cap u_{\mathbb{L}}(\nu)) \otimes \mathbb{Q} \\ &= j_*([X]_{\mathbb{L}} \otimes \mathbb{Q}) \cap (u_{\mathbb{L}}(\nu) \otimes \mathbb{Q}) = j_*L_*(X) \cap (L^*(\nu)^{-1} \cup u_{\mathbb{Q}}) \\ &= j_*L_*(X) \cap (u_{\mathbb{Q}} \cup L^*(\nu)^{-1}) = (j_*L_*(X) \cap u_{\mathbb{Q}}) \cap L^*(\nu)^{-1} \\ &= (g^!L_*(X)) \cap L^*(\nu)^{-1}. \end{aligned}$$

(Note that all involved classes lie in even degrees and hence no signs come in.) □

Example 3.19. For the top L -class, Theorem 3.18 implies ($n = \dim X$, $m = \dim Y$)

$$\begin{aligned} g^![X] &= g^!L_n(X) = (L^*(\nu) \cap L_*(Y))_m \\ &= ((1 + L^1(\nu) + \dots) \cap (L_m(Y) + L_{m-4}(Y) + \dots))_m \\ &= ((1 + L^1(\nu) + \dots) \cap ([Y] + L_{m-4}(Y) + \dots))_m \\ &= 1 \cap [Y] = [Y], \end{aligned}$$

i.e. Gysin maps fundamental classes to fundamental classes.

4. Application: The L -class of singular Schubert varieties

As an application of the L -class Gysin Theorem 3.18, we compute explicitly the Goresky-MacPherson L -class of some singular Schubert varieties. The Chern-Schwartz-MacPherson classes of Schubert varieties were computed by Aluffi and Mihalcea in [2]. We are not presently aware of any existing computation of L -classes for such varieties in the literature. We shall follow the notation of [2], and work over the complex number field \mathbb{C} . For $n \geq k$, let $G_k(\mathbb{C}^n)$ denote the Grassmann variety of k -dimensional linear subspaces of \mathbb{C}^n . Let F_* be a complete flag of subspaces of \mathbb{C}^n ,

$$F_0 = \{0\} \subset F_1 \subset F_2 \subset \dots \subset F_n = \mathbb{C}^n,$$

$\dim F_j = j$, and let d be a nonnegative integer. A *partition* a of d is a non-increasing sequence of nonnegative integers $a = (a_1 \geq a_2 \geq \dots \geq a_k)$ such that $\sum a_i = d$. The *Schubert variety* X_a of a partition a is the subvariety $X_a \subset G_k(\mathbb{C}^n)$ given by

$$X_a = \{P \in G_k(\mathbb{C}^n) \mid \dim_{\mathbb{C}}(P \cap F_{a_{k+1-i}+i}) \geq i\}.$$

Its dimension is $\dim_{\mathbb{C}} X_a = \sum a_i$. In our notation, we will often omit trailing zeros, e.g. we may write X_1 for $X_{1,0}$. If $b = (b_1 \geq \cdots \geq b_k)$ is another partition (not necessarily of d , but possibly of a different integer), then we write $b \leq a$ if and only if $b_i \leq a_i$ for all i . If $b \leq a$, then there is a closed embedding $X_b \subset X_a$. The Chow homology $A_*(X_a)$ of X_a is freely generated by the Schubert classes $[X_b]$ for all $b \leq a$, and the cycle map from Chow to Borel-Moore homology is an isomorphism, as Schubert varieties possess cellular decompositions. If Y is a subvariety of a variety X , we shall denote its fundamental class in the homology of X by $[Y]_X$. For example, if $X_b \subset X = X_a \subset G = G_k(\mathbb{C}^n)$, then X_b determines a class $[X_b]_X$ in the homology of X_a and a class $[X_b]_G$ in the homology of G . Most Schubert varieties are singular and do in fact not satisfy (local or even just global) Poincaré duality. The singular set of a Schubert variety can be computed efficiently using a result of Lakshmibai-Weyman [37, Theorem 5.3, p. 203].

Let us compute the L -class

$$L_2(X_{2,1}) \in H_2(X_{2,1}; \mathbb{Q}) = \mathbb{Q}[X_1]$$

of the 3-dimensional Schubert variety $X_{2,1}$, a hypersurface of the Grassmannian $G := G_2(\mathbb{C}^4) = X_{2,2}$, $\dim G = 4$. Since the homology in degree 2 is one-dimensional, generated by the fundamental class of the Schubert variety $X_1 \subset X_{2,1}$, there exists a unique $\alpha \in \mathbb{Q}$ with $L_2(X_{2,1}) = \alpha \cdot [X_1]$. We must determine this coefficient α . Under the Plücker embedding, G is realized as a nonsingular quadric hypersurface in \mathbb{P}^5 . Composing, we obtain a closed embedding of $X_{2,1}$ in \mathbb{P}^5 . With $(x_0 : \cdots : x_5)$ homogeneous coordinates on \mathbb{P}^5 , $X_{2,1}$ is contained in the hyperplane $\mathbb{P}^4 = \{x_5 = 0\} \subset \mathbb{P}^5$. Then $X_{2,1} \subset \mathbb{P}^4$ is the quadric hypersurface given by

$$X_{2,1} = \{x_2x_3 - x_1x_4 = 0\}$$

with singular set $\{(1 : 0 : 0 : 0 : 0)\}$, a point. Let H be the hyperplane

$$H := \{x_0 - x_1 = 0\} \subset \mathbb{P}^4.$$

The salient feature is that H is transverse to $X_{2,1}$ and to X_1 in \mathbb{P}^4 ; any other such plane would do just as well. Note that the singular point $(1 : 0 : 0 : 0 : 0)$ of $X_{2,1}$ is not on H . Let Y be the transverse intersection

$$Y = H \cap X_{2,1} = \{x \mid x_0 = x_1, x_2x_3 - x_1x_4 = 0\},$$

a nonsingular surface. Via the Segre embedding, $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$. In particular, the signature of Y vanishes, $\sigma(Y) = 0$. The normally nonsingular codimension 1 embedding $g : Y \hookrightarrow X_{2,1}$ has an associated Gysin map

$$g^! : H_2(X_{2,1}; \mathbb{Q}) \longrightarrow H_0(Y; \mathbb{Q}).$$

Let ν_Y be the normal bundle of Y in $X_{2,1}$. By transversality, $\nu_Y = \nu_H|_Y$, where ν_H is the normal bundle of H in \mathbb{P}^4 (which is the restriction of the

hyperplane line bundle $\mathcal{O}(1)$ on \mathbb{P}^4 to H). By the L -class Gysin formula (Theorem 3.18),

$$\begin{aligned} g^!L_2(X_{2,1}) &= ((1 + L^1(\nu_Y)) \cap ([Y] + \sigma(Y)[\text{pt}]))_0 \\ &= L^1(\nu_Y) \cap [Y] = \frac{1}{3}c_1(\mathcal{O}(1))^2|_Y, \end{aligned}$$

where c_1 is the first Chern class. It remains to compute $g^![X_1]$. By transversality, $g^![X_1] = [X_1 \cap Y]_Y = [X_1 \cap H]_Y$. In coordinates, X_1 is given as the line $X_1 = \{(x_0 : x_1 : 0 : 0 : 0)\} \subset \mathbb{P}^4$. Hence, $X_1 \cap Y = \{(1 : 1 : 0 : 0 : 0)\}$ is a point and $g^![X_1] = [\text{pt}]_Y \in H_0(Y)$. We arrive thus at the following result:

$$L_2(X_{2,1}) = \alpha[X_1], \quad \alpha = \langle L^1(\nu_H|_{H \cap X_{2,1}}), [H \cap X_{2,1}] \rangle. \tag{4.1}$$

Here, $\langle -, - \rangle$ denotes the Kronecker product, i.e. evaluation of a degree d cohomology class on a degree d homology class. The above Kronecker product can be computed using standard methods; one finds that $\alpha = \frac{2}{3}$.

As a second example, we work out the L -class $L_6(X) \in H_6(X; \mathbb{Q})$ of the 5-dimensional singular Schubert variety $X = X_{3,2}$. While the previous example had an isolated singularity, the singular set of $X_{3,2}$ is 2-dimensional, given by

$$\text{Sing}(X_{3,2}) = X_{1,1} \subset X.$$

The class $L_6(X)$ can be uniquely written as a linear combination

$$L_6(X) = \lambda[X_3] + \mu[X_{2,1}], \quad \lambda, \mu \in \mathbb{Q}$$

and we must determine the coefficients λ and μ . We work inside of the Grassmannian $G = G_2(\mathbb{C}^5)$. Fix a complete flag F_* in \mathbb{C}^5 , say the standard flag. We make the following notational convention: The symbol X_a will always refer to the Schubert variety associated to the partition a and the flag F_* , $X_a = X_a(F_*)$, whereas X'_a will refer to Schubert varieties in $G_2(\mathbb{C}^5)$ that are associated to a but possibly different flags F'_* .

If M is any nonsingular subvariety of G which is transverse to $X = X_{3,2}$, then the (possibly singular) variety $Y = M \cap X$ is normally nonsingular in X and its normal bundle is $\nu_Y = \nu_M|_Y$, where ν_M is the normal bundle of M in the Grassmannian. The inclusion $g : Y \hookrightarrow X$ thus has an associated Gysin restriction $g^! : H_*(X; \mathbb{Q}) \rightarrow H_{*-2c}(Y; \mathbb{Q})$, where c is the complex codimension of Y in X . Our method will be to choose M in G so that it is transverse to the representatives X_3 and $X_{2,1}$ of the generators $[X_3], [X_{2,1}]$ of $H_6(X; \mathbb{Z})$, in addition to the requirement that M be transverse to X .

Consider the variety $X_{2,2} = X_{2,2}(F_*)$ in G . The group $\text{GL}_5(\mathbb{C})$ acts transitively on G . By the Kleiman transversality theorem, there exists an element $\gamma \in \text{GL}_5(\mathbb{C})$ such that $M := \gamma \cdot X_{2,2}$ satisfies

$$M \pitchfork X, \quad M \pitchfork X_3, \quad M \pitchfork X_{2,1}, \tag{4.2}$$

where the symbol \pitchfork denotes transversality. Note that $X_{2,2}$ and its translate M are nonsingular. The translate M is again a Schubert variety: Applying

γ to the flag F_* yields a complete flag $F'_* = \gamma \cdot F_*$ such that

$$M = \gamma \cdot X_{2,2}(F_*) = X_{2,2}(F'_*) =: X'_{2,2}.$$

The transverse intersection $Y = M \cap X = X'_{2,2} \cap X_{3,2}$ has dimension $\dim Y = 3$. By Schubert calculus, there exists a flag F''_* such that $Y = X_{2,1}(F''_*)$. We will write $X'_{2,1}$ for $X_{2,1}(F''_*)$. We need to determine the associated Gysin homomorphism

$$g^! : H_6(X_{3,2}; \mathbb{Q}) \longrightarrow H_2(X'_{2,1}; \mathbb{Q}).$$

Now the second homology of $Y = X'_{2,1}$ has rank 1 generated by the rational curve $X'_1 \subset X'_{2,1}$: $H_2(X'_{2,1}; \mathbb{Q}) = \mathbb{Q}[X'_1]_{X'_{2,1}}$. So $g^!$ is a linear map $g^! : \mathbb{Q}[X_3]_X \oplus \mathbb{Q}[X_{2,1}]_X \rightarrow \mathbb{Q}[X'_1]_{X'_{2,1}}$. By the above transversality properties (4.2) of M in G , together with standard Schubert calculus (Pieri's and Giambelli's formula), $g^![X_3]_X = 0$ and $g^![X_{2,1}]_X = [X'_1]_Y$. In particular, $g^!L_6(X) = \mu[X'_1]_Y$. On the other hand, by Theorem 3.18,

$$\begin{aligned} g^!L_6(X) &= (((1 + L^1(\nu_Y) + L^2(\nu_Y) + \dots) \cap (L_6(Y) + L_2(Y)))_2 \\ &= L_2(Y) + L^1(\nu_Y) \cap [Y], \end{aligned}$$

from which we infer that

$$\mu[X'_1]_Y = L_2(Y) + L^1(\nu_{X'_{2,2}}|_Y) \cap [Y] \in H_2(Y; \mathbb{Q}), \quad Y = X'_{2,1}.$$

The class $L_2(X'_{2,1})$ has already been calculated in the previous example of this section, see (4.1).

It remains to determine the coefficient λ . For this, one needs to look at a different Gysin map, coming from a different nonsingular M . Consider the nonsingular variety $X_3 = X_3(F_*)$ in $G = G_2(\mathbb{C}^5)$. By Kleiman transversality, there exists a $\gamma \in \text{GL}_5(\mathbb{C})$ such that $M := \gamma \cdot X_3$ satisfies the transversality requirements (4.2). Applying γ to the flag F_* yields a flag F'_* such that $M = X_3(F'_*) =: X'_3$. The transverse intersection $Y = M \cap X = X'_3 \cap X_{3,2}$ has dimension $\dim Y = 2$. By Schubert calculus, $Y = X_2(F''_*)$ for some flag F''_* in \mathbb{C}^5 . We will write X'_2 for $X_2(F''_*)$. We determine the Gysin restriction

$$g^! : H_6(X_{3,2}; \mathbb{Q}) \longrightarrow H_0(X'_2; \mathbb{Q})$$

associated to $g : Y = X'_2 \hookrightarrow X = X_{3,2}$. This is a linear map $g^! : \mathbb{Q}[X_3]_X \oplus \mathbb{Q}[X_{2,1}]_X \rightarrow \mathbb{Q}[X'_{(\emptyset)}]$, where (\emptyset) denotes the empty partition with $X'_{(\emptyset)}$ a point in X'_2 . By the transversality properties of M in G , together with Schubert calculus, $g^![X_3]_X = [X'_{(\emptyset)}]_Y = [\text{pt}]_Y$ and $g^![X_{2,1}]_X = 0$. In particular, $g^!L_6(X) = \lambda[\text{pt}]_Y$. On the other hand by Theorem 3.18,

$$\begin{aligned} g^!L_6(X) &= (((1 + L^1(\nu_Y) + L^2(\nu_Y) + \dots) \cap (L_4(Y) + L_0(Y)))_0 \\ &= \sigma(Y)[\text{pt}] + L^1(\nu_Y) \cap [Y], \end{aligned}$$

and hence

$$\lambda[\text{pt}]_Y = \sigma(Y)[\text{pt}]_Y + L^1(\nu_{X'_3}|_Y) \cap [Y] \in H_0(Y; \mathbb{Q}), \quad Y = X'_2.$$

Since $Y = X'_2 \cong \mathbb{P}^2$, the signature is given by $\sigma(Y) = \sigma(\mathbb{P}^2) = 1$. Let $\omega \in H^2(X'_{2,1}; \mathbb{Q})$ be the unique cohomology class such that $\langle \omega, [X'_1] \rangle = +1$. Combining with (4.1), we obtain the following result:

$$L_6(X_{3,2}) = \lambda[X_3] + \mu[X_{2,1}],$$

where λ, μ are the rational numbers

$$\lambda = 1 + \langle L^1(\nu_{X'_3}|), [X'_2] \rangle,$$

$$\mu = \langle L^1(\nu_H|), [H \cap X_{2,1}] \rangle + \langle \omega \cup L^1(\nu_{X'_{2,2}}|), [X'_{2,1}] \rangle, \quad \langle \omega, [X'_1] \rangle = +1.$$

We may call this a *normally nonsingular expansion* of $L_6(X)$. The Kronecker products appearing in this expansion can then be computed further using standard methods.

5. Hodge-theoretic characteristic classes

For an algebraic variety X , let $K_0^{\text{alg}}(X)$ denote the Grothendieck group of the abelian category of coherent sheaves of \mathcal{O}_X -modules. When there is no danger of confusion with other K -homology groups, we shall also write $K_0(X) = K_0^{\text{alg}}(X)$. Let $K^0(X) = K_{\text{alg}}^0(X)$ denote the Grothendieck group of the exact category of algebraic vector bundles over X . The tensor product $\otimes_{\mathcal{O}_X}$ induces a *cap product*

$$\cap : K^0(X) \otimes K_0(X) \longrightarrow K_0(X), \quad [E] \cap [\mathcal{F}] = [E \otimes_{\mathcal{O}_X} \mathcal{F}].$$

Thus,

$$-\cap[\mathcal{O}_X] : K^0(X) \longrightarrow K_0(X) \tag{5.1}$$

sends a vector bundle $[E]$ to its associated (locally free) sheaf of germs of local sections $[E \otimes \mathcal{O}_X]$. If X is smooth, then $-\cap[\mathcal{O}_X]$ is an isomorphism.

