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Minimal genus and simplified classes in rational manifolds

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Abstract. This note studies the minimal genus problem for classes which are equivalent, via the geometric diffeomorphism group, to a simplified class in $\mathbb{C}P^2\#k\overline{\mathbb{C}P^2}$. It is shown that the orbit structure for primitive classes is basically determined by the self-intersection number. Making use of this result, an upper bound for the minimal genus for each orbit is determined and it is shown that for k large enough, then genus stabilizes at either 0 or 1.

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1. Introduction

The minimal genus problem asks what the smallest possible genus of a smoothly embedded connected surface representing the class $A \in H_2(M, \mathbb{Z})$ in a given smooth manifold M is. In this note, this problem is studied for classes of $M = \mathbb{C}P^2\#k\overline{\mathbb{C}P^2}$ whose orbit under the action of the geometric automorphism group $D(M)$ contains a simplified class.

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The homology of a smooth oriented four manifold M comes endowed with a symmetric bilinear form $Q : H_2(M,\mathbb{Z}) \times H_2(M,\mathbb{Z}) \to \mathbb{Z}$. Orientation preserving diffeomorphisms of M act by automorphisms on $H_2(M,\mathbb{Z})$ and these preserve the form Q. To study the minimal genus problem, it is natural to ask, for a class $A \in H_2(M,\mathbb{Z})$, what do the orbits O_A under this geometric automorphism group $D(M)$ look like? What can be said about $H_2(M,\mathbb{Z})/D(M)$?

These questions are studied for rational manifolds $M = \mathbb{C}P^2\#k\overline{\mathbb{C}P^2}$ $(k \geq 2)$, which have second homology $H_2(M,\mathbb{Z}) \simeq \mathbb{Z}^{k+1}$ and $Q = diag(1,-1,...,-1)$. In this manifold, the questions above can be viewed from a variety of aspects. Aside from the geometric view stated above, the pair $(H_2(M,\mathbb{Z}),Q)$ form a lattice, hence results from this area as well as on quadratic forms can be applied. Moreover, there exists a Kac-Moody algebra with Q as its generalized Cartan matrix and with root lattice $H_2(M,\mathbb{Z})$. These structures are reviewed in Section 2.

In the following, for a class $A \in H_2(M,\mathbb{Z})$, the notation $A^2 = Q(A,A) = n$ will be used. For classes with $A^2 \geq -2$, results by Kac ([11], 5.11) describe the orbit structure of the $D(M)$-action completely for $3 \leq k \leq 10$ and some results are known if $n \geq -16$, see Section 2. For classes with $A^2 < -16$ no systematic results are known to the author.

The main result of this note is the following, for definitions of terms used see Section 2.

Theorem 1.1. *Let $M = \mathbb{C}P^2\#k\overline{\mathbb{C}P^2}$ $(k \geq 3)$ and $A \in H_2(M,\mathbb{Z})$ with $A^2 = -n < 0$. Assume that A is primitive and the $D(M)$-orbit O_A of A contains a simplified class. Then for each (k,n), $n > 0$, the following is true.*

(1) *If A is ordinary, then O_A is the unique primitive ordinary orbit.*
(2) *If $n \equiv_4 k-1$ and A is characteristic, then O_A is the unique primitive characteristic orbit.*

For $k = 2$, this is known to be false by an example of C.T.C. Wall, see [25]. In Section 7, Wall's example is put into a broader context and the existence of multiple orbits is related to binary quadratic forms of indefinite type.

It is possible to determine a representative for each primitive orbit, see Cor. 3.2. This makes it possible to determine the minimal genus in the ordinary case.

Lemma 1.2. *Let $M = \mathbb{C}P^2\#k\overline{\mathbb{C}P^2}$ $(k \geq 3)$ and $A \in H_2(M,\mathbb{Z})$ with $A^2 < 0$. Assume that A is primitive ordinary and the $D(M)$-orbit O_A of A contains a simplified class. Then it has minimal genus $g_A = 0$. Moreover, for each class there exists a complex orientation-compatible structure on M such that the curve can be chosen holomorphic and a symplectic orientation-compatible structure on M such that the curve is symplectic.*

If A is primitive characteristic this is a bit more complicated. In particular, the minimal genus can only be determined for sufficiently many blow-ups, a stabilization-type result.

