Good functions for translations

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ABSTRACT. We examine ways to describe the good functions for a.e. convergence of sequences of translations in the real line. For sequences, it is well-known that translations are generically bad pointwise a.e., while for any integrable function there is a subsequence which is good pointwise a.e. We construct various examples of when the sequence $f(x + t_n)$ does not converge a.e. or when it does converge a.e. for a sequence (t_n) tending to zero. In particular, let $f \in L^\infty[0, 1]$. We show that if for any sequence (t_n) tending to zero, the sequence $f(x + t_n)$ converges for a.e. x, then f must be equal a.e. to a Riemann integrable function, and conversely. We discuss other techniques, issues, and questions related to sequences in the real line.

CONTENTS

1. Introduction

We consider a sequence (t_n) of non-zero real numbers converging to 0. Let $T_{t_n} f(x) = f(t_n + x)$ for $f : R \to \mathbb{C}$. It is well-known that these operators are good in norm on $L^p(\mathbb{R}), 1 \le p < \infty$. That is, for all $f \in L^p(\mathbb{R}), \|T_{t_n} f - f\|_p \to 0$ as $n \to \infty$.

However, these operators are not good pointwise. In particular, we have that

(a) given (t_n) going to zero, with $t_n \ne 0$ for all n, there is a dense G_δ set $\mathcal{G}$ in $L^p(\mathbb{R}), 1 \le p < \infty$, with respect to the L^p-norm topology, such that for all $f \in \mathcal{G}$, one has for a.e. x, $\limsup |T_{t_n} f(x)| = \infty$;

(b) and in addition, for each sequence described above, there exists a dense G_δ set, $\mathcal{B}$, of Lebesgue measurable sets in $[0, 1]$, such that for every $E \in \mathcal{B}$, $T_{t_n} \mathbb{1}_E(x)$ fails to converge for a.e. x. In fact, it can be arranged so that every $E \in \mathcal{B}$ has, for a.e. x, both

$$\limsup_{n \to \infty} T_{t_n} \mathbb{1}_E(x) = 1 \text{ and } \liminf_{n \to \infty} T_{t_n} \mathbb{1}_E(x) = 0.$$

So while translations converge in norm, they generically fail to converge a.e. Indeed, even their averages generically fail to converge a.e.– see Bellow [4] and Bourgain [6]. We refer the reader to Karagulyan [16] for the general version where one considers measures with discrete support.

These two results – convergence in norm and the failure to converge pointwise in general – provide necessary background for the discussion that follows. We refer to them as **Result 1** and **Result 2** respectively[1].

But what functions nonetheless are, for every sequence of translations converging to 0, pointwise good a.e? Clearly, if f is continuous at x, then $f(x + t_n) \to f(x)$ for all translates $t_n \to 0$. So if f bounded and Riemann integrable on each bounded interval, then it is a good function for pointwise a.e. convergence of translations. Actually, one only needs that such f is equal a.e. to a Riemann integrable function for the a.e. convergence to hold. Even more generally, we say that a Lebesgue measurable f is *relatively continuous off a null set* if there is a Lebesgue null set N such that f restricted to $\mathbb{R} \backslash N$ is real-valued and continuous in the relative topology. If f is relatively continuous off a null set, then for any sequence of translations $t_n \to 0$, one has for a.e. x, $f(x + t_n)$ converges to $f(x)$.

Of course, if f is not continuous at x, then $f(x + t_n)$ will fail to converge to $f(x)$ for a correctly chosen $t_n \to 0$. But for a fixed (t_n), it is not clear what structural property is needed for the convergence to fail on a set of positive measure. This could be a larger class than the functions that are relatively continuous off a null set. However, we will show the following:

Theorem. *Let f be a bounded, real-valued function on a bounded interval. Then the translates $f(x + t_n)$ converge (as t_n tends to zero) for all (t_n) if and only if f is locally equal to a Riemann integrable function a.e.*

This theorem, given in context below as Proposition 3.2 and **Result 7**, also serves to illustrate the delicate placement of quantifiers in these results. For example, **Result 5** demonstrates that there is no single "litmus test" sequence for which a.e. convergence of the translates necessarily implies that the function is equal to a Riemann integrable function a.e.

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[1]In order to provide rapid and easy reference to the major propositions that comprise the core of the narrative, along with associated issues, most of the results and questions in this paper can be found by searching for Result or Question, which will be bold face in the text. We have identified in this fashion 10 Results and 7 Questions; some of these are well-known standard facts and the rest are new to this article. Reading these first, one can get a clear overview of some of the complications surrounding the subject and the content of this article.

Remark 1.1. This bad pointwise behavior of subsequences depends critically on the discreteness of the support of the operators. If one instead considers a sequence of absolutely continuous measures– that is, an approximate identity for $L^1(\mathbb{R})$– then there are subsequences that do behave well a.e. on all of $L^1(\mathbb{R})$. This was considered in Rosenblatt [24], but the best theorem was not obtained there: Kostyukovsky and Olevskii actually earlier gave the best subsequence theorem. In [20], they show a subsequence exists that will actually give a.e. convergence on $L^1(\mathbb{R})$. It does not seem to be completely resolved yet exactly what property of a sequence of convolutions by Borel measures on $\mathbb{R}$ will guarantee that there is a subsequence that converges a.e. on $L^p(\mathbb{R})$ for some or all $p, 1 \leq p \leq \infty$.

In Section 2 we look at some basic examples that illustrate the ideas and methods to follow, and some of the constraints on these examples. Then, in Section 3 we extend this analysis to show essentially that the Riemann integrable functions are the ones pointwise good a.e. for all sequences of translations converging to zero. Following that, in Section 4, given an integrable function, we examine some questions concerning what rate (t_n) needs to converge to zero for pointwise a.e. convergence for the t_n translates of f. In Section 5 we extend the overall thinking of the previous sections to provide a concept of boundary that applies best in this context. Lastly, in Section 6 we consider a famous question due to Erdős regarding the universal behavior of similar copies of translates converging to zero and provide a possible approach to solving this problem.

2. Basic examples

Consider now a compact set $K \subset [0, 1]$ of positive measure but with no interior. Let U be its dense, open complement. We will inductively construct a sequence (t_n) converging to 0 such that for all $x \in K$, we have $t_n + x \in K^c$ infinitely often. Once that is done, we note that $\mathbb{1}_{-t_n+K} \to \mathbb{1}_K$ pointwise a.e. along a subsequence of (t_n); consequentially, for a.e. $x \in L$, we also have $t_n + x \in K$ infinitely often. So, for a.e. x, we have both conditions and there is a null set $N \subset K$ such that on $K \backslash N$, we do not have convergence of $T_{t_n} \mathbb{1}_K = \mathbb{1}_{t_n+K}$ as $n \to \infty$. This also means, in the notation introduced in Section 5, that $K \backslash N$ is a subset of the *edge set* $\mathcal{E}_{\{t_n : n \geq 1\}}(K)$.

Here is a basic fact that anticipates some later theorems:

Proposition 2.1 (**Result 3**). *Given $E \subset [0, 1]$ which is compact, has no interior, and has positive measure, there is a sequence (t_n) decreasing to 0 such that $T_{t_n} \mathbb{1}_K = \mathbb{1}_{K-t_n}$ fails to converge for a.e. $x \in K$.*

Proof. We will create $N_j \leq N_{j+1}$ and inductively choose t_n for n with $N_j \leq n < N_{j+1}$. It will be clear what the construction is if we just describe the inductive step at the j-th stage. Suppose N_j has been already fixed.

First, if $x \in K$ and $\epsilon > 0$, then because K^c is an open dense set, there must be a non-empty open interval in K^c within ϵ of x. Hence, given $\epsilon_j > 0$, for some $N_{j+1} > N_j$, we can choose a finite cover of K by open intervals $(I_n : N_j \leq n <$

N_{j+1}) which are of such small length that for each of them there is a t_n with $I_n + t_n \subset K^c$ and $|t_n| \leq \epsilon_j$ for all $n, N_j \leq n < N_{j+1}$. This construction may require a choice of N_{j+1} that is large, but that does not matter as long as it can be fixed. This defines $(t_n : n < N_{j+1})$ and completes the inductive step.

By making the size ϵ_j smaller than all previous terms and reordering the translates on the block, we can arrange that the sequence (t_n) is decreasing with all $t_n > 0$. This gives us a sequence (t_n) converging to 0 such that for all $x \in K$, we have $t_n + x \in K^c$ infinitely often. $\square$

Remark 2.2. It is not immediately clear how to use a similar construction to show that every bounded set that is not Jordan measurable will have a bad sequence associated with it. The difficulty is that the structure of such sets can be much more complicated than the case of a compact set with no interior. Indeed, it is easy to show that in the symmetry pseudo-metric on the measurable sets, the generic set E with have the property that both it and its complement intersect every non-empty open set in a set of positive measure.

Question 2.3. *Can we extend this type of construction to functions that are not Riemann integrable?*

This question is addressed in Section 3.

In addition, we can see that any sequence t_n converging to zero has associated with it functions f for which $T_{t_n} f \to f$ a.e. as $n \to \infty$. Of course, this is immediate for characteristic functions of intervals. But the picture is less clear if we require $f = \mathbb{1}_E$, with E compact and without interior, as in Proposition 2.1.

To demonstrate, we will first need this lemma:

Lemma 2.4 (Result 4). *Suppose (t_n) is a bounded sequence in $\mathbb{R}$. Then $U(\epsilon) = \bigcup_{n=1}^{\infty} t_n + (0, \epsilon)$ has $m(U(\epsilon)) \to 0$ as $\epsilon \to 0$ if and only if the closure of $\{t_n : n \geq 1\}$ is a Lebesgue null set.*

Proof. Let $U^s(\epsilon) = \bigcup_{n=1}^{\infty} t_n + (-\epsilon, \epsilon)$. Let K be the closure of $\{t_n : n \geq 1\}$. We have $K \subset U^s(\epsilon)$ for any $\epsilon > 0$, since, otherwise, there is $x \in K$ such that $|x - t_n| \geq \epsilon$ for all n. This is impossible since (t_n) contains a subsequence such that $t_{n_j} \to x$ as $j \to \infty$.

Since (t_n) is bounded, K is compact. Hence, if V is open and $K \subset V$, then there is some $\delta > 0$ such that $|x - y| \geq \delta$ for all $x \in K$ and $y \notin V$. So there must be some $\epsilon > 0$ such that $K \subset U^s(\epsilon) \subset V$.

Now by outer regularity of m, if K is a Lebesgue null set and $\delta > 0$, then there exists V open such that $K \subset V$ and $m(V) < \delta$. Take $\epsilon > 0$ such that $K \subset U^s(\epsilon) \subset V$. Hence, $m(U^s(\epsilon)) < \delta$. Since $U(\epsilon) \subset U^s(\epsilon)$, we have that $m(U(\epsilon)) < \delta$, too. Since the sets $U(\epsilon)$ are decreasing as ϵ decreases, this means that if K is a Lebesgue null set, then $m(U(\epsilon)) \to 0$ as $\epsilon \to 0$.