Let X be a complex algebraic variety and E an algebraic vector bundle over X . For a nonnegative integer p , let $\Lambda^p(E)$ denote the p -th exterior power of E . The *total λ -class* of E is by definition

$$\lambda_y(E) = \sum_{p \geq 0} \Lambda^p(E) \cdot y^p,$$

where y is an indeterminate functioning as a bookkeeping device. This construction induces a homomorphism $\lambda_y(-) : K_{\text{alg}}^0(X) \longrightarrow K_{\text{alg}}^0(X)[y]$ from the additive group of $K^0(X)$ to the multiplicative monoid of the polynomial ring $K^0(X)[y]$. Now let X be a smooth variety, let TX denote its holomorphic tangent bundle and T^*X its holomorphic cotangent bundle. Then $\Lambda^p(T^*X)$ is the vector bundle of holomorphic p -forms on X . Its associated sheaf of sections is denoted by Ω_X^p . Thus

$$[\Lambda^p(T^*X)] \cap [\mathcal{O}_X] = [\Omega_X^p]$$

and hence

$$\lambda_y(T^*X) \cap [\mathcal{O}_X] = \sum_{p=0}^{\dim X} [\Omega_X^p] y^p.$$

Let X be a complex algebraic variety and let $MHM(X)$ denote the abelian category of M. Saito's algebraic mixed Hodge modules on X . Totaro observed in [69] that Saito's construction of a pure Hodge structure on the intersection homology of compact varieties implicitly contains a definition of certain characteristic homology classes for singular algebraic varieties. The following definition is based on this observation and due to Brasselet, Schürmann and Yokura, [15], see also the expository paper [62].

Definition 5.1. The *motivic Hodge Chern class transformation*

$$MHC_y : K_0(MHM(X)) \rightarrow K_0^{\text{alg}}(X) \otimes \mathbb{Z}[y^{\pm 1}]$$

is defined by

$$MHC_y[M] = \sum_{i,p} (-1)^i [\mathcal{H}^i(Gr_{-p}^F DR[M])] (-y)^p.$$

Here, $Gr_p^F DR : D^b MHM(X) \rightarrow D_{\text{coh}}^b(X)$, with $D_{\text{coh}}^b(X)$ the bounded derived category of sheaves of \mathcal{O}_X -modules with coherent cohomology sheaves, denotes the functor of triangulated categories constructed by M. Saito, see [56, §2.3], [60, §1], [57, §3.10], obtained by taking a suitable filtered de Rham complex of the filtered holonomic D -module underlying a mixed Hodge module. For every $p \in \mathbb{Z}$, these functors induce functors between the associated Grothendieck groups.

A flat morphism $f : X \rightarrow Y$ gives rise to a flat pullback $f^* : \text{Coh}(Y) \rightarrow \text{Coh}(X)$ on coherent sheaves, which is exact and hence induces a flat pullback $f_K^* : K_0^{\text{alg}}(Y) \rightarrow K_0^{\text{alg}}(X)$. This applies in particular to smooth morphisms and is then often called smooth pullback. An arbitrary algebraic morphism $f : X \rightarrow Y$ (not necessarily flat) induces a homomorphism

$$f^* : K_0(MHM(Y)) \longrightarrow K_0(MHM(X))$$

which corresponds under the forgetful functor

$$\text{rat} : D^b MHM(-) \rightarrow D_c^b(-; \mathbb{Q})$$

to f^{-1} on constructible complexes of sheaves. (Additional remarks on rat are to be found further below.) We record Schürmann's [62, Cor. 5.11, p. 459]:

Proposition 5.2. (*Verdier-Riemann-Roch for smooth pullbacks.*) For a smooth morphism $f : X \rightarrow Y$ of complex algebraic varieties, the Verdier Riemann-Roch formula

$$\lambda_y(T_{X/Y}^*) \cap f_K^* MHC_y[M] = MHC_y(f^*[M]) = MHC_y[f^*M]$$

holds for $M \in D^b MHM(Y)$, where $T_{X/Y}^*$ denotes the relative cotangent bundle of f .

Let E be a complex vector bundle and let a_i denote the Chern roots of E . In [34], Hirzebruch introduced a cohomological characteristic class

$$T_y^*(E) = \prod_{i=1}^{\text{rk } E} Q_y(a_i),$$

where y is an indeterminate, coming from the power series

$$Q_y(a) = \frac{a(1+y)}{1-e^{-a(1+y)}} - ay \in \mathbb{Q}[y][[a]].$$

If R is an integral domain over \mathbb{Q} , then a power series $Q(a) \in R[[a]]$ is called *normalized* if it starts with 1, i.e. $Q(0) = 1$. With $R = \mathbb{Q}[y]$, we have $Q_y(0) = 1$, so $Q_y(a)$ is normalized. For $y = 0$,

$$T_0^*(E) = \prod_{i=1}^{\text{rk } E} \frac{a_i}{1-e^{-a_i}} = \text{td}^*(E) \tag{5.2}$$

is the classical Todd class of E , while for $y = 1$,

$$T_1^*(E) = \prod_{i=1}^{\text{rk } E} \frac{a_i}{\tanh a_i} = L^*(E) \tag{5.3}$$

is the Hirzebruch L -class of the vector bundle E , as in Section 2. We shall also need a certain unnormalized version of $Q_y(a)$: Let

$$\tilde{Q}_y(a) = \frac{a(1+ye^{-a})}{1-e^{-a}} \in \mathbb{Q}[y][[a]]$$

and set

$$\tilde{T}_y^*(E) = \prod_{i=1}^{\text{rk } E} \tilde{Q}_y(a_i).$$

Note that $\tilde{Q}_y(0) = 1+y \neq 1$, whence $\tilde{Q}_y(a)$ is unnormalized. The relation

$$(1+y)Q_y(a) = \tilde{Q}_y((1+y)a)$$

implies:

Proposition 5.3. *If E is a complex vector bundle of complex rank r , then for the degree $2i$ components:*

$$\tilde{T}_y^i(E) = (1+y)^{r-i} T_y^i(E).$$

More conceptually, we have the following formula for the unnormalized class:

Proposition 5.4. *For any complex vector bundle E , we have*

$$\tilde{T}_y^*(E) = \text{td}^*(E) \cup \text{ch}^*(\lambda_y(E^*)).$$

Proof. The Chern character is given by

$$\text{ch}^*(\lambda_y(E^*)) = \prod_{i=1}^{\text{rk } E} (1 + ye^{-a_i}),$$

with a_i the Chern roots of E , see [35, p. 11]. Thus

$$\begin{aligned} \tilde{T}_y^*(E) &= \prod_i \frac{a_i(1 + ye^{-a_i})}{1 - e^{-a_i}} = \prod_i \frac{a_i}{1 - e^{-a_i}} \prod_i (1 + ye^{-a_i}) \\ &= \text{td}^*(E) \cup \text{ch}^*(\lambda_y(E^*)). \end{aligned}$$

□

Let $\tau_* : K_0(X) \rightarrow H_{2*}^{\text{BM}}(X) \otimes \mathbb{Q}$ denote the Todd class transformation of Baum, Fulton, MacPherson. We review, to some extent, construction and properties of this transformation. Let

$$\alpha^* : K_{\text{alg}}^0(X) \rightarrow K_{\text{top}}^0(X)$$

be the forget map which takes an algebraic vector bundle to its underlying topological vector bundle. Composing with the Chern character, one obtains a transformation

$$\tau^* = \text{ch}^* \circ \alpha^* : K_{\text{alg}}^0(X) \rightarrow H^{2*}(X; \mathbb{Q}),$$

see [10, p. 180]. Baum, Fulton and MacPherson construct a corresponding homological version

$$\alpha_* : K_0^{\text{alg}}(X) \rightarrow K_0^{\text{top}}(X)$$

for quasi-projective varieties X . Composing with the homological Chern character

$$\text{ch}_* : K_0^{\text{top}}(X) \rightarrow H_{2*}^{\text{BM}}(X; \mathbb{Q}),$$

where H_*^{BM} denotes Borel-Moore homology, they obtain a transformation

$$\tau_* = \text{ch}_* \circ \alpha_* : K_0^{\text{alg}}(X) \rightarrow H_{2*}^{\text{BM}}(X; \mathbb{Q}).$$

This transformation is in fact available for any algebraic scheme over a field and generalizes the Grothendieck Riemann-Roch theorem to singular varieties.

Remark 5.5. Let $A_*(V)$ denote Chow homology of a variety V , i.e. algebraic cycles in V modulo rational equivalence. Then there is a transformation

$$\tau_* : K_0^{\text{alg}}(X) \rightarrow A_*(X) \otimes \mathbb{Q}$$

such that

$$\begin{array}{ccc} K_0^{\text{alg}}(X) & & \\ \tau_* \downarrow & \searrow \tau_* & \\ A_*(X) \otimes \mathbb{Q} & \xrightarrow{\text{cl}} & H_{2*}^{\text{BM}}(X; \mathbb{Q}) \end{array}$$

commutes, where cl is the cycle map; see the first commutative diagram on p. 106 of [9, (0.8)]. The construction of τ_* to Chow homology is described in Fulton’s book [29, p. 349]. Thus Todd classes are algebraic cycles that are well-defined up to rational equivalence over \mathbb{Q} .

According to [10, Theorem, p. 180], τ_* and τ^* are compatible with respect to cap products, i.e. the diagram

$$\begin{CD} K^0(X) \otimes K_0(X) @>\tau^* \otimes \tau_*>> H^*(X; \mathbb{Q}) \otimes H_*^{\text{BM}}(X; \mathbb{Q}) \\ @V \cap VV @VV \cap V \\ K_0(X) @>\tau_*>> H_*^{\text{BM}}(X; \mathbb{Q}) \end{CD}$$

commutes. Thus, if E is a vector bundle and \mathcal{F} a coherent sheaf on X , then

$$\tau_*([E] \cap [\mathcal{F}]) = \text{ch}^*(E) \cap \tau_*[\mathcal{F}]. \tag{5.4}$$

For smooth X ,

$$\tau_*[\mathcal{O}_X] = \text{td}^*(TX) \cap [X] = T_0^*(TX) \cap [X].$$

So if E is a vector bundle on a smooth variety, then

$$\tau_*([E] \cap [\mathcal{O}_X]) = (\text{ch}^*(E) \cup \text{td}^*(TX)) \cap [X]. \tag{5.5}$$

For locally complete intersection morphisms $f : X \rightarrow Y$, Gysin maps

$$f_{\text{BM}}^* : H_*^{\text{BM}}(Y) \longrightarrow H_{*-2d}^{\text{BM}}(X)$$

have been defined by Verdier [70, §10], and Baum, Fulton and MacPherson [9, Ch. IV, §4], where d denotes the (complex) virtual codimension of f . Thus for a regular closed embedding g , there is a Gysin map g_{BM}^* on Borel-Moore homology, which we shall also write as $g^!$, and for a smooth morphism f of relative dimension r , there is a smooth pullback $f_{\text{BM}}^* : H_*^{\text{BM}}(Y) \rightarrow H_{*+2r}^{\text{BM}}(X)$. Baum, Fulton and MacPherson show:

Proposition 5.6. (*Verdier-Riemann-Roch for smooth pullbacks.*) *For a smooth morphism $f : X \rightarrow Y$ of complex algebraic varieties and $[\mathcal{F}] \in K_0^{\text{alg}}(Y)$,*

$$\text{td}^*(T_{X/Y}) \cap f_{\text{BM}}^* \tau_*[\mathcal{F}] = \tau_*(f_K^*[\mathcal{F}]).$$

Yokura [73] twisted τ_* by a Hirzebruch-type variable y :

Definition 5.7. The *twisted Todd transformation*

$$\text{td}_{1+y} : K_0(X) \otimes \mathbb{Z}[y^{\pm 1}] \longrightarrow H_{2*}^{\text{BM}}(X) \otimes \mathbb{Q}[y^{\pm 1}, (1+y)^{-1}]$$

is given by

$$\text{td}_{1+y}[\mathcal{F}] := \sum_{k \geq 0} \tau_k[\mathcal{F}] \cdot \frac{1}{(1+y)^k},$$

where the Baum-Fulton-MacPherson transformation τ_* is extended linearly over $\mathbb{Z}[y^{\pm 1}]$, and τ_k denotes the degree $2k$ -component of τ_* .

Remark 5.8. Regarding the transformation τ_* as taking values in Chow groups $A_*(-) \otimes \mathbb{Q}$ (cf. Remark 5.5), the above definition yields a twisted Todd transformation

$$\mathrm{td}_{1+y} : K_0(X) \otimes \mathbb{Z}[y^{\pm 1}] \longrightarrow A_*(X) \otimes \mathbb{Q}[y^{\pm 1}, (1+y)^{-1}],$$

which commutes with the Borel-Moore twisted Todd transformation under the cycle map.

The definition of the motivic Hirzebruch class transformation below is due to Brasselet, Schürmann and Yokura [15], see also Schürmann's expository paper [62].

Definition 5.9. The *motivic Hirzebruch class transformation* is

$$MHT_{y^*} := \mathrm{td}_{1+y} \circ MHC_y : K_0(MHM(X)) \longrightarrow H_{2*}^{\mathrm{BM}}(X) \otimes \mathbb{Q}[y^{\pm 1}, (1+y)^{-1}].$$

For the intersection Hodge module IC_X^H on a complex variety X of pure dimension n , we use the convention

$$IC_X^H := j_{!*}(\mathbb{Q}_U^H[n]),$$

which agrees with [62, p. 444] and [48, p. 345]. Here, $U \subset X$ is smooth, of pure dimension n , Zariski-open and dense, and $j_{!*}$ denotes the intermediate extension of mixed Hodge modules associated to the open inclusion $j : U \hookrightarrow X$. The underlying perverse sheaf is $\mathrm{rat}(IC_X^H) = IC_X$, the intersection chain sheaf, where $\mathrm{rat} : MHM(X) \rightarrow \mathrm{Per}(X) = \mathrm{Per}(X; \mathbb{Q})$ is the faithful and exact functor that sends a mixed Hodge module to its underlying perverse sheaf. Here, $\mathrm{Per}(X)$ denotes perverse sheaves on X which are constructible with respect to *some* algebraic stratification of X . This functor extends to a functor $\mathrm{rat} : D^b MHM(X) \rightarrow D_c^b(X) = D_c^b(X; \mathbb{Q})$ between bounded derived categories. For every object of $D_c^b(X)$ there exists *some* algebraic stratification with respect to which the object is constructible, and these stratifications will generally vary with the object. Recall that a functor F is *conservative*, if for every morphism ϕ such that $F(\phi)$ is an isomorphism, ϕ is already an isomorphism. Faithful functors on balanced categories (such as abelian or triangulated categories) are conservative. According to [59, p. 218, Remark (i)], $\mathrm{rat} : D^b MHM(X) \rightarrow D_c^b(X)$ is not faithful. But:

Lemma 5.10. *The functor $\mathrm{rat} : D^b MHM(X) \rightarrow D_c^b(X)$ is conservative.*

Using cones, this lemma appears embedded in the proof of [21, Lemma 5.3, p. 1752], see also Exercise 11.2.1 in Maxim's book [41]. The module IC_X^H is the unique simple object in the category $MHM(X)$ which restricts to $\mathbb{Q}_U[n]$ over U . As U is smooth and pure n -dimensional, $\mathbb{Q}_U^H[n]$ is pure of weight n . Since the intermediate extension $j_{!*}$ preserves weights, IC_X^H is pure of weight n . There is a duality isomorphism (polarization) $\mathbb{D}_X^H IC_X^H \cong IC_X^H(n)$. Taking rat , this isomorphism induces a self-duality isomorphism

$$\mathbb{D}_X IC_X = \mathbb{D}_X \mathrm{rat} IC_X^H \cong \mathrm{rat} \mathbb{D}_X^H IC_X^H \cong \mathrm{rat} IC_X^H(n) \cong IC_X,$$

if an isomorphism $\mathbb{Q}_U(n) \cong \mathbb{Q}_U$ is chosen.

Definition 5.11. ([15].) The *intersection generalized Todd class* (or *intersection Hirzebruch characteristic class*) is

$$IT_{y^*}(X) := MHT_{y^*}[IC_X^H[-n]] \in H_{2*}^{\text{BM}}(X) \otimes \mathbb{Q}[y^{\pm 1}, (1+y)^{-1}].$$

Taking up expectations formulated in [17], this class started to be investigated in detail in [22].

Remark 5.12. The intersection characteristic class $IT_{y^*}(X)$ is represented by an algebraic cycle by Remark 5.8.

