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The geodesic complexity of n-dimensional Klein bottles

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Abstract. The geodesic complexity of a metric space X is the smallest k for which there is a partition of $X \times X$ into locally compact sets $E_0, \ldots, E_k$ on each of which there is a continuous choice of minimal geodesic $\sigma(x_0, x_1)$ from x_0 to x_1. We prove that the geodesic complexity of an n-dimensional Klein bottle K_n equals $2n$. The topological complexity of K_n remains unknown for $n > 2$.

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1. Introduction

The motion planning problem is of central importance in the field of robotics. The geodesic complexity $\mathrm{GC}(X)$ of a metric space (X, d) is a measure of the minimal instability of any optimal motion planner on X (optimal in the sense that the motions are always along shortest paths). It was introduced recently by the second author in [6], inspired by Farber's topological complexity $\mathrm{TC}(X)$ [3], which is a measure of the minimal instability over all motion planners on X, not necessarily along shortest paths. In fact, $\mathrm{TC}(X)$ is defined for all topological spaces and is a homotopy invariant, while $\mathrm{GC}(X)$ depends on the metric [6].

Let $X^I = PX \to X \times X$ denote the free path fibration, which maps each path γ in X to the pair $(\gamma(0), \gamma(1))$. Furthermore, let $GX \subset PX$ consist of the minimal geodesics. This is paths whose length equals the distance between the endpoints: $\ell(\gamma) = d(\gamma(0), \gamma(1))$. The restriction of the free path

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fibration to GX defines a map $\pi : GX \to X \times X$, which is not a fibration in general.

The following definition of topological complexity differs slightly from the version most commonly used in the literature. However, Farber gave it as an alternative definition in [4, Prop. 4.12] and showed that it is equivalent to the original definition from [3] under mild assumptions on the space X. As remarked in [6], the definition from [3] would not work in the geodesic setting, which is why the alternative version needs to be used here.

Definition 1.1. *The* topological complexity *of a space X, $\mathrm{TC}(X)$, is defined to be the smallest k for which there exists a decomposition into $k+1$ pairwise disjoint locally compact sets $X \times X = \bigsqcup_{i=0}^{k} E_i$ such that there are local sections $s_i \colon E_i \to PX$ of $PX \to X \times X$.*

Definition 1.2 ([6])**.** *The* geodesic complexity *of a metric space (X, d), $\mathrm{GC}(X, \mathrm{d})$, is defined to be the smallest k for which there exists a decomposition into $k+1$ pairwise disjoint locally compact sets $X \times X = \bigsqcup_{i=0}^{k} E_i$ such that there are local sections $s_i \colon E_i \to GX$ of π.*

It was shown in [6] that these numbers are different in general, even though they agree in many cases.

The higher Klein bottles K_n were introduced by the first author in the course of his work on planar polygon spaces [2]:

$$K_n = (S^1)^n / (z_1, \dots, z_{n-1}, z_n) \sim (\bar{z}_1, \dots, \bar{z}_{n-1}, -z_n).$$

These spaces are a generalization of the standard Klein bottle, which corresponds to K_2. We consider the metric space (K_n, d) with the flat metric coming from the universal covering $\mathbb{R}^n$.

The topological complexity of the Klein bottle K_2 was an open problem for over a decade, even after the topological complexity had been determined for all other orientable and nonorientable surfaces. In 2017, $\mathrm{TC}(K_2)$ was finally computed by Cohen and Vandembroucq [1] and later by Iwase, Sakai and Tsutaya [5]. The geodesic complexity $\mathrm{GC}(K_2)$ (with the flat metric) is equal to the topological complexity $\mathrm{TC}(K_2)$, as was shown by the second author in [6].

The topological complexity of K_n is currently unknown for $n \geq 3$, and a proof seems to be out of reach at this point. The geodesic complexity is computed in this article:

Theorem 1.3. *The geodesic complexity of the flat higher Klein bottle K_n is given by*

$$\mathrm{GC}(K_n) = 2n.$$

Most of the work in the proof consists in understanding the *total cut locus* of K_n. Briefly, the total cut locus of a metric space X consists of all pairs of points (x, y) in $X \times X$ such that there are at least two shortest paths between x and y. In other words, it is the union over all x in X of the cut locus of each x, in the usual sense.

It is conceivable that, in fact, $\mathrm{TC}(K_n) = \mathrm{GC}(K_n)$ for all n could be shown by a general argument, in which case this work would yield the values $\mathrm{TC}(K_n)$ as an application of the previous theorem.

2. Upper bound

In this section, we prove $\mathrm{GC}(K_n) \leq 2n$ by demonstrating explicit geodesic motion planning rules, postponing many details to later sections.

Let $\sim$ be the equivalence relation on $\mathbb{R}^n$ generated by $x \sim x + e_i$, $1 \leq i \leq n-1$, where e_i is the unit vector in the ith coordinate, and

$$(x_1, \ldots, x_{n-1}, x_n) \sim (1 - x_1, \ldots, 1 - x_{n-1}, x_n + 1). \tag{2.1}$$

The higher Klein bottle K_n is the quotient space, with $p : \mathbb{R}^n \to K_n$ the quotient map. The metric d_K on K_n is defined by

$$d_K(y, y') = \min\{d(x, x') : x \in p^{-1}(y), x' \in p^{-1}(y')\},$$

where d is the Euclidean metric on $\mathbb{R}^n$. If $P, Q \in \mathbb{R}^n$, we say that Q is K-*close* to P if $d_K(p(P), p(Q)) = d(P, Q)$. If $\sigma_{P,Q}$ denotes the uniform linear path from P to Q, then the geodesics in K_n are paths of the form $p \circ \sigma_{P,Q}$ such that Q is K-close to P.

Let $P = (x_1, \ldots, x_n)$ be a point in the n-cube I^n, where I stands for the closed unit interval. We will define a set $C(P)$, which contains a subset of points in $p^{-1}(p(P)) - \{P\}$ which are near P. Specifically, the set $C(P)$ contains the points $P \pm e_1, \ldots, P \pm e_{n-1}$ and also the points $P \pm 2e_n$ and $(y_1, \ldots, y_{n-1}, x_n \pm 1)$, where

$$y_i \in \begin{cases} \{-x_i, 1 - x_i\} & 0 < x_i < \frac{1}{2} \\ \{x_i\} & x_i \in \{0, \frac{1}{2}, 1\} \\ \{1 - x_i, 2 - x_i\} & \frac{1}{2} < x_i < 1. \end{cases} \tag{2.2}$$

Let $\mathcal{R}(P)$ denote the intersection of the closed half-spaces containing P bounded by the perpendicular bisectors of the segments from P to each of the points of $C(P)$. Note that the polytope $\mathcal{R}(P)$ consists of those points Q which are K-close to P, and that $(p \times p)(\{(P, Q) : P \in [0,1)^n, Q \in \partial(\mathcal{R}(P))\})$ is the total cut locus of K_n.

For $0 \leq j \leq n$, let $R_j(P)$ denote the set of interiors of j-dimensional faces of $\mathcal{R}(P)$. In particular, the set $R_n(P)$ consists of the single set $\mathrm{int}(\mathcal{R}(P))$. The equivalence relation $\sim$ induces one on each set $R_j(P)$. We consider subsets $\mathcal{D} := D_1 \times \cdots \times D_n \subset \mathbb{R}^n$, where

$$D_i = \begin{cases} (0, \frac{1}{2}) \cup (\frac{1}{2}, 1) \text{ or } \{0, \frac{1}{2}\} & 1 \leq i \leq n-1 \\ (0, 1) \text{ or } \{0\} & i = n. \end{cases} \tag{2.3}$$

Note that for each $\mathcal{D}$, $p : \mathcal{D} \to p(\mathcal{D})$ is a homeomorphism.

Most of our work goes into proving the following theorem.

Theorem 2.1. *For $n \geq 2$, each of the sets $\mathcal{D}$ listed above can be partitioned into finitely many subsets M_α, which are analytic spaces of varying dimensions, on which the polytopes $\mathcal{R}(P)$ vary continuously and bijectively, preserving $\sim$. That is, if we choose a point $P_0 \in M_\alpha$, there exist polytope-equivalences $\theta_P : \mathcal{R}(P_0) \to \mathcal{R}(P)$ preserving $\sim$ for all $P \in M_\alpha$, which vary continuously with P (i.e., if $x \in \mathcal{R}(P_0)$, then $P \mapsto \theta_P(x)$ is continuous). Also, $\theta_{P_0} = 1$.*

Moreover, (a) if $\dim(M_\alpha) \leq \dim(M_{\alpha'})$ and $M_\alpha \neq M_{\alpha'}$, then the closure of M_α is disjoint from $M_{\alpha'}$ or any set equivalent to $M_{\alpha'}$, and (b) if F is a j-face in $\mathcal{R}(P_0)$ for $P_0 \in M_\alpha$ and $\langle P_i \rangle \to P$ for $P_i \in M_\alpha$ and $P \in I^n$, then $\lim \theta_{P_i}(F)$ is contained in a j'-face of $\mathcal{R}(P)$ for $j' \leq j$.

By *analytic space*, we mean an open subset of a real variety (the set of solutions of a set of polynomial (for us, quadratic) equations over $\mathbb{R}$). Our analytic spaces may possibly have singularities, but they have dimension, like a manifold. The following corollary is half of Theorem 1.3.

Corollary 2.2. $\mathrm{GC}(K_n) \leq 2n$.