Conversely, $K \subset U^s(\epsilon/2)$ for any $\epsilon > 0$. Now $m(U^s(\epsilon/2)) = m(U(\epsilon))$ since $U^s(\epsilon/2) + \epsilon/2 = U(\epsilon)$. Hence, if $m(U(\epsilon)) \to 0$ as $\epsilon \to 0$, we must have $m(K) = 0$. $\square$

Remark 2.5. We have, as an immediate consequence of Lemma 2.4, that if $t_n \to$ 0 as $n \to \infty$, then $m(U(\epsilon)) \to 0$ as $\epsilon \to 0$. Similarly, we also have that

$$m\left(\bigcup_{n=1}^{\infty} t_n + (a, b)\right) \to 0 \text{ as } b \to a.$$

We are now ready to show that no sequence of translates, with t_n going to zero, fails to converge for every compact set without interior.

Proposition 2.6 (**Result 5**). *Given a sequence t_n going to zero, there exists a compact set K of positive measure and no interior such that $T_{t_n} \mathbb{1}_K \to \mathbb{1}_K$ a.e. as $n \to \infty$.*

Proof. We want to construct a compact set K and an increasing sequence N_k such that for all $k \geq 1$

$$m\left(\bigcup_{n \geq N_k} (-t_n + K\Delta K)\right) \leq 1/2^k.$$

Then for a.e. x, we would have, for a large enough $k = k(x)$, $x \notin \bigcup_{n \geq N_k} (-t_n + K\Delta K)$. So for all $n \geq N_k$ either $x \in -t_n + K \cap K$ or $x \notin -t_n + K \cup K$. That is, either $x \in K$ and $t_n + x \in K$, or $x \notin K$ and $t_n + x \notin K$. So $\mathbb{1}_K(x) = \mathbb{1}_K(t_n + x)$ for all $n \geq N_k$. Hence, for a.e. x, $\mathbb{1}_K(t_n + x) \to \mathbb{1}_K(x)$ as $n \to \infty$.

We proceed by induction. Starting with $K_0 = [0, 1]$, for $k \geq 1$, we then create a compact set K_k by removing a small open interval, U_k, from K_{k-1}. The compact set K will be $\bigcap_{k=1}^{\infty} K_k$. By construction, the complement of K will be small enough so that $m(K) > 0$. Furthermore, the small open intervals U_k will be chosen in a manner that guarantees that $[0, 1]\backslash K$ is dense in $[0, 1]$.

We will arrange that we have the following inequalities. First, that

$$m(K_k \backslash K) < 1/2^k. \tag{2.1}$$

Also, there exists N_k such that

$$m\left(\bigcup_{n \geq N_k} (-t_n + K_k \Delta K_k)\right) < 1/2^k. \tag{2.2}$$

Finally, there exists N_k such that for all $l \geq k$,

$$m\left(\bigcup_{n \geq N_k} (-t_n + K_l \Delta K_l)\right) < 1/2^k. \tag{2.3}$$

Notice that (2.3) includes (2.2). But it is easier to explain how (2.2) holds and then how it can be modified to show that, in fact, (2.3) is also true.

The basic reason that these estimates hold is that K_k is a finite union of closed intervals. Consequently, we can use Lemma 2.4 to show that the inductive step does not change any relevant previous estimates.

To arrange that (2.1) holds, note that

$$K_k \backslash K = K_k \backslash K_{k+1} \cup \cdots \cup K_{k+d-1} \backslash K_{k+d} \cup K_{k+d} \backslash K \text{ for all } d \geq 1.$$

Taking the limit as $d \to \infty$ gives

$$K_k \backslash K = \bigcup_{i=1}^{\infty} K_{k+i-1} \backslash K_{k+i}.$$

Hence, by making $m(K_{k+i-1} \backslash K_{k+i}) = m(U_{k+i})$ small enough, we can guarantee that $m(K) > 0$ and that (2.1) holds for all k.

At the same time, because K_k is a finite union of closed intervals and $t_n \to 0$ as $n \to \infty$, we know that there is some N_k such that (2.2) holds.

In addition, Lemma 2.4 tell us that for any $l \geq 1$, in choosing U_l, we can arrange that (2.3) holds for all $k, 1 \leq k < l$. Actually, when removing the small interval U_l at an inductive step we will need to use repeatedly the following general inequality:

$$\bigcup_{i=1}^{\infty}(s_i + (V \backslash U) \Delta (V \backslash U)) \subset \bigcup_{i=1}^{\infty}(s_i + V \Delta V) \cup \bigcup_{i=1}^{\infty} s_i + U.$$

It is important to note here that it is the value of $m(U_l)$ that allows us to arrange (2.3); we have the freedom to place U_l anywhere we like within $[0, 1]$. This is what enables us to guarantee that K will not have any interior.

Finally, let us now use (2.1) and (2.3) to arrive at the conclusion. From (2.3), it follows that for each $M \geq N_k$, and $l \geq k$, we have

$$m \left(\bigcup_{M \geq n \geq N_k} (-t_n + K_l \Delta K_l) \right) < 1/2^k.$$

Letting $l \to \infty$ and (2.1), we see that

$$m \left(\bigcup_{M \geq n \geq N_k} (-t_n + K \Delta K) \right) \leq 1/2^k.$$

Now let $M \to \infty$, to get $m \left(\bigcup_{n \geq N_k} (-t_n + K \Delta K) \right) \leq 1/2^k$.
This is the sequence of inequalities that we wanted at the outset.   □

Remark 2.7. This type of example shows how wide the exceptional class is for a given sequence (t_n) converging to 0. While generically $T_{t_n} f$ fails to converge a.e., there are many functions in the first category exceptional class where a.e. convergence does hold– including ones that are far from being continuous. Examples like $\mathbb{1}_K$ in Proposition 2.6 are not only discontinuous at every point in K: they aren't even equal a.e. to a function that is continuous on a set of positive measure.

Remark 2.8. In the proof above, we constructed K and an increasing sequence N_k such that

$$m\left(\bigcup_{n \geq N_k}(-t_n + K\Delta K)\right) \leq 1/2^k \text{ for all } k \geq 1.$$

This condition is not unexpected. Suppose that $K \subset [0, 1]$ and $\mathbb{1}_K(t_n + x) \to \mathbb{1}_K(x)$ for a.e. x. Then for a.e. x, there is N such that for all $n \geq N$, $x \notin -t_n + K\Delta K$. So, given $\epsilon > 0$, for large enough N, one must have

$$m\left(\bigcup_{n \geq N}(-t_n + K\Delta K)\right) < \epsilon.$$

Hence, for any k, there exists N_k such that

$$m\left(\bigcup_{n \geq N_k}(-t_n + K\Delta K)\right) < 1/2^k.$$

It should be noted here that the union in the construction above, $\cup_{n \geq N_k}(-t_n + K\Delta K)$, allows for significant overlap in the individual sets. This, coupled with the choice of N_k, is critical, as illustrated by Proposition 2.9, provided by T. Adams [1].

Proposition 2.9 (**Result 6**). *Let (t_n) be a sequence of non-negative real numbers going to zero. Then (t_n) is summable if and only if there exists a bounded Lebesgue measurable $A \subset \mathbb{R}$ of positive measure, such that*

$$\sum_{n=1}^{\infty} m(-t_n + A\Delta A) < \infty.$$

Before we begin the proof, we will need a lower bound on $m(-t_n + A\Delta A)$ that increases with t_n. Here are two alternatives to produce such a bound. We are indebted to R. Kaufman for bringing these facts to our attention.

Let $f \in L^1[0, 1]$ be non-zero and consider the rate at which $\|T_t f - f\|_1$ goes to zero as t goes to zero. Here we take $[0, 1]$ with the usual Lebesgue measure m, and we use addition modulo one, so essentially we are considering the behavior of functions and rotations on the circle.

Proposition 2.10. *If f is not constant, then*

$$\liminf_{|t| \to 0} \frac{\|T_t f - f\|_1}{|t|} > 0.$$

Proof. We use a Fourier coefficient argument. Suppose f is not constant. We have, for $k \in \mathbb{Z}$,

$$\widehat{f}(k) = \int_0^1 f(x) \exp(-2\pi i k x)\, dm(x).$$

We also know that $|\widehat{f}(k)| \le \|f\|_1$ for all k. Now $\widehat{T_t f - f}(k) = (\exp(2\pi i k t) - 1)\widehat{f}(k)$. Since f not constant, taking $k \ne 0$ such that $c = \widehat{f}(k)$ is not zero, then we have, for a constant $C > 0$,

$$\frac{\|T_t f - f\|_1}{|t|} \ge |c| \frac{|(\exp(2\pi i k t) - 1)|}{|t|} \ge C > 0$$

as $t \to 0$. $\qquad\square$

Another approach is to use a subadditive function limit theorem.

Definition 2.11. We say that $N(s)$ defined for $s > 0$ is subadditive if

$$N(s_1 + s_2) \le N(s_1) + N(s_2) \text{ for all } s_1, s_2 > 0.$$

For the original versions of the following proposition, see Fekete [13] and Hammersley [14].

Proposition 2.12. *If $N(s)$ is continuous and subadditive then*

$$\lim_{t \to 0^+} \frac{N(t)}{t} = \sup_{s > 0} \frac{N(s)}{s}.$$

Proof. Take any $s > 0$ and k_t whole numbers such that $k_t t \to s$ as $t \to 0$. By subadditivity, we have $N(k_t t) \le k_t N(t)$. Hence,

$$\frac{N(k_t t)}{k_t t} \le \frac{N(t)}{t}.$$

But, by continuity, we see that

$$\lim_{t \to 0^+} \frac{N(k_t t)}{k_t t} = \frac{N(s)}{s}.$$

Apply this with a sequence t_n in place of t with

$$\lim_{n \to \infty} \frac{N(t_n)}{t_n} = \liminf_{t \to 0^+} \frac{N(t)}{t}.$$

We have k_{t_n} as above and $k_{t_n} t_n \to s$ as $n \to \infty$. Then $N(k_{t_n} t_n)/k_{t_n} t_n \to N(s)/s$ as $n \to \infty$. So we get

$$\liminf_{t \to 0^+} \frac{N(t)}{t} = \lim_{n \to \infty} \frac{N(t_n)}{t_n} \ge \lim_{n \to \infty} \frac{N(k_{t_n} t_n)}{k_{t_n} t_n} = \frac{N(s)}{s}.$$

Since s was arbitrary, we have

$$\liminf_{t \to 0^+} \frac{N(t)}{t} \ge \sup_{s > 0} \frac{N(s)}{s}.$$

But then

$$\liminf_{t \to 0^+} \frac{N(t)}{t} \ge \sup_{s > 0} \frac{N(s)}{s} \ge \limsup_{t \to 0^+} \frac{N(t)}{t} \ge \liminf_{t \to 0^+} \frac{N(t)}{t}.$$

This proves the proposition, including the case that the value of $\sup_{s > 0} N(s)/s$ is infinite. $\qquad\square$

Now apply the proposition to $N(s) = \|T_s f - f\|_p$ where $1 \le p \le \infty$. We have

$$\|T_{s_1+s_2} f - f\|_p \le \|T_{s_1+s_2} f - T_{s_2} f\|_p + \|T_{s_2} f - f\|_p = \|T_{s_1} f - f\|_p + \|T_{s_2} f - f\|_p,$$

yielding the following corollary.

Corollary 2.13. *For all $f \in L_1[0,1]$ and $1 \le p \le \infty$, we have*

$$\lim_{t \to 0^+} \|T_t f - f\|_p / t = \sup_{s>0} \|T_s f - f\|_p / s.$$

If f is not constant then $\|T_s f - f\|_p \ne 0$ for some $s > 0$. So Corollary 2.13 gives a better outcome than Proposition 2.10

Corollary 2.14. *Given a measurable set $A \subset [0,1]$ with the Lebesgue measure $m(A) > 0$, the limit*

$$\lim_{t \to 0} \frac{m(A + t\Delta A)}{|t|}$$

exists and is strictly positive.