6. Hodge classes and normally nonsingular inclusions

We embark on establishing a Verdier-Riemann-Roch type formula

$$g^!IT_{1^*}(X) = L^*(N) \cap IT_{1^*}(Y)$$

for appropriately normally nonsingular regular algebraic embeddings $g : Y \hookrightarrow X$ of complex algebraic varieties. Here, $g^!$ denotes Verdier’s Gysin map on Borel-Moore homology for closed regular algebraic embeddings, and N is the algebraic normal bundle of g . Following Verdier’s construction of $g^!$, one must first understand how $IT_{1^*}(X)$ behaves under specialization to homology of the algebraic normal bundle. This then reduces the problem to establishing the desired formula in the special case where g is the zero section embedding into an algebraic vector bundle. Philosophically, one may view the specialization map Sp_{BM} as an algebro-geometric substitute for the simple topological operation of “restricting a Borel-Moore cycle to an open tubular neighborhood of Y ”. From this point of view, one expects that $\text{Sp}_{\text{BM}} IT_{1^*}(X) = IT_{1^*}(N)$, and this is what we do indeed prove (Proposition 6.26). That proof rests on three ideas: First, in the context of deformation to the normal cone, the specialization map to the central fiber can itself be expressed in terms of a *hypersurface* Gysin restriction. Second, results of Cappell-Maxim-Schürmann-Shaneson [20] explain that a global hypersurface Gysin restriction applied to the motivic Hirzebruch class transformation agrees with first taking Hodge nearby cycles, and then executing the Hirzebruch transformation. Third, we show that Saito’s Hodge nearby cycle functor takes the intersection Hodge module on the deformation space to the intersection Hodge module of the special fiber N (Proposition 6.20). This requires in particular an analysis of the behavior of the Hodge intersection module both under $g^!$ for topologically normally nonsingular closed algebraic embeddings g (Lemma 6.19), and under smooth pullbacks (Lemma 6.13). Vietoris-Begle techniques are being used after careful premeditation of constructibility issues. The remaining step is then to understand why the relation $k^!IT_{1^*}(N) = L^*(N) \cap IT_{1^*}(Y)$ holds for the zero section embedding $k : Y \hookrightarrow N$ of an algebraic vector bundle $N \rightarrow Y$. We achieve this in Proposition 6.28. In the case of a zero section embedding k , the Gysin restriction $k^!$ is, by the Thom isomorphism theorem, inverse to smooth pullback under the vector bundle projection, and we find it easier to establish a relation for

the latter (Proposition 6.23). Schürmann's MHC_y -Verdier-Riemann-Roch theorem also enters.

Since algebraic normal bundles of regular algebraic embeddings need not faithfully reflect the normal topology near the subvariety, the main result, Theorem 6.30, requires a tightness assumption, which holds automatically in transverse situations (Proposition 6.3). Furthermore, our methods require that the exceptional divisor in the blow-up of $X \times \mathbb{C}$ along $Y \times 0$ be normally nonsingular. We do not know at present whether the latter condition, related to the “clean blow-ups” of Cheeger, Goresky and MacPherson, is necessary. Again, it holds in transverse situations (Corollary 6.7).

As regular algebraic embeddings need not be topologically normally nonsingular, we define:

Definition 6.1. A closed regular algebraic embedding $Y \hookrightarrow X$ of complex algebraic varieties is called *tight*, if its underlying topological embedding (in the complex topology) is normally nonsingular and compatibly stratifiable (Definition 3.4), with topological normal bundle $\pi : E \rightarrow Y$ as in Definition 3.3, and $E \rightarrow Y$ is isomorphic (as a topological vector bundle) to the underlying topological vector bundle of the algebraic normal bundle $N_Y X$ of Y in X .

Example 6.2. A closed embedding $g : M \hookrightarrow W$ of smooth complex algebraic varieties is tight because the normal bundles can be described in terms of tangent bundles, and the smooth tubular neighborhood theorem applies to provide normal nonsingularity (with respect to the intrinsic stratification consisting of only the top stratum).

Closed subschemes $X, Y \subset S$ of a scheme S are called *Tor-independent* if

$$\mathrm{Tor}_i^{\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_Y) = 0 \text{ for all } i > 0.$$

Proposition 6.3. *Let $M \hookrightarrow W$ be a closed algebraic embedding of smooth complex algebraic varieties. Let $X \subset W$ be a (possibly singular) algebraic subvariety, equipped with an algebraic Whitney stratification and set $Y = X \cap M$. If*

- *each stratum of X is transverse to M , and*
- *X and M are Tor-independent in W ,*

then the embedding $g : Y \hookrightarrow X$ is tight.

Proof. The topological aspects of the proof proceed along the lines of [30, p. 48, Proof of Thm. 1.11]. By smoothness, the closed embedding $M \hookrightarrow W$ is regular with algebraic normal bundle $N_M W$. The Tor-independence of X and M ensures that the closed embedding $Y \hookrightarrow X$ is also regular (see [36, Lemma 1.7]), and that the excess normal bundle vanishes, i.e. the canonical closed embedding $N_Y X \rightarrow j^* N_M W$ is an isomorphism of algebraic vector bundles, where j is the embedding $j : Y \hookrightarrow M$. The bundle $\pi : E \rightarrow Y$

in Definition 6.1 may then be taken to be the underlying topological vector bundle of the restriction of $N_M W$ to Y . \square

Let $V \hookrightarrow U$ be a closed regular embedding of complex varieties. Then $\beta : \text{Bl}_V U \rightarrow U$ will denote the blow-up of U along V . The exceptional divisor $E = \beta^{-1}(V) \subset \text{Bl}_V U$ is the projectivization $\mathbb{P}(N)$ of the algebraic normal bundle N of V in U .

Definition 6.4. Let $X \hookrightarrow W \hookleftarrow M$ be closed algebraic embeddings of algebraic varieties with M, W smooth. We say that these embeddings are *upwardly transverse*, if X and M are Tor-independent in W , there exists an algebraic Whitney stratification of X which is transverse to M in W , and there exists a (possibly non-algebraic) Whitney stratification on the strict transform of $X \times \mathbb{C}$ in $\text{Bl}_{M \times 0}(W \times \mathbb{C})$ which is transverse to the exceptional divisor.

Definition 6.5. A tight embedding $Y \hookrightarrow X$ is called *upwardly normally nonsingular* if the inclusion $E \subset \text{Bl}_{Y \times 0}(X \times \mathbb{C})$ of the exceptional divisor E is topologically normally nonsingular.

This notion is related to the *clean blow-ups* of Cheeger, Goresky and MacPherson [26, p. 331]. A monoidal transformation $\pi : \tilde{X} \rightarrow X$ with nonsingular center $Y \subset X$ is called a clean blow-up if $E \rightarrow Y$ is a topological fibration, where $E = \pi^{-1}(Y)$ is the exceptional divisor, and the inclusion $E \subset \tilde{X}$ is normally nonsingular.

Proposition 6.6. *Let $X \hookrightarrow W \hookleftarrow M$ be Tor-independent closed algebraic embeddings of algebraic varieties with M, W smooth. If there exists an algebraic Whitney stratification of X which is transverse to M , and a Whitney stratification on the strict transform of X in $\text{Bl}_M W$ which is transverse to the exceptional divisor, then the inclusion $Y = X \cap M \hookrightarrow X$ is tight and the inclusion $E' \subset \text{Bl}_Y X$ of the exceptional divisor is topologically normally nonsingular. The corresponding topological normal vector bundle of $E' \subset \text{Bl}_Y X$ is then isomorphic to the restriction to E' of the tautological line bundle $\mathcal{O}_E(-1)$ over the exceptional divisor $E \subset \text{Bl}_M W$.*

Proof. Let $\beta : \text{Bl}_M W \rightarrow W$ be the blow-up of W along M . As M is a smoothly embedded smooth subvariety of the smooth variety W , $\text{Bl}_M W$ is smooth and the exceptional divisor $E = \beta^{-1}(M) = \mathbb{P}(N)$ is a smooth variety smoothly embedded in $\text{Bl}_M W$. By assumption, the strict transform $\beta^{-1}(X)$ can be equipped with a Whitney stratification transverse to E . Then the intersection $\tilde{Y} = \beta^{-1}(X) \cap E$ is an oriented pseudomanifold Whitney stratified by its intersection with the strata of $\beta^{-1}(X)$, the inclusion $\tilde{Y} \hookrightarrow \beta^{-1}(X)$ is normally nonsingular with oriented topological normal bundle isomorphic to the restriction of the topological normal bundle ν_E of E in $\text{Bl}_M W$. Now this normal bundle is the tautological bundle $\nu_E = \mathcal{O}_E(-1)$.

Since X and M are Tor-independent in W , and $M \hookrightarrow W$ is a regular embedding, the embedding $Y = X \cap M \hookrightarrow X$ is regular as well, and the blow-up $\text{Bl}_Y X$ of X along $Y = X \cap M$ is given by

$$\text{Bl}_Y X = X \times_W \text{Bl}_M W = \beta^{-1}(X),$$

[36, Lemma 1.7]. The exceptional divisor E' of $\beta| : \text{Bl}_Y X \rightarrow X$ is

$$E' = \beta^{-1}(Y) = \beta^{-1}(X \cap M) = \beta^{-1}(X) \cap \beta^{-1}(M) = \beta^{-1}(X) \cap E = \tilde{Y}.$$

The inclusion $Y \hookrightarrow X$ is tight by Proposition 6.3. □

Corollary 6.7. *If $X \hookrightarrow W \hookleftarrow M$ are upwardly transverse embeddings, then the embedding $Y = X \cap M \hookrightarrow X$ is upwardly normally nonsingular. The corresponding topological normal vector bundle of the exceptional divisor $E' \subset \text{Bl}_{Y \times 0}(X \times \mathbb{C})$ is then isomorphic to the restriction to E' of the tautological line bundle $\mathcal{O}_E(-1)$ over the exceptional divisor $E \subset \text{Bl}_{M \times 0}(W \times \mathbb{C})$.*

Proof. There exists an algebraic Whitney stratification \mathcal{X} of X such that each stratum of X is topologically transverse to M . Since X and M are Tor-independent in W , the embedding $Y = X \cap M \hookrightarrow X$ is regular, [36, Lemma 1.7]. Thus by Proposition 6.3, the embedding $Y \hookrightarrow X$ is tight. Equip $X' := X \times \mathbb{C}$ with the product stratification $\mathcal{X}' = \mathcal{X} \times \mathbb{C}$. Then \mathcal{X}' is an algebraic Whitney stratification of X' in $W' := W \times \mathbb{C}$. Since X is Whitney transverse to M in W , X' is Whitney transverse to $M' := M \times 0$ in W' . The Tor-independence of X and M in W implies Tor-independence of X' and M' in W' , since

$$\text{Tor}_n^{R[t]}(A[t], B) = \text{Tor}_n^R(A, B)$$

for R -modules A, B ; R a \mathbb{C} -algebra. Since $X \hookrightarrow W \hookleftarrow M$ are upwardly transverse embeddings, there exists a Whitney stratification on the strict transform of X' in $\text{Bl}_{M'}(W')$ which is transverse to the exceptional divisor. Hence, we may apply Proposition 6.6 to $X' \hookrightarrow W' \hookleftarrow M'$. It follows that the inclusion

$$Y' = X' \cap M' = (X \times \mathbb{C}) \cap (M \times 0) = (X \cap M) \times 0 = Y \times 0 \hookrightarrow X' = X \times \mathbb{C}$$

is tight and the inclusion $E' \subset \text{Bl}_{Y'} X'$ of the exceptional divisor is topologically normally nonsingular. In fact, the corresponding topological normal vector bundle of $E' \subset \text{Bl}_{Y'} X'$ is then isomorphic to the restriction to E' of the tautological line bundle $\mathcal{O}_E(-1)$ over the exceptional divisor $E \subset \text{Bl}_{M'} W'$. □

Let $Y \hookrightarrow X$ be a regular embedding with normal bundle $N = N_Y X$. We recall briefly the technique of *deformation to the (algebraic) normal bundle*. The embedding of Y in X gives rise to an embedding $Y \times 0 \hookrightarrow X \times 0 \hookrightarrow X \times \mathbb{C}$ of $Y \times 0$ in $X \times \mathbb{C}$. Let $Z = \text{Bl}_{Y \times 0}(X \times \mathbb{C})$ be the blow-up of $X \times \mathbb{C}$ along $Y \times 0$, with exceptional divisor $\mathbb{P}(N \oplus 1)$. The second factor projection

$X \times \mathbb{C} \rightarrow \mathbb{C}$ induces a flat morphism $p_Z : Z \rightarrow \mathbb{C}$, whose special fiber $p_Z^{-1}(0)$ is given by

$$p_Z^{-1}(0) = \text{Bl}_Y X \cup_{\mathbb{P}(N)} \mathbb{P}(N \oplus 1).$$

Let $Z^\circ = Z - \text{Bl}_Y X$. Then p_Z restricts to a morphism $p : Z^\circ \rightarrow \mathbb{C}$, whose special fiber is $p^{-1}(0) = N$ and whose general fiber is $p^{-1}(t) \cong X \times \{t\}$, $t \in \mathbb{C}^*$. This construction deforms the given regular embedding of Y in X to the zero section embedding of Y in the normal bundle N .

Proposition 6.8. *Let $g : Y \hookrightarrow X$ be an upwardly normally nonsingular embedding of compact complex algebraic varieties with associated deformation $p : Z^\circ \rightarrow \mathbb{C}$ to the (algebraic) normal bundle $N = N_Y X = p^{-1}(0)$. Then there exists an open neighborhood (in the complex topology) of $p^{-1}(0)$ in Z° which is homeomorphic to $F|_N$, where $F \rightarrow \mathbb{P}(N \oplus 1)$ is the topological normal bundle to the exceptional divisor in Z .*

Proof. As g is upwardly normally nonsingular, g is tight and the inclusion $\mathbb{P}(N \oplus 1) \subset \text{Bl}_{Y \times 0}(X \times \mathbb{C}) = Z$ of the exceptional divisor is normally nonsingular. In particular, g is normally nonsingular and thus there is a locally cone-like topological stratification $\mathcal{X} = \{X_\alpha\}$ of X such that $\mathcal{Y} := \{Y_\alpha := X_\alpha \cap Y\}$ is a locally cone-like topological stratification of Y , and there exists a topological vector bundle $\pi : E \rightarrow Y$ together with a topological embedding $j : E \rightarrow X$ such that $j(E)$ is open in X , $j|_Y = g$, and the homeomorphism $j : E \xrightarrow{\cong} j(E)$ is stratum preserving, where the open set $j(E)$ is endowed with the stratification $\{X_\alpha \cap j(E)\}$ and E is endowed with the stratification $\mathcal{E} = \{\pi^{-1}Y_\alpha\}$.

As the inclusion $\mathbb{P}(N \oplus 1) \subset \text{Bl}_{Y \times 0}(X \times \mathbb{C}) = Z$ is normally nonsingular, there exists a topological vector bundle $\pi_F : F \rightarrow \mathbb{P}(N \oplus 1)$ together with a topological embedding $J : F \rightarrow Z$ such that $J(F)$ is open in Z and $J|_{\mathbb{P}(N \oplus 1)}$ is the inclusion $\mathbb{P}(N \oplus 1) \subset Z$. As N is open in $\mathbb{P}(N \oplus 1)$, the total space $F|_N$ is an open subset of F , and hence $J(F|_N)$ is open in $J(F)$, which is open in Z . Thus $J(F|_N)$ is open in Z .

Let d be a metric on Z , whose metric topology agrees with the complex topology on Z . Let $r : \mathbb{P}(N \oplus 1) \rightarrow \mathbb{R}_{\geq 0}$ be the continuous function defined by $r(x) = \frac{1}{2}d(x, \text{Bl}_Y X)$. If $x \in N \subset \mathbb{P}(N \oplus 1)$, then $r(x) > 0$ since $\text{Bl}_Y X$ is compact. If $x \in \mathbb{P}(N) = \mathbb{P}(N \oplus 1) - N$, then $r(x) = 0$ since $\mathbb{P}(N) \subset \text{Bl}_Y X$. Endow F with the unique metric such that J becomes an isometry. Let $F_r \subset F$ be the open subset given by all vectors $v \in F$ of length $|v| := d(0, v) < r(\pi_F(v))$. Given a vector v in $F_r \cap \pi_F^{-1}(N) = F_r$, a triangle inequality argument shows that v has positive distance from every point in $\text{Bl}_Y X$, from which we conclude that $J(F_r|_N)$ and $\text{Bl}_Y X$ are disjoint. Hence $J(F_r|_N) \subset Z^\circ$. As $F_r|_N$ is open in $F|_N$, we can find an open disc bundle $F' \subset F_r|_N$ over N . Then $J(F')$ is an open neighborhood of $N = p^{-1}(0)$ in Z° and the composition of J with a fiber-preserving homeomorphism $F|_N \cong F'$ over N yields a homeomorphism $F|_N \cong F' \cong J(F')$. \square

We shall next stratify $F|_N$ in a topologically locally cone-like fashion:

Proposition 6.9. *Assumptions and notation as in Proposition 6.8. Let $\tilde{\pi} : F|_N \rightarrow N$ denote the bundle projection. Let $\mathcal{Y} = \{Y_\alpha = X_\alpha \cap Y\}$ be the locally cone-like topological stratification of Y guaranteed by the normal nonsingularity of Y in X . Then the strata*

$$S_\alpha := \tilde{\pi}^{-1}(\pi_N^{-1}Y_\alpha)$$

form a locally cone-like topological stratification $\mathcal{S} = \{S_\alpha\}$ of $F|_N$, where π_N denotes the vector bundle projection $\pi_N : N \rightarrow Y$.