Proof. Fix M_α and $P_0 \in M_\alpha$. Choose a representative of each equivalence class of $R_j(P_0)$, and let $R'_j(P_0)$ denote their union. Let $R'_j(P) = \theta_P(R'_j(P_0))$, and

$$\mathcal{S}_{\alpha,j} = \{(P, Q) : P \in M_\alpha,\ Q \in R'_j(P)\}.$$

The bijective function $(p \times p)|\mathcal{S}_{\alpha,j}$ has a continuous inverse. ⟦It suffices to show that there does not exist a sequence $(P_i, Q_i) \in \mathcal{S}_{\alpha,j}$ converging to a point (P, Q) not in $\mathcal{S}_{\alpha,j}$ which is equivalent ($\sim$) to a point of $\mathcal{S}_{\alpha,j}$. Assume, for the sake of contradiction, that such a sequence exists. Since the equivalence relation is on the second component, we must have $P \in M_\alpha$. Using the functions θ_{P_i} and θ_P, we may assume that the convergence $Q_i \to Q$ is taking place in $\mathcal{R}(P_0)$, where disjointness of interiors of j-faces, together with the fact that equivalence of faces preserves dimensions, implies that the contemplated situation cannot occur.⟧ The function $\mathcal{S}_{\alpha,j} \to (\mathbb{R}^n)^I$ defined by $(P, Q) \mapsto \sigma_{P,Q}$ is continuous, hence so is the composite

$$(p \times p)(\mathcal{S}_{\alpha,j}) \xrightarrow{(p\times p)^{-1}} \mathcal{S}_{\alpha,j} \to (\mathbb{R}^n)^I \to (K_n)^I,$$

giving a continuous choice on $(p \times p)(\mathcal{S}_{\alpha,j})$ of geodesics in K_n. This is called a geodesic motion planning rule on $(p \times p)(\mathcal{S}_{\alpha,j})$. Using all α and all j, the sets $(p \times p)(\mathcal{S}_{\alpha,j})$ give a partition of $K_n \times K_n$ into subsets on which continuous choice of geodesics can be found. However, some of the $\mathcal{S}_{\alpha,j}$'s can be combined:

We claim that if $\dim(M_\alpha) + j = \dim(M_{\alpha'}) + j'$, then $(p \times p)(\mathcal{S}_{\alpha,j})$ and $(p \times p)(\mathcal{S}_{\alpha',j'})$ are topologically disjoint (or equal). This implies that, for $0 \leq i \leq 2n$, we can use as our sets the union of all $(p \times p)(\mathcal{S}_{\alpha,j})$ for which $\dim(M_\alpha) + j = i$, establishing the corollary, since we have $2n + 1$ subsets admitting continuous choices of geodesics.

To prove the claim, suppose $(p(P_i), p(Q_i))$ converges to $(p(P), p(Q))$, where $P_i \in M_\alpha$, $Q_i \in \theta_{P_i}(R'_j(P_0))$, $P \in M_{\alpha'}$, and $Q \in \theta_P(R'_{j'}(P_0))$. Then a subsequence, which we also call $\langle P_i \rangle$, converges to a point equivalent to P. Then part (a) of the theorem implies that either $M_{\alpha'} = M_\alpha$ or $\dim(M_{\alpha'}) < \dim(M_\alpha)$. In the first case, then $j' = j$, and so $\mathcal{S}_{\alpha,j} = \mathcal{S}_{\alpha',j'}$. In the second case, $j' > j$. Since $R_j(P_0)$ is finite, we may assume that there is a j-face F of $\mathcal{R}(P_0)$ such that $Q_i \in \theta_{P_i}(F)$ for all i, and since the number of equivalence classes under p is finite, then we may assume a sequence of Q_i's converges, so by (b), $j' \le j$, a contradiction.

By separating $x_n \in \{0,1\}$ from $x_n \in (0,1)$, we remove any concern about continuity when $x_n = 0$, and similarly for $x_i \in \{0, \frac{1}{2}, 1\}$. □

The sets M_α are quite simple for $n \le 6$, as described in the following example, which is a consequence of Theorem 4.2. See especially material immediately following Figure 4.5. Here we initiate the practice, continued throughout, of using $(a_1, \ldots, a_n)$ for the coordinates of P, the point away from which we will be moving, while $(x_1, \ldots, x_n)$ will be used for coordinates in $\mathcal{R}(P)$, the points toward which we move from P.

Example 2.3. *If $n \le 4$, Theorem 2.1 holds using as M_α exactly the sets $\mathcal{D}$ described prior to the theorem. For $n = 5$, it holds with the one change that $\mathcal{D} = ((0, \frac{1}{2}) \cup (\frac{1}{2}, 1))^4 \times D_5$ be replaced by $\mathcal{E} = \{\frac{1}{4}, \frac{3}{4}\}^4 \times D_5$ and $\mathcal{D} - \mathcal{E}$. If $n = 6$, $\mathcal{D} = ((0, \frac{1}{2}) \cup (\frac{1}{2}, 1))^5 \times D_6$ must be replaced by*

$$\mathcal{E} = \{(a_1, \ldots, a_5) : \sum_{i=1}^{5} \min((a_i - \tfrac{1}{4})^2, (a_i - \tfrac{3}{4})^2) = \tfrac{1}{16}\} \times D_6$$

and $\mathcal{D} - \mathcal{E}$, and also $\mathcal{D} = ((0, \frac{1}{2}) \cup (\frac{1}{2}, 1))^4 \times \{0, \frac{1}{2}\} \times D_6$ must be replaced by $\mathcal{E} = \{\frac{1}{4}, \frac{3}{4}\}^4 \times \{0, \frac{1}{2}\} \times D_6$ and $\mathcal{D} - \mathcal{E}$, and similarly for permutations of the first five factors.

3. Examples when $n = 2$ or 3

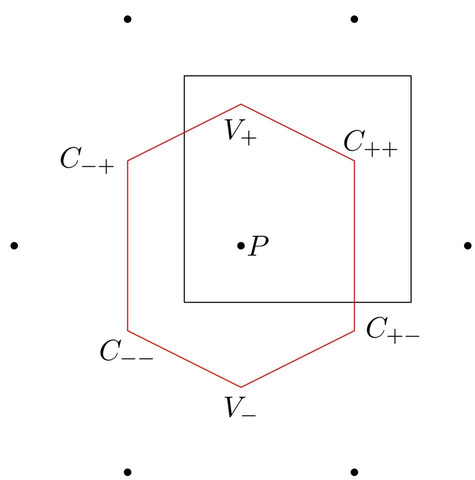
In this section, we illustrate the sets $\mathcal{R}(P)$ when $n = 2$ or 3, when we can actually picture them. For $0 < a \le \frac{1}{2}$, let

$$\Delta_a = a - 2a^2 = \tfrac{1}{8} - 2(a - \tfrac{1}{4})^2. \tag{3.1}$$

This formula is very important throughout the paper.

We begin with the case $n = 2$. Let $P = (a_1, a_2)$ with $0 < a_1 < \frac{1}{2}$. For $\varepsilon \in \{0,1\}$, the equation of the perpendicular bisector of the segment between (a_1, a_2) and $(\varepsilon - a_1, a_2 + 1)$ is $x_2 = a_2 + \frac{1}{2} + (2a_1 - \varepsilon)(x_1 - \frac{1}{2}\varepsilon)$. It is easy to check that $\mathcal{R}(a_1, a_2)$ is a hexagon as pictured in Figure 3.1, with $V_\pm = (\frac{1}{2} - a_1, a_2 \pm (\frac{1}{2} + \Delta_{a_1}))$ and $C_{\pm,\pm'} = (a_1 \pm \frac{1}{2}, a_2 \pm' (\frac{1}{2} - \Delta_{a_1}))$. We have $V_+ \sim C_{--} \sim C_{+-}$, and $V_- \sim C_{-+} \sim C_{++}$. Also $V_+C_{-+} \sim C_{--}V_-$ and $V_+C_{++} \sim C_{+-}V_-$ and $C_{-+}C_{--} \sim C_{++}C_{+-}$. For $\frac{1}{2} < a_1 < 1$, the shape is the same with formulas modified slightly.

Figure 3.1. Polygon when $n = 2$



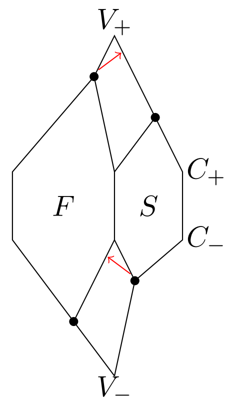
If $a_1 \in \{0, \frac{1}{2}, 1\}$, then $\mathcal{R}(a_1, a_2)$ is a unit square centered at (a_1, a_2), with vertical sides equivalent, horizontal sides equivalent (in the opposite order), and all four vertices equivalent. The continuous bijective variation of $\mathcal{R}(P)$, preserving $\sim$, for $P = (a_1, a_2)$ in any one of the domains $(0, \frac{1}{2}) \times (0, 1)$, $(\frac{1}{2}, 1) \times (0, 1)$ (hence in $((0, \frac{1}{2}) \cup (\frac{1}{2}, 1)) \times (0, 1)$), $\{0, \frac{1}{2}\} \times (0, 1)$, $(0, 1) \times \{0\}$, etc., is clear. As an example of a motion planning rule, we might choose, for $P_0 \in M_\alpha = (0, \frac{1}{2}) \times (0, 1)$, the interiors of the segments $C_{-+}V_+$, V_+C_{++}, and $C_{++}C_{+-}$ as representatives of equivalence classes of $R_1(P_0)$, with $R'_1(P_0)$ being their union, and choose $R'_0(P_0) = \{V_+, C_{++}\}$. The motion planning rule for $\{(P, Q) : P \in M_\alpha, Q \in \theta_P(R'_j(P_0))\}$ is $(p(P), p(Q)) \mapsto (p \times p)(\sigma_{P,Q})$, $j \in \{0, 1, 2\}$.