Remark 2.15. It is not hard to see that the limit in Corollary 2.13 can be infinite. For example, take f to be the characteristic function on a sequence of closed, pairwise disjoint intervals converging to 0. More concretely, take $A = \bigcup_{k=1}^{\infty} [1/4^{2k+1}, 1/4^{2k}]$. It would be interesting to characterize when the limit is infinite.

Corollary 2.13 and Corollary 2.14 are *saturation limits*, that is, rates which are optimal. It is reasonable to ask if there are saturation limit theorems for convolutions with other approximate identities. In generally, the answer to this question isn't immediately apparent. But in the special classical case of the Lebesgue derivative, there is such a rate estimate in line with Proposition 2.10.

Proposition 2.16. *Let $\Phi_t f(x)$ denote the Lebesgue derivative:*

$$\Phi_t f(x) = \frac{1}{t} \int\limits_0^t f(x + s) \, dm(s) = \phi_t * f(x),$$

where $\phi_t = \frac{1}{t} 1_{[-t,0]}$.

Given a non-constant $f \in L^1[0,1]$, there is some constant $C > 0$ such that

$$\liminf_{|t| \to 0} \frac{\|\Phi_t f(x) - f\|_1}{|t|} \ge C.$$

Proof. We use the argument in Proposition 2.10. Take $k \ne 0$ such that $\widehat{f}(k)$ is not zero. For $t \ne 0$, we have

$$\|\Phi_t f - f\|_1 \ge |\widehat{\phi}_t(k) - 1| |\widehat{f}(k)|,$$

where $\widehat{\phi}_t(k) = \frac{1}{t} \int_0^t \exp(2\pi i s k) \, dm(s)$.

For $k \neq 0$, this gives

$$\widehat{\phi}_t(k) = \frac{1}{2\pi i k t}(\exp(2\pi i k t) - 1) = 1 + \sum_{l=1}^{\infty} \frac{(2\pi i k t)^l}{(l+1)!}.$$

So $|\widehat{\phi}_t(k) - 1| = \pi|tk| + o(|t|)$ as $|t| \to 0$. This gives the conclusion with any $C < \pi|k||\widehat{f}(k)|$. □

We now proceed with the proof of Proposition 2.9:

Proof of Proposition 2.9. If (t_n) is summable, then we may let A be any bounded non-trivial interval because

$$\sum_{n=1}^{\infty} m\left(-t_n + A\Delta A\right) \leq \sum_{n=1}^{\infty} 2t_n < \infty.$$

Conversely, suppose $\sum_{n=1}^{\infty} t_n = \infty$ and let A be any bounded set of positive measure in $\mathbb{R}$.

Applying Corollary 2.14, we have that there is a positive constant c so that for n sufficiently large, $m\left((-t_n + A)\Delta A\right) > ct_n$. Thus

$$\sum_{n=1}^{\infty} m\left((-t_n + A)\Delta A\right) > c\sum_{n \geq N} t_n,$$

which diverges. □

Proposition 2.9 raises an interesting question.

Question 2.17. *Suppose (t_n) is summable. Can we construct K, with positive positive and no interior, such that $\sum_{n=1}^{\infty} m(t_n + K\Delta K) < \infty$?*

For simplicity, suppose that $t_n > 0$ are decreasing. We could do this if we could construct K_n decreasing, as above, with

$$\sum_{n=1}^{\infty} m(K_n\Delta K) < \infty, \text{ and} \tag{2.4}$$

$$\sum_{n=1}^{\infty} m(t_n + K_n\Delta K_n) < \infty. \tag{2.5}$$

Then

$$\sum_{n=1}^{\infty} m(t_n + K\Delta K) \leq \sum_{n=1}^{\infty} m(t_n + K\Delta t_n + K_n) + \sum_{n=1}^{\infty} m(t_n + K_n\Delta K_n) + \sum_{n=1}^{\infty} m(K_n\Delta K)$$

$$= 2\sum_{n=1}^{\infty} m(K_n\Delta K) + \sum_{n=1}^{\infty} m(t_n + K_n\Delta K_n) < \infty.$$

The construction gives (2.4). The issue is whether or not we can keep the successive insertion of gaps small enough and placed correctly to have (2.5) at the same time. This might be possible with $\sum_{n=1}^{\infty} t_n < \infty$.

In addition, we have this question:

Question 2.18. *Does there exist a compact set $K \subset [0,1]$ with positive measure and no interior such that, for all (t_n) with $\sum_{n=1}^{\infty} |t_n| < \infty$, for a.e. x, $T_{t_n} \mathbb{1}_K(x) \to \mathbb{1}_K(x)$ as $n \to \infty$?*

This question should be considered in light of Result 7, and compared with the more general Question 3.16. In addition, an answer to Question 2.18 would give further insights into Results 5 and 8.

3. Translation construction

Denote by $f \in L_c^{\infty}(\mathbb{R})$ the Lebesgue measurable functions that are essentially bounded on every bounded interval. We first observe the easiest part– see Corollary 3.14 for how the following proposition can be extended after some intermediate analysis.

Proposition 3.1. *Suppose $f \in L_c^{\infty}(\mathbb{R})$ and on every bounded interval I, $f = g$ a.e. where g is bounded and Riemann integrable on I. Then for every sequence of translations (t_n) with $\lim_{n\to\infty} t_n = 0$, we have*

$$\lim_{n\to\infty} f(x + t_n) = f(x) \text{ for a.e. } x.$$

Proof. Assume that $f \in L^{\infty}[0,1]$ and $f = g$ a.e., where g is bounded and Riemann integrable on $[0,1]$. For simplicity, extend both these functions to be 0 off $[0,1]$. The continuity a.e. property of g, and the fact that $f = g$ a.e., shows that there is a null set N such that for any $x_n \to x$ as $n \to \infty$, with $x \notin N$ and $x_n \notin N$ for all n, we have $f(x_n) \to f(x)$ as $n \to \infty$. But we have for a.e. x, both $x \notin N$ and also $x \notin N - t_n$ for all n. Hence, with $x_n = x + t_n$, the above shows that $f(x + t_n) \to f(x)$ as $n \to \infty$. So $f(t_n + x) \to f(x)$ a.e. The same argument extends to any bounded interval I in place of $[0,1]$ and proves the proposition. □

Proposition 3.1 yields one direction in the following theorem, which is a major extension of Proposition 2.1:

Proposition 3.2 (**Result 7**)**.** *Suppose that $f \in L_c^{\infty}(\mathbb{R})$. Then we have $f(t_n + x)$ converging for a.e. x for every (t_n) converging to 0 if and only if on every bounded interval I there is a g, bounded and Riemann integrable on I, such that $f = g$ a.e. on I.*

It will take more effort to prove the other implication in Proposition 3.2. We need to show that for $f \in L_c^{\infty}(\mathbb{R})$, if all translations $f(x + t_n)$ converge a.e., then on every bounded interval f is equal a.e. to a bounded Riemann integrable function. We prove this by contrapositive: we assume that, on some bounded interval, f is not equal a.e. to a bounded Riemann integrable function, then construct a sequence of translates that fail to converge a.e. on some set of positive measure. The proof of this is given in Section 3.1, Section 3.2, and Section 3.3; it is, in some sense, a generalization of the construction for $f = \mathbb{1}_E$

in Section 2. After this, in Section 3.4, we discuss how Proposition 3.2 can likewise be generalized to address other functions f besides those in $L_c^\infty(\mathbb{R})$.

3.1. Some preparation. To prove the steps leading to Proposition 3.2, we need to consider the essential limits at a point for bounded functions. We let $EUf(x)$ denote the *essential limit supremum* of $f(y)$ as $y \to x$. That is, first, given a bounded measurable F, we let ess sup F denote the essential supremum. This can be defined de novo, or it can be just taken as $\|F + M\|_\infty - M$ where M is any constant so that $F + M \geq 0$ a.e. Then,

$$EUf(x) = \lim_{\delta \to 0^+} \text{ess sup } f \mathbb{1}_{[x-\delta, x+\delta]}.$$

We similarly define the *essential limit infimum $ELf(x)$*. Or, if we want, we can just define

$$ELf(x) = -EU(-f)(x) \text{ for all } x \in [0, 1].$$

It is worth observing the following principle:

Proposition 3.3. *Given f bounded and Lebesgue measurable on $[0, 1]$, and some $x \in [0, 1]$, there is a null set $N_x \subset [0, 1]$ such that*

$$EUf(x) = \limsup_{y \to x} f|_{[0,1] \setminus N_x}(y), \text{ and}$$
$$ELf(x) = \liminf_{y \to x} f|_{[0,1] \setminus N_x}(y).$$

Remark 3.4. The central theorem of this section consists of two parts. We first observe that if $EUf = ELf$ a.e., then there is a Riemann integrable function g such that $f = g$ a.e. This is addressed in Section 3.2. Then, in Section 3.3, we show that if this fails to happen, then there is a sequence of translations $f(x + t_n)$ with $t_n \to 0$ and a measurable set E with positive measure such that the translates fail to converge for all $x \in E$.

Here is a structural fact about EUf and ELf that follows from Proposition 3.3:

Proposition 3.5. *We have $EUf(x) = ELf(x)$ if and only if there is a null set, N_x, depending on x, such that*

$$\lim_{\substack{y \to x \\ y \notin N_x}} f(y) = EUf(x).$$

Remark 3.6. Of course, if f is Riemann integrable, then a.e. x is a point of continuity for f and so, for a.e. x, $\limsup_{y \to x} f(y) = \liminf_{y \to x} f(y)$. Thus, there is no need for y to avoid a null set for good limit behavior a.e. On the other hand, if $f = g$ a.e., where g is Riemann integrable, then there is one null set, N, independent of x, which can be excluded in the essential supremum and essential infimum calculations when showing $f(y) \to f(x)$ as $y \to x$ off a null set.

3.2. The positive theorem. Our goal in this section and in Section 3.3 is to show that, for a bounded and Lebesgue measurable f with compact support, $EUf(x) = ELf(x)$ if and only if there is a Riemann integrable g such that $f(x) = g(x)$ a.e. We discuss the positive direction here; together, these two sections will enable us to address the remaining direction in the proof of Proposition 3.1.

Proposition 3.7. *Suppose that f is Lebesgue measurable and essentially bounded on $[0, 1]$. If $EUf(x) = ELf(x)$ for a.e. x, then there is a Riemann integrable function g on $[0, 1]$ such that $f = g$ a.e.*

Proof. Extend f to be zero on $\mathbb{R}\backslash[0, 1]$ and let

$$g(x) = \limsup_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^\epsilon f(x + t)\, dt.$$

We take the limit supremum here because the limit may not exist for every x.

However, by Lebesgue's Differentiation Theorem, we in fact have that

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^\epsilon f(x + t)\, dt = f(x) \text{ for a.e. } x.$$

Hence, $g(x) = f(x)$, except on a null set.

Let W be such a null set; that is, $g(x) = f(x)$ for $x \notin W$.

Now, suppose N is a Lebesgue null set such that for any $x \notin N$, we have $EUf(x) = ELf(x)$. Fix such an x. Applying Proposition 3.5, we have that

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^\epsilon f(x + t)\, dt = EUf(x),$$

since the integral is unchanged by the values of $f(x + t)$ on a null set of values t.