Proof. The strata S_α are topological manifolds since the Y_α are topological manifolds and the total space of a (locally trivial) vector bundle over a topological manifold is again a topological manifold. Using local triviality of vector bundles and the fact that \mathcal{Y} is locally cone-like, one constructs filtration preserving homeomorphisms that show that \mathcal{S} is locally cone-like. \square

In order for Lemma 6.11, concerning the constructibility of nearby cycle complexes, to become applicable, we must refine the stratification so that the central fiber becomes a union of strata:

Lemma 6.10. *The refinement of the locally cone-like topological stratification \mathcal{S} of Proposition 6.9 given by $\mathcal{S}' := \{S_\alpha - N\} \cup \{S_\alpha \cap N\}$ is again a locally cone-like topological stratification of $F|_N$. (Here we have identified N with the zero section of $F|_N$.)*

Proof. Away from the zero section N , the strata of \mathcal{S}' agree with the strata of \mathcal{S} . So it suffices to prove that points $v \in N$ on the zero section have cone-like neighborhoods in a stratum preserving fashion. According to Proposition 6.9, v has an open neighborhood W together with a homeomorphism $W \cong \mathbb{R}^{2+2r+i} \times cL$, where $cL = \text{cone}^\circ L$ denotes the open cone on L . This homeomorphism is stratum preserving if we endow W with the stratification induced from \mathcal{S} . It is, however, *not* stratum preserving if we endow W with the stratification induced from \mathcal{S}' . Let L' be the join $L' = S^1 * L$, a compact space. Composing the homeomorphism

$$\begin{aligned} \mathbb{R}^{2+2r+i} \times cL &= \mathbb{R}^{2r+i} \times (\mathbb{R}^2 \times cL) \cong \mathbb{R}^{2r+i} \times (cS^1 \times cL) \\ &\cong \mathbb{R}^{2r+i} \times c(S^1 * L) \cong \mathbb{R}^{2r+i} \times cL' \end{aligned}$$

with the above homeomorphism, we obtain a homeomorphism $W \cong \mathbb{R}^{2r+i} \times cL'$. We shall now stratify L' in such a way that this homeomorphism is stratum preserving if W is equipped with the stratification induced from \mathcal{S}' . This will finish the proof. Let A and B be compact spaces with stratifications $\mathcal{A} = \{A_\alpha\}$, $\mathcal{B} = \{B_\beta\}$, respectively. The product stratification of $cA \times cB$ is given by

$$\begin{aligned} cA \times cB &= \{(0, 1) \times A_\alpha \times (0, 1) \times B_\beta\} \cup \{(0, 1) \times A_\alpha \times \{cB\}\} \\ &\cup \{\{cA\} \times (0, 1) \times B_\beta\} \cup \{(cA, cB)\}. \end{aligned}$$

The join $A * B = cA \times B \cup_{A \times B} A \times cB$ is canonically stratified by

$$\mathcal{J} = \{(0, 1) \times A_\alpha \times B_\beta\} \cup \{\{c_A\} \times B_\beta\} \cup \{A_\alpha \times \{c_B\}\}.$$

Therefore, the cone $c(A * B)$ has the canonical stratification

$$\begin{aligned} \mathcal{CJ} = & \{(0, 1) \times (0, 1) \times A_\alpha \times B_\beta\} \cup \{(0, 1) \times A_\alpha \times \{c_B\}\} \\ & \cup \{(0, 1) \times \{c_A\} \times B_\beta\} \cup \{(c_A, c_B)\}, \end{aligned}$$

which agrees with $\mathcal{CA} \times \mathcal{CB}$ under the homeomorphism $cA \times cB \cong c(A * B)$. So this homeomorphism is stratum preserving if we stratify as indicated. We apply this with $A = S^1$, $B = L$, and $\mathcal{A} = \{S^1\}$ (one stratum). Then, using the above method, the open disc $cA = cS^1 = D^{\circ 2}$ receives the stratification

$$\mathcal{CA} = \{(0, 1) \times S^1, c_A = 0\},$$

where $0 \in D^{\circ 2}$ denotes the center of the disc. In the stratification \mathcal{S} this disc is stratified with precisely one stratum (the entire disc), while in the refined stratification \mathcal{S}' , the disc must be stratified with two strata, namely the center and its complement. As we have seen, this is achieved automatically by the above cone stratification procedure. Thus if we endow $L' = S^1 * L$ with the canonical join stratification \mathcal{J} described above, then the stratification \mathcal{CJ} will agree with $\mathcal{CA} \times \mathcal{CB}$ under the homeomorphism $cL' \cong cS^1 \times cL = D^{\circ 2} \times cL$ and $D^{\circ 2} \times cL$ contains $\{0\} \times cL$ as a union of strata, as required by \mathcal{S}' . □

Via the homeomorphism of Proposition 6.8, the locally cone-like topological stratification \mathcal{S}' of Lemma 6.10 induces a locally cone-like topological stratification \mathcal{S}_U of a neighborhood U of $p^{-1}(0) = N$ in Z° . In \mathcal{S}_U , the central fiber N is a union of strata. Hence, Lemma 6.11 below is applicable to the stratification \mathcal{S}_U . We will apply the Lemma in this manner in proving Proposition 6.20 on the Hodge nearby cycle functor applied to the intersection Hodge module of the deformation space. Saito’s Hodge nearby cycle functor ψ_f^H is a functor

$$\psi_f^H : MHM(V) \longrightarrow MHM(F),$$

where $f : V \rightarrow \mathbb{C}$ is an algebraic function with central fiber $F = f^{-1}(0)$. Deligne’s nearby cycle functor ψ_f does not preserve perverse sheaves, but the shifted functor $\psi_f[-1] : \text{Per}(V) \rightarrow \text{Per}(F)$ does. Saito thus often uses the notation ${}^p\psi_f := \psi_f[-1]$ for the shifted functor. Then the diagram

$$\begin{array}{ccc} MHM(V) & \xrightarrow{\psi_f^H} & MHM(F) \\ \text{rat} \downarrow & & \downarrow \text{rat} \\ \text{Per}(V) & \xrightarrow{\psi_f[-1]} & \text{Per}(F) \end{array}$$

commutes. It is also customary to write $\psi_f'^H := \psi_f^H[1]$. Then one gets a commutative diagram

$$\begin{array}{ccc}
 D^bMHM(V) & \xrightarrow{\psi_f'^H} & D^bMHM(F) \\
 \text{rat} \downarrow & & \downarrow \text{rat} \\
 D_c^b(V) & \xrightarrow{\psi_f} & D_c^b(F)
 \end{array} \tag{6.1}$$

So $\psi_f'^H$ lifts ψ_f to the derived category of mixed Hodge modules. In proving Proposition 6.20 below, we shall use Schürmann’s [61, Lemma 4.2.1, p. 247] in the following form:

Lemma 6.11. (*Schürmann.*) *Let V be a topological space endowed with a locally cone-like topological stratification and let $p : V \rightarrow \mathbb{C}$ be a continuous function such that the subspace $F = p^{-1}(0)$ is a union of strata. If \mathcal{F} is a constructible complex of sheaves on V , then $\psi_p\mathcal{F}$ is constructible with respect to the induced stratification of F , i.e. the restrictions of the cohomology sheaves to strata are locally constant.*

Let $g : Y \hookrightarrow X$ be a regular closed algebraic embedding with algebraic normal bundle $N = N_Y X$. The associated deformation to the normal bundle $p : Z^\circ \rightarrow \mathbb{C}$ comes with the commutative diagram

$$\begin{array}{ccccccc}
 & & \{0\} & \hookrightarrow & \mathbb{C} & \longleftarrow & \mathbb{C}^* \\
 & & \uparrow & & \uparrow p_Y & & \uparrow \text{pr}_2 \\
 & & Y \times 0 & \xrightarrow{i_Y} & Y \times \mathbb{C} & \xleftarrow{j_Y} & Y \times \mathbb{C}^* & \xrightarrow{\text{pr}_1} & Y \\
 & & \downarrow k & & \downarrow G & & \downarrow g \times \text{id} & & \downarrow g \\
 Y & \xleftarrow{\pi_N} & N & \xrightarrow{i} & Z^\circ & \xleftarrow{j} & X \times \mathbb{C}^* & \xrightarrow{\text{pr}_1} & X \\
 & & \downarrow & & \downarrow p & & \downarrow \text{pr}_2 & & \\
 & & \{0\} & \hookrightarrow & \mathbb{C} & \longleftarrow & \mathbb{C}^* & &
 \end{array}$$

The map $\pi_N : N \rightarrow Y$ is the bundle projection and $k : Y \times 0 \hookrightarrow N$ its zero section. The inclusions i, i_Y are closed, while the embeddings j, j_Y are open. The map G is the closed embedding of $Y \times \mathbb{C}$ into the deformation space Z° .

According to Saito [57, p. 269], the specialization functor $\psi_p^H j_!(- \boxtimes \mathbb{Q}_{\mathbb{C}^*}^H[1])$ from mixed Hodge modules on X to mixed Hodge modules on N induces the identity on $MHM(Y)$, that is, the canonical morphism

$$k^* \psi_p^H [1](j_! \text{pr}_1^* M) \longrightarrow g^* M \tag{6.2}$$

for $M \in D^bMHM(X)$ is an isomorphism in $D^bMHM(Y)$. Indeed, Verdier’s property “(SP5) Restrictions aux sommets” ([71, p. 353]) asserts that upon applying the functor rat to (6.2), the underlying morphism is an isomorphism

in $D_c^b(Y)$. Since the functor rat is conservative by Lemma 5.10, it follows that (6.2) is an isomorphism in $D^b\text{MHM}(Y)$.

The behavior of the intersection Hodge module under normally nonsingular pullback and normally nonsingular restriction is treated in the next lemmas. We recall [62, p. 443, Prop. 4.5]:

Lemma 6.12. *Let $\pi : X \rightarrow Y$ be a morphism of algebraic varieties. If π is a smooth morphism of pure fiber dimension r , then there is a natural isomorphism of functors*

$$\pi^! = \pi^*[2r](r) : D^b\text{MHM}(Y) \rightarrow D^b\text{MHM}(X).$$

The following result will be applied later in the case where the smooth morphism is the projection of an algebraic vector bundle.

Lemma 6.13. *Let X and Y be pure-dimensional complex algebraic varieties and let $\pi : X \rightarrow Y$ be a smooth algebraic morphism of pure fiber dimension r . Then*

$$\pi^*[IC_Y^H[r]] = [IC_X^H]$$

under the smooth pullback $\pi^* : K_0(\text{MHM}(Y)) \rightarrow K_0(\text{MHM}(X))$.

Proof. According to Saito [57, p. 257], $\text{MHM}(-)$ is stable under smooth pullbacks. There is thus a functor $\pi^*[r] : \text{MHM}(Y) \rightarrow \text{MHM}(X)$ and by Lemma 6.12, $\pi^![-r] = \pi^*r$. This functor is exact, which can be shown by an argument similar to the one used to prove Lemma 6.18 below. Let $V \subset Y$ be a Zariski-open, smooth, dense subset with inclusion $j : V \hookrightarrow Y$. The preimage $j_U : U = \pi^{-1}(V) \hookrightarrow X$ is again Zariski-open, smooth, and dense. The restriction $\pi_U : U \rightarrow V$ of π is again smooth of pure fiber dimension r , so that in particular $\pi_U^![-r] = \pi_U^*r$. By [57, p. 323, (4.4.3)], the cartesian square

$$\begin{array}{ccc} U & \xrightarrow{j_U} & X \\ \pi_U \downarrow & & \downarrow \pi \\ V & \xrightarrow{j} & Y \end{array}$$

has associated base change natural isomorphisms $j_{U*}\pi_U^! \cong \pi^!j_*$ and $j_{U!}\pi_U^* \cong \pi^*j_!$. Using these, and the exactness of $\pi^*[r]$, we obtain isomorphisms

$$\pi^*[r](H^0j_! \rightarrow H^0j_*) = H^0(j_{U!}\pi_U^*[r]) \rightarrow H^0(j_{U*}\pi_U^*[r]).$$

Substitution of $\mathbb{Q}_V^H[n]$, where $n = \dim_{\mathbb{C}} Y$, gives

$$\begin{aligned} \pi^*[r]IC_Y^H &= \pi^*[r] \operatorname{im}(H^0 j_! \mathbb{Q}_V^H[n] \rightarrow H^0 j_* \mathbb{Q}_V^H[n]) \\ &= \operatorname{im} \pi^*[r](H^0 j_! \mathbb{Q}_V^H[n] \rightarrow H^0 j_* \mathbb{Q}_V^H[n]) \\ &= \operatorname{im}(H^0(j_{U!} \pi_U^*[r] \mathbb{Q}_V^H[n]) \rightarrow H^0(j_{U*} \pi_U^*[r] \mathbb{Q}_V^H[n])) \\ &= \operatorname{im}(H^0(j_{U!} \mathbb{Q}_U^H[n+r]) \rightarrow H^0(j_{U*} \mathbb{Q}_U^H[n+r])) \\ &= IC_X^H. \end{aligned}$$

□

Given an algebraic stratification \mathcal{S} of a complex algebraic variety X , let $D_c^b(X, \mathcal{S})$ denote the full subcategory of $D_c^b(X)$ consisting of all complexes on X which are constructible with respect to \mathcal{S} . Similarly, we define $\operatorname{Per}(X, \mathcal{S})$ to be the full subcategory of $\operatorname{Per}(X)$ consisting of all perverse sheaves on X which are constructible with respect to \mathcal{S} . The category $\operatorname{Per}(X, \mathcal{S})$ is abelian and the inclusion functor $\operatorname{Per}(X, \mathcal{S}) \rightarrow \operatorname{Per}(X)$ is exact. (For example, a kernel in $\operatorname{Per}(X)$ of a morphism of \mathcal{S} -constructible perverse sheaves is itself \mathcal{S} -constructible and a kernel in $\operatorname{Per}(X, \mathcal{S})$.) Perverse truncation and cotruncation, and hence perverse cohomology, restricts to \mathcal{S} -constructible objects:

$$\begin{array}{ccc} D_c^b(X) & \xrightarrow{pH^k} & \operatorname{Per}(X) \\ \uparrow & & \uparrow \\ D_c^b(X, \mathcal{S}) & \xrightarrow{pH^k} & \operatorname{Per}(X, \mathcal{S}). \end{array} \tag{6.3}$$

More generally, we may consider $D_c^b(X, \mathcal{S})$ and $\operatorname{Per}(X, \mathcal{S})$ on any space X equipped with a locally cone-like topological stratification \mathcal{S} .

Lemma 6.14. *Let X be an even-dimensional space equipped with a locally cone-like topological stratification \mathcal{S} whose strata are all even-dimensional. Let $g : Y \hookrightarrow X$ be a normally nonsingular topological embedding of even real codimension $2c$ with respect to \mathcal{S} . Let \mathcal{T} be the locally cone-like stratification of Y induced by \mathcal{S} . Then the functor $g^! [c] = g^* [-c] : D_c^b(X, \mathcal{S}) \rightarrow D_c^b(Y, \mathcal{T})$ restricts to a functor $g^! [c] = g^* [-c] : \operatorname{Per}(X, \mathcal{S}) \rightarrow \operatorname{Per}(Y, \mathcal{T})$, which is exact.*

Proof. First, as g is normally nonsingular with respect to \mathcal{S} (see Definition 3.3) of real codimension $2c$, we have $g^! = g^* [-2c] : D_c^b(X, \mathcal{S}) \rightarrow D_c^b(Y, \mathcal{T})$, see e.g. [5, p. 163, proof of Lemma 8.1.6]. Let S_α be the strata of \mathcal{S} and $s_\alpha : S_\alpha \hookrightarrow X$ the corresponding stratum inclusions. By (1) of Definition 3.3, a locally cone-like topological stratification of Y is given by $\mathcal{T} = \{T_\alpha\}$ with $T_\alpha = S_\alpha \cap Y$ and inclusions $t_\alpha : T_\alpha \hookrightarrow Y$. By (3) of Definition 3.3, there is a topological vector bundle $\pi : E \rightarrow Y$ and a topological embedding $j : E \hookrightarrow X$ onto an open subset $j(E)$ of X . The restricted homeomorphism $j : E \cong j(E)$ is stratum preserving, i.e. restricts further to

a homeomorphism $j| : \pi^{-1}(T_\alpha) \cong S_\alpha \cap j(E)$. Therefore,

$$\dim_{\mathbb{R}} T_\alpha = \dim_{\mathbb{R}} S_\alpha - 2c. \tag{6.4}$$

The category $\text{Per}(X, \mathcal{S})$ of \mathcal{S} -constructible perverse sheaves on X is the heart of $({}^pD^{\leq 0}(X, \mathcal{S}), {}^pD^{\geq 0}(X, \mathcal{S}))$, the perverse t-structure with respect to \mathcal{S} . For $A^\bullet \in {}^pD^{\geq 0}(X, \mathcal{S})$, one uses (6.4) to verify that $g^! [c] A^\bullet \in {}^pD^{\geq 0}(Y, \mathcal{T})$. In particular, the functor $g^! [c] = g^*[-c] : D_c^b(X, \mathcal{S}) \rightarrow D_c^b(Y, \mathcal{T})$ is left-t-exact with respect to the perverse t-structure. Similarly, $A^\bullet \in {}^pD^{\leq 0}(X, \mathcal{S})$ implies that $g^! [c] A^\bullet \in {}^pD^{\leq 0}(Y, \mathcal{T})$. Hence, $g^! [c] = g^*[-c]$ is also right-t-exact, and thus t-exact. It follows that $g^! [c] = g^*[-c] : D_c^b(X, \mathcal{S}) \rightarrow D_c^b(Y, \mathcal{T})$ preserves hearts. Moreover, ${}^pH^0(g^! [c]) = g^! [c]$, and this functor is exact on the category of perverse sheaves, for example by [5, Prop. 7.1.15, p. 151]. \square

For a complex algebraic variety X endowed with an algebraic stratification \mathcal{S} , $MHM(X, \mathcal{S})$ denotes the full subcategory of $MHM(X)$ whose objects are those mixed Hodge modules M on X such that $\text{rat}(M) \in \text{Ob Per}(X, \mathcal{S})$.