Lemma 1.3. *Let $M = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ ($k \geq 3$) and $A \in H_2(M, \mathbb{Z})$ with $A^2 < 0$. Assume that A is primitive characteristic, $A^2 = -(8\gamma + k - 1)$ ($\gamma > 0$) and the $D(M)$-orbit O_A of A contains a simplified class.*

(1) *If $1 \leq k \leq 2\gamma - 1$, then the minimal genus of any embedded surface representing A is bounded above by $2\gamma - k$.*
(2) *If γ is odd and $k \geq 2\gamma - 1$, then the minimal genus of any embedded surface representing A is 1.*
(3) *If γ is even and $k \geq 2\gamma$, then the minimal genus of any embedded surface representing A is 0.*

The outline of the paper is as follows. Section 2 introduces notation and basic definitions. Section 3 reduces the question in $k \geq 4$ to a question in $k = 3$. The core of this note is the proof of Theorem 1.1 in the case $k = 3$. This begins in Section 4, where it is shown that all classes $A = (a, b, c, d)$ with $A^2 = -n < 0$ and $|a - d| = 1$ or 2 are equivalent under the $D(M)$-action (Theorem 4.3). Section 5 relates this equivalence to a class with difference $|a-d| = 1$ or 2 to an appropriate system of Diophantine equations (Theorem 5.1). Finally, Section 6 proves that this system has a solution over the integers. This then completes the proof of Theorem 1.1. Section 7 consider the case $k = 2$ and places the example of Wall into a broader context. The minimal genus result are described in Section 8.

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2. Notation

Let $M = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ ($k \geq 2$). In this note, the standard basis for $H_2(M, \mathbb{Z})$ is given by $\{H, E_1, .., E_k\}$ with $H^2 = Q(H, H) = 1$ and $E_i^2 = Q(E_i, E_i) = -1$. In this basis, a class will be denoted $A = aH - \sum_i b_i E_i = (a, b_1, ..., b_k)$.

$H_2(M, \mathbb{Z})$ can be viewed from three different viewpoints with corresponding automorphism (sub)groups:

(1) The homology of the manifold M. The geometric automorphism group is given by
$$D(M) = \{\sigma \in Aut(H_2(M, \mathbb{Z})) : \sigma = f_* \text{ for some } f \in Diff^+(M)\}. \quad (1)$$
(2) An integer lattice L with quadratic form Q together with the orthogonal group $O(L)$ of lattice automorphisms preserving Q.
(3) ([29]) The lattice $H_2(M, \mathbb{Z})$ is the root lattice of a Kac-Moody algebra, Q is the generalized Cartan matrix, and the Weyl group W is the subgroup of $O(L)$ generated by reflections on classes with $Q(x, x) \in \{-1, -2\}$. A reflection of B on the class A is given by
$$r_A(B) = B - \frac{2Q(A, B)}{A^2} A.$$
Note that for $k \leq 9$ this is a hyperbolic Kac-Moody algebra.

Clearly, $D(M) \subset O(L)$. More is true if $k \leq 9$:

Theorem 2.1 ([8],[17], [29]). *If $k \leq 9$, then $W = D(M) = O(L)$. If $k \geq 10$, then $D(M)$ is a proper subgroup of W.*

The generators of these groups will be of interest in the following sections. For this reason, consider the following maps with regard to the standard basis.

- T_H: Reflection on the class H. This has the effect of a sign change on the H-coordinate.
- T_i: Reflection on the class E_i. This has the effect of a sign change on the E_i-coordinate.
- T_{ij}: Reflection on the class $E_i - E_j$. This has the effect of interchanging the elements E_i and E_j.
- ($k = 2$) Reflection on the class $H - E_1 - E_2$:
$$S = \begin{pmatrix} 3 & -2 & -2 \\ 2 & -1 & -2 \\ 2 & -2 & -1 \end{pmatrix}$$
- ($k = 3$) Reflection on the class $H - E_1 - E_2 - E_3$:
$$R = \begin{pmatrix} 2 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & -1 \\ 1 & -1 & -1 & 0 \end{pmatrix}$$
Observe that RT_iR is the map S applied to the class A with the b_i term ignored.
- ($k \geq 4$) R acts only on the components (H, E_i, E_j, E_k), acting by identity on the remaining terms, i.e. there is a family of operators R_{ijk} generated by R.