So, for $x \notin N$, we have $g(x) = EUf(x)$. And for $x \notin W \cup N$, $f(x) = g(x) = EUf(x)$.

We now wish to show that g is continuous for every $x \notin N$ and thus Riemann integrable.

Indeed, fixing such $x \notin N$, by Proposition 3.5 we have that, for any $\epsilon_0 > 0$, there is a null set N_x and $\delta > 0$ such that if $|x - w| \le \delta$ and $w \notin N_x$, then $|F(x) - f(w)| \le \epsilon_0$.

Suppose $|x - y| \leq \delta/3$ and $\epsilon < \delta/3$ and let $U_y = \{t : y + t \notin N_x\}$. We have

$$
\begin{aligned}
\left| F(x) - \frac{1}{\epsilon} \int_0^\epsilon f(y + t)\, dt \right| &= \left| \frac{1}{\epsilon} \int_0^\epsilon F(x) - f(y + t) dt \right| \\
&= \left| \frac{1}{\epsilon} \int_{[0,\epsilon] \cap U_y} F(x) - f(y + t) dt \right| \\
&\leq \frac{1}{\epsilon} \int_{[0,\epsilon] \cap U_y} |F(x) - f(y + t)|\, dt \\
&\leq \epsilon_0.
\end{aligned}
$$

Letting $\epsilon \to 0^+$, this shows that for $x \notin N$, if $|x - y| \leq \delta/3$, then $|F(x) - g(y)| \leq \epsilon_0$. Hence, for $x \notin N$, if $|x - y| \leq \delta/3$, we have $|g(x) - g(y)| \leq \epsilon_0$. Since ϵ_0 is arbitrary, g is continuous at x. $\qquad\square$

Remark 3.8. The same type of argument can be used to show the following: there is a Riemann-integrable g such that $f = g$ a.e. if and only if there is a null set $N \subset [0,1]$ such that $f|_{N^c}$, the function f restricted to the set $N^c = [0,1]\backslash N$, is continuous on N^c. However, this characterization is not as useful for our purposes as the one in Proposition 3.7

3.3. The negative theorem. We now need to show that when f is not equal a.e. to a bounded Riemann integrable function– and so $EUf(x)$ fails to be equal to $ELf(x)$ a.e.– we can construct a bad sequence of translates. This is the content of the following proposition:

Proposition 3.9. *Suppose that f is Lebesgue measurable and essentially bounded on $[0,1]$. If $EUf(x) > ELf(x)$ on a set of x of positive measure in $[0,1]$ then there is a sequence (t_n) converging to zero and a set E of positive measure such that $f(x + t_n)$ fails to converge for all $x \in E$.*

Proof. For convenience, let $f = 0$ off $[0,1]$. We assume there is a measurable set E of positive measure and $\gamma > 0$ such that for $x \in E$, we have $EUf(x) \geq ELf(x) + \gamma$. On E, the values of $EUf(x)$ are in $[-\|f\|_\infty, \|f\|_\infty]$. So there is some positive measure subset E_0 of E such that on E_0, the values of $EUf(x)$ differ by no more than $\gamma/100$. We can reduce E_0 again so the same constraint holds for the values of $ELf(x)$. This is actually not necessary for the construction, but it makes the argument a little more straightforward. That is, we may assume that there is a set of positive measure E such that

(a) $EUf \geq ELf + \gamma$ on E,
(b) all values of EUf are within $\gamma/100$ of each other on E, and
(c) all values of ELf are within $\gamma/100$ of each other on E.

We will inductively construct non-zero, distinct (t_n) as a union of disjoint finite blocks B_k. The blocks will have all terms in B_{k+1} smaller than all terms in B_k. In addition, for odd k, one has for a subset of x in E of measure at least $(1 - 1/2^{4k})m(E)$, there is some $t \in B_k$ such that $f(x + t)$ is within $2\gamma/100$

of $EUf(x)$. But for even k, one has for a subset of x in E of measure at least $(1 - 1/2^{4k})m(E)$, there is some $t \in B_k$ such that $f(x + t)$ is within $2\gamma/100$ of $ELf(x)$. So, indexing $\bigcup_{k=1}^{\infty} B_k$ as a sequence (t_n), we have that $f(x + t_n)$ fails to converge for all $x \in E$ because of conditions (a), (b), and (c), above.

We assume that the blocks B_l with $l < k$ have been constructed, and now describe how to choose B_k.

Assume that k is odd and fix $\epsilon_k > 0$. For each $z \in E$, there is y, with $|y - z| \le \epsilon_k$, such that
$$f(y) \ge EUf(z) - \gamma/100.$$
Indeed, because we are using the essential supremum, there is a set Y_z of positive measure such that all $y \in Y_z$ have this property. If, for $x \in E$, we may choose t and a $y \in Y_z$ such that $x = y - t$, then by condition (b) above,
$$f(x + t) = f(y) \ge EUf(z) - \frac{\gamma}{100} \ge EUf(x) - \frac{2\gamma}{100}.$$

Now, we can choose a finite set $C(k)$ such that the union of all the $Y_z - t$ over $t \in C(k)$ covers a subset of E of measure at least $(1 - 1/2^{4k})m(E)$. But perhaps for $t \in C(k)$, we would not have $|t| < \epsilon_k$. However, if we constrain the translates $t \in C(k)$ so that $|t|$ are all small, then we can only accomplish the same thing for $J \cap E$, where J is a small interval centered on z. We solve this problem by moving to a different section of E, and carrying out the same process there to keep the translates small in absolute value.

That is, there is a natural number L_k so that, for $i = 1, \dots, L_k$, we have $z_i \in E$, sets Y_{z_i} of positive measure, and finite sets $C_i(k)$ of non-zero numbers, for which the following hold.

(1) For all i and $y \in Y_{x_i}$, $f(y) > EUf(z_i) - \gamma/100$.
(2) The union of $Y_{z_i} - t$ with $t \in C_i(k)$ and $i = 1, \dots, L_k$ covers a subset F_k^o of E of measure at least $(1 - 1/2^{4k})m(E)$.
(3) Finally, $|t| \le \epsilon$ for all $t \in C_i(k), i = 1, \dots, L_k$.

Now we take B_k to be the union of all $C_i(k), i = 1, \dots, L_k$. Then, if $x \in F_k^o$, there exists $i, y \in Y_{z_i}$, and $t_i \in C_i(k)$ such that $x = y - t_i$. So $f(x + t_i) = f(y) \ge EUf(z_i) - \gamma/100 \ge EUf(x) - 2\gamma/100$.

Thus, we have constructed B_k, a union of finite sets $C_i(k)$ with all values $|t| \le \epsilon$ for $t \in B_k$, such that for some set $F_k^o \subset E$ with $m(F_k) \ge (1 - 1/2^{4k})m(E)$, we have for $x \in F_k^o$ there is some $t \in B_k$ such that $f(x + t) \ge EUf(x) - 2\gamma/100$.

On the other hand, if k is even, we use the same type of construction, but instead apply condition (c). Proceeding in a similar fashion, we may show that there is some $F_k^e \subset E$ with $m(F_k^e) \ge (1 - 1/2^{4k})m(E)$, such that for $x \in F_k^o$ there is some $t \in B_k$ such that $f(x + t) \le ELf(x) + 2\gamma/100$.

Our choice of ϵ for each k in the construction above ensures not only that $t \to 0$, but also that it may do so as rapidly as we might like: all values of $t \in B_k$ can be made smaller, even much smaller, than any value of $t \in B_l, 1 \le l < k$. This only requires that the $t \in B_l, 1 \le l < k$ are not zero, which can clearly be part of the inductive construction.

Now, by the trivial direction of the Borel-Cantelli Lemma, we have that for a.e. $x \in E$, there is some constant K such that $x \in F_k^o$ for all odd $k \geq K$, and $x \in F_k^e$ for all even $k \geq K$.

But then for a.e. $x \in E$, we must have both $f(x+t) \geq EUf(x) - 2\gamma/100$ and $f(x+t) \leq ELf(x) + 2\gamma/100$ infinitely often for $t \in \bigcup_{k=1}^\infty B_k$. Since we have also condition (a) above, for a.e. $x \in E$ the translates $f(x+t)$ fail to converge as $t \to 0$ with $t \in \bigcup_{k=1}^\infty B_k$.

The construction, therefore, gives a sequence, (t_n), with $|t_n|$ decreasing to 0, such that $f(x+t_n)$ does not converge for all $x \in E$. □

Remark 3.10. These arguments provide a proof of a somewhat surprising theorem for bounded Lebesgue measurable functions on a bounded interval. We have shown that in order for there to be a.e. convergence for all translations going to zero, we have to restrict the functions very strongly: the function must be locally a.e. equal to a Riemann integrable function. In a measure-theoretic sense, this means that for all translations to behave well, there has to be continuity for the function at a.e. point.

3.4. Summary and extensions. We will now summarize the proof of Proposition 3.2:

Proof of Proposition 3.2. We use Proposition 3.1 for one direction of the proof. For the converse, we may assume that f is Lebesgue measurable and essentially bounded on $[0,1]$; the argument will clearly extend to any bounded interval I. Suppose that f is not equal a.e. on $[0,1]$ to a bounded Riemann integrable function g. Then Proposition 3.7 and Proposition 3.9 show that there is a sequence (t_n) converging to zero such that $f(t_n + x)$ fails to converge for all x in a set of positive measure. □

We would like to extend this theorem to the largest reasonable class: $L^0(\mathbb{R})$, the Lebesgue measurable functions on $\mathbb{R}$ that are a.e. real-valued. To do this, we use *truncations*.

Definition 3.11. Given a function $f : \mathbb{R} \to \mathbb{R}$, the truncation of f at height $M \in [0, \infty)$ is given by

$$f^M = f\mathbb{1}_{\{|f| \leq M\}} + M\mathbb{1}_{\{f > M\}} - M\mathbb{1}_{\{f < -M\}}.$$

Proposition 3.12. *Suppose $f \in L^0(\mathbb{R})$. Then for every (t_n) converging to 0, we have $f(t_n + x)$ converging for a.e. x if and only if all for all M, the truncation f^M of f has the same property.*

Proof. In a loose sense, the main idea is that convergence puts one in a set where $f(x+t_n) = f^M(t_n+x)$ for some sufficiently large M and some sufficiently large n.

To be precise, assume first that for every (t_n) converging to 0, we have $f(t_n + x)$ converges to $f(x)$ for a.e. x. Fix (t_n) and M, and let N be the null set for which there is convergence off N. If $x \notin N$ and $|f(x)| < M$, then eventually

$|f(t_n + x)| < M$. So, for large n, $f^M(t_n + x) = f(t_n + x)$ and $f^M(x) = f(x)$, and hence $f^M(t_n + x)$ converges to $f^M(x)$.

If $x \notin N$ and $|f(x)| \geq M$, then for any $\epsilon > 0$, we eventually have that either $f(t_n + x) \geq M - \epsilon$ or $f(t_n + x) \leq -M + \epsilon$. So, again, we have that that $f^M(t_n + x) \to f^M(x)$.