Lemma 6.15. *The category $MHM(X, \mathcal{S})$ is abelian and the inclusion functor $MHM(X, \mathcal{S}) \rightarrow MHM(X)$ is exact.*

Proof. We use the following general category-theoretic fact: Let $F : A \rightarrow B$ be an exact functor between abelian categories. Let B' be a full subcategory of B such that B' is abelian and the inclusion functor $B' \rightarrow B$ is exact. Then the full subcategory A' of A given by

$$\text{Ob } A' = \{X \in \text{Ob } A \mid \exists X' \in \text{Ob } B' : F(X) \cong X'\}$$

is an abelian category and the inclusion functor $A' \rightarrow A$ is exact. In particular, if B' is in addition isomorphism-closed in B , then A' with

$$\text{Ob } A' = \{X \in \text{Ob } A \mid F(X) \in \text{Ob } B'\}$$

is abelian with $A' \rightarrow A$ exact. We apply this to the exact functor $F = \text{rat} : MHM(X) \rightarrow \text{Per}(X)$ and $B' = \text{Per}(X, \mathcal{S})$. We noted earlier that $\text{Per}(X, \mathcal{S})$ is abelian and $\text{Per}(X, \mathcal{S}) \rightarrow \text{Per}(X)$ is exact. Quasi-isomorphisms of complexes of sheaves preserve \mathcal{S} -constructibility. Thus $\text{Per}(X, \mathcal{S})$ is isomorphism closed in $\text{Per}(X)$. The statement of the lemma follows since A' as described above agrees in the application $F = \text{rat}$ with $MHM(X, \mathcal{S})$. \square

By definition, the functor $\text{rat} : MHM(X) \rightarrow \text{Per}(X)$ restricts to a functor $\text{rat} : MHM(X, \mathcal{S}) \rightarrow \text{Per}(X, \mathcal{S})$. Since $\text{Per}(X, \mathcal{S})$ is isomorphism-closed in $\text{Per}(X)$, the subcategory $MHM(X, \mathcal{S})$ is isomorphism-closed in $MHM(X)$. The functor $\text{rat} : MHM(X, \mathcal{S}) \rightarrow \text{Per}(X, \mathcal{S})$ is exact and faithful. Let $DM(X, \mathcal{S})$ denote the full subcategory of $D^bMHM(X)$ whose objects M^\bullet satisfy $\text{rat } M^\bullet \in \text{Ob } D_c^b(X, \mathcal{S})$. Thus by definition, $\text{rat} : D^bMHM(X) \rightarrow D_c^b(X)$ restricts to

$$\text{rat} : DM(X, \mathcal{S}) \rightarrow D_c^b(X, \mathcal{S}),$$

which is still conservative. We shall momentarily give an alternative description of $DM(X, \mathcal{S})$ via cohomological restrictions. We will use the following

constructibility principle: If $C^\bullet \in \text{Ob } D_c^b(X)$ is a complex such that ${}^pH^k(C^\bullet)$ is \mathcal{S} -constructible for every k , then C^\bullet is \mathcal{S} -constructible.

Lemma 6.16. *The subcategory $DM(X, \mathcal{S}) \subset D^bMHM(X)$ equals the full subcategory of $D^bMHM(X)$ whose objects M^\bullet satisfy*

$$H^k(M^\bullet) \in \text{Ob } MHM(X, \mathcal{S})$$

for all k .

Proof. Let M^\bullet be an object of $DM(X, \mathcal{S})$. Thus $\text{rat } M^\bullet$ is an object of $D_c^b(X, \mathcal{S})$. It follows that ${}^pH^k(\text{rat } M^\bullet) \in \text{Ob } \text{Per}(X, \mathcal{S})$, by (6.3). As ${}^pH^k(\text{rat } M^\bullet) = \text{rat } H^k(M^\bullet)$, the latter is an object of $\text{Per}(X, \mathcal{S})$. By the definition of $MHM(X, \mathcal{S})$, $H^k(M^\bullet)$ is in $MHM(X, \mathcal{S})$. Conversely, let M^\bullet be an object of $D^bMHM(X)$ such that $H^k(M^\bullet) \in \text{Ob } MHM(X, \mathcal{S})$ for all k . Then $\text{rat } H^k(M^\bullet) \in \text{Ob } \text{Per}(X, \mathcal{S})$ for all k . So ${}^pH^k(\text{rat } M^\bullet) \in \text{Ob } \text{Per}(X, \mathcal{S})$ for all k , and this implies, by the remark preceding the lemma, that $\text{rat } M^\bullet \in \text{Ob } D_c^b(X, \mathcal{S})$. Hence $M^\bullet \in \text{Ob } DM(X, \mathcal{S})$. \square

Lemma 6.16 implies:

Lemma 6.17. *The MHM -cohomology functor*

$$H^k : D^bMHM(X) \rightarrow MHM(X)$$

restricts to a functor $H^k : DM(X, \mathcal{S}) \rightarrow MHM(X, \mathcal{S})$,

$$\begin{array}{ccc} D^bMHM(X) & \xrightarrow{H^k} & MHM(X) \\ \uparrow & & \uparrow \\ DM(X, \mathcal{S}) & \xrightarrow{H^k} & MHM(X, \mathcal{S}). \end{array}$$

The diagram

$$\begin{array}{ccc} D^bMHM(X) & \xrightarrow{H^k} & MHM(X) \\ \text{rat} \downarrow & & \downarrow \text{rat} \\ D_c^b(X) & \xrightarrow{{}^pH^k} & \text{Per}(X) \end{array}$$

commutes ([48, Lemma 14.5, p. 341]), whence the restricted diagram

$$\begin{array}{ccc} DM(X, \mathcal{S}) & \xrightarrow{H^k} & MHM(X, \mathcal{S}) \\ \text{rat} \downarrow & & \downarrow \text{rat} \\ D_c^b(X, \mathcal{S}) & \xrightarrow{{}^pH^k} & \text{Per}(X, \mathcal{S}) \end{array} \tag{6.5}$$

commutes as well.

Lemma 6.18. *Let X be a complex algebraic variety and let $g : Y \hookrightarrow X$ be a closed algebraic embedding of complex codimension c , whose underlying topological embedding is normally nonsingular and compatibly stratifiable.*

Let \mathcal{S} be an algebraic stratification of X compatible with the normal nonsingularity of the embedding and such that the induced stratification \mathcal{T} on Y is again algebraic. Then the functor $g^*[-c] : D^bMHM(X) \rightarrow D^bMHM(Y)$ restricts to a functor $g^*[-c] : MHM(X, \mathcal{S}) \rightarrow MHM(Y, \mathcal{T})$, which is exact. A similar statement applies to $g^1[c]$.

Proof. We start out by showing that

$$g^*[-c] : D^bMHM(X) \rightarrow D^bMHM(Y)$$

restricts to a functor $g^*[-c] : DM(X, \mathcal{S}) \rightarrow DM(Y, \mathcal{T})$. If M^\bullet is an object of $DM(X, \mathcal{S})$, then $\text{rat } M^\bullet \in \text{Ob } D_c^b(X, \mathcal{S})$ and thus $g^*[-c](\text{rat } M^\bullet) \in \text{Ob } D_c^b(Y, \mathcal{T})$. Now $g^*[-c](\text{rat } M^\bullet) = \text{rat}(g^*[-c]M^\bullet)$, from which we conclude that $g^*[-c]M^\bullet \in \text{Ob } DM(Y, \mathcal{T})$.

Let $P \in \text{Per}(X, \mathcal{S})$ be a perverse sheaf on X . By Lemma 6.14, $g^*[-c]P \in \text{Per}(Y, \mathcal{T})$ and hence $g^*[-c]P = {}^pH^0 g^*[-c]P$, while ${}^pH^k(g^*[-c]P) = 0$ for $k \neq 0$.

The exact functor $\text{rat} : MHM(X) \rightarrow \text{Per}(X)$ induces degreewise a functor $D^bMHM(X) \rightarrow D^b\text{Per}(X)$. The “realization” functor

$$\text{real} : D^b\text{Per}(X) \rightarrow D_c^b(X)$$

of BBD [11, p. 82, 3.1.9 and Prop. 3.1.10] satisfies $\text{real} \circ [-c] = [-c] \circ \text{real}$, see [11, p. 82, (3.1.9.3)]. Saito defines $\text{rat} : D^bMHM(X) \rightarrow D_c^b(X)$ as the composition

$$D^bMHM(X) \longrightarrow D^b\text{Per}(X) \xrightarrow{\text{real}} D_c^b(X).$$

(See [57, p. 222, Theorem 0.1].) Thus the diagram

$$\begin{array}{ccc} D^bMHM(X) & \xrightarrow{[-c]} & D^bMHM(X) \\ \text{rat} \downarrow & & \downarrow \text{rat} \\ D_c^b(X) & \xrightarrow{[-c]} & D_c^b(X) \end{array} \tag{6.6}$$

commutes. Let $M \in MHM(X, \mathcal{S})$ be a single mixed Hodge module, thought of as an object in $DM(X, \mathcal{S}) \subset D^bMHM(X)$ concentrated in degree 0. Applying the functor $g^*[-c]$, we obtain an object $g^*[-c]M \in DM(Y, \mathcal{T})$. By (6.5), $\text{rat } H^k(g^*[-c]M) = {}^pH^k(\text{rat}(g^*[-c]M))$. Since g^* on D^bMHM lifts g^* on D_c^b , we have ${}^pH^k(\text{rat}(g^*[-c]M)) = {}^pH^k(g^* \text{rat}([-c]M))$. By the commutativity of diagram (6.6), ${}^pH^k(g^* \text{rat}([-c]M)) = {}^pH^k(g^*[-c] \text{rat } M)$. Now $P = \text{rat } M$ is a perverse (\mathcal{S} -constructible) sheaf on X and hence, as observed above, ${}^pH^k(g^*[-c] \text{rat } M) = 0$ for $k \neq 0$. We conclude that

$$\text{rat } H^k(g^*[-c]M) = 0 \text{ for } k \neq 0.$$

Since $\text{rat} : MHM(Y) \rightarrow \text{Per}(Y)$ is faithful, $H^k(g^*[-c]M) = 0$ for $k \neq 0$. So in $DM(Y, \mathcal{T})$, there is a natural isomorphism $H^0(g^*[-c]M) = g^*[-c]M$, given by composing the natural quasi-isomorphisms

$$\tau_{\geq 0}\tau_{\leq 0}g^*[-c]M \longrightarrow \tau_{\geq 0}g^*[-c]M \longleftarrow g^*[-c]M.$$

This shows that $g^*[-c]M$ is canonically quasi-isomorphic to the single mixed Hodge module $H^0(g^*[-c]M) \in MHM(Y, \mathcal{T})$.

Let

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{A} & \mathcal{A}' \\ F \downarrow & & \downarrow F' \\ \mathcal{B} & \xrightarrow{B} & \mathcal{B}' \end{array}$$

be a commutative diagram of additive functors between abelian categories with F, F' exact and F' faithful. If B is exact, then A is exact. Applying this to the commutative diagram of functors

$$\begin{array}{ccc} MHM(X, \mathcal{S}) & \xrightarrow{g^*[-c]} & MHM(Y, \mathcal{T}) \\ \text{rat}_X \downarrow & & \downarrow \text{rat}_Y \\ \text{Per}(X, \mathcal{S}) & \xrightarrow{g^*[-c]} & \text{Per}(Y, \mathcal{T}), \end{array}$$

with $\text{rat}_Y, \text{rat}_X$ faithful and exact, and $g^*[-c]$ on perverse sheaves exact, we conclude that $g^*[-c] : MHM(X, \mathcal{S}) \rightarrow MHM(Y, \mathcal{T})$ is exact. The argument for $g^! [c]$ is entirely analogous. \square

Lemma 6.19. *Let X, Y be pure-dimensional complex algebraic varieties. Let $g : Y \hookrightarrow X$ be a closed algebraic (not necessarily regular) embedding of complex codimension c , whose underlying topological embedding is normally nonsingular and compatibly stratifiable. Then there is an isomorphism $g^*IC_X^H[-c] = IC_Y^H$.*

Proof. By compatible stratifiability, there exists an algebraic stratification \mathcal{S} of X such that g is normally nonsingular with respect to \mathcal{S} , and the induced stratification \mathcal{T} of Y is again algebraic. Let $U \subset X$ be the top stratum of \mathcal{S} . Since \mathcal{S} is algebraic and X is pure-dimensional, U is a Zariski-open, smooth, dense subset of X . Let $j : U \hookrightarrow X$ be the corresponding inclusion. The intersection $V = U \cap Y \hookrightarrow Y$ is the top stratum of \mathcal{T} and hence also Zariski-open, smooth (as a variety), and dense in Y . Let $j_V : V \hookrightarrow Y$ be the corresponding inclusion. By (3)(c) of Definition 3.3, the restriction $g_V : V \rightarrow U$ of g is again (algebraic and) normally nonsingular (with respect to the intrinsic stratification consisting of one stratum) of codimension c . By [57, p. 323, (4.4.3)], the cartesian square

$$\begin{array}{ccc} V & \xrightarrow{j_V} & Y \\ g_V \downarrow & & \downarrow g \\ U & \xrightarrow{j} & X \end{array}$$

has associated base change natural isomorphisms $j_{V*}g_V^! \cong g^!j_*$ and $j_{V!}g_V^* \cong g^*j_!$. Let $m = \dim_{\mathbb{C}} X$ and $n = \dim_{\mathbb{C}} Y$ so that $c = m - n$. The complexes $j_!\mathbb{Q}_U[m]$ and $j_*\mathbb{Q}_U[m]$ are \mathcal{S} -constructible, e.g. by [14, Cor. 3.11.(iii), p.