Now we consider the case $n = 3$. Consider first the points $P = (a_1, a_2, a_3)$ with $0 < a_1, a_2 < \frac{1}{2}$. The planes bisecting the lines from P to $P \pm e_1$ and $P \pm e_2$ yield a region bounded by four vertical walls, rising above and below the square in the x_1x_2-plane with vertices at $(a_1 \pm \frac{1}{2}, a_2 \pm' \frac{1}{2})$. These walls will be capped above and below by pyramids with vertices $V_\pm = (\frac{1}{2} - a_1, \frac{1}{2} - a_2, a_3 \pm (\frac{1}{2} + \Delta_{a_1} + \Delta_{a_2}))$. The top vertex V_+ is the intersection of four planes with equations, for $\varepsilon_i \in \{0, 1\}$,

$$x_3 = a_3 + \tfrac{1}{2} + \sum_{i=1}^{2} (2a_i - \varepsilon_i)(x_i - \tfrac{1}{2}\varepsilon_i).$$

Each of these planes comes down and intersects two of the walls. In Figure 3.2, we depict schematically part of one of these regions $\mathcal{R}(P)$. The letters F and S denote a front and side wall. There is also a back wall and another side wall. Three other quadrilaterals come down from V_+.

Figure 3.2. Polytope when $n = 3$



The points at the $C_\pm$ level are at height $a_3 \pm (\frac{1}{2} - \Delta_{a_1} - \Delta_{a_2})$. Since $\Delta_a \le \frac{1}{8}$, the sides of the upper pyramid do not intersect those of the lower pyramid. For $n \ge 6$, such an intersection complicates the analysis. The points indicated by $\bullet$s have coordinates $(\frac{1}{2} - a_1, a_2 \pm \frac{1}{2}, a_3 \pm' (\frac{1}{2} + \Delta_{a_1} - \Delta_{a_2}))$ or $(a_1 \pm \frac{1}{2}, \frac{1}{2} - a_2, a_3 \pm' (\frac{1}{2} - \Delta_{a_1} + \Delta_{a_2}))$. Points on the walls are equivalent to points on the opposite walls. Points on the slanted quadrilaterals are equivalent to points on the vertically displaced quadrilateral, but in opposite non-vertical directions. For example, points in the red arrow in the top portion of Figure 3.2 are equivalent to corresponding points on the lower arrow. The continuous dependence of this region $\mathcal{R}(P)$ on P is clear from these formulas. If a_1 or a_2 (or both) is in $(\frac{1}{2}, 1)$, the description is similar, with $\frac{1}{2} - a_i$ replaced by $\frac{3}{2} - a_i$, and $\Delta_a = \frac{1}{8} - 2(a - \frac{3}{4})^2$. The change at $a_i = \frac{1}{2}$ is not a problem for continuity since $\frac{1}{2}$ has been removed from the domain.

If $a_1 \in \{0, \frac{1}{2}\}$, then $\mathcal{R}(P) = [a_1 - \frac{1}{2}, a_1 + \frac{1}{2}] \times \mathcal{R}(a_2, a_3)$, where $\mathcal{R}(a_2, a_3)$ is the 2-dimensional region described in the discussion for $n = 2$, and $(a_1 - \frac{1}{2}, x_2, x_3) \sim (a_1 + \frac{1}{2}, x_2, x_3)$. To see this, we observe that the planes bounding $\mathcal{R}(P)$ are $\mathbb{R} \times \mathcal{P}$, where $\mathcal{P}$ bounds $\mathcal{R}(a_2, a_3)$, and $\{a_1 \pm \frac{1}{2}\} \times \mathbb{R}^2$. After reminding the reader that the case when $a_3 = 0$ (or 1) is separated from the case when $a_3 \in (0, 1)$, but is similar in nature, we conclude that we have completed the claimed description of the required regions when $n = 3$.

4. Vertices of a polytope $\mathcal{R}(P)$

In this section, we determine the vertices of the polytopes $\mathcal{R}(P)$ when $0 < a_i < \frac{1}{2}$ for $1 \le i \le n-1$ and $a_n = 0$. The restriction to $a_n = 0$ is just to simplify formulas. Just add a_n to all x_n values for the general case. We let $\widehat{\mathcal{R}}(P)$ denote an approximation of $\mathcal{R}(P)$ which does not take into account truncating due to the points $P \pm 2e_n$ of $C(P)$.

Definition 4.1. *Let $P = (a_1, \dots, a_{n-1}, 0)$ with $0 < a_1, \dots, a_{n-1} < \frac{1}{2}$. Let $\widehat{\mathcal{R}}(P)$ denote the polytope in $\mathbb{R}^n$ which is the intersection of the following half-spaces:*

$$x_i \le a_i + \tfrac{1}{2},\ i = 1, \dots, n-1 \tag{4.1}$$
$$x_i \ge a_i - \tfrac{1}{2},\ i = 1, \dots, n-1 \tag{4.2}$$
$$x_n \le \tfrac{1}{2} + \sum_{i=1}^{n-1}(2a_i - \delta_i)(x_i - \tfrac{1}{2}\delta_i),\ \delta_i \in \{0,1\} \tag{4.3}$$
$$x_n \ge -\tfrac{1}{2} - \sum_{i=1}^{n-1}(2a_i - \delta_i)(x_i - \tfrac{1}{2}\delta_i),\ \delta_i \in \{0,1\}. \tag{4.4}$$

Intuitively, $\widehat{\mathcal{R}}(P)$ can be thought of as walls capped by a pyramid on top and another on the bottom. The vertices will occur where a j-face of a pyramid intersects an $(n-j)$-dimensional wall. However, when $n \ge 6$, some parts of the top pyramid might intersect a wall at a negative value of x_n. Such a vertex will be outside the lower pyramid; it will not satisfy (4.4). This vertex, and a corresponding vertex on the bottom pyramid, will be replaced by a number of "middle" vertices on the intersection, at $x_n = 0$, of the lower and upper pyramids.

When $n \ge 6$, some vertices of $\widehat{\mathcal{R}}(P)$ will lie above the hyperplane $x_n = 1$, which is the perpendicular bisector of the segment connecting P to $P + 2e_n$. The desired polytope $\mathcal{R}(P)$ is the intersection of $\widehat{\mathcal{R}}(P)$ with the region $-1 \le x_n \le 1$.

We define a *labeled set* S to be a set S together with a function $\varepsilon_S : S \to \{0,1\}$, $i \mapsto \varepsilon_i$. Also, $|S|$ is the cardinality of S, and $[\![n-1]\!] = \{1, \dots, n-1\}$. With Δ as in (3.1), let $\Delta(S) = \sum_{i\in S} \Delta_{a_i}$, and let $\widetilde{S} = [\![n-1]\!] - S$. The quantity

$$K(S) = \tfrac{1}{2} - \Delta(S) + \Delta(\widetilde{S}) \tag{4.5}$$

will play a central role in our results. Note that, for fixed S, $K(S)$ is a quadratic function of $a_1, \dots, a_{n-1}$. It depends only on the set S, not the labeling. It satisfies

$$K(S) + K(\widetilde{S}) = 1, \tag{4.6}$$

which will be useful later.

The vertices of $\mathcal{R}(P)$ are described as follows.

Theorem 4.2. *For $P = (a_1, \dots, a_{n-1}, 0)$ with $0 < a_1, \dots, a_{n-1} < \frac{1}{2}$, $\mathcal{R}(P)$ has vertices of possibly three types.*

- *For labeled sets* $S \subset [\![n-1]\!]$ *satisfying* $0 \le K(S) \le 1$, *there are "standard" vertices* $v_S^{\pm}$ *with*

$$x_i = \begin{cases} a_i - \frac{1}{2} + \varepsilon_i & i \in S \\ \frac{1}{2} - a_i & i \in \widetilde{S} \\ \pm K(S) & i = n. \end{cases} \tag{4.7}$$

  If $K(S) = 0$, *then* $v_S^+ = v_S^-$.
- *For labeled sets* S *satisfying* $K(S) < 0 < K(S - \{k\})$, *there are "middle" vertices* $v_{S,k}^0$ *with*

$$x_i = \begin{cases} a_i - \frac{1}{2} + \varepsilon_i & i \in S - \{k\} \\ \frac{1}{2} - a_i & i \in \widetilde{S} \\ a_k - \frac{1}{2} + \varepsilon_k - \frac{K(S)}{2a_k - \varepsilon_k} & i = k \\ 0 & i = n. \end{cases} \tag{4.8}$$

- *For labeled sets* $S \cup \{k\}$ *satisfying* $K(S \cup \{k\}) < 1 < K(S)$, *there are "truncating" vertices* $v_{S,k}^{\pm}$ *satisfying*

$$x_i = \begin{cases} a_i - \frac{1}{2} + \varepsilon_i & i \in S \\ \frac{1}{2} - a_i & i \in \widetilde{S} - \{k\} \\ a_k - \frac{1}{2} + \varepsilon_k + \frac{1 - K(S \cup \{k\})}{2a_k - \varepsilon_k} & i = k \\ \pm 1 & i = n. \end{cases} \tag{4.9}$$

There are no other vertices of $\mathcal{R}(P)$.

Note that by (4.6), $\mathcal{R}(P)$ has truncating vertices for $\widetilde{S}$ iff it has middle vertices for S. Truncating vertices are distinguished from standard vertices by having a second subscript.