Conversely, suppose for each M and (t_n) converging to 0, there is a null set N such that $f^M(t_n + x) \to f^M(x)$ for all $x \notin N$. Consequentially, there is a null set N such that, for all $M \in \mathbb{N}$ and all $x \notin N$, $f^M(t_n + x) \to f^M(x)$. But since $f \in L^0(\mathbb{R})$, for a.e. x there exists an $M \in \mathbb{N}$ so that $|f(x)| < M$. Thus, for large n, $f^M(t_n + x) = f(t_n + x)$ and $f^M(x) = f(x)$. Hence, we also have $f(t_n + x) \to f(x)$ for all $x \notin N$. □

Proposition 3.2 and Proposition 3.12 give this extension of our basic theorem.

Proposition 3.13. *Given $f \in L^0(\mathbb{R})$, we have, for every (t_n) converging to 0, $f(t_n + x)$ converging for a.e. x if and only if for every M and bounded interval I, f^M is equal a.e. to a bounded Riemann integrable function on I.*

These facts, in turn, lead to the following extension of Proposition 3.2:

Corollary 3.14 (**Result 8**). *Consider a real-valued function f on $\mathbb{R}$. Then $f(t_n + x)$ converges a.e. for all (t_n) going to zero if and only if there is a Lebesgue null set $N \subset \mathbb{R}$ such that f restricted to $\mathbb{R} \backslash N$ is continuous in the relative topology on $\mathbb{R} \backslash N$.*

Proof. It is easy to see the translation property follows from the existence of N, just as we have argued above, for example in Proposition 3.1. For the converse we use Proposition 3.13.

Take $I_s = [s - 1, s + 1]$ over all integers s and consider f^M on I_s. Then f^M is equal to a Riemann integrable function, R_s, on I_s a.e.

Now let $F(M, s)$ of be the set of full measure in I_s where f^M equals R_s and where R_s is continuous. Define

$$F(s) = \cap_{M \in \mathbb{N}} F(M, s) \text{ and } N = \cup_{s \in \mathbb{N}} I_s \backslash F(s).$$

Note that N is a null set.

Now, restrict f to $W = \mathbb{R} \backslash N$ and take a sequence in W converging to a point in W; say, $x_n \to x$. This x is interior to some $I(s)$ and, because f is real-valued, $|f(x)| < M$ for some M. Also, $f^M(x_n) \to f(x)$ because $x_n, x \in F_s \subset F(M, s)$.

Since $f^M(x_n)$ is eventually close to $f(x)$, $|f^M(x_n)|$ is close to $|f(x)|$, which is less than M. We cannot then have $f^M(x_n) = \pm M$. Hence, $|f(x_n)|$ is also eventually less than M. It follows that, for all n large enough, $f^M(x_n) = f(x_n)$. And hence $f(x_n) \to f(x)$. □

Remark 3.15. The proof of Corollary 3.14 does not change if assume that f is real-valued except for a Lebesgue null set. Also, the property guarantees that f itself is the limit a.e. for each sequence $(f(t_n + x) : n \geq 1)$.

Proposition 3.2 suggests this question:

Question 3.16. *Can we describe conditions on a class $\mathcal{T}$ of sequences (t_n) converging to zero such that, if $f \in L^\infty[0,1]$ and $T_{t_n} f$ converges a.e. for all $(t_n) \in \mathcal{T}$, then $f = g$ a.e. for some g that is bounded and Riemann integrable on $[0,1]$? That is, what conditions on $\mathcal{T}$ guarantee that a.e. convergence of $T_{t_n} f$ for all (t_n) going to zero forces the function f to be regular (relatively continuous off a null set)?*

It would be very interesting to know if the class of summable sequences is not large enough *for this to happen.*

3.5. Differentiation and moving averages.
The theorems above allow us to determine the good functions for all moving derivatives in the line.

Let $D(\epsilon, t)f(x) = \frac{1}{\epsilon} \int_t^{t+\epsilon} f(x+y)\,dy$ where $\epsilon > 0$ and $t \in \mathbb{R}$. We consider $D(\epsilon_n, t_n)f$ where (ϵ_n, t_n) is always assumed to have $\epsilon_n \to 0$ and $t_n \to 0$ as $n \to \infty$. There are a number of papers that address when the choice of (ϵ_n, t_n) allows one to conclude that for all $f \in L^1(\mathbb{R})$, one has for a.e. x, $D(\epsilon_n, t_n)f(x) \to f(x)$ as $n \to \infty$; see, for example, Nagel and Stein [21]. There are also a number of papers about the analogous idea in the context of averages of maps in dynamical systems. See Bellow, Jones, and Rosenblatt [5].

Without some control of the sequence (ϵ_n, t_n), the moving derivative fails to converge a.e. for a dense G_δ set of functions in $L^1(\mathbb{R})$. So, in the spirit of both facts discussed earlier in this paper and the facts in Parrish and Rosenblatt [23], one can ask, "What are the *good functions*?" That is, which functions yield a.e. convergence no matter what sequence (ϵ_n, t_n) is chosen, as long as $\epsilon_n \to 0$ and $t_n \to 0$ as $n \to \infty$?

In the analogous context of dynamical systems, where we have fixed the ergodic mapping τ, there is an interesting answer to this question. Indeed, for moving averages, this a.e. convergence requirement is equivalent to a strong convergence condition on the usual ergodic averages $A_n^\tau f = \frac{1}{n} \sum_{k=1}^{n} f \circ \tau^k$.

Proposition 3.17. *let $f \in L^1(X)$ be a mean-zero function. For f, we have all moving averages with respect to an ergodic map τ converge a.e. if and only if for all $\gamma > 0$,*

$$\sum_{n=1}^{\infty} m(\{|A_n^\tau f| \geq \gamma\}) < \infty.$$

Remark 3.18. Consequently, it is not hard to see that there are no functions other than constants that are good functions for all moving averages, for all ergodic mappings. See Adams and Rosenblatt [3] and Adams and Rosenblatt [2].

It is likewise very restrictive to require that all moving derivatives converge a.e.:

Proposition 3.19. *A function $f \in L^\infty(\mathbb{R})$ is good for all moving derivatives $D(\epsilon_n, t_n)$ if and only if it is a bounded, locally Riemann integrable function.*

Proof. Knowing that $f = g$, with g locally Riemann integrable, easily gives the positive direction, because Riemann integrable functions are continuous a.e.

The proof of the other direction could just capitalize on the same density arguments that give Proposition 3.9, but there is a simpler approach. Given $f \in L^\infty(\mathbb{R})$ that is not equivalent to a locally Riemann integrable function, we know from Proposition 3.9 that there is a sequence (t_n) tending to 0 such that $f(x + t_n)$ fails to converge on a set x of positive measure.

For each n, choose $\epsilon_n > 0$ such that $\|D(\epsilon_n, t_n)f - T_{t_n}f\|_1 \le 1/2^n$. Since $\sum_{n=1}^\infty \|D(\epsilon_n, t_n)f - T_{t_n}f\|_1$ is finite, we have $\lim_{n\to\infty} D(\epsilon_n, t_n)f(x) - T_{t_n}f(x) = 0$ for a.e. x. Hence, $D(\epsilon_n, t_n)f$ must also fail to converge on some set of positive measure. $\qquad\square$

4. Rates in special cases

The construction of a bad sequence of translates in Section 3 clearly uses a finite, but potentially very large, number of translates in each B_k. Something of this type is generally necessary because we know that, given f, for every sequence (t_n) converging to 0, there is a subsequence (t_{n_m}) such that $f(x + t_{n_m}) \to f(x)$ for a.e. x as $m \to \infty$.

Question 4.1. *Given f not equal a.e. to a Riemann integrable function, how quickly must (t_n) to converge to 0 in order to have $f(x + t_n) \to f(x)$ for a.e. x? What analytical property of f can be used to describe this rate?*

There is certainly some issue here: in addition to the above remark about a.e. convergent subsequences, a Baire category argument shows that no matter what sequence (t_n) we take, and no matter how quickly it converges to zero, the generic bounded measurable function f will have $f(x + t_n)$ failing to converge a.e.

We should at least remark on this general principle, which is stated less simply than it could be to make the points in Remark 4.3 more clear.

Proposition 4.2. *Let $f \in L^1(\mathbb{R})$ and take a summable sequence δ_n. Choose step functions ϕ_n such that $\|f - \phi_n\|_1 \le \delta_n$. Then choose t_n such that $\|\phi_n - T_{t_n}(\phi_n)\|_1 \le \delta_n$. It follows that $f(t_n + x) \to f(x)$ a.e. as $n \to \infty$.*

Proof. We actually have strong convergence of $T_{t_n}f$ to f, i.e.

$$\int_{\mathbb{R}} \sum_{n=1}^\infty |f - T_{t_n}f| \, dm = \sum_{n=1}^\infty \|f - T_{t_n}f\|_1 < \infty.$$

This is because

$$\|f - T_{t_n}f\|_1 \le \|f - \phi_n\|_1 + \|\phi_n - T_{t_n}(\phi_n)\|_1 + \|T_{t_n}(\phi_n) - T_{t_n}f\|_1 \le 3\delta_n.$$

Thus, we have a.e. convergence of the translates. $\qquad\square$

Remark 4.3. In Proposition 4.2, we could just appeal to the L^1-norm continuity of translations to directly choose t_n, thereby ensuring that $\|f - T_{t_n}f\|_1 \le \delta_n$ for all n.

Instead, we have chosen the indirect phrasing to emphasize two interesting approximation issues. The first issue is the question of what properties of f

allow us to explicitly derive the step functions ϕ_n above. Once these are chosen, the second issue arises: given a particular step function ϕ_n, we must determine how to quantify the choice of t_n so that $\|\phi - T_{t_n}(\phi_n)\|_1$ is small.

Based on properties of f, we would hopefully know what to do to decrease $\|f - \phi_n\|_1$. But to achieve this, generally we would have to increase the number of intervals on which ϕ_n is constant while decreasing their lengths. This will force the choice of the t_n to be correspondingly smaller.

Making this specific will require information about the *variability* of f.

Let us consider in greater detail some more explicit examples. Note that there is specific information available in the literature about the failure of a.e. convergence of translates for the generic set (or function); consider, for example, the theorems in Ellis and Friedman [9] and later in Ellis [9]. The facts presented earlier by Newman [22] are also of interest. These theorems show the following lemma.

Lemma 4.4. *Given any $\epsilon > 0$ and a sequence (t_n) in $[0, 1]$ with distinct terms converging to 0, there is an open dense set $U \subset [0, 1]$ with $m(U) < \epsilon$ such that $[0, 1] \subset \bigcup_{n=1}^{\infty} U + t_n$.*

Remark 4.5. The regularity of Lebesgue measure m shows that it is enough to prove that there is a measurable set $E \subset [0, 1]$ with $m(E) < \epsilon$ such that $[0, 1] \setminus \bigcup_{n=1}^{\infty} E + t_n$ is a null set.