79]. Thus the objects $j_! \mathbb{Q}_U^H[m]$ and $j_* \mathbb{Q}_U^H[m]$ of $D^bMHM(X)$ belong in fact to $DM(X, \mathcal{S})$. Consequently, the canonical morphism

$$H^0 j_! \mathbb{Q}_U^H[m] \longrightarrow H^0 j_* \mathbb{Q}_U^H[m]$$

is in the abelian category $MHM(X, \mathcal{S})$. Its image $IC_X^H \in MHM(X, \mathcal{S})$ is the intersection Hodge module on X . The exactness of the functor $g^*[-c] : MHM(X, \mathcal{S}) \rightarrow MHM(Y, \mathcal{T})$ provided by Lemma 6.18 ensures that in $MHM(Y, \mathcal{T})$,

$$\begin{aligned} g^*[-c]IC_X^H &= \text{im } g^*[-c](H^0 j_! \mathbb{Q}_U^H[m] \longrightarrow H^0 j_* \mathbb{Q}_U^H[m]) \\ &= \text{im}(H^0 g^* j_! \mathbb{Q}_U^H[m-c] \longrightarrow H^0 g^* j_* \mathbb{Q}_U^H[m-c]). \end{aligned}$$

We shall show that the normal nonsingularity of g implies that the natural morphism $g^* j_* \mathbb{Q}_U^H \rightarrow j_{V*} g_V^* \mathbb{Q}_U^H$ in $DM(Y, \mathcal{T})$ is an isomorphism. As $\text{rat} : DM(Y, \mathcal{T}) \rightarrow D_c^b(Y, \mathcal{T})$ is conservative, it suffices to prove that the underlying morphism

$$g^* j_* \mathbb{Q}_U \rightarrow j_{V*} g_V^* \mathbb{Q}_U$$

is an isomorphism in $D_c^b(Y, \mathcal{T})$. As g is normally nonsingular, $g^! [c] = g^*[-c]$ on $D_c^b(X, \mathcal{S})$ and, as g_V is normally nonsingular, $g_V^! [c] = g_V^*[-c]$ on $D_c^b(U, \mathcal{S} \cap U)$. Using the above base change isomorphism $g^! j_* \cong j_{V*} g_V^!$, we get a composition of isomorphisms

$$g^* j_* \mathbb{Q}_U = g^! j_* \mathbb{Q}_U[2c] \cong j_{V*} g_V^! \mathbb{Q}_U[2c] = j_{V*} g_V^* \mathbb{Q}_U$$

which factors $g^* j_* \mathbb{Q}_U \rightarrow j_{V*} g_V^* \mathbb{Q}_U$. This establishes the claim. We deduce that the image above can be written as

$$\text{im}(H^0 j_{V!} g_V^* \mathbb{Q}_U^H[n] \rightarrow H^0 j_{V*} g_V^* \mathbb{Q}_U^H[n]),$$

which, as $g_V^* \mathbb{Q}_U^H = \mathbb{Q}_V^H$, is

$$\text{im}(H^0 j_{V!} \mathbb{Q}_V^H[n] \rightarrow H^0 j_{V*} \mathbb{Q}_V^H[n]) = IC_Y^H.$$

□

Let X and Y be pure-dimensional varieties and $g : Y \hookrightarrow X$ be a closed regular algebraic embedding whose underlying topological embedding is normally nonsingular and compatibly stratifiable. Take $M = IC_X^H[1]$ in the isomorphism (6.2), shifted by $[-c]$, to obtain an isomorphism

$$k^* \psi_p^H[1](\text{pr}_1^* IC_X^H[1])[-c] \cong g^*[-c]IC_X^H[1] \cong IC_Y^H[1], \tag{6.7}$$

using Lemma 6.19.

Proposition 6.20. *Let X, Y be pure-dimensional compact complex algebraic varieties. Let $Y \hookrightarrow X$ be an upwardly normally nonsingular embedding (Definition 6.5) with algebraic normal bundle $N = N_Y X$ and associated deformation to the normal bundle $p : Z^\circ \rightarrow \mathbb{C}$. Then*

$$\psi_p^H IC_{Z^\circ}^H = IC_N^H.$$

Proof. By Lemma 6.13, $\text{pr}_1^* IC_X^H[1] = IC_{X \times \mathbb{C}^*}^H$. Using the isomorphism (6.7), which is applicable here as $Y \hookrightarrow X$ is tight (X, Y pure-dimensional), and tight embeddings are regular and topologically normally nonsingular in a compatibly stratifiable manner, we obtain an isomorphism

$$k^* \psi_p^H(IC_{X \times \mathbb{C}^*}^H)[-c] \cong IC_Y^H.$$

In $D^bMHM(N)$, we have the adjoint relation

$$\text{Hom}_{D^bMHM(N)}(\pi_N^* M_1, M_2) = \text{Hom}_{D^bMHM(X)}(M_1, \pi_{N*} M_2),$$

see [62, p. 441, Thm. 4.1]. Thus there is an adjunction morphism

$$\pi_N^* \pi_{N*} \psi_p^H(IC_{X \times \mathbb{C}^*}^H) \xrightarrow{\text{adj}} \psi_p^H(IC_{X \times \mathbb{C}^*}^H) \tag{6.8}$$

in $D^bMHM(N)$. Taking rat , one obtains the adjunction morphism

$$\pi_N^* \pi_{N*} \psi_p[-1](IC_{X \times \mathbb{C}^*}) \xrightarrow{\text{adj}} \psi_p[-1](IC_{X \times \mathbb{C}^*}) \tag{6.9}$$

in $D_c^b(N)$. As $Y \hookrightarrow X$ is upwardly normally nonsingular, and X, Y compact, Propositions 6.8, 6.9, and Lemma 6.10 all apply. We obtain an open neighborhood U of $N = p^{-1}(0)$ in Z° together with a locally cone-like topological stratification \mathcal{S}_U of U such that the central fiber $N \subset U$ is a union of strata, and those strata are given by $\mathcal{S}_\alpha \cap N = \pi_N^{-1} Y_\alpha$. Taking nearby cycles is a local operation: if $p' : U \rightarrow \mathbb{C}$ denotes the restriction of p to U , then $\psi_p(-) = \psi_{p'}(-|_U)$. In particular, $\psi_p[-1](IC_{X \times \mathbb{C}^*}) = \psi_{p'}[-1](IC_U)$. The complex IC_U is constructible with respect to the locally cone-like topological stratification \mathcal{S}_U , by topological invariance of intersection homology, see also [14, V, Cor. 4.18, p. 95]. Thus by Lemma 6.11, $\psi_{p'}[-1](IC_U)$ (and hence also $\psi_p[-1](IC_{X \times \mathbb{C}^*})$) is constructible with respect to the strata $\mathcal{S}_\alpha \cap N = \pi_N^{-1} Y_\alpha$. In particular, $\psi_p[-1](IC_{X \times \mathbb{C}^*})$ is cohomologically locally constant with respect to the strata $\pi_N^{-1} Y_\alpha$ so that Vietoris-Begle implies that (6.9) is an isomorphism, [14, p. 164, Lemma 10.14(i)]. Since rat is conservative on $D^bMHM(N)$ by Lemma 5.10, the adjunction morphism (6.8) is an isomorphism

$$\pi_N^* \pi_{N*} \psi_p^H(IC_{X \times \mathbb{C}^*}^H) \cong \psi_p^H(IC_{X \times \mathbb{C}^*}^H).$$

Applying $k^*[-c]$ yields isomorphisms

$$k^* \pi_N^* \pi_{N*} \psi_p^H(IC_{X \times \mathbb{C}^*}^H)[-c] \cong k^* \psi_p^H(IC_{X \times \mathbb{C}^*}^H)[-c] \cong IC_Y^H.$$

Since $k^* \pi_N^*$ is the identity, this is an isomorphism

$$\pi_{N*} \psi_p^H(IC_{X \times \mathbb{C}^*}^H) \cong IC_Y^H[c].$$

Applying π_N^* , we get an isomorphism

$$\pi_N^* \pi_{N*} \psi_p^H(IC_{X \times \mathbb{C}^*}^H) \cong \pi_N^* IC_Y^H[c].$$

By Lemma 6.13, $\pi_N^* IC_Y^H[c] = IC_N^H$. Hence $\psi_p^H(IC_{X \times \mathbb{C}^*}^H) \cong IC_N^H$. □

Recall that a flat morphism $f : X \rightarrow Y$ gives rise to a flat pullback $f_K^* : K_0^{\text{alg}}(Y) \rightarrow K_0^{\text{alg}}(X)$.

Proposition 6.21. *Let Y be a complex algebraic variety and $\pi : N \rightarrow Y$ an algebraic vector bundle over Y . For any coherent sheaf \mathcal{F} on Y ,*

$$T_y^*(T_\pi) \cap \pi_{\text{BM}}^* \text{td}_{1+y}[\mathcal{F}] = \text{td}_{1+y}(\lambda_y(T_\pi^*) \cap \pi_K^*[\mathcal{F}]).$$

Proof. We define an Adams-type operation ψ^j , which operates on a cohomology class ξ of degree $2j$ by $\psi^j(\xi) = (1 + y)^j \cdot \xi$. Similarly, a homological Adams-type operation is given by $\psi_k(x) = (1 + y)^{-k} \cdot x$ on a degree- $2k$ homology class x . The behavior of these operations in a cap product of a degree- $2(j - k)$ class ξ and a degree $2j$ -class x is described by the formula

$$\psi_k(\xi \cap x) = \psi^{j-k}(\xi) \cap \psi_j(x).$$

Let r be the complex rank of N . Note that if x has degree $2(j - r)$, then $\pi_{\text{BM}}^*(x)$ has degree $2j$. Under smooth pullback, one then has

$$(1 + y)^r \psi_j \pi_{\text{BM}}^*(x) = \pi_{\text{BM}}^*(\psi_{j-r}x).$$

By the definition of td_{1+y} and (5.4),

$$\begin{aligned} \text{td}_{1+y}(\lambda_y(T_\pi^*) \cap \pi_K^*[\mathcal{F}]) &= \sum_{k \geq 0} \psi_k(\tau_*(\lambda_y(T_\pi^*) \cap \pi_K^*[\mathcal{F}]))_k \\ &= \sum_{k \geq 0} \psi_k(\text{ch}^*(\lambda_y(T_\pi^*)) \cap \tau_* \pi_K^*[\mathcal{F}]))_k. \end{aligned}$$

By BFM-VRR for smooth pullbacks (Proposition 5.6), this equals

$$\sum_{k \geq 0} \psi_k(\text{ch}^*(\lambda_y(T_\pi^*)) \cup \text{td}^*(T_\pi) \cap \pi_{\text{BM}}^* \tau_*[\mathcal{F}]))_k,$$

which by Proposition 5.4 is

$$\sum_{k \geq 0} \psi_k(\tilde{T}_y^*(T_\pi) \cap \pi_{\text{BM}}^* \tau_*[\mathcal{F}]))_k.$$

Computing the degree- $2k$ component in this expression, we get

$$\sum_{k, j \geq 0} \psi_k \left(\tilde{T}_y^{j-k}(T_\pi) \cap (\pi_{\text{BM}}^* \tau_*[\mathcal{F}])_j \right) = \sum_{k, j \geq 0} \psi^{j-k} \tilde{T}_y^{j-k}(T_\pi) \cap \psi_j(\pi_{\text{BM}}^* \tau_*[\mathcal{F}]))_j.$$

According to Proposition 5.3, this can be written in terms of T_y^* as

$$\begin{aligned}
 & \sum_{k,j \geq 0} \psi^{j-k} (1+y)^{r-(j-k)} T_y^{j-k}(T_\pi) \cap \psi_j(\pi_{\text{BM}}^* \tau_*[\mathcal{F}])_j \\
 &= \sum_{k,j \geq 0} (1+y)^r T_y^{j-k}(T_\pi) \cap \psi_j(\pi_{\text{BM}}^* \tau_*[\mathcal{F}])_j \\
 &= \sum_{k,j \geq 0} T_y^{j-k}(T_\pi) \cap (1+y)^r \psi_j(\pi_{\text{BM}}^* \tau_*[\mathcal{F}])_j \\
 &= \sum_{k,j \geq 0} T_y^{j-k}(T_\pi) \cap \pi_{\text{BM}}^*(\psi_{j-r} \tau_{j-r}[\mathcal{F}]) \\
 &= \sum_{i \geq 0} T_y^i(T_\pi) \cap \pi_{\text{BM}}^* \sum_{k \geq 0} \psi_k \tau_k[\mathcal{F}] \\
 &= T_y^*(T_\pi) \cap \pi_{\text{BM}}^* \text{td}_{1+y}[\mathcal{F}].
 \end{aligned}$$

□

Theorem 6.22. *Let Y be a complex algebraic variety and $\pi : N \rightarrow Y$ an algebraic vector bundle over Y . For $M \in D^b\text{MHM}(Y)$,*

$$T_y^*(T_\pi) \cap \pi_{\text{BM}}^* \text{MHT}_{y^*}[M] = \text{MHT}_{y^*}(\pi_{\text{MHM}}^* M).$$

Proof. This follows from Schürmann’s MHC_y -Verdier-Riemann-Roch formula (Proposition 5.2) and Proposition 6.21:

$$\begin{aligned}
 \text{MHT}_{y^*}(\pi_{\text{MHM}}^* M) &= \text{td}_{1+y} \text{MHC}_y(\pi_{\text{MHM}}^* M) \\
 &= \text{td}_{1+y}(\lambda_y(T_\pi^*) \cap \pi_K^* \text{MHC}_y[M]) \\
 &= T_y^*(T_\pi) \cap \pi_{\text{BM}}^* \text{td}_{1+y} \text{MHC}_y[M] \\
 &= T_y^*(T_\pi) \cap \pi_{\text{BM}}^* \text{MHT}_{y^*}[M].
 \end{aligned}$$

□

Proposition 6.23. *If Y is a pure-dimensional complex algebraic variety and $\pi : N \rightarrow Y$ an algebraic vector bundle over Y , then*

$$T_y^*(T_\pi) \cap \pi_{\text{BM}}^* \text{IT}_{y^*}(Y) = \text{IT}_{y^*}(N).$$

Proof. Using Theorem 6.22 and Lemma 6.13,

$$\begin{aligned}
 T_y^*(T_\pi) \cap \pi_{\text{BM}}^* \text{IT}_{y^*}(Y) &= T_y^*(T_\pi) \cap \pi_{\text{BM}}^* \text{MHT}_{y^*}[\text{IC}_Y^H[-n]] \\
 &= \text{MHT}_{y^*}(\pi_{\text{MHM}}^* [\text{IC}_Y^H[-n]]) \\
 &= \text{MHT}_{y^*}[\text{IC}_N^H[-m]] = \text{IT}_{y^*}(N).
 \end{aligned}$$

□

Given a closed algebraic embedding $Y \hookrightarrow X$, a specialization map

$$\text{Sp}_{\text{BM}} : H_*^{\text{BM}}(X) \longrightarrow H_*^{\text{BM}}(N_Y X), \tag{6.10}$$

on Borel-Moore homology, where $N = N_Y X$ is the normal cone of Y in X , has been constructed by Verdier in [70, §8]. As before, let $p : Z^\circ \rightarrow \mathbb{C}$ be

the deformation to the normal cone, obtained by restricting $p_Z : Z \rightarrow \mathbb{C}$. It will be convenient to embed the family Z as a Zariski open dense subset into the following family W : The embedding of Y in X gives rise to an embedding $Y \times 0 \hookrightarrow X \times 0 \hookrightarrow X \times \mathbb{P}^1$. Let $W = \text{Bl}_{Y \times 0}(X \times \mathbb{P}^1)$. There is a flat morphism $p_W : W \rightarrow \mathbb{P}^1$, whose special fiber is given by

$$p_W^{-1}(0) = p_Z^{-1}(0) = \text{Bl}_Y X \cup_{\mathbb{P}(N)} \mathbb{P}(N \oplus 1).$$

Let $W^\circ = W - \text{Bl}_Y X$. Then p_W restricts to a morphism $p : W^\circ \rightarrow \mathbb{P}^1$, whose special fiber is $p^{-1}(0) = N$. The open complement $\mathbb{P}^1 - \{0\} \cong \mathbb{C}$ has preimage $p^{-1}(\mathbb{C}) \cong X \times \mathbb{C}$. As blow-ups are determined locally, the open dense embedding $X \times \mathbb{C} \subset X \times \mathbb{P}^1$ induces an open dense embedding $Z \subset W$ and an open dense embedding $Z^\circ \subset W^\circ$. The advantage of W over Z is that the open complement \mathbb{C} of $\{0\}$ in \mathbb{P}^1 is contractible and has the structure of a complex vector space, while neither is true for \mathbb{C}^* . The factor projection $\text{pr}_1 : X \times \mathbb{C} \rightarrow X$ induces a smooth pullback $\text{pr}_{1,\text{BM}}^* : H_*^{\text{BM}}(X) \rightarrow H_{*+2}^{\text{BM}}(X \times \mathbb{C})$ on Borel-Moore homology. (We continue to use real, not complex, indexing for Borel-Moore homology.) By the Thom isomorphism theorem, this suspension map is an isomorphism. Thus we may invert it and define

$$\lim_{t \rightarrow 0} := \text{Sp}_{\text{BM}} \circ (\text{pr}_{1,\text{BM}}^*)^{-1} : H_{*+2}^{\text{BM}}(X \times \mathbb{C}) \longrightarrow H_*^{\text{BM}}(N). \tag{6.11}$$

The closed embedding $i : N \hookrightarrow W^\circ$ is regular (with trivial algebraic normal bundle pulled up from the trivial normal bundle of $\{0\}$ in \mathbb{P}^1). Thus there is a Gysin homomorphism

$$i^! = i_{\text{BM}}^* : H_{*+2}^{\text{BM}}(W^\circ) \longrightarrow H_*^{\text{BM}}(N).$$

The following proposition is a special case of Schürmann’s specialization for motivic Hirzebruch classes ([64]), a proof is sketched in [44, Theorem 4.6, Corollary 4.7].

Proposition 6.24. *Let $Y \hookrightarrow X$ be a closed algebraic embedding with normal cone $N = N_Y X$ and associated deformation to the normal cone $p : W^\circ \rightarrow \mathbb{P}^1$. Then the diagram*

$$\begin{array}{ccc} K_0\text{MHM}(W^\circ) & \xrightarrow{\psi_p^H} & K_0\text{MHM}(N) \\ \text{MHT}_{1*} \downarrow & & \downarrow \text{MHT}_{1*} \\ H_{*+2}^{\text{BM}}(W^\circ; \mathbb{Q}) & \xrightarrow{i^!} & H_*^{\text{BM}}(N; \mathbb{Q}) \end{array}$$

commutes. (Actually, this holds more generally for MHT_{y} , but we use it only for $y = 1$.)*

For a complex variety V , $A_k(V)$ denotes the Chow group of algebraic k -cycles in V modulo rational equivalence.