Proof. Let $t = |\widetilde{S}|$. For $\widehat{\mathcal{R}}(P)$, we seek solutions of the inequalities of Definition 4.1 satisfying $x_i = a_i - \frac{1}{2} + \varepsilon_i$ for $i \in S$ which have equality in $t+1$ independent equations of type (4.3) and (4.4). The following fact is very useful. For $\varepsilon \in \{0, 1\}$ and $0 < a < \frac{1}{2}$,

$$(2a - \varepsilon)(x - \tfrac{1}{2}\varepsilon) = \begin{cases} \Delta_a & x = \frac{1}{2} - a \\ -\Delta_a & x = a - \frac{1}{2} + \varepsilon \\ -\Delta_a + 2a & \varepsilon = 0,\ x = a + \frac{1}{2} \\ -\Delta_a + 1 - 2a & \varepsilon = 1,\ x = a - \frac{1}{2}. \end{cases}$$

Because of this, if, for $i \in S$, the inequality (4.3) is satisfied for $\delta_i = 0$ and 1 (with other δ_j's fixed) with equality in one, then it must be the case that $\delta_i = \varepsilon_i$ and $(2a_i - \delta_i)(x_i - \frac{1}{2}\delta_i) = -\Delta_{a_i}$. After setting $x_i = a_i - \frac{1}{2} + \varepsilon_i$ for

$i \in S$, the relevant inequalities of Definition 4.1 become

$$x_n \leq \tfrac{1}{2} - \Delta(S) + \sum_{i \in \widetilde{S}} (2a_i - \delta_i)(x_i - \tfrac{1}{2}\delta_i) \qquad (4.10)$$

$$x_n \geq -\tfrac{1}{2} + \Delta(S) - \sum_{i \in \widetilde{S}} (2a_i - \delta_i)(x_i - \tfrac{1}{2}\delta_i). \qquad (4.11)$$

Note that v_S^+ satisfies equality in (4.10) for every choice of δ_i for $i \in \widetilde{S}$, and it satisfies the inequalities (4.11) iff $K(S) \geq 0$. Note also that the system of equations associated with (4.10) has rank $t+1$, as it is easily seen to reduce to the equations $x_i = \frac{1}{2} - a_i$, $i \in \widetilde{S}$, and $x_n = \frac{1}{2} - \Delta(S) + \sum_{\widetilde{S}} 2a_i x_i$. Thus, assuming $K(S) \geq 0$, v_S^+ is a vertex of $\widehat{\mathcal{R}}(P)$, and similarly so is v_S^-. There can be no other vertices associated to S with $t+1$ independent equations of type (4.10) because $t+1$ linearly independent equations in $t+1$ variables can have at most one solution.

Next we consider the truncation. If $K(S \cup \{k\}) < 1 < K(S)$, then v_S^+ lies above the hyperplane $x_n = 1$, as does v_T^+ for any $T \subset S$. We obtain new vertices as the intersection of the edge between $v_{S\cup\{k\}}^+$ and v_S^+ with the hyperplane $x_n = 1$. Note that for each labeled set S, there are two labeled sets $S \cup \{k\}$, corresponding to the two possible labels on k. The points $v_{S\cup\{k\}}^+$ and v_S^+ differ only in the kth and nth components, which are $(a_k - \frac{1}{2} + \varepsilon_k, K(S \cup \{k\}))$ for $v_{S\cup\{k\}}^+$, and $(\frac{1}{2} - a_k, K(S \cup \{k\}) + 2\Delta_{a_k})$ for v_S^+. The point on the segment between them with $x_n = 1$ has

$$x_k = a_k - \tfrac{1}{2} + \varepsilon_k + \frac{1 - 2a_k - \varepsilon_k}{2\Delta_{a_k}}(1 - K(S \cup \{k\})), \qquad (4.12)$$

which simplifies as claimed.

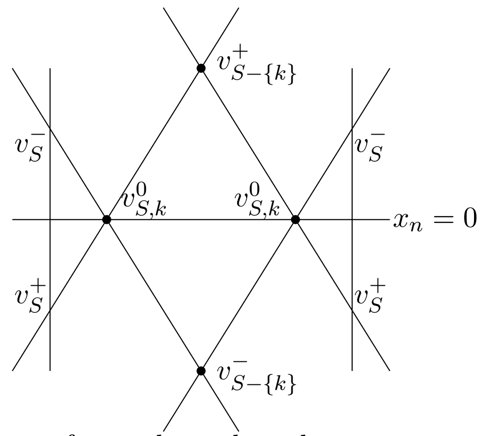
The middle vertices are obtained similarly by computing the point where the segment from v_S^+ to $v_{S-\{k\}}^+$ meets the hyperplane $x_n = 0$. Since k is in the labeled set S, there are two vertices $v_{S,k}^0$ connected to each vertex $v_{S-\{k\}}^+$. As explained in the paragraph after Definition 4.1, these occur when the intersection of the slant hyperplanes associated to (4.3) with the walls associated to (4.1) and (4.2) corresponding to the labeled set S meet at a negative value $K(S)$ of x_n. The slant hyperplanes here have $\delta_i = \varepsilon_i$ for $i \in S$, so (4.10) and (4.11) apply.

When $K(S) < 0$, the vertex v_S^+ satisfies the wall inequalities for the labeled set S and (4.10) for $\delta_i = 0$ or 1 when $i \in \widetilde{S}$, but does not satisfy (4.11). Choose any k having $K(S - \{k\}) > 0$, and replace the wall hyperplane $x_k = a_k - \frac{1}{2} + \varepsilon_k$ by any hyperplane associated to (4.11). Comparing this equation with the corresponding equation in (4.10) yields $x_n = 0$. The new x_k value is nicely found by linearity similarly to (4.12).

The diagram in Figure 4.3 is quite representative. The horizontal axis is x_k and the vertical axis is x_n. The vertical lines represent the walls

$x_k = a_k \pm \frac{1}{2}$, and the horizontal line represents the hyperplane $x_n = 0$. The "vertices" v_S^+ (resp. v_S^-) are not on the polytope since they lie outside the hyperplanes coming up from $v_{S-\{k\}}^-$ (resp. down from $v_{S-\{k\}}^+$).

Figure 4.3. Formation of middle vertices



In the remainder of this proof, we show that there are no additional vertices obtained as intersections of hyperplanes of type (4.10) and type (4.11).[1] First note that any intersection of the two types of hyperplanes must occur at $x_n = 0$. This follows since if for all vectors $\delta = \langle \delta_i \rangle$

$$\tfrac{1}{2} - \Delta(S) + \sum_{\widetilde{S}} (2a_i - \delta_i)(x_i - \tfrac{1}{2}\delta_i) \ge x_n$$

and

$$\tfrac{1}{2} - \Delta(S) + \sum_{\widetilde{S}} (2a_i - \delta_i)(x_i - \tfrac{1}{2}\delta_i) \ge -x_n,$$

with equality in the first for some δ and equality in the second for some δ, then $x_n = 0$.

We are looking for $t+1$ linearly independent hyperplanes associated to (4.10) and (4.11) intersecting at $x_n = 0$. It is enough to consider that all but one of them are of type (4.10). To see this, as just noted, one hyperplane of type (4.11) forces $x_n = 0$, which then implies that points $(x_1, \ldots, x_n)$ which satisfy any other equation of type (4.11) also satisfy the corresponding equation of type (4.10), and so all but one of the type-(4.11) hyperplanes can be replaced by type-(4.10) hyperplanes.

As already noted, if $K(S) < 0$, the putative v_S^+ does not satisfy (4.11), but if also $K(S - \{k\}) > 0$, there will be a vertex $v_{S-\{k\}}^+$, and $v_{S,k}^0$ can be found as above. If $K(S - \{k, \ell\}) > K(S - \{k\}) > 0 > K(S)$, there will not be an additional vertex obtained by intersecting a segment from v_S^+ to

[1]When we talk about hyperplanes of type (4.10) or (4.11), we mean that the $\le$ or $\ge$ is replaced by $=$.

$v^+_{S-\{k,\ell\}}$ with $x_n = 0$ because it would lie on an edge from $v^0_{S,k}$ to

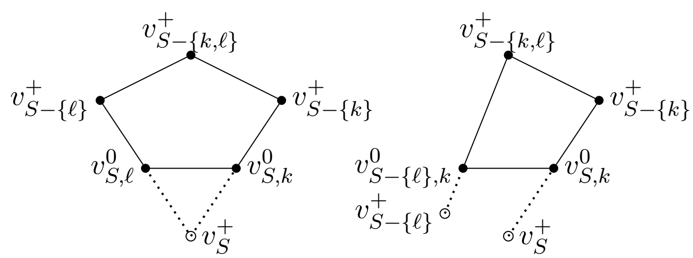
$$\begin{cases} v^0_{S,\ell} & \text{if } K(S-\{\ell\}) > 0 \\ v^0_{S-\{\ell\},k} & \text{if } K(S-\{\ell\}) < 0. \end{cases}$$

These vertices all have $x_i = \frac{1}{2} - a_i$ for $i \in \widetilde{S}$, and $x_i = a_i - \frac{1}{2} + \varepsilon_i$ for $i \in S - \{k,\ell\}$, and lie in the plane (for simplicity, let $\varepsilon_k = \varepsilon_\ell = 0$)

$$x_n = \tfrac{1}{2} + 2a_k x_k + 2a_\ell x_\ell - \Delta(S - \{k,\ell\}) + \Delta(\widetilde{S}).$$

See Figure 4.4. □

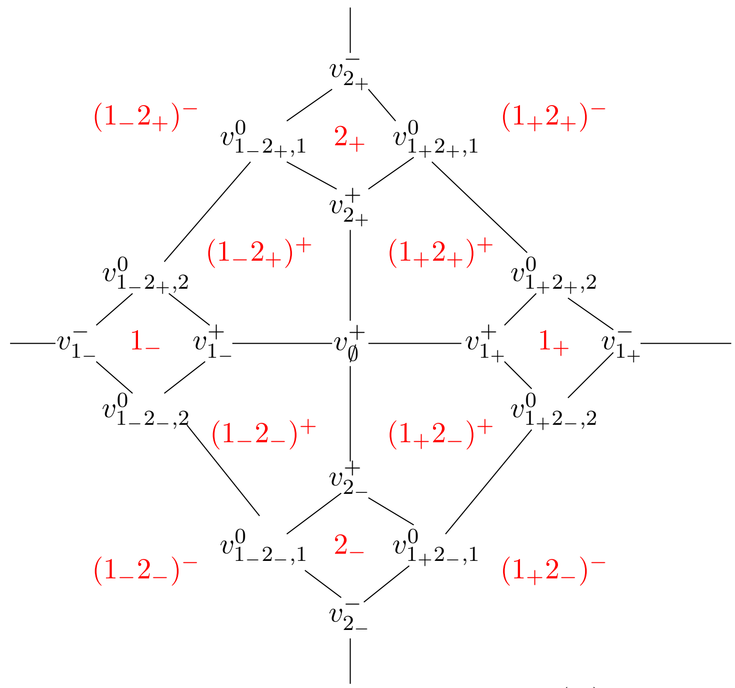
Figure 4.4. Face containing middle vertices