Corollary 4.6. *Given any $\epsilon > 0$ and a sequence of distinct (t_n) converging to 0, there is an open dense set $U \subset [0, 1]$ with $m(U) < \epsilon$ such that for all $N \geq 1$, $[0, 1] \subset \bigcup_{n=N}^{\infty} U + t_n$.*

Proposition 4.7. *Given any sequence (t_n) converging to zero, there is a compact set $K \subset [0, 1]$ such that a.e. $T_{t_n} \mathbb{1}_K$ does not converge.*

Proof. We use Corollary 4.6 and take $K = U^c$. Then for all $x \in [0, 1]$, we have $x \notin K - t_n$ infinitely often. But $\mathbb{1}_{K - t_{n_m}}$ must converge to $\mathbb{1}_K$ a.e. along some subsequence, (n_m). $\qquad \square$

Question 4.8. *Given a compact $K \subset [0, 1]$, perhaps with no interior, what can be said about the rate at which (t_n) must converge to zero if $\mathbb{1}_{K + t_n}$ is to converge a.e.?*

One approach to this question would be to consider whether this rate can be characterized by aspects of what we might call the geometry of K. For example, one might be able to approach the question via *geometric entropy*:

Definition 4.9. The geometric entropy of a compact set K is the function

$$g(\ell) = \min_{(s_n)} \left\{ m \; : \; K \subseteq \bigcup_{n=1}^{m} \left(0, \frac{1}{\ell}\right) + s_n \right\},$$

where the minimum is taken over all real sequences. That is, $g(\ell)$ is the smallest number of open intervals of length $\frac{1}{\ell}$ whose union contains K.

Alternately, it could be that one needs more information about specific numbers, sizes, and locations of the intervals in the complement $[0,1]\backslash K$.

We present two seemingly different approach to answering the question. Both use a strong convergence criterion given by series to get a positive conclusion; that is, we give conditions on (t_n) such that

$$\sum_{n=1}^{\infty} m(K\Delta K + t_n) < \infty.$$

But then, since $\|\mathbb{1}_K - \mathbb{1}_{K+t_n}\|_1 = m(K\Delta K + t_n)$, this series condition means that, for a.e. x, we must eventually have $\mathbb{1}_K(x) = \mathbb{1}_{K+t_n}(x)$. The a.e. convergence of $\mathbb{1}_{K+t_n}$ to $\mathbb{1}_K$ follows. Because this series approach is rather a strong constraint, no doubt the rates can be improved by using different methods.

Method A. We assume that K is not Jordan measurable. For this first approach, it is easier to consider $U = [0,1]\backslash K$. Here U is an open set written as a disjoint union of open intervals I_k:

$$U = \bigcup_{k=1}^{\infty} I_k$$

We assume for concreteness that the lengths $m(I_k)$ are non-increasing.

Position the intervals I_k in such a way that their union, U, is open and dense in $[0,1]$. For a fixed j, let $\mathcal{T}_j = \bigcup_{k=j+1}^{\infty} I_k$. Then we can take an increasing sequence j_n such that $\sum_{n=1}^{\infty} m(\mathcal{T}_{j_n}) < \infty$.

Choose a decreasing sequence (t_n) so that

$$t_n < m(I_{j_n}) \text{ and } \sum_{n=1}^{\infty} j_n t_n < \infty.$$

Let $\mathcal{S}_j = \bigcup_{k=1}^{j} I_k$. Then for all n, we have $U = \mathcal{S}_{j_n} \cup \mathcal{T}_{j_n}$. Because of the first requirement on our chosen sequence, $m(I_k) > t_n$ for all $k \leq j_n$. So we have

$$m(U\Delta U + t_n) = \|\mathbb{1}_{\mathcal{S}_{j_n}} + \mathbb{1}_{\mathcal{T}_{j_n}} - \mathbb{1}_{\mathcal{S}_{j_n}+t_n} - \mathbb{1}_{\mathcal{T}_{j_n}+t_n}\|_1$$

$$\leq 2m(\mathcal{T}_{j_n}) + \sum_{k=1}^{j_n} m(I_k \Delta I_k + t_n)$$

$$\leq 2m(\mathcal{T}_{j_n}) + \sum_{k=1}^{j_n} 2t_n.$$

Hence, by the second requirement on (t_n), we have

$$\sum_{n=1}^{\infty} m(U\Delta U + t_n) < \infty.$$

Example 4.10. Say $m(I_k) = C/k^2$ for some constant C. Then $j_n = n^2$ is sufficiently large for $\sum_{n=1}^{\infty} m(\mathcal{T}_{j_n}) < \infty$. Also, then $t_n \leq 1/n^4$ allows for both requirements on the sequence to hold.

Example 4.11. Suppose $m(I_k) = C/2^k$ for some constant C. Then $j_n = n$ is sufficiently large and $t_n \leq 1/2^n$ allows for the rest of the conditions to hold.

This construction also yields the following.

Proposition 4.12 (**Result 9**). *There exists a compact set $K \subset [0, 1]$ of positive measure and no interior such that for all (t_n) such that $|t_n| = O(\rho^m)$ for some $\rho < 1$, for a.e. x, $T_{t_n} \mathbb{1}_K(x) \to \mathbb{1}_K(x)$ as $n \to \infty$.*

Method B. Here is a somewhat different approach using the geometric entropy. The constraints are similar to those in Method A.

For a compact set K, consider the ϵ neighborhood $K(\epsilon) = \{x \,:\, \text{there exists } y \in K, |x - y| \leq \epsilon\}$. Since for every $x \notin K$ we have $x \notin K(\epsilon)$ for small enough ϵ, we have $m(K(\epsilon) \backslash K) \to 0$ as $\epsilon \to 0^+$.

For a fixed ϵ, let N_ϵ be the smallest number of intervals $(x_k - \epsilon, x_k + \epsilon)$, such that

$$K \subset \bigcup_{k=1}^{N_\epsilon} (x_k - \epsilon, x_k + \epsilon).$$

We must have $\bigcup_{k=1}^{N_\epsilon} (x_k - \epsilon, x_k + \epsilon) \subset K(2\epsilon)$. So, for any $\delta > 0$, there is ϵ small enough so that

$$\bigcup_{k=1}^{N_\epsilon} (x_k - \epsilon, x_k + \epsilon) \backslash K \leq \delta.$$

We now choose a infinite sequence of the variables δ, ϵ, and N_ϵ as follows. First, assume that we have $\delta_n > 0$ such that $\sum_{n=1}^{\infty} \delta_n < \infty$. Then we choose ϵ_n and N_{ϵ_n} so that

$$\bigcup_{k=1}^{N_{\epsilon_n}} (x_k - \epsilon_n, x_k + \epsilon_n) \backslash K \leq \delta_n.$$

Now we impose conditions on a decreasing sequence (t_n) similar to those in Method A, namely:

$$t_n < 2\epsilon_n \text{ and } \sum_{n=1}^{\infty} N_{\epsilon_n} t_n < \infty.$$

To have (t_n) converging to zero as slowly as possible here, it seems reasonable that we take δ_n and ϵ_n as large as possible, knowing that there is no maximum choice possible in this situation.

Now let $W_n = \bigcup_{k=1}^{N_{\epsilon_n}} (x_k - \epsilon_n, x_k + \epsilon_n)$. We know that

$$W_n \Delta W_n + t_n \subset \bigcup_{k=1}^{N_{\epsilon_n}} (x_k - \epsilon_n, x_k + \epsilon_n) \Delta (t_n + x_k - \epsilon_n, t_n + x_k + \epsilon_n),$$

so if $t_n \leq 2\epsilon_n$, then $m(W_n \Delta W_n + t_n) \leq 2N_{\epsilon_n} t_n$.

Hence, we have

$$\sum_{n=1}^{\infty} \|\mathbb{1}_K - \mathbb{1}_{K+t_n}\|_1 \leq \sum_{n=1}^{\infty} \|\mathbb{1}_K - \mathbb{1}_{W_n}\|_1 + \sum_{n=1}^{\infty} \|\mathbb{1}_{W_n} - \mathbb{1}_{W_n+t_n}\|_1$$

$$+ \sum_{n=1}^{\infty} \|\mathbb{1}_{W_n+t_n} - \mathbb{1}_{K+t_n}\|_1$$

$$\leq \sum_{n=1}^{\infty} 2\delta_n + \sum_{n=1}^{\infty} \|\mathbb{1}_{W_n} - \mathbb{1}_{W_n+t_n}\|_1$$

$$\leq \sum_{n=1}^{\infty} 2\delta_n + \sum_{n=1}^{\infty} 2N_{\epsilon_n} t_n.$$

Once again, $\sum_{n=1}^{\infty} \|\mathbb{1}_K - \mathbb{1}_{K+t_n}\|_1 < \infty$ and so a.e. $\mathbb{1}_{K+t_n} \to \mathbb{1}_K$ as $n \to \infty$.

5. Edges

Consider a bounded set E. It is well-known that the characteristic function $\mathbb{1}_E$ is Riemann-integrable if and only if its topological boundary, $\partial_{\mathbb{R}}(E)$, has Lebesgue measure zero– i.e. E is *Jordan measurable*. We would like to generalize the concept of boundary with translates in mind. The goal is to give a way to distinguish levels of good functions for a.e. convergence of translates that are converging to zero.

Consider a set S in $\mathbb{R}$. Formally it is arbitrary, but think of it as having zero in its set of accumulation points. That is, zero is in its closure and is not an isolated point in the closure. Denote the complement $\mathbb{R} \backslash E$ by E^c.

Definition 5.1. The *base points $E \subset \mathbb{R}$ relative to S*, denoted by $\mathcal{B}_S(E)$ is the set of points x such that for some $\delta > 0$, we have $x + (S \cap (-\delta, \delta)) \subset E$.

That is, the base points are the values x such that $x + s \in E$ when $s \in S$ is close enough to zero. If S is an interval containing zero, then the base points give the usual topological interior of a set E.

If the base points of the set E are analogous to its interior, then the set of base points of its complement, $\mathcal{B}_S(E^c)$, is analogous to the interior of E^c. We thus arrive at our notion of a boundary relative to a set S:

Definition 5.2. The *edge of E relative to S* is the set

$$\mathcal{E}_S(E) = (\mathcal{B}_S(E) \cup \mathcal{B}_S(E^c))^c$$

The edge points are, then, the points that are neither base points for E nor for E^c.

Example 5.3. For example, consider S to be the set of points in a sequence (t_n) of non-zero real numbers converging to 0. Then the base points of E relative to S consists of all x such that $x + t_n \in E$ for all large enough n, and the base points of E^c relative to S consists of all x such that $x + t_n \in E^c$ for all large enough n. So the edge points relative to S are all x such that $x + t_n$ is in both

E and E^c infinitely often. That is, if $x \notin \mathcal{E}_S(E)$, then $x + t_n$ must eventually lie entirely in either E or E^c.

Example 5.4. Another interesting type of set to consider is one homeomorphic to the usual Cantor set: i.e., when S is a compact, totally disconnected set. If $0 \in S$, then the notions of base points and edge points relative to S again address where $x + s$ lies for $s \in S$ near zero.

Suppose that $\mathcal{S}$ is a class of sets S. We are interested in the behavior of the edge points $\mathcal{E}_S(E)$ is as $S \in \mathcal{S}$ varies. In addition to the question of how the individual sets of edges grow as $S \in \mathcal{S}$ varies, it's useful to have an handle on the growth of the edges of all the sets in $\mathcal{S}$; that is, to know the size of

$$\bigcup_{S \in \mathcal{S}} \mathcal{E}_S(E).$$

Consider the following basic principal.

Proposition 5.5. *Suppose E is bounded set and $\mathcal{E}_S(E)$ has Lebesgue measure zero. Then for a.e. x, we have, for any sequence (t_n) in S tending to zero,*

$$\lim_{n \to \infty} \mathbb{1}_{-t_n + E}(x) = \mathbb{1}_E(x).$$

Proof. Noting that a.e. $x \notin \mathcal{E}_S(E)$, we have, for a.e. x, either $x + s$ is in E for all values of $s \in S$ sufficiently close to zero, or $x + s$ is in E^c for all values of $s \in S$ sufficiently close to zero.