Lemma 6.25. *Let $Y \hookrightarrow X$ be a closed algebraic embedding with normal cone $N = N_Y X$ and associated deformation $p : W^\circ \rightarrow \mathbb{P}^1$ to the normal cone. Then the diagram*

$$\begin{array}{ccc}
 H_{*+2}^{\text{BM}}(W^\circ) & \xrightarrow{i^!} & H_*^{\text{BM}}(N) \\
 & \searrow^{j_{\text{BM}}^*} & \nearrow^{\lim_{t \rightarrow 0}} \\
 & & H_{*+2}^{\text{BM}}(X \times \mathbb{C})
 \end{array}$$

commutes on algebraic cycles, where j_{BM}^ denotes restriction of a Borel-Moore cycle to an open subset, i.e. the diagram commutes on the image of the cycle map $\text{cl} : A_{*+1}(W^\circ) \rightarrow H_{2*+2}^{\text{BM}}(W^\circ)$.*

Proof. There is a short exact sequence

$$A_{*+1}(N) \xrightarrow{i_*} A_{*+1}(W^\circ) \xrightarrow{j_A^*} A_{*+1}(X \times \mathbb{C}) \longrightarrow 0,$$

where the map i_* is proper pushforward under the proper map $i : N \hookrightarrow W^\circ$, and j_A^* is restriction to an open subset. Let $i_A^! : A_{*+1}(W^\circ) \rightarrow A_*(N)$ denote the Gysin map for divisors. Then the composition $i_A^! \circ i_*$ is zero, since the algebraic normal bundle of N in W° is trivial. (Intuitively, the triviality of the normal bundle implies that any cycle in N can be pushed off of N in W° and thus its transverse intersection with N is zero.) By exactness, we may identify $A_{*+1}(X \times \mathbb{C})$ with the cokernel of i_* . Then $\text{im } i_* \subset \ker i_A^!$ implies that $i_A^!$ induces uniquely a map

$$\lim_{t \rightarrow 0}^A : A_{*+1}(X \times \mathbb{C}) \longrightarrow A_*(N)$$

such that

$$\begin{array}{ccc}
 A_{*+1}(W^\circ) & \xrightarrow{i_A^!} & A_*(N) \\
 & \searrow^{j_A^*} & \nearrow^{\lim_{t \rightarrow 0}^A} \\
 & & A_{*+1}(X \times \mathbb{C})
 \end{array} \tag{6.12}$$

commutes. Note that this is the diagram in the statement of the lemma, only on Chow instead of Borel-Moore. To finish the proof, one uses that the Gysin map of a regular embedding, as well as smooth pullback, commute with the cycle map from Chow to Borel-Moore. The Chow level specialization map $\text{Sp}_A : A_*(X) \rightarrow A_*(N)$ is defined to be the composition

$$A_*(X) \xrightarrow{\text{pr}_{1,A}^*} A_{*+1}(X \times \mathbb{C}) \xrightarrow{\lim_{t \rightarrow 0}^A} A_*(N),$$

see [29, p. 89, Proof of Prop. 5.2] or [63, p. 15, (25)], or [70, p. 198], and is known to commute with the cycle map. \square

Proposition 6.26. *Let X, Y be pure-dimensional compact complex algebraic varieties. If $g : Y \hookrightarrow X$ is an upwardly normally nonsingular embedding with algebraic normal bundle $N_Y X$, then*

$$\mathrm{Sp}_{\mathrm{BM}} IT_{1*}(X) = IT_{1*}(N_Y X).$$

Proof. By Definition (6.11), $\mathrm{Sp}_{\mathrm{BM}} IT_{1*}(X) = \lim_{t \rightarrow 0} \mathrm{pr}_{1, \mathrm{BM}}^* IT_{1*}(X)$. We regard $\pi := \mathrm{pr}_1 : X \times \mathbb{C} \rightarrow X$ as the projection of the trivial line bundle 1_X over X . Then $T_\pi = \pi^*(1_X) = 1_{X \times \mathbb{C}}$ and hence, using (5.3), $T_1^*(T_\pi) = T_1^*(1_{X \times \mathbb{C}}) = L^*(1_{X \times \mathbb{C}}) = 1$. By Proposition 6.23,

$$\pi_{\mathrm{BM}}^* IT_{1*}(X) = 1 \cap \pi_{\mathrm{BM}}^* IT_{1*}(X) = T_1^*(T_\pi) \cap \pi_{\mathrm{BM}}^* IT_{1*}(X) = IT_{1*}(X \times \mathbb{C}).$$

With $n = \dim_{\mathbb{C}} X$, we thus have

$$\mathrm{Sp}_{\mathrm{BM}} IT_{1*}(X) = \lim_{t \rightarrow 0} IT_{1*}(X \times \mathbb{C}) = \lim_{t \rightarrow 0} MHT_{1*}(IC_{X \times \mathbb{C}}^H[-n-1]).$$

Let j denote the open embedding $j : X \times \mathbb{C} \hookrightarrow W^\circ$ associated to the deformation to the normal bundle; $j^{-1}IC_{W^\circ}^H = IC_{X \times \mathbb{C}}^H$. Since the transformation MHT_{y*} commutes with restriction to open subsets,

$$\mathrm{Sp}_{\mathrm{BM}} IT_{1*}(X) = \lim_{t \rightarrow 0} j_{\mathrm{BM}}^* MHT_{1*}(IC_{W^\circ}^H[-n-1]).$$

By Remark 5.12, the class $MHT_{1*}(IC_{W^\circ}^H[-n-1]) = IT_{1*}(W^\circ)$ is algebraic. Hence, by Lemma 6.25,

$$\lim_{t \rightarrow 0} j_{\mathrm{BM}}^* MHT_{1*}(IC_{W^\circ}^H[-n-1]) = i^! MHT_{1*}(IC_{W^\circ}^H[-n-1]).$$

By Proposition 6.24,

$$i^! MHT_{1*}(IC_{W^\circ}^H[-n-1]) = MHT_{1*} \psi_p'^H(IC_{W^\circ}^H[-n-1]).$$

Finally, by Proposition 6.20 (which requires upward normal nonsingularity of the embedding, pure-dimensionality and compactness),

$$\psi_p'^H(IC_{W^\circ}^H[-n-1]) = \psi_p^H[1](IC_{W^\circ}^H[-n-1]) = \psi_p^H(IC_{W^\circ}^H[-n]) = IC_N^H[-n].$$

We conclude that $\mathrm{Sp}_{\mathrm{BM}} IT_{1*}(X) = MHT_{1*}(IC_N^H[-n]) = IT_{1*}(N)$, since $\dim_{\mathbb{C}} N = n$. □

The following cap product formula for homological Gysin maps is standard, see e.g. [12, Ch. V, §6.2 (c), p. 35] or [7, Lemma 5, p. 613].

Lemma 6.27. *Let Y be a complex algebraic variety and let $\pi : N \rightarrow Y$ be an algebraic vector bundle projection. If $\eta \in H^*(Y)$ and $a \in H_*^{\mathrm{BM}}(Y)$ are classes in even degrees, then*

$$\pi_{\mathrm{BM}}^*(\eta \cap a) = \pi^*(\eta) \cap \pi_{\mathrm{BM}}^*(a).$$

Proposition 6.28. *Let Y be a pure-dimensional complex algebraic variety and let $k : Y \hookrightarrow N$ be the zero section of an algebraic vector bundle projection $\pi : N \rightarrow Y$. Then*

$$k^! IT_{y*}(N) = T_y^*(N) \cap IT_{y*}(Y).$$

Proof. By the Thom isomorphism theorem, the Gysin pullback $k^! = k_{\text{BM}}^!$ and the smooth pullback π_{BM}^* are inverse isomorphisms on Borel-Moore homology, see Chriss-Ginzburg [27, Prop. 2.6.43, p. 107]. The relative tangent bundle of π is given by $T_\pi = \pi^*N$. Since T_y^* is a natural characteristic class in cohomology, $T_y^*(T_\pi) = \pi^*T_y^*(N)$. Thus, using Proposition 6.23 and Lemma 6.27, we get

$$\begin{aligned} k_{\text{BM}}^! IT_{y*}(N) &= k_{\text{BM}}^!(T_y^*(T_\pi) \cap \pi_{\text{BM}}^* IT_{y*}(Y)) \\ &= k_{\text{BM}}^!(\pi^* T_y^*(N) \cap \pi_{\text{BM}}^* IT_{y*}(Y)) \\ &= k_{\text{BM}}^! \pi_{\text{BM}}^*(T_y^*(N) \cap IT_{y*}(Y)) = T_y^*(N) \cap IT_{y*}(Y). \end{aligned}$$

□

Lemma 6.29. *Let $g : Y \hookrightarrow X$ be a closed regular embedding of possibly singular varieties. Let $N = N_Y X$ denote the algebraic normal bundle and let c be the complex codimension of Y in X . The Gysin map $g^! : H_*^{\text{BM}}(X) \rightarrow H_{*-2c}^{\text{BM}}(Y)$ factors as*

$$\begin{array}{ccc} H_*^{\text{BM}}(X) & \xrightarrow{g^!} & H_{*-2c}^{\text{BM}}(Y) \\ & \searrow \text{Sp}_{\text{BM}} & \nearrow k^! \\ & H_*^{\text{BM}}(N), & \end{array}$$

where $k^!$ is the Gysin restriction to the zero section and Sp_{BM} is Verdier’s Borel-Moore specialization map (6.10).

Proof. This is simply Verdier’s description of the Gysin map as given in [70, p. 222], observing that $(\pi_{\text{BM}}^*)^{-1} = k_{\text{BM}}^!$ according to the Thom isomorphism theorem on Borel-Moore homology. □

Theorem 6.30. *Let X, Y be pure-dimensional compact complex algebraic varieties and let $g : Y \hookrightarrow X$ be an upwardly normally nonsingular embedding (Definition 6.5). Let $N = N_Y X$ be the algebraic normal bundle of g and let ν denote the topological normal bundle of the topologically normally nonsingular inclusion underlying g . Then*

$$g^! IT_{1,*}(X) = L^*(N) \cap IT_{1,*}(Y) = L^*(\nu) \cap IT_{1,*}(Y).$$

Proof. By Lemma 6.29, $g^! IT_{1,*}(X) = k^! \text{Sp}_{\text{BM}} IT_{1,*}(X)$. Proposition 6.26, which requires (pure-dimensionality and) upward normal nonsingularity of the embedding, yields $k^! \text{Sp}_{\text{BM}} IT_{1,*}(X) = k^! IT_{1,*}(N)$, while by Proposition 6.28, $k^! IT_{1,*}(N) = T_1^*(N) \cap IT_{1,*}(Y)$. Finally, we recall that $T_1^*(N) = L^*(N)$, (5.3). Since g is tight, there is a bundle isomorphism $N \cong \nu$ of topological vector bundles. Hence $L^*(N) = L^*(\nu)$. □

References

- [1] AKIN, ETHAN. Manifold phenomena in the theory of polyhedra. *Trans. Amer. Math. Soc.* **143** (1969), 413–473. MR0253329, Zbl 0195.53702, doi:10.1090/S0002-9947-1969-0253329-1. 1288
- [2] ALUFFI, PAOLO; MIHALCEA, LEONARDO C. Chern classes of Schubert cells and varieties. *J. Algebraic Geom.* **18** (2009), no. 1, 63–100. MR2448279, Zbl 1163.14004, arXiv:math/0607752, doi:10.1090/S1056-3911-08-00482-7. 1274, 1276, 1301
- [3] BANAGL, MARKUS. Extending intersection homology type invariants to non-Witt spaces. *Mem. Amer. Math. Soc.* **160** (2002), no. 760, x+83 pp. MR1937924, Zbl 1030.55005, doi:10.1090/memo/0760. 1274
- [4] BANAGL, MARKUS. The L -class of non-Witt spaces. *Annals of Math. (2)* **163** (2006), no. 3, 743–766. MR2215133, Zbl 1101.57013, doi:10.4007/annals.2006.163.743. 1274
- [5] BANAGL, MARKUS. Topological invariants of stratified spaces. Springer Monographs in Mathematics. *Springer, Berlin*, 2007. xii+259 pp. ISBN: 978-3-540-38585-1; 3-540-38585-1. MR2286904, Zbl 1108.55001, doi:10.1007/3-540-38587-8. 1320, 1321
- [6] BANAGL, MARKUS. Topological and Hodge L -classes of singular covering spaces and varieties with trivial canonical class. *Geometriae Dedicata* **199** (2019), 189–224. MR3928798, Zbl 07045068, arXiv:1706.03578, doi:10.1007/s10711-018-0345-2. 1277
- [7] BANAGL, MARKUS; CAPPELL, SYLVAIN E.; SHANESON, JULIUS L. Computing twisted signatures and L -classes of stratified spaces. *Math. Ann.* **326** (2003), no.3, 589–623. MR1992279, Zbl 1034.32021, doi:10.1007/s00208-003-0439-4. 1280, 1331
- [8] BANAGL, MARKUS; LAURES, GERD; MCCLURE, JAMES E. The L -homology fundamental class for IP -spaces and the stratified Novikov conjecture. *Selecta Math. (N.S.)* **25** (2019), no. 1, Paper No. 7, 104 pp. MR3908854, Zbl 1416.55001, doi:10.1007/s00029-019-0458-y, arXiv:1404.5395. 1275, 1281, 1287, 1288, 1297, 1298, 1301
- [9] BAUM, PAUL F.; FULTON, WILLIAM; MACPHERSON, ROBERT D. Riemann–Roch for singular varieties. *Inst. Hautes Études Sci. Publ. Math.* **45** (1975), 101–145. MR0412190, Zbl 0332.14003, doi:10.1007/BF02684299. 1309
- [10] BAUM, PAUL; FULTON, WILLIAM; MACPHERSON, ROBERT. Riemann–Roch and topological K -theory for singular varieties. *Acta Math.* **143** (1979), no. 3-4, 155–192. MR0549773, Zbl 0474.14004, doi:10.1007/BF02392091. 1308, 1309
- [11] BEILINSON, ALEKSANDR A.; BERNSTEIN, JOSEPH N.; DELIGNE, PIERRE. Faisceaux pervers. *Analysis and topology on singular spaces, I* (Luminy, 1981), 5–171, Astérisque, 100, *Soc. Math. France, Paris*, 1982. MR0751966, Zbl 0536.14011. 1323
- [12] BOARDMAN, J. M. Stable homotopy theory. (Summary). Mimeographed notes, *University of Warwick*, 1965–66. 1280, 1331
- [13] DE BOBADILLA, J. F.; PALLARES, I. The Brasselet–Schürmann–Yokura conjecture on L -classes of singular varieties. Preprint, 2020. arXiv:2007.11537. 1274
- [14] BOREL, ARMAND; ET AL. Intersection cohomology. Notes on the seminar held at the University of Bern, Bern, 1983. Progress in Mathematics, 50. Swiss seminars. *Birkhäuser Boston, Inc. Boston, MA*,, 1984. x+234 pp. ISBN: 0-8176-3274-3. MR0788171, Zbl 0553.14002. 1325, 1326
- [15] BRASSELET, JEAN-PAUL; SCHÜRMAN, JÖRG; YOKURA, SHOJI. Hirzebruch classes and motivic Chern classes for singular spaces. *J. Topol. Anal.* **2** (2010), no. 1, 1–55. MR2646988, Zbl 1190.14009, doi:10.1142/S1793525310000239. 1273, 1274, 1276, 1306, 1310, 1311
- [16] BUONCRISTIANO, SANDRO; ROURKE, COLIN P.; SANDERSON, BRIAN J. A geometric approach to homology theory. London Mathematical Society Lecture Note Series, 18. *Cambridge University Press, Cambridge-New York-Melbourne*, 1976. iii+149