$K(S-\{\ell\}) > 0$ $K(S-\{\ell\}) < 0$

The diagram in Figure 4.5 of a fictitious polytope in $\mathbb{R}^3$ can be very useful in visualizing the middle vertices as they sit in the whole polytope. This polytope is fictitious in two ways. First, the actual polytopes associated to K_3 do not have middle vertices, and second, whenever middle vertices occur, then truncating vertices occur, too, but that is not the case in our diagram. This represents a polytope with $K(\{1\}) > 0$, $K(\{2\}) > 0$, and $K(\{1,2\}) < 0$. Subscripts on the numbers 1 and 2 refer to the labeling, $-$ if $\varepsilon_i = 0$, and $+$ if $\varepsilon_i = 1$. Thus, for example, $v^0_{1_+2_+,2}$ is the middle vertex $v^0_{S,2}$ with $S = \{1,2\}$ and $\varepsilon_1 = \varepsilon_2 = 1$. Heights (i.e., x_3 values) decrease as the distance from $v^+_\emptyset$ increases in the diagram. The bottom vertex $v^-_\emptyset$ is the point at ∞. All edges and faces are indicated. The labeling for faces is that, for example, 1_- means the wall $x_1 = a_1 - \frac{1}{2}$, while $(1_-2_+)^\pm$ means the hyperplane $x_3 = \pm(\frac{1}{2} + 2a_1x_1 + (2a_2 - 1)(x_2 - \frac{1}{2}))$. The diagram shows very clearly how vertices are intersections of three independent hyperplanes, and how the intersection of a slant hyperplane with its negative gives an edge at height 0.

Figure 4.5. Fictitious polytope



We illustrate Theorem 4.2 by describing the vertices of $\mathcal{R}(P)$ for $P \in (0, \frac{1}{2})^{n-1} \times \{0\}$ and $n \le 6$. Since $0 < \Delta_a \le \frac{1}{8}$, $K(S) \ge 0$ is satisfied if $|S| \le 4$. Thus for $n \le 5$, all standard vertices exist and there are no middle or truncating vertices. The only slight deviation is that if $n = 5$, $|S| = 4$, and $a_1 = a_2 = a_3 = a_4 = \frac{1}{4}$, then $x_n = 0$, so $v^+_S = v^-_S$.

For $n = 6$, $\mathcal{R}(P)$ depends on $Z := \sum_{i=1}^{5}(a_i - \frac{1}{4})^2$. If $Z > \frac{1}{16}$, then standard vertices $v^\pm_S$ exist for all labeled sets S. There are $2 \cdot 3^5$ vertices, since for $i \in [\![5]\!]$, either $\varepsilon_i = 0$ or 1 or $i \notin S$. If $Z = \frac{1}{16}$, there are standard vertices for all S except that if $|S| = 5$, then $v^+_S = v^-_S$. If $Z < \frac{1}{16}$, then $0 < K(S) < 1$ for $1 \le |S| \le 4$, while $K(S) > 1$ if $|S| = 0$, and $K(S) < 0$ if $|S| = 5$. There are standard vertices $v^\pm_S$ for all S with $1 \le |S| \le 4$, and middle vertices $v^0_{[\![5]\!],k}$ for all labelings of $[\![5]\!]$. There are truncating vertices $v^\pm_{\emptyset,k}$ for the ten labeled sets $\{k\}$.

Using (4.6), Theorem 4.2 shows that the types of vertices that occur in $\mathcal{R}(P)$ depend on the sign of $K(S)$ for each $S \subset [\![n-1]\!]$, and, for a fixed vertex type, the coordinates of the vertices are continuous functions of the a_i's. We now show that equivalence of vertices depends just on the sets S (without regard for labeling) and whether $K(S) = 0$.

Proposition 4.6. *If $P \in (0, \frac{1}{2})^{n-1} \times I$, the only equivalences of vertices of $\mathcal{R}(P)$ are*

a. *If $S = S'$ as sets, then $v^+_S \sim v^+_{S'}$, $v^-_S \sim v^-_{S'}$, $v^0_{S,k} \sim v^0_{S',k}$, $v^+_{S,k} \sim v^+_{S',k}$, and $v^-_{S,k} \sim v^-_{S',k}$, and if also $K(S) = 0$ or 1, then $v^+_S \sim v^-_{S'}$.*

b. *If $S' = \widetilde{S}$ as sets, then $v_S^+ \sim v_{S'}^-$, and if also $K(S) = 0$ or 1, then $v_S^+ \sim v_{S'}^+$ and $v_S^- \sim v_{S'}^-$.*

c. *If $k \in S$ and $K(S) < 0 < K(S - \{k\})$, then $v_{S,k}^0 \sim v_{\widetilde{S},k}^{\pm}$ for any labelings.*

Proof. The general rule, which is immediate from the definition, is that vertices $(x_1, \dots, x_n)$ and $(x_1', \dots, x_n')$ of $\mathcal{R}(P)$ are equivalent iff either $x_i - x_i' \in \mathbb{Z}$ for $1 \le i \le n-1$ and $x_n - x_n' \in 2\mathbb{Z}$, or $x_i + x_i' \in \mathbb{Z}$ for $1 \le i \le n-1$ and $x_n - x_n' \in 2\mathbb{Z}+1$. Recall that $v_S^+ = (x_1, \dots, x_{n-1}, a_n + K(S))$ and $v_S^- = (x_1, \dots, x_{n-1}, a_n - K(S))$, and similarly for $v_{S'}^{\pm}$. First note that $x_i - x_i' \in \mathbb{Z}$ for $i \le n-1$ iff the sets S and S' are equal, and $x_i + x_i' \in \mathbb{Z}$ for $i \le n-1$ iff $S' = \widetilde{S}$. Next note that

$$K(S) - K(S') = \begin{cases} 0 & S = S' \text{ as sets} \\ 2K(S) - 1 & S' = \widetilde{S}, \end{cases}$$

while the difference of the nth components of v_S^+ and $v_{S'}^-$ is given by

$$K(S) + K(S') = \begin{cases} 2K(S) & S = S' \text{ as sets} \\ 1 & S' = \widetilde{S}. \end{cases}$$

The proposition for standard vertices now follows easily from the general rule noted above, as do the other equivalences in part (a).

Part (c) is true since the sum of the x_i values for the two vertices is an integer for all $i < n$, and the x_n values differ by 1. We show this for the x_k values by noting that the sum of the relevant x_k values from (4.8) and (4.9) is

$$\begin{aligned} & 2a_k - 1 + 2\varepsilon_k + \frac{K(S - \{k\}) - K(S)}{2a_k - \varepsilon_k} \\ = \quad & -1 + 2\varepsilon_k + \frac{2a_k(2a_k - \varepsilon_k) + 2\Delta_{a_k}}{2a_k - \varepsilon_k} \\ = \quad & -1 + 2\varepsilon_k + \frac{2a_k(1 - \varepsilon_k)}{2a_k - \varepsilon_k} \\ = \quad & \varepsilon_k \end{aligned}$$

for $\varepsilon_k \in \{0, 1\}$. Other possible equivalences involving middle vertices are eliminated by showing that under the hypotheses of (4.8), x_k cannot equal $a_k - \frac{1}{2} + \varepsilon_k$ or $\frac{1}{2} - a_k$. □

An analogous result holds for $((0, \frac{1}{2}) \cup (\frac{1}{2}, 1))^{n-1}$.

5. Proof of Theorem 2.1

We begin by showing how $\mathcal{D} = (0, \frac{1}{2})^{n-1} \times (0, 1)$ can be partitioned into subsets M_α as claimed in Theorem 2.1. The partitioning is done by the quadric surfaces $K(S) = 0$ for subsets S of $[\![n-1]\!]$. The partitioning just depends on S as a set; the labeling on S is used in describing vertices. Each set M_α is uniquely determined by a specification of whether, for each set S, $K(S)$ is positive, negative, or 0. We have described the rather simple partitioning when $n = 6$ in Example 2.3. We will now discuss the more-typical case $n = 7$, as a precursor to the general proof.

When $n = 7$ and $0 < a_i < \frac{1}{2}$,

$$K(S) = \tfrac{5-|S|}{4} + \sum_{i\in S} 2(a_i - \tfrac{1}{4})^2 - \sum_{i\in\widetilde{S}} 2(a_i - \tfrac{1}{4})^2.$$

Since $0 \le (a_i - \frac{1}{4})^2 < \frac{1}{16}$, $K(S) > 0$ if $|S| \le 4$, but $K(S)$ can be positive or negative if $|S| = 5$ or 6. Note that by (4.6), $K(\widetilde{S}) > 1$ iff $K(S) < 0$, so truncating vertices associated to $\widetilde{S}$ will occur exactly when middle vertices associated to S occur. If $K(S) < 0 < K(S - \{k\})$, there are $2^{|S|}$ middle vertices $v^0_{S,k}$ and $2^{|\widetilde{S}|+2}$ truncating vertices $v^\pm_{\widetilde{S},k}$, taking the labeling into account. Domains in which one or more $K(S)$ equal 0 will have dimension less than 7, and will bound other regions. Recall that the nth coordinate of points in $\mathcal{D}$ does not play an important role in this part of the analysis. The description of the vertices of $\mathcal{R}(P)$ assumed $a_n = 0$; for arbitrary a_n, just add that amount to the nth coordinate of all vertices described in Theorem 4.2. The domain M_α will be a product with $(0, 1)$ as its last factor.