Hence, for a sequence (t_n) in S converging to zero, we would have either $x + t_n \in E$ or $x + t_n \in E^c$ for all $n > N$ with N large enough; the particular N depending on x. Thus, for a.e. x, $\mathbb{1}_{-t_n+E}(x)$ converges as n converges to ∞.

But we know that $\mathbb{1}_{-t_n+E}$ converges in L_1-norm to $\mathbb{1}_E$. Hence, for some subsequence $(t_{n_k} : k \geq 1)$, we have $\mathbb{1}_{-t_{n_k}+E}(x) \to \mathbb{1}_E(x)$ for a.e. x. We must therefore have that, for a.e. x, $\mathbb{1}_{-t_n+E}(x) \to \mathbb{1}_E(x)$ as $n \to \infty$. $\qquad \square$

Corollary 5.6. *If $\mathcal{S}$ is the set all sequences (t_n) converging to 0, and for all $S \in \mathcal{S}$ we have $\mathcal{E}_S(E)$ has Lebesgue measure zero, then for every sequence (t_n) converging to zero, we have*

$$\lim_{n \to \infty} \mathbb{1}_{t_n+E}(x) = \mathbb{1}_E(x), \text{ for a.e. } x.$$

Remark 5.7. It is not clear how different it would be to require that $\bigcup_{S \in \mathcal{S}} \mathcal{E}_S(E)$ has Lebesgue measure zero. This is possibly a much stronger statement: it means that a.e. x will give the conclusion for every $S \in \mathcal{S}$. See Corollary 5.8 where this would be true.

This approach gives us an alternative proof of a fact that we already know.

Corollary 5.8. *If E is Jordan measurable, then for a.e. x, we have*

$$\lim_{n \to \infty} \mathbb{1}_{t_n+E}(x) = \mathbb{1}_E(x)$$

for every sequence (t_n) converging to zero.

Proof. Although we always have $\mathcal{E}_S(E) \subset \mathcal{E}_{\mathbb{R}}(E)$, a simpler way to observe this corollary is to note that $\mathbb{1}_E$ must be continuous a.e. (at any point not an edge point); convergence must hold at any point of continuity. □

In addition, we have this converse principle.

Proposition 5.9. *Let $S = \{t_n : n \geq 1\}$, where (t_n) converges to zero. If*

$$\lim_{n \to \infty} \mathbb{1}_{-t_n + E}(x) = \mathbb{1}_E(x), a.e.,$$

then $\mathcal{E}_S(E)$ must have Lebesgue measure zero.

Question 5.10. *The core question concerns what types of sets have the property in Corollary 5.6. It is clear that the class must include the Jordan measurable sets, but it might be larger– even much larger.*
More generally, suppose that, for every sequence (t_n) tending to zero drawn from S, we have $m\left(\mathcal{E}_{\{-t_n:n\geq 1\}}(E)\right) = 0$. If it follows that $\mathcal{E}_S(E)$ must also have measure zero, what does this reveal about S?

We close this section with a few observations about the nature of edge points.

Lemma 5.11. *Suppose $S_0 \subset S$. Then $\mathcal{B}_S(E) \subset \mathcal{B}_{S_0}(E)$. Consequentially, $\mathcal{E}_{S_0}(E) \subset \mathcal{E}_S(E) \subset \mathcal{E}_{\mathbb{R}}(E)$.*

Proof. It is clear from the definitions that $\mathcal{B}_S(E) \subset \mathcal{B}_{S_0}(E)$ and also $\mathcal{B}_S(E^c) \subset \mathcal{B}_{S_0}E^c$. But then $\mathcal{E}_{S_0}(E) = \mathcal{B}_{S_0}(E)^c \cap \mathcal{B}_{S_0}(E^c)^c \subset \mathcal{B}_S(E)^c \cap \mathcal{B}_S(E^c)^c = \mathcal{E}_S(E)$. □

Lemma 5.12. *Suppose $x \in \mathcal{E}_{\mathbb{R}}(E)$, then there is a sequence (t_n) tending to zero such that $x \in \mathcal{E}_{\{t_n:n\geq 1\}}(E)$.*

Proof. There are points in E and in E^c arbitrarily close to x. So we can choose a_n with absolute values decreasing to zero so that $x + a_n \in E$ for all n, and b_n with absolute values decreasing to zero that that $x + b_n \in E^c$ for all n. Intertwine these into a sequence (t_n) tending to zero; we have that $x \in \mathcal{E}_{\{t_n:n\geq 1\}}(E)$. Note that we may order elements from (a_n) and (b_n) in such a way that $|t_n| \to 0$ monotonically if desired. □

Remark 5.13. There are other interesting issues that can be phrased in terms of base points and edge points. When we have a particular class of sets $\mathcal{S}$, then having a non-trivial set of base points for all of the sets $S \in \mathcal{S}$ gives some structural information. For example, consider a fixed sequence (u_n) converging to 0. Let $\mathcal{S}$ consist of the sets of terms in the sequences $(s\,u_n)$ where $s > 0$ is arbitrary. Then knowing that $x \in \mathcal{E}_S(E)$ for some $S \in \mathcal{S}$ says that a similar copy of tail $(u_n : n \geq N)$ is in E. This is the famous unsolved Erdős similarity problem: there should be no (u_n) for which this is true for all sets E of positive Lebesgue measure. But currently not even the case where $u_n = \dfrac{1}{2^n}$ for all n is known. See the discussion in Section 6 and the references [7, 12, 15, 17, 18, 26].

6. Erdős similarity problem

Erdős asked: does there exist a sequence $(x_k : k \geq 1)$ of non-zero real numbers decreasing to zero such that every set $E \subset \mathbb{R}$ with positive Lebesgue measure contains a similar copy of $(x_k : k \geq 1)$? We call such a sequence a *universal* sequence (for the Erdős' Similarity Problem). There are a number of articles that have addressed this problem since Erdős' question appeared in print. See Erdős [10], problem 433.7* (the asterisk meaning it was asked in a problem session at the 5th Balkan Mathematical Congress). But at this time, the Erdős Similarity Problem remains unsolved.

The general belief is that the answer to this question is negative. The theorem in Falconer [12] and Eigen [7] shows that a universal sequence cannot have $\lim_{k \to \infty} x_k/x_{k+1} = 1$. This is why one should consider possible universal sequence like $x_k = 1/2^k$ for all k. We do not know at this time if this is a universal sequence or not.

We give here some ideas on how one might approach a construction that would show there are no universal sequences. We note that Erdős' question could also be stated where we replace sets E of positive Lebesgue measure with sets E that are second Baire category. But the method of construction we are going to consider is definitely measure-theoretic.

One aspect of the similarity question is clear: it is necessary to use the scaling factor s. This is well-known: there exists a Lebesgue measurable set of positive measure E such that for all x, the elements $x_k + x \notin E$ for infinitely many k. For example, see Komjáth [19]. Indeed, $E \subset [0, 1]$ can be chosen to be compact and have measure as close to 1 as we like. So, if one restricts the similarity to allow for s only in some countable set, then there is not universal sequence. The theorem in Kolountzakis [18] is stronger: there is always a compact set E of positive Lebesgue measure such that for a.e. $s > 0$ and all x, we have $sx_k + x \notin E$ for infinitely many k.

It is perhaps also worth pointing out here that as always the order of the quantifiers matters. Indeed, fix $s > 0$. One can always construct a sequence of non-zero real numbers $(x_k : k \geq 1)$ decreasing to zero such that for a.e. $x \in E$, eventually $sx_k + x \in E$. This is because for any fixed sequence $(x_k : k \geq 1)$ tending to zero, and any Lebesgue measurable set of finite measure, $1_{sx_k+E} \to 1_E$ in L_1-norm.

One can ask why this problem has resisted solution so far. The optimistic view is that there has just not been a flexible enough construction found. The pessimistic view is that the problem is actually an axiomatic issue. Indeed, the two parameter nature of similarity, and the facts mentioned here, are why there is a faint odor of the problem perhaps being independent of ZF(C), like some of the technical issues involved in even the two variable Fubini Theorem.

In addition, one could reasonably change the Similarity Problem to be just that every set of positive measure contains a similar copy of the tail of the sequence. That is, for every E of positive Lebesgue measure there exists $K, s > 0$,

and x such that $sx_k + x \in E$ for all $k \geq K$. It is not clear if this weaker form of universality is actually equivalent to the one where $K = 1$. Also, one could also ask that $x \in E$. If E is closed, then it must be. But formally this is not necessary in general. However, since every set E of positive Lebesgue measure contains a compact set K of positive Lebesgue measure, one can assume that $x \in E$ anyway.

Moreover, a universal sequence would have the property that for a.e. $x \in E$, there must be some $s > 0$ such that $sx_k + x \in E$ for all $k \geq 1$ (or for all large enough k). To prove this, consider the set $\mathcal{B}$ of $x \in E$ for which there is no such $s > 0$ with this property. If $\mathcal{B}$ is of positive Lebesgue measure, then universality implies that there is some $x \in \mathcal{B}$ and $s > 0$ such that $sx_k + x \in \mathcal{B}$ for all $k \geq 1$. But then these elements are also in E, contradicting the choice of $x \in \mathcal{B} \subset E$ because of the definition of $\mathcal{B}$. Hence, $\mathcal{B}$ is of Lebesgue measure zero, which established the property.

The possible issue with this argument is that it is not clear if $\mathcal{B}$ is Lebesgue measurable because of the use of the continuous parameters s and x in the definition. But we can fix this. First, there is no harm in assuming that E is a Borel set. Consider the subset $\mathcal{E}_k$ of $(0, \infty) \times \mathbb{R}$ of all (s, x) such that for all k, we have $sx_k + x \in E$. Since $(s, x) \to sx_k + x$ is continuous, and E is a Borel set, we have $\mathcal{E}_k$ is a Borel set. Hence, so is $\mathcal{G} = \bigcap_{k=1}^{\infty} \mathcal{E}_k$. But then $((0, \infty) \times \mathbb{R}) \setminus \mathcal{G}$ is a Borel set and its projection $\mathcal{B}$ on the second coordinate is an analytic set. Analytic sets are universally measurable and so this projection is a Lebesgue measurable set.

6.1. The Erdős similarity problem and depth.

Our starting point is an argument that shows for any set of positive Lebesgue measure E and a finite set S, there is a similar copy of S in E. This is of course very well known, going back at least to Steinhaus [25]. But the construction here brings up the role of certain parameters that leads us to the concept of *depth*.

Proposition 6.1. *Given a set E of positive Lebesgue measure and $x_1, \dots, x_K$, there is $s > 0$ and x such that $sx_k + x \in E$ for $k = 1, \dots, K$.*

Proof. Take a Lebesgue point $z \in E$. Then for any $\epsilon > 0$, there is a $\delta > 0$ such that $m([z, z + \delta] \cap E)/\delta > 1 - \epsilon$. Hence, $m([0, \delta] \cap (E - z))/\delta > 1 - \epsilon$. But then for any $\sigma > 0$, we have $m([0, \sigma\delta] \cap \sigma(E - z))/\sigma\delta = m([0, \delta] \cap (E - z))/\delta > 1 - \epsilon$. Although we had to choose δ possibly very small, we see that here we can take σ as large as we like.