- pp. MR0413113, Zbl 0315.55002, doi:10.1017/CBO9780511662669. 1275, 1281, 1287, 1288, 1289, 1290, 1292, 1294
- [17] CAPPELL, SYLVAIN E.; SHANESON, JULIUS L. Genera of algebraic varieties and counting lattice points. *Bull. Amer. Math. Soc. (N.S.)* **30** (1994), no. 1, 62–69. MR1217352, Zbl 0847.14010, arXiv:math/9401219, doi:10.1090/S0273-0979-1994-00436-7. 1274, 1311
- [18] CAPPELL, SYLVAIN E.; SHANESON, JULIUS L. Stratifiable maps and topological invariants. *J. Amer. Math. Soc.* **4** (1991), no. 3, 521–551. MR1102578, Zbl 0746.32016, doi:10.1090/S0894-0347-1991-1102578-4. 1280
- [19] CAPPELL, SYLVAIN E.; SHANESON, JULIUS L.; WEINBERGER, SHMUEL. Classes topologiques caractéristiques pour les actions de groupes sur les espaces singuliers. *C. R. Acad. Sci. Paris Sér. I Math.* **313** (1991), no. 5, 293–295. MR1126399, Zbl 0742.57023.
- [20] CAPPELL, SYLVAIN E.; MAXIM, LAURENTIU G.; SCHÜRMAN, JÖRG; SHANESON, JULIUS L. Characteristic classes of complex hypersurfaces, *Adv. Math.* **225** (2010), no. 5, 2616–2647. MR2680178, Zbl 1209.32016, doi:10.1016/j.aim.2010.05.007. 1277, 1311
- [21] CAPPELL, SYLVAIN E.; MAXIM, LAURENTIU G.; SCHÜRMAN, JÖRG; SHANESON, JULIUS L. Equivariant characteristic classes of singular complex algebraic varieties. *Comm. Pure Appl. Math.* **65** (2012), no. 12, 1722–1769. MR2982640, Zbl 1276.14009, arXiv:1004.1844, doi:10.1002/cpa.21427. 1277, 1310
- [22] CAPPELL, SYLVAIN E.; MAXIM, LAURENTIU G.; SHANESON, JULIUS L. Hodge genera of algebraic varieties. I. *Comm. Pure Appl. Math.* **61** (2008), no. 3, 422–449. MR2376848, Zbl 1129.14018, arXiv:math/0606655. doi:10.1002/cpa.20202. 1311
- [23] CAPPELL, SYLVAIN E.; WEINBERGER, SHMUEL. Classification de certains espaces stratifiés. *C. R. Acad. Sci. Paris Sér. I Math.* **313** (1991), no. 6, 399–401. MR1126422, Zbl 0733.57011. 1274
- [24] CASSON, ANDREW J. Generalisations and applications of block bundles. *The Hauptvermutung book*, 33–67, *K-Monogr. Math.*, 1, *Kluwer Acad. Publ., Dordrecht*, 1996. MR1434102, Zbl 0871.57022. 1282
- [25] CHEEGER, JEFF. Spectral geometry of singular Riemannian spaces. *J. Differential Geom.* **18** (1983), no. 4, 575–657. MR0730920, Zbl 0529.58034, doi:10.4310/jdg/1214438175. 1278
- [26] CHEEGER, JEFF; GORESKEY, MARK; MACPHERSON, ROBERT. L^2 -cohomology and intersection homology of singular algebraic varieties. *Seminar on Differential Geometry*, pp. 303–340, *Ann. of Math. Stud.*, 102, *Princeton Univ. Press, Princeton, N.J.*, 1982. MR0645745, Zbl 0503.14008, doi:10.1515/9781400881918. 1313
- [27] CHRIS, NEIL; GINZBURG, VICTOR. Representation theory and complex geometry. Reprint of the 1997 edition. *Modern Birkhäuser Classics*, *Birkhäuser Boston, Ltd., Boston, MA*, 2010. x+495 pp. ISBN: 978-0-8176-4937-1. MR2838836, Zbl 1185.22001, doi:10.1007/978-0-8176-4938-8. 1332
- [28] CURRAN, STEPHEN J. Intersection homology and free group actions on Witt spaces. *Michigan Math. J.* **39** (1992), no. 1, 111–127. MR1137893, Zbl 0755.57017, doi:10.1307/mmj/1029004459. 1294
- [29] FULTON, WILLIAM. Intersection theory. Second edition. *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*, 2. *Springer-Verlag, Berlin*, 1998. xiv+470 pp. ISBN: 3-540-62046-X; 0-387-98549-2. MR1644323, Zbl 0885.14002, doi:10.1007/978-1-4612-1700-8. 1309, 1330

- [30] GORESKEY, MARK; MACPHERSON, ROBERT. Stratified Morse theory. Ergebnisse der Mathematik und ihrer Grenzgebiet. 3. Folge Band 14. *Springer-Verlag, Berlin*, 1988. xiv+272 pp. ISBN: 3-540-17300-5. MR0932724, Zbl 0639.14012. 1312
- [31] GORESKEY, MARK; MACPHERSON, ROBERT. Intersection homology theory. *Topology* **19** (1980), no. 2, 135–162. MR0572580, Zbl 0448.55004, doi:10.1016/0040-9383(80)90003-8. 1273, 1274
- [32] GORESKEY, MARK; MACPHERSON, ROBERT. Intersection homology. II. *Invent. Math.* **72** (1983), no. 1, 77–129. MR0696691, Zbl 0529.55007, doi:10.1007/BF01389130. 1278
- [33] GORESKEY, MARK; SIEGEL, PAUL H. Linking pairings on singular spaces. *Comment. Math. Helv.* **58** (1983), no. 1, 96–110. MR0699009, Zbl 0529.55008. 1297
- [34] HIRZEBRUCH, FRIEDRICH. Neue topologische Methoden in der algebraischen Geometrie. Ergebnisse der Mathematik und ihrer Grenzgebiete, Heft 9. *Springer-Verlag, Berlin-Göttingen-Heidelberg*, 1956. viii+165 pp. MR0082174, Zbl 0070.16302. 1274, 1307
- [35] HIRZEBRUCH, FRIEDRICH.; BERGER, THOMAS; JUNG, RAINER. Manifolds and modular forms. Aspects of Math., E20. *Friedr. Vieweg & Sohn, Braunschweig*, 1992. xii+211 pp. ISBN: 3-528-06414-5. MR1189136, Zbl 0767.57014. 1308
- [36] ILLUSIE, LUC; TEMKIN, MICHAEL. Exposé X. Gabber’s modification theorem (log smooth case). Travaux de Gabber sur l’uniformisation locale et la cohomologie étale des schémas quasi-excellents. *Astérisque*, No. 363-364 (2014), 167–212. ISBN: 978-2-85629-790-2. MR3329779, Zbl 1327.14072. 1312, 1314
- [37] LAKSHMIBAI, VENKATRAMANI; WEYMAN, JERZY M. Multiplicities of points on a Schubert variety in a minuscule G/P . *Adv. Math.* **84** (1990), no. 2, 179–208. MR1080976, Zbl 0729.14037, doi:10.1016/0001-8708(90)90044-N. 1302
- [38] LASHOF, RICHARD K.; ROTHENBERG, MELVIN G. Microbundles and smoothing. *Topology* **3** (1965), 357–388. MR0176480, Zbl 0141.40601 doi:10.1016/0040-9383(65)90003-0. 1282
- [39] LAURES, GERD; MCCLURE, JAMES E. Multiplicative properties of Quinn spectra. *Forum Math.* **26** (2014), no.4, 1117–1185. MR3228927, Zbl 1311.55013, arXiv:0907.2367, doi:10.1515/forum-2011-0086. 1288
- [40] LIBGOBER, ANATOLY; MAXIM, LAURENȚIU. Hodge polynomials of singular hyper-surfaces. *Michigan Math. J.* **60** (2011), no. 3, 661–673. MR2861093, Zbl 1229.32019, doi:10.1307/mmj/1320763053. 1277
- [41] MAXIM, LAURENȚIU. Intersection homology & perverse sheaves (with applications to singularities), Graduate Texts in Mathematics, 281. *Springer International Publishing*, 2019. xv+270 pp. ISBN: 978-3-030-27643-0; 978-3-030-27644-7. Zbl 07105770. doi:10.1007/978-3-030-27644-7. 1310
- [42] MAXIM, LAURENȚIU; SAITO, MORIHIKO; SCHÜRMAN, JÖRG. Symmetric products of mixed Hodge modules. *J. Math. Pures Appl.* **96** (2011), no. 5, 462–483. MR2843222, Zbl 1238.32021, doi:10.1016/j.matpur.2011.04.003. 1277
- [43] MAXIM, LAURENȚIU; SCHÜRMAN, JÖRG. Characteristic classes of singular toric varieties. *Comm. Pure Appl. Math.* **68** (2015), no. 12, 2177–2236. MR3417881, Zbl 1327.14032, arXiv:1303.4454, doi:10.1002/cpa.21553. 1275, 1277
- [44] MAXIM, LAURENȚIU; SCHÜRMAN, JÖRG. Characteristic classes of mixed Hodge modules and applications. *Schubert varieties, equivariant cohomology and characteristic classes — IMPANGA 15*, 159–202, EMS Ser. Congr. Rep., *Eur. Math. Soc., Zurich*, 2018. MR3754192, Zbl 1387.14032. 1329
- [45] MCCRORY, CLINT. Cone complexes and PL transversality. *Trans. Amer. Math. Soc.* **207** (1975), 269–291. MR0400243, Zbl 0272.57009, doi:10.1090/S0002-9947-1975-0400243-7. 1287
- [46] MILNOR, JOHN W. Microbundles. I. *Topology* **3** (1964), Suppl. 1, 53–80. MR0161346, Zbl 0124.38404 doi:10.1016/0040-9383(64)90005-9. 1282

- [47] PARDON, WILLIAM L. Intersection homology, Poincaré spaces and the characteristic variety theorem. *Comment. Math. Helv.* **65** (1990), no. 2, 198–233. MR1057240, Zbl 0707.57017. 1297
- [48] PETERS, CHRIS A.M.; STEENBRINK, JOSEPH H.M. Mixed Hodge structures. *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*, 52. *Springer-Verlag Berlin*, 2008. xiv+470 pp. ISBN: 978-3-540-77015-2. MR2393625, Zbl 1138.14002 doi:10.1007/978-3-540-77017-6. 1310, 1322
- [49] RANICKI, ANDREW A. Algebraic L -theory and topological manifolds. *Cambridge Tracts in Mathematics*, 102. *Cambridge University Press, Cambridge*, 1992. viii+358 pp. ISBN: 0-521-42024-5. MR1211640, Zbl 0767.57002. 1295, 1301
- [50] RANICKI, ANDREW A. The total surgery obstruction. *Algebraic topology, Aarhus, 1978* (Proc. Sympos., Univ. Aarhus, Aarhus, 1978), 275–316, *Lect. Notes in Math.*, 763, *Springer, Berlin*, 1979. MR0561227, Zbl 0428.57012, doi:10.1007/BFb0088091. 1286
- [51] ROURKE, COLIN P.; SANDERSON, BRIAN J. Block bundles. *Bull. Amer. Math. Soc.* **72** (1966), 1036–1039. MR0214079, Zbl 0147.42501, doi:10.1090/S0002-9904-1966-11635-X. 1282
- [52] ROURKE, COLIN P.; SANDERSON, BRIAN J. Block bundles. I. *Ann. of Math. (2)* **87** (1968), 1–28. MR0226645, Zbl 0215.52204 doi:10.2307/1970591. 1282, 1290, 1292
- [53] ROURKE, COLIN P.; SANDERSON, BRIAN J. Block bundles. III. Homotopy theory. *Ann. of Math. (2)* **87** (1968), 431–483. MR0232404, Zbl 0215.52302, doi:10.2307/1970714. 1282, 1292
- [54] ROURKE, COLIN P.; SANDERSON, BRIAN J. On topological neighbourhoods. *Compositio Math.* **22** (1970), 387–424. MR0298671, Zbl 0218.57005. 1282
- [55] RUDYAK, YULI B. On Thom spectra, orientability, and cobordism. *Springer Monographs in Mathematics. Springer-Verlag Berlin*, 1998. xii+587 pp. ISBN: 3-540-62043-5. MR1627486, Zbl 0906.55001. 1281, 1284
- [56] SAITO, MORIHIKO. Modules de Hodge polarisables. *Publ. Res. Inst. Math. Sci.* **24** (1988), no. 6, 849–995 (1989). MR1000123, Zbl 0691.14007, doi:10.2977/prims/1195173930. 1277, 1306
- [57] SAITO, MORIHIKO. Mixed Hodge modules. *Publ. Res. Inst. Math. Sci.* **26** (1990), no. 2, 221–333. MR1047415, Zbl 0727.14004 doi:10.2977/prims/1195171082. 1306, 1318, 1319, 1323, 1324
- [58] SAITO, MORIHIKO. Introduction to mixed Hodge modules. *Actes du Colloque de Théorie de Hodge (Luminy, 1987). Astérisque No. 179-180* (1989), 10, 145–162. MR1042805, Zbl 0753.32004.
- [59] SAITO, MORIHIKO. Extension of mixed Hodge modules. *Compositio Math.* **74** (1990), no. 2, 209–234. MR1047741, Zbl 0726.14007. 1310
- [60] SAITO, MORIHIKO. Mixed Hodge complexes on algebraic varieties. *Math. Ann.* **316** (2000), no. 2, 283–331. MR1741272, Zbl 0976.14011 arXiv:math/9906088, doi:10.1007/s002080050014. 1306
- [61] SCHÜRMAN, JÖRG. Topology of singular spaces and constructible sheaves. *Instytut Matematyczny Polskiej Akademii Nauk. Monografie Matematyczne (New Series)*, 63. *Birkhäuser Verlag, Basel*, 2003. x+452 pp. ISBN: 3-7643-2189-X. MR2031639, Zbl 1041.55001, doi:10.1007/978-3-0348-8061-9. 1279, 1318
- [62] SCHÜRMAN, JÖRG. Characteristic classes of mixed Hodge modules. *Topology of Stratified Spaces*, 419–470, *Math. Sci. Res. Inst. Publ.*, 58. *Cambridge Univ. Press, Cambridge*, 2011. MR2796417, Zbl 1235.14010. 1277, 1306, 1310, 1319, 1326
- [63] SCHÜRMAN, JÖRG. A generalized Verdier-type Riemann–Roch theorem for Chern–Schwartz–MacPherson classes. Preprint, 2002. arXiv:math/0202175. 1274, 1330

- [64] SCHÜRMAN, JÖRG. Specialization of motivic Hodge–Chern classes. Preprint, 2009. arXiv:0909.3478. 1329
- [65] SHANESON, JULIUS L. Characteristic classes, lattice points, and Euler–MacLaurin formulae. *Proceedings of the International Congress of Mathematicians, Vol. 1, 2*, (Zürich, 1994), 612–624, *Birkhäuser, Basel*, 1995. MR1403962, Zbl 0846.57018. 1274
- [66] SIEBENMANN, LAURENCE C. Deformation of homeomorphisms on stratified sets. I, II. *Comment. Math. Helv.* **47** (1972), 123–163. MR0319207, Zbl 0252.57012. 1279
- [67] SIEGEL, PAUL H. Witt spaces: A geometric cycle theory for KO -homology at odd primes. *Amer. J. of Math.* **105** (1983), no. 5, 1067–1105. MR0714770, Zbl 0547.57019, doi:10.2307/2374334. 1274, 1278, 1280
- [68] SWITZER, ROBERT M. Algebraic topology—homotopy and homology. Die Grundlehren der math. Wissenschaften, Band 212, *Springer-Verlag, New York-Heidelberg*, 1975. xii+526 pp. MR0385836, Zbl 0305.55001, doi:10.1007/978-3-642-61923-6. 1289, 1294
- [69] TOTARO, BURT. Chern numbers for singular varieties and elliptic homology. *Ann. of Math. (2)* **151** (2000), no. 2, 757–791. MR1765709, Zbl 1050.14500. 1276, 1306
- [70] VERDIER, JEAN-LOUIS. Le théorème de Riemann–Roch pour les intersections complètes. *Séminaire de Géométrie analytique (École Norm. Sup., Paris, 1974–75)*, pp. 189–228. Astérisque, no. 36–37, *Soc. Math. France, Paris*, 1976. MR0444657, Zbl 0334.14026. 1309, 1328, 1330, 1332
- [71] VERDIER, JEAN-LOUIS. Spécialisation de faisceaux et monodromie modérée. *Analysis and topology on singular spaces, II, III* (Luminy, 1981), 332–364, Astérisque, no. 101–102, *Soc. Math. France, Paris*, 1983. MR0737938, Zbl 0532.14008. 1318
- [72] WEINBERGER, SHMUEL. The topological classification of stratified spaces. Chicago Lectures in Mathematics. *University of Chicago Press, Chicago, IL*, 1994. xiv+283 pp. ISBN: 0-226-88566-6; 0-226-88567-4. MR1308714, Zbl 0826.57001. 1274
- [73] YOKURA, SHOJI. A singular Riemann–Roch for Hirzebruch characteristics. *Singularities Symposium—Lojasiewicz 70* (Kraków, 1996; Warsaw, 1996), 257–268, Banach Center Publ., 44, *Polish Acad. Sci. Inst. Math., Warsaw*, 1998. MR1677403, Zbl 0915.14005. 1309

(Markus Banagl) MATHEMATISCHES INSTITUT, UNIVERSITÄT HEIDELBERG, IM NEUENHEIMER FELD 205, 69120 HEIDELBERG, GERMANY
banagl@mathi.uni-heidelberg.de

This paper is available via <http://nyjm.albany.edu/j/2020/26-52.html>.