We now describe the domains M_α in $(0, \frac{1}{2})^6$ when $n = 7$, omitting the 7th factor. Let $b_i = a_i - \frac{1}{4}$, so $b_i \in (-\frac{1}{4}, \frac{1}{4})$, and $\frac{1}{2}K(S) = \frac{5-|S|}{8} + \sum_S b_i^2 - \sum_{\widetilde{S}} b_i^2$. Let $Z = \sum_{i=1}^6 b_i^2$. In the following list, $K(S) > 0$ unless mentioned to the contrary. The regions M_α are of the following six types. For each, we tell (a) conditions on b_i, (b) conditions on $K(S)$, (c) types of vertices, and (d) dimension (incorporating also the $(0, 1)$ last factor) and number of regions.

a. $Z > \frac{1}{8}$. $K(S) > 0\ \forall S$. Have $v^\pm_S\ \forall S$. One such M_α of dimension 7.
b. $Z = \frac{1}{8}$. $K([\![6]\!]) = 0$. Have $v^\pm_S\ \forall S$ except $v^+_{[\![6]\!]} = v^-_{[\![6]\!]}$. One such region of dimension 6.
c. $Z < \frac{1}{8}$; all $b_k^2 < \frac{1}{2}Z$. $K([\![6]\!]) < 0$. Have $v^\pm_S$ for $1 \le |S| \le 5$, $v^0_{[\![6]\!],k}\ \forall k$, and $v^\pm_{\emptyset,k}\ \forall k$. One such region of dimension 7.
d. $Z < \frac{1}{8}$; one $b_k^2 = \frac{1}{2}Z$. $K([\![6]\!]) < 0$ and $K([\![6]\!] - \{k\}) = 0$. Have $v^\pm_S$ for $1 \le |S| \le 5$ with $v^+_{[\![6]\!]-\{k\}} = v^-_{[\![6]\!]-\{k\}}$. Also $v^0_{[\![6]\!],i}$ and $v^\pm_{\emptyset,i}$ for $i \ne k$. Six such regions of dimension 6.
e. $Z < \frac{1}{8}$; $b_k^2 = b_\ell^2 = \frac{1}{2}Z$; all other $b_i = 0$. $K([\![6]\!]) < 0$, $K([\![6]\!] - \{k\}) = K([\![6]\!] - \{\ell\}) = 0$. Have $v^\pm_S$ for $1 \le |S| \le 5$ with $v^+_{[\![6]\!]-\{k\}} = v^-_{[\![6]\!]-\{k\}}$ and $v^+_{[\![6]\!]-\{\ell\}} = v^-_{[\![6]\!]-\{\ell\}}$. Also $v^0_{[\![6]\!],i}$ and $v^\pm_{\emptyset,i}$ for $i \ne k, \ell$. Fifteen such regions of dimension 2.
f. $Z < \frac{1}{8}$; one $b_k^2 > \frac{1}{2}Z$. $K([\![6]\!]) < K([\![6]\!] - \{k\}) < 0$. Have $v^\pm_S$ for $S \ne [\![6]\!], [\![6]\!] - \{k\}, \{k\}, \emptyset$, also $v^0_{[\![6]\!]-\{k\},\ell}$ and $v^\pm_{\{k\},\ell}$ for all $\ell \ne k$. Also $v^0_{[\![6]\!],i}$ and $v^\pm_{\emptyset,i}$ for $i \ne k$. Six such regions of dimension 7.

Continuing to let $b_i = a_i - \frac{1}{4}$, and ignoring the nth component, for arbitrary n regions M_α are determined by, for all subsets S of $[\![n-1]\!]$, whether

$$\tfrac{n+3-2|S|}{16} + \sum_S b_i^2 - \sum_{\widetilde{S}} b_i^2$$

is positive, negative, or zero. Which vertices of each type occur for $\mathcal{R}(P)$ for P in a region M_α is determined by the signs of the various $K(S)$, which are unique to the region. The coordinates of the vertices of $\mathcal{R}(P)$ of various types are continuous functions of the coordinates $(a_1, \ldots, a_n)$ of P. Moreover, equivalence of vertices is also determined by the sets S and which $K(S) = 0$, which is determined by M_α. The maps θ_P of Theorem 2.1 send the various vertices of $\mathcal{R}(P_0)$ in Theorem 4.2 to the corresponding vertices of $\mathcal{R}(P)$. The same formulas for x_i apply, just to different values of a_i.

We need to "know" the faces of the polytopes $\mathcal{R}(P)$. All of our vertices are obtained as intersections of at least n of the hyperplanes which defined the polytope in Definition 4.1 together with $x_n = a_n \pm 1$. All we need to say about the faces is that a j-face, for $j < n$, is a j-dimensional subset which is the intersection of the polytope with exactly $n - j$ independent bounding hyperplanes. The face equals the convex hull of the vertices which lie in that intersection of hyperplanes. This justifies the claims of the first paragraph of Theorem 2.1 for $\mathcal{D} = (0, \frac{1}{2})^{n-1} \times (0, 1)$.

For $\mathcal{D} = (\frac{1}{2}, 1)^{n-1} \times (0, 1)$, all formulas are extremely similar to those for $(0, \frac{1}{2})^{n-1} \times (0, 1)$, as noted near the end of Section 3. We combine them into $((0, \frac{1}{2}) \cup (\frac{1}{2}, 1))^{n-1} \times (0, 1)$, although we could consider them separately. For $((0, \frac{1}{2}) \cup (\frac{1}{2}, 1))^{n-1} \times \{0\}$, the polygons $\mathcal{R}(P)$ are essentially the same as those we have been considering. We separate it to avoid continuity problems due to the relation (2.1).

After permuting variables, a general $\mathcal{D}$ can be written as

$$\{0, \tfrac{1}{2}\}^j \times ((0, \tfrac{1}{2}) \cup (\tfrac{1}{2}, 1))^{n-1-j} \times D_n.$$

For $P = (a_1, \ldots, a_n)$ in this $\mathcal{D}$,

$$\mathcal{R}(P) = \prod_{i=1}^{j} [a_i - \tfrac{1}{2}, a_i + \tfrac{1}{2}] \times \mathcal{R}(a_{j+1}, \ldots, a_n)$$

with $a_i - \frac{1}{2} \sim a_i + \frac{1}{2}$ and $\mathcal{R}(a_{j+1}, \ldots, a_n)$ as previously described. This can be seen by observing that the inequalities (4.3) and (4.4) will not have the terms for $i \leq j$ because the segment from P to points of $C(P)$ as in (2.2) will involve no change in x_i. The domains M_α here will be $\{x\} \times M'_\alpha$, where $x \in \{0, \frac{1}{2}\}^j$ and M'_α is a domain that works for $\mathcal{R}(a_{j+1}, \ldots, a_n)$.

We now explain why part (a) of Theorem 2.1 is true. Suppose a sequence $P_i \in M_\alpha \subset \mathcal{D}$ converges to $P \in M_{\alpha'} \subset \mathcal{D}'$. We wish to show that if $\alpha' \neq \alpha$, then $\dim M_{\alpha'} < \dim M_\alpha$.

Case 1. $\mathcal{D}' = \mathcal{D}$: Let $N = n-1$. To simplify formulas, let $\mathcal{D} = \{P = (b_1, \ldots, b_N) : -\frac{1}{4} < b_i < \frac{1}{4}\}$. For $S \subset [\![N]\!]$, define $K_S : \mathcal{D} \to \mathbb{R}$ by

$$K_S(P) = \tfrac{N+4-2|S|}{16} + \sum_S b_i^2 - \sum_{\widetilde{S}} b_i^2.$$

This equals $K(S)$ defined in (4.5). Note that $S \subsetneqq S'$ implies $K_{S'}(P) < K_S(P)$. Let $V_S = \{P : K_S(P) = 0\}$.

For a function $\alpha : \mathcal{P}([\![N]\!]) \to \{-1, 0, 1\}$ which satisfies that if $S \subsetneqq S'$, then $\alpha(S') \le \alpha(S)$ with $\alpha(S') < \alpha(S)$ if either is 0, let

$$M_\alpha = \{P \in \mathcal{D} : \operatorname{sgn}(K_S(P)) = \alpha(S)\ \forall S\}.$$

Note that M_α is an open subset of the analytic set $V(\alpha) := \bigcap_{\alpha(S)=0} V_S$. Let $d = \dim(V(\alpha)) = \dim(M_\alpha)$.

Proposition 5.1. *If $\langle P_i \rangle$ in $M_\alpha \subset \mathcal{D}$ approaches $P \in M_{\alpha'} \subset \mathcal{D}$, then either $\alpha = \alpha'$ or $\dim(M_{\alpha'}) < d$.*

Proof. Since $K_S(P)$ is a continuous function of P, if $\alpha(S) = 0$, then $\alpha'(S) = 0$, while if $\alpha(S) \ne 0$, then $\alpha'(S) = \alpha(S)$ or 0. Thus if $\alpha' \ne \alpha$, there must be a set S_0 with $\alpha'(S_0) = 0$ and $\alpha(S_0) \ne 0$. The points $(b_1, \ldots, b_N)$ in $V(\alpha)$ are those which satisfy a linear system in b_i^2, with a row for each $S \in \alpha^{-1}(0)$. This linear system can be row reduced to a matrix which (after permuting variables) expresses $b_1^2, \ldots, b_{N-d}^2$ in terms of $b_{N-d+1}^2, \ldots, b_N^2$. Use these reduced rows to eliminate the first $N-d$ variables from the equation $K_{S_0}(P) = 0$. If the obtained equation involving $b_{N-d+1}^2, \ldots, b_N^2$ is not identically 0, then it can be used to eliminate another variable, showing that $\dim(V(\alpha')) < d$. If it is identically 0, this says that any P which satisfies $K_S(P) = 0$ for all $S \in \alpha^{-1}(0)$ must satisfy $K_{S_0}(P) = 0$. Thus all $P \in M_\alpha$ have $K_{S_0}(P) = 0$, and so $\alpha(S_0) = 0$, contradiction. $\square$

Case 2. $\mathcal{D}'$ is formed from $\mathcal{D}$ by changing $(0,1)$ in the last factor to $\{0\}$. This case is easy, since the M_α's for $\mathcal{D}'$ are obtained from those of $\mathcal{D}$ by changing the last factor from $(0,1)$ to $\{0\}$. Our considerations of M_α have generally been just with regard to the first $n-1$ variables. Since all 0's of M_α hold for $M_{\alpha'}$, the dimension with respect to the first $n-1$ variables of $M_{\alpha'}$ is $\le$ that of M_α, but its dimension is smaller due to the last factor.