Observe that $T_H T_1 T_2 T_3 = -id$, $T_i^t = T_i$, $T_{ij}^t = T_{ij}$, $S^t = T_1 T_2 S T_1 T_2$ and $R^t = T_1 T_2 T_3 R T_1 T_2 T_3$ ($\cdot^t$ denotes the transpose.).

Lemma 2.2 ([25], [19], [29]). *W is generated by T_i, T_{ij} and R. $D(M)$ is generated by these operators and T_H.*

Definition 2.3. *Two classes $A, B \in H_2(M, \mathbb{Z})$ are called $D(M)$-equivalent if there is a $\sigma \in D(M)$ such that $\sigma(A) = B$. Denote by O_A the orbit of the class A under the action of $D(M)$.*

Homology classes in rational manifolds under $D(M)$-action exhibit two special classes. The first, reduced, arise as the elements of the fundamental chamber C, see [11] and [25] . The second are simplified classes, which form the counterpart to the reduced classes, see [18].

Definition 2.4. *Let $A \in H_2(M, \mathbb{Z})$.*

(1) *A class $A = (a, b_1, ..., b_k)$ is called reduced if $a \geq 0$, $b_1 \geq ... \geq b_k \geq 0$ and*
   (a) *$a \geq b_1$ ($k = 1$),*
   (b) *$a \geq b_1 + b_2$ ($k = 2$) or*

(c) $a \geq b_1 + b_2 + b_3$ *($k \geq 3$)*

(2) *A class* $A = (a, b_1, ..., b_k)$ *is called simplified if* $a \geq 0, \quad b_1 \geq ... \geq b_k \geq 0$ *and*

(a) $2a \leq b_1 + b_2$ *($k = 2$) or*

(b) $3a \leq b_1 + b_2 + b_3$ *($k \geq 3$)*

It is not hard to see that if $k \leq 9$ and $A^2 < 0$, then A may be equivalent to a simplified class, but not a reduced one. For $k \geq 10$, $A^2 < 0$, A may be equivalent to either type. For each fixed value $A^2 = -n$, the set of simplified classes is always finite.

Lemma 2.5 ([15], [18], [29]). *Let* $M = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ *and* $A \in H_2(M, \mathbb{Z})$ *with* $A \neq 0$.

(1) *A is $D(M)$-equivalent to a reduced or simplified class.*

(2) *Each orbit O_A contains either a simplified class or a reduced class, never both.*

(3) *If A is $D(M)$-equivalent to a reduced class, then this is the unique reduced class in O_A.*

This lemma implies that to understand orbits containing reduced classes, it suffices to study reduced classes as generators of the orbits. In the simplified case, each orbit may contain many such classes and thus a list of simplified classes does not enumerate the orbits. It is shown in [29], that each orbit contains a minimal simplified class. However, it is not clear how to decide if a given simplified class is minimal in its orbit or how to list only the minimal simplified classes.

Denote for $n \in \mathbb{Z}$ the set

$$Z_n = \{A \in H_2(M, \mathbb{Z}) \mid A^2 = n\}.$$

The following summarizes known results for the structure of $Z_n/D(M)$.

(1) Reduced Classes : The collection of all reduced classes (when $k \geq 10$ this will also include classes with $A^2 < 0$) forms the fundamental chamber C of the Kac-Moody algebra generated by the root lattice $H_2(M, \mathbb{Z})$. The action of the Weyl group W on this fundamental chamber produces the Tits cone X and C is a fundamental domain for this action ([11], 3.12 and 5.10). The structure of $Z_n/D(M)$ is then given by Lemma 2.5.

(2) $n \in \{0, -1, -2\}$, $3 \leq k \leq 10$: Example 5.11 in [11] shows that in this case Theorem 1.1 holds. In fact, this result can be extended with the aid of simplified classes, see below. See also [18]. In particular, for $n = 0$ the following is true.