Now we have $m(\bigcap_{k=1}^{K} [0, \sigma\delta] - x_k)/(\sigma\delta)$ as close to 1 as we like by making σ large. So we claim that by making ϵ smaller, choosing δ as needed, and then taking σ sufficiently large, we would have $m(\bigcap_{k=1}^{K} \sigma(E - z) - x_k) > 0$.

Here are the precise estimate that verify this. We have

$$m\left(\bigcap_{k=1}^{K}\sigma(E-z)-x_k\right) \geq m\left(\bigcap_{k=1}^{K}[0,\sigma\delta]-x_k\right) - Km([0,\sigma\delta]\setminus\sigma(E-z)).$$

This estimate is because for any sets A_k and $B_k, k = 1, \dots, K$, we have

$$\bigcap_{k=1}^{K} B_k \subset \bigcap_{k=1}^{K} A_k \cup \bigcup_{k=1}^{K} B_k\setminus A_k$$

and so

$$m(\bigcap_{k=1}^{K} B_k) \leq m(\bigcap_{k=1}^{K} A_k) + \sum_{k=1}^{K} m(B_k\setminus A_k).$$

Apply this with $A_k = \sigma(E-z) - x_k$ and $B_k = [0,\sigma\delta] - x_k$ for all k.

Now let $M = \max_{1\leq k\leq K} |x_k|$. Using the same principle above, but with $A_k = [0,\sigma\delta] - x_k$ and $B_k = [0,\sigma\delta]$ for all k, we see that

$$m\left(\bigcap_{k=1}^{K}[0,\sigma\delta] - x_k\right) \geq m([0,\sigma\delta]) - KM = \sigma\delta - KM.$$

But also because

$$m([0,\sigma\delta]\cap\sigma(E-z))/\sigma\delta > 1 - \epsilon,$$

we have

$$m([0,\sigma\delta]\setminus\sigma(E-z)) \leq \epsilon\sigma\delta.$$

Hence,

$$m\left(\bigcap_{k=1}^{K}\sigma(E-z) - x_k\right) \geq \sigma\delta - KM - K\epsilon\sigma\delta$$

So we see that if $\epsilon < 1/10K$, and then σ is chosen large enough for $KM/\sigma\delta < 1/10$, then

$$m\left(\bigcap_{k=1}^{K}\sigma(E-z) - x_k\right)/\sigma\delta \geq 1 - 2/10 > 0.$$

Since $m(\bigcap_{k=1}^{K} \sigma(E-z)-x_k) > 0$. we certainly know that $\bigcap_{k=1}^{K} \sigma(E-z)-x_k$ is not empty. So there are $e_k \in E, k = 1, \dots, K$, and some y such that $\sigma(e_k-z)-x_k = y$ for all $k = 1, \dots, K$. That is, $\frac{1}{\sigma}x_k + \frac{1}{\sigma}y + z = e_k$ for all k. Or, with $s = 1/\sigma$ and $x = \frac{1}{\sigma}y + z$, we have $sx_k + x \in E$ for all $k = 1, \dots, K$. $\qquad\square$

In the proof of Proposition 6.1, we have to choose σ possibly large, and so $s = 1/\sigma$ is possibly very small. Depending on $x_1, \dots, x_K$ and z, and of course E, we get $m(\bigcap_{k=1}^{K} \sigma(E-z) - x_k) > 0$. What we really needed was only that this

intersection be non-empty. It may be that σ could be smaller than what was chosen. But if E is bounded and $x_1, \dots, x_K$ are distinct, then σ cannot be very small. This suggests this definition.

Definition 6.2. Fix some sequence $(x_k : k \geq 1)$ and some $K \geq 1$. Define the *depth* of E at z, denoted by $D_K(E, z)$, to be the infimum of values of σ such

$$\bigcap_{k=1}^{K} \sigma(E - z) - x_k \text{ is not empty.}$$

Remark 6.3. At least if E is compact, then the infimum in this definition is actually a minimum i.e. the depth is given by some value of σ. The depth of course depends on $(x_k : k \geq 1)$ and K. Clearly, $K \to D_K(E, z)$ is non-decreasing as K increases; that is $D_{K+1}(E, z) \geq D_K(E, z)$ for all $K \geq 1$.

Remark 6.4. If (x_k) is universal, then in fact we would have some s and x such that $sx_k + x \in E$ for all k. Hence, $0 \in \bigcap_{k=1}^{\infty} \frac{1}{s}(E - x) - x_k$. But it is not clear if actually also, even with E compact, that $m(\bigcap_{k=1}^{K} \frac{1}{s}(E - x) - x_k) > 0$ for all K. But perhaps the right question is: if (x_k) is universal, does there exist x and some $s > 0$ such that for all $K \geq 1$, we have $m\left(\bigcap_{k=1}^{K} \frac{1}{s}(E - x) - x_k \right) > 0$?

We propose to construct a compact set E of positive measure that shows that $(x_k : k \geq 1)$ cannot be universal. We propose to do this be constructing E such that for all $z \in E$, the depth $D_K(E, z)$ goes to infinity as K goes to infinity. We can see that this will accomplish what we want. Although the our proposed approach here is only tentative, it has its interest anyway.

So then here is our **Result 10**: the solution of the Erdős Similarity Problem can be phrased in terms of the concept of depth: a sequence $(x_k : k \geq 1)$, converging to 0, is universal if and only if for all compact sets E of positive measure, there is $x \in E$ and an L such that the depth $D_K(E, x) \leq L$ for all K.

Indeed, suppose $(x_k : k \geq 1)$ is universal. Then if E is compact and positive measure, there exists $x \in E$ with $s > 0$ such that $sx_k + x \in E$ for all $k \geq 1$. So we would have $0 \in \frac{1}{s}(E - x) - x_k$ for all $k = 1, \dots, K$. But then $\bigcap_{k=1}^{K} \frac{1}{s}(E - x) - x_k$ is not empty. This means that the depth $D_K(E, x) \leq \frac{1}{s}$ for all K.

Conversely, suppose there is some $x \in E$ such that for some L the depth $D_K(E, x) \leq L$ for all K. Then for all K, there exists $y_K \in \bigcap_{k=1}^{K} \sigma_K(E - x) - x_k$ for some $\sigma_K \leq L$. We then have $y_K/\sigma_K + x_k/\sigma_K + x \in E$ for all $k \leq K$. But there is a subsequence of y_K converging to y and a subsequence of $\sigma_K \to \sigma \leq L$. Since E is compact, it follows that with $z = y/\sigma + x$, we would have $z + x_k/\sigma \in E$ for all k. Hence, $(x_k : k \geq 1)$ is universal.

Remark 6.5. We can modify this idea to weaken the universal nature of $(x_k : k \geq 1)$ to be that every set of positive Lebesgue measure contains a similar copy of some tail of the sequence. Then we would modify the notion of depth accordingly.

Remark 6.6. We can also consider depth only for a.e. $z \in E$. This is because if $(x_k : k \geq 1)$ is universal, then for a.e. $x \in E$, there exists s such that $sx_k + x \in E$ for all k. Indeed, it is enough to prove this for compact sets E by inner regularity of the measure.

To see how this holds, consider $P = \bigcap_{k=1}^{\infty} \{(s, x) : x \in E, sx_k + x \in E\}$. The sets being intersected are the inverse images via the map $(s, x) \to sx_k + x$, which is continuous. So P is a Borel set because E is compact and so a Borel set. Hence, the set of $x \in E$ such that there is s with $sx_k + x \in E$ for all k is the projection of a Borel set on the second coordinate. But continuous projections of Borel sets are analytic, and hence are Lebesgue measurable. So this set of $x \in E$ that realizes the similarity property is a Lebesgue measurable set.

Now if this projection of P on the second coordinate is not of full measure in E, then there is a compact set $F \subset E$ of positive measure that misses it. This is where measurability of the projection is used. But then universality means there is $x \in F \subset E$ and s such that $sx_k + x \in F \subset E$ for all k. This is a contradiction because F misses the projection of P on the second coordinate.

So if $(x_k : k \geq 1)$ is universal, then for a compact set E of positive measure, the depth $D_K(E, x)$ is bounded as K varies for a.e. x.

What this means is that we can get a counterexample to the universality if we can construct a compact set E of positive measure that has for a.e. $x \in E$, the depth $D_K(E, x) \to \infty$ as $K \to \infty$.

Note: the same type of argument can be used for this weaker form of universality: for all compact sets E of positive measure, there is k_0, $x \in E$, and s such that $sx_k + x \in E$ for all $k \geq k_0$. If this holds, then one can argue that for a.e. $x \in E$, there is k_0 and s such that $sx_k + x \in E$ for all $k \geq k_0$. So "all" we have to do is construct a compact set E of positive measure such that the *tail depth* goes to infinity for a.e. $x \in E$.

Remark 6.7. Here is a proposed scheme for the construction of a compact set E of positive Lebesgue measure whose depth at (almost) every point in it tends to infinity, with respect to the sequence $(x_k : k \geq 1)$. We start with any set E_0, including for example the whole interval $[0, 1]$. We will inductively choose sets S_n with $\sum_{n=1}^{\infty} m(S_n) < \frac{1}{2} m(E_0)$. Let $E_n = E_{n-1} \backslash S_n$ for $n \geq 1$. We take E to be the limit of E_n i.e. $E = E_0 \backslash \left(\bigcup_{n=1}^{\infty} S_n \right)$. Without loss of generality in the construction, we may assume that the sets S_n^n are also open. So the set E will be a compact set.

The choice of S_n is designed to increase the overall depth $D_K(E_n, z)$ for $z \in E_n$. Indeed, suppose $z \in E_n = E_{n-1} \cap ([0,1] \backslash S_n)$. Let $S_n^c = [0,1] \backslash S_n$. Now

$$\bigcap_{k=1}^{K} \sigma(E_n - z) - x_k = \left(\bigcap_{k=1}^{K} \sigma(E_{n-1} - z) - x_k \right) \bigcap \left(\bigcap_{k=1}^{K} \sigma(S_n^c - z) - x_k \right)$$

$$\subset \bigcap_{k=1}^{K} \sigma(S_n^c - z) - x_k$$

Hence, for any z, $D_K(S_n^c, z) \leq D_K(E_n, z)$. This tells us that if we have the depth $D_K(S_n^c, z)$ large for all $z \in S_n^c$, then we will have increased the depth $D_K(E_{n-1}, z)$ overall by passing to $D_K(E_n, z)$.

The proof of Proposition 6.1 suggests that to increase the depth $D_K(S_n^c, z)$, we need to take S_n^c such that the Lebesgue derivatives $m(S_n^c \cap [z, z+\delta)/\delta$ converge to 1 slowly for a.e. $z \in S_n^c$. But in some sense this thinking does not matter. We just need to get the depth $D_K(S_n^c, z)$ large for some, presumably large, K.

Also, we know that if there is indeed a universal sequence $(x_k : k \geq 1)$, then for a.e. $z \in E$, there is some s such that $sx_k + z \in E$ for all k. But we are generally expecting that we will need to use very small values of s. Correspondingly, depth is used to show that $s = 1/\sigma$ and σ must be generally large. But if not, would we have a universal $(x_k : k \geq 1)$ such that for some s_0, for a.e. $z \in E$, we would have $sx_k + z \in E$ for all k, but with $s \geq s_0$. Of course, s cannot be too large, because our sets $E \subset [0,1]$.

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This paper is available via `http://nyjm.albany.edu/j/2022/28-36.html`.