Case 3. $\mathcal{D}'$ is formed from $\mathcal{D}$ by changing one or more $(0, \frac{1}{2})$ factors to $\{0\}$. (Other cases similar.) For simplicity, let's say that it is just the $(n-1)^{\text{st}}$ factor of $(0, \frac{1}{2})$ which is changed to 0. We consider the subsets S of $[\![n-2]\!]$ which determine vertices of polytopes for P in $(0, \frac{1}{2})^{n-2} \times \{0\} \times (0,1)$, and via the function K, determine the regions $M_{\alpha'}$ of $(0, \frac{1}{2})^{n-2} \times \{0\} \times (0,1)$. The key point is that, since $\lim_{a \to 0} \Delta_a = 0$, if a sequence $P_i \in M_\alpha$ converges to $P \in M_{\alpha'}$, then for any $S \subset [\![n-1]\!]$,

$$\lim_{P_i \to P} K(S)(P_i) = K(S \cap [\![n-2]\!])(P).$$

Since, for $S \subset [\![n-2]\!]$, $K(S) > K(S \cup \{n-1\})$, it cannot happen that two different $S \subset [\![n-1]\!]$ have $K(S) = 0$ in M_α but yield only one such when intersected with $[\![n-2]\!]$. So, similarly to Case 1, $\mathcal{D}'$ has as many restricting equations as does $\mathcal{D}$, but in a domain one dimension smaller.

If instead P is in a set equivalent to $M_{\alpha'} \subset \mathcal{D}'$, then it is similar to Case 2 or 3 above with one or more factors changed to $\{1\}$ instead of $\{0\}$. Our M_α's have been defined to be in various $\mathcal{D} \subset [0,1)^n$. For example, if

$$\mathcal{D} = \{0, \tfrac{1}{2}\} \times \{0, \tfrac{1}{2}\} \times ((0, \tfrac{1}{2}) \cup (\tfrac{1}{2}, 1))^{n-3} \times (0,1),$$

the M_α's in the subset with 0 in the first two coordinates admit homeomorphic and equivalent $M_{\alpha'}$'s with the first two coordinates changed to 1, and the argument in Case 2 above shows that if a sequence in some M_α approaches a point $P = (1, 1, a_3, \dots, a_n)$, then the dimension of the new $M_{\alpha'}$ containing P is less than that of M_α. Similarly, if P, the limit of a sequence in M_α, has final component 1, then it lies in a set $M_{\alpha'}$ which is homeomorphic (under a $x \leftrightarrow 1-x$ correspondence in other components) to one of the $M_{\alpha'}$'s already considered, and its dimension is less than that of M_α by the argument in Case 3 above. This completes the proof of part (a) of Theorem 2.1.

The proof of part (b) of Theorem 2.1 involves two issues: (i) that vertices of $\mathcal{R}(P_i)$ converge to vertices of $\mathcal{R}(P)$, and (ii) that faces approach faces, of equal or smaller dimension. (i) is trivial if $P \in M_\alpha$. If P is in the same $\mathcal{D}$-set as M_α, but a different M_α, then it must be the case that, for some S, either $K(S)(P_i)$ or $K(S - \{k\})(P_i)$ approaches 0 in (4.8) or $K(S)(P_i)$ or $K(S \cup \{k\})(P_i)$ approaches 1 in (4.9). By computing the limiting value of x_k in each case, we see that $v^0_{S,k} \to v^+_S = v^-_S$, $v^0_{S,k} \to v^+_{S-\{k\}} = v^-_{S-\{k\}}$, $v^\pm_{S,k} \to v^\pm_S$, or $v^\pm_{S,k} \to v^\pm_{S-\{k\}}$ in the four cases, so a "standard" vertex is obtained in the limit. If P_i approaches a point with some component equal to 0 or $\frac{1}{2}$, then, in the formula $K(S) = \frac{1}{2} - \Delta(S) + \Delta(\widetilde{S})$, the limit as a_i approaches 0 or $\frac{1}{2}$ equals the value it would have if the ith component of S was omitted, so a vertex is obtained in the limit.

Regarding (ii): Vertices in a j-face F of $\mathcal{R}(P_0)$ satisfy $n-j$ independent equalities in Definition 4.1, perhaps including also $x_n = a_n \pm 1$ or $x_n = 0$. The corresponding vertices of $\mathcal{R}(P_i)$ satisfy the same equalities. For example, if the vertices of F satisfy $x_n = \frac{1}{2} + \sum 2a_j(P_0)x_j$, then vertices of $\mathcal{R}(P_i)$ satisfy $x_n = \frac{1}{2} + \sum 2a_j(P_i)x_j$ by the definition of θ_{P_i}, and since $a_j(P_i)$ approaches $a_j(P)$, the same is true in $\mathcal{R}(P)$. So the vertices of $\mathcal{R}(P)$ satisfy at least $n-j$ independent equalities of the appropriate form, and hence span a face of dimension $\le j$.

We amplify on the word "independent" by showing that the rank of a system of such equations at the limit point cannot be less than that for the points in the sequence, by the nature of the equations. After inserting values $x_i = a_i - \frac{1}{2} + \varepsilon_i$ of wall variables, and then subtracting one equation (associated to (4.10)) from all the others, the system is effectively a matrix of 0's and 1's determined by the differences of the values of δ_i's from (4.10) in the equations being subtracted. These do not change under taking limits. The system at the limit point could involve more equations than those in the sequence, for example if a sequence of points on a slant hyperplane approach a point on a wall, and this would decrease the dimension of the face.

6. Lower bounds

Theorem 6.1. *The geodesic complexity of the higher Klein bottle K_n satisfies*

$$\mathrm{GC}(K_n) \geq 2n.$$

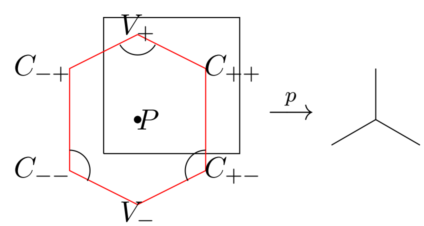
Proof. Assume that there exists a decomposition $X \times X = \bigsqcup_{i=0}^{k} E_i$ satisfying Definition 1.2. We need to show that $k \geq 2n+1$. We will do this in three steps.

First part: $k \geq n+1$

Fix a point P in the universal cover of K_n. Let $\mathcal{R}$ be the polytope associated to P. Fix a vertex V of $\mathcal{R}$. Let U_ϵ be a ball of (sufficiently small) radius ϵ around the projection of V in K_n.

The ball U_ϵ is homeomorphic to a ball in $\mathbb{R}^n$ centered around the origin. This ball is divided into open chambers, which correspond to the projection of the interior of $\mathcal{R}$. For example, if $V = V_+$ in Figure 6.2, which is a rendering of Figure 3.1, the chambers are projections of the small sectors at V_+, C_{--}, and C_{+-} in the interior of the hexagon. The diagram also indicates their projection in K_2. The chambers are divided by $(n-1)$-dimensional walls. The walls are the projection of part of the $(n-1)$-skeleton of $\mathcal{R}$. Those walls intersect along lower dimensional walls, which are the projections of the skeleta of the corresponding dimension.

Figure 6.2. Chambers



Choose a point in an open chamber. Let Q denote its unique preimage in the interior of the polytope $\mathcal{R}$. There is a unique shortest path between P and Q, and its projection to K_n is the unique shortest path between $p(P)$ and $p(Q)$.

Now choose a point x in the interior of an $(n-1)$-wall. Every neighborhood of x intersects the two chambers meeting at this wall. Suppose we choose two sequences $\langle x_i^1 \rangle$ and $\langle x_i^2 \rangle$ converging to x, each sequence contained in a different chamber. For every x_i^1 there is a unique shortest path over $(p(P), x_i^1)$, and analogously for x_i^2. However, the shortest paths over $(p(P), x_i^1)$ and over $(p(P), x_i^2)$ converge to two different shortest paths over $(p(P), x)$. This is because approaching x from each chamber corresponds to approaching two distinct points in the polytope in the universal cover, which are the two preimages of x.

Assume, for the sake of contradiction, that there is a neighborhood W_ϵ of x such that $\{p(P)\} \times W_\epsilon$ is contained in E_1. Then we may assume that E_1 contains both sequences $(p(P), x_i^1)$ and $(p(P), x_i^2)$, as well as $(p(P), x)$ itself. By the preceding paragraph, this implies that there cannot exist a continuous choice of shortest path over E_1. However, a local section is required to exist over each E_i. This shows that every neighborhood of $(p(P), x)$ intersects nontrivially at least two different sets E_i. Equivalently, $(p(P), x)$ lies in the closure of at least two E_i.

Fix the point P throughout this proof. Denote $\tilde{E}_i = E_i \cap (\{p(P)\} \times K_n)$ so as to omit the first coordinate from the notation.

The rest of the argument proceeds by induction: Assume that every point y which is in the interior of an $(n-k+1)$-wall lies in the closure of at least k many $\tilde{E}_i$. We want to show that every point x in the interior of an $(n-k)$-wall lies in the closure of at least $k+1$ many $\tilde{E}_i$. Choose a sufficiently small neighborhood W_ϵ of x. Assume, for the sake of contradiction, that W_ϵ is contained in $\tilde{E}_1 \sqcup \tilde{E}_2 \sqcup \cdots \sqcup \tilde{E}_k$. Furthermore assume that x is in $\tilde{E}_1$.

For the following argument it is useful to picture a tiling of the universal covering space $\mathbb{R}^n$ by copies of the polytope $\mathcal{R}$, according to the deck transformations (translations and reflections). From this point of view, the chambers can be thought of as the projections of (pieces of) those polytopes which are adjacent to the vertex V.

Let L be the intersection of W_ϵ with an $(n-k+1)$-wall which contains x in its closure. The point x is in the closure of several $(n-k+1)$-walls and no chamber contains all of those $(n-k+1)$-walls in its boundary.

Choose a sequence $\langle y_i \rangle$ in L converging to x. Since L is contained in $\tilde{E}_1 \sqcup \tilde{E}_2 \sqcup \cdots \sqcup \tilde{E}_k$, by the induction hypothesis each y_i lies in the closure of $\tilde{E}_1$ (as well as $\tilde{E}_2, \ldots, \tilde{E}_k$). Therefore, there exists a sequence in $\tilde{E}_1$ converging to y_i, which might be constant. By a diagonal argument we can find a sequence $\langle x_i \rangle$ converging to x such that the entire sequence is either contained in L or contained in the interior of one chamber which is adjacent to L.

Now consider the shortest path $s_1(p(P), x)$ given by the local section over E_1. By continuity, $s_1(p(P), x)$ needs to coincide with the limit of $s_1(p(P), x_i)$, as i goes to infinity. This implies that the path $s_1(p(P), x)$ needs to intersect a chamber which is adjacent to L, by the previous paragraph. Note that if the x_i's were all in L, then a subsequence of them has $s_1(p(P), x_i)$ intersecting a single chamber adjacent to L.

However, we could repeat the argument with any other $(n-k+1)$-wall which has x in its closure, reaching an analogous conclusion. As we noted, those $(n-k+1)$-walls can be chosen so that they do not have a chamber in common, which yields a contradiction.

This concludes the induction argument. Because V is a vertex (or 0-wall) it follows that the pair $(p(P), p(V))$ lies in the closure of at least $n+1$ many E_i. We say that at least $n+1$ sets E_i accumulate at every pair $(p(P), p(V))$.

Second part $k \geq 2n$

Consider a point $P = (a_1, a_2, \dots, a_n)$ in the universal cover of K_n with coordinates $0 < a_i < \frac{1}{2}$ for $1 \leq i \leq n-2$, $a_{n-1} = \frac{1}{2}$ and $a_n = 0$. Assume that all the a_i are very close to $\frac{1}{2}$ so that there are no middle or truncating vertices or unusual equivalences between vertices in $\mathcal{R}(P)$.

Let S be the labeled subset of $[\![n-2]\!]$ associated to some vertex V_0 of $\mathcal{R}(P)$ and assume that V_0 has coordinate $x_{n-1} = 0$.

Now fix all coordinates as above except that $0 < a_{n-1} < \frac{1}{2}$. As a_{n-1} approaches $\frac{1}{2}$ from the left, the polytopes of the corresponding points have two vertices approaching the vertex V_0. Those vertices correspond to the labeled sets S and $S \cup \{n-1\}$ as subsets of $[\![n-1]\!]$ (with $\varepsilon_{n-1} = 0$). We will call them V_1 and V_2 respectively. Keep in mind that V_1 and V_2 vary as a_{n-1} changes.

Concretely, V_1 and V_2 only differ at x_{n-1} and x_n. The vertex V_1 has $x_{n-1} = \frac{1}{2} - a_{n-1}$ and the vertex V_2 has coordinate $x_{n-1} = a_{n-1} - \frac{1}{2}$. As a_{n-1} approaches $\frac{1}{2}$, both x_{n-1} coordinates approach 0, which is precisely the x_{n-1} coordinate of V_0. The x_n coordinates of V_1 and V_2 differ by $2\Delta_{a_{n-1}}$ which goes to 0 as a_{n-1} goes to $\frac{1}{2}$, and the other Δ_{a_i} summands in (4.5) are equal in V_1, V_2, and V_0.

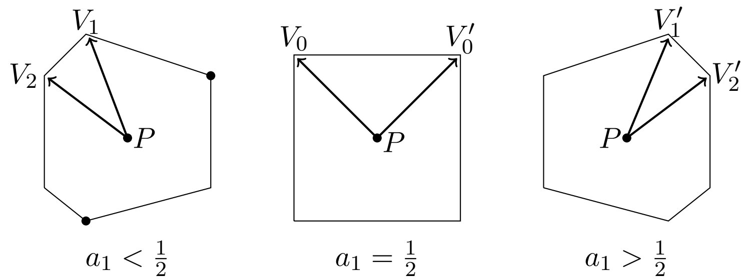
How many ways are there to make a continuous choice of shortest paths between the projections of the $P = (a_1, a_2, \dots, a_n)$ and the projections of those three types of vertices? It boils down to choosing paths between the point P and other vertices of $\mathcal{R}(P)$ which are equivalent to V_1, V_2 and V_0 respectively.

We assumed that all vertices are regular and have a small x_n coordinate. By Proposition 4.6, such vertices are equivalent to all vertices which have subscript either the same set S with the same x_n or the complementary set $\widetilde{S}$ and inverted x_n. For example, in Figure 6.3, V_2 is equivalent to the two vertices indicated with $\bullet$s.

While there are several consistent choices of shortest paths over the three equivalence classes, all those paths need to go in negative direction in the $n-1$ coordinate. This is because in order for two representatives of the equivalence classes of V_1 and V_2 to approach each other, one of the two vertices needs to have $x_{n-1} = \frac{1}{2} - a_{n-1}$ and the other one $x_{n-1} = a_{n-1} - \frac{1}{2}$, both of which are much less than $a_{n-1} = \frac{1}{2}$. This is illustrated in the case $n = 2$ in the left half of Figure 6.3. As a_1 approaches $\frac{1}{2}$ from the left, and

V_1 and V_2 approach the vertex V_0, the only continuous choice of minimal geodesics requires all geodesics to go the left. Similarly for a_1 approaching $\frac{1}{2}$ from the right, V_1' and V_2' approach V_0', and geodesics go to the right.

Figure 6.3. Geodesics when approaching from left or right



Repeating the whole procedure except with $\frac{1}{2} < a_{n-1} < 1$ approaching $\frac{1}{2}$ from the right instead is completely analogous. However, the conclusion in that case is completely opposite: in all possible continuous choices of shortest paths over the three types of vertices, the paths need to go in positive direction in the $n-1$ coordinate. This is because in that case the coordinates are $x_{n-1} = 1 - (a_{n-1} - \frac{1}{2})$ and $x_{n-1} = a_{n-1} + \frac{1}{2}$.

Now assume, for the sake of contradiction, that there is a ball U_ϵ around $(p(P_0), p(V_0))$ which is contained in $E_1 \sqcup \cdots \sqcup E_{n+1}$ and that $(p(P_0), p(V_0))$ lies in E_1. In the first part of the proof we showed that at least $n+1$ sets E_i accumulate at every pair of the form $(p(P), p(V))$. In particular, E_1 accumulates at every pair $(p(P), p(V))$ contained in U_ϵ.

If ϵ is small enough, the paths in the image of the section over $E_1 \cap U_\epsilon$ need to be very close to the path $s_1(p(P_0), p(V_0))$. In particular, they need to go in the same direction in the $n-1$ coordinate. Because E_1 accumulates at all the pairs $(p(P), p(V_1))$ and $(p(P), p(V_2))$ in U_ϵ, this means that there are shortest paths with endpoints very close to $p(V_1)$ and $p(V_2)$ which are close to $s_1(p(P_0), p(V_0))$ and thus close to each other. As we saw above, if two shortest paths from $p(P)$ to $p(V_1)$ and $p(V_2)$ respectively are close to each other they need to go in negative $n-1$ direction if $0 < a_{n-1} < \frac{1}{2}$ and in positive $n-1$ direction if $\frac{1}{2} < a_{n-1} < 1$. This implies that the paths cannot all go in the same direction.

This yields a contradiction. Therefore, at least $n+2$ many E_i intersect every neighborhood of every pair of the form $(p(P), p(V))$ with $0 < a_i < \frac{1}{2}$ for $1 \leq i \leq n-2$ and $a_{n-1} = \frac{1}{2}$. In other words, at least $n+2$ sets E_i accumulate at every pair of that form.

Note that the assumption $0 < a_i < \frac{1}{2}$ can be relaxed to $0 < a_i < \frac{1}{2}$ for some $1 \leq i \leq n-2$ and $\frac{1}{2} < a_i < 1$ for other $1 \leq i \leq n-2$, and the argument above still goes through.

The rest of the argument proceeds by induction. The next induction step is to let the coordinate a_{n-2} approach $\frac{1}{2}$ from the left and the right and show that yet another set E_i is needed in every neighborhood of the corresponding vertex.

After letting all coordinates a_i with $1 \leq i \leq n-1$ approach $\frac{1}{2}$, one by one, we conclude that every vertex of the polytope $\mathcal{R}(P)$ for $P = (\frac{1}{2}, \dots, \frac{1}{2}, 0)$ is in the closure of $(n+1) + (n-1) = 2n$ many E_i.

Third part $k \geq 2n+1$

Let S_{2n} denote the set of all pairs $(p(\frac{1}{2}, \dots, \frac{1}{2}, a_n), p(1, \dots, 1, a_n + \frac{1}{2}))$. We just proved that every such pair of points lies in the closure of $2n$ sets E_i.

Assume that $k = 2n$, i.e., there are exactly $2n$ sets E_i in the decomposition of $K_n \times K_n$. Then S_{2n} is contained in the closure of every E_i.

Because the E_i are locally compact, they have to be locally closed, which means that they are open in their closure. In particular all the intersections $E_i \cap S_{2n}$ need to be open in S_{2n}, as well as disjoint. Because S_{2n} is connected, this implies that if $E_i \cap S_{2n} \neq \emptyset$, then $S_{2n} \subset E_i$. Therefore there is a continuous choice of geodesics over S_{2n}. However, to make a continuous choice of path for all pairs $(p(\frac{1}{2}, \dots, \frac{1}{2}, a_n), p(1, \dots, 1, a_n + \frac{1}{2}))$ it is necessary to make a consistent choice of left or right in the first coordinate, say. This is of course not possible for all a_n, because as a_n goes from 0 to 1, the directions switch. □

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