Lemma 2.6 ([11], Cor 5.11; [15]). *Let* $M = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$, $k \in \{2, 3\}$, *and assume that* $A \in H_2(M, \mathbb{Z})$ *satisfies* $A^2 = 0$. *Then A is either* 0 *or* $D(M)$-*equivalent to the class* $a(H - E_1)$, $a \in \mathbb{Z}$.

(3) $n = -16, k = 2$: Wall [25] showed that there are two primitive ordinary classes with distinct orbits. In Section 7, this example will be presented and a more general result shown to be true. See also Prop.1, [18].
(4) $0 > n \geq -16, 2 \leq k \leq 9$: The main results in [18] addresses the case for $k = 3$ and these results are then extended to $4 \leq k \leq 9$ proving a version of Theorem 1.1. The case of $k = 2$ is a combined result of [26], [14] and [18].

Definition 2.7. *(1) A class $A \in H_2(M, \mathbb{Z})$ is called characteristic if $Q(A, B) \equiv_2 Q(B, B)$ for all $B \in H_2(M, \mathbb{Z})$. It is called ordinary if it is not characteristic. (2) A class $A \in H_2(M, \mathbb{Z})$ is called divisible if $A = dB$ for some $d \in \mathbb{Z}$. A class is primitive if it is not divisible.*

The condition $Q(A, B) \equiv_2 Q(B, B)$ implies that in $M = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ each a, b_i is odd.

These considerations lead to the following decomposition: fix (k, n) and consider the $D(M)$-orbits in Z_n for $M = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$.

(1) Reduced Orbits: Each orbit contains a single reduced class. This class may be primitive, characteristic or divisible. For fixed n there may be more than one orbit, even when restricted to primitive non-characteristic classes.
(2) Simplified Orbits: Each orbit contains at least one simplified class.
    (a) Orbits of Divisible Classes: Only if n is divisible by a square will such orbits exist. Moreover, if $n = p^2\tau$, then the structure of these orbits in $Z_n/D(M)$ will be determined by the structure of $Z_\tau/D(M)$.
    (b) Primitive Characteristic Orbits: All classes in these orbits have purely odd entries. These can be considered rare as for fixed k they only occur for $n = 8\gamma + k - 1, \gamma \in \mathbb{N}$.
    (c) Primitive Ordinary Orbits: The remaining primitive and non - characteristic classes form the last set of orbits.

In this paper, the focus will be on simplified classes, especially in $k = 2$ and $k = 3$. The constructions in the following sections are generally valid also for reduced classes in $k = 3$, but they are of little significance for $k \geq 4$ in contrast to the simplified case. Moreover, it will be assumed throughout that the class A is primitive.

The action of $D(M)$ on a class is almost parity preserving, as the following lemma shows.

Lemma 2.8. *Let $n = 4\tau + i$, $\tau \in \mathbb{Z}$ and assume A is primitive with $A^2 = n$. Abbreviate even=e and odd=o and list the parities of a and b_i in order a, b_1, b_2, b_3 by a 4-tuple of e's and o's.*

(1) $i = 1$: *A has parities oeee or eooo.*
(2) $i = 2$: *A has parities eeoo or oooo.*
(3) $i = 3$: *A has parities eeeo or oooe.*
(4) $i = 4$: *A has parity ooee.*

Moreover, the action by $D(M)$ preserves the parities if $i = 2, 4$ and, if $i = 1, 3$, may swap them.

Proof. Clearly any elementary transformation does not change any of the parities. With regard to the map R, consider the case $i = 1$:

$$\tilde{a} = 2a - b_1 - b_2 - b_3 \text{ and } \tilde{b}_i = a - b_j - b_k.$$

If one b_* is odd, then $\tilde{a}$ is odd and two of the $\tilde{b}_*$ are odd, the other even. If two of b_* are even, then $\tilde{a}$ is even, two of the b_* are even and the remaining one is odd. So the map R interchanges the two possible parity-tuples for $i = 1$.

The other cases are similar. □

3. The case $k \geq 4$

The aim of the next sections is to prove the following result.

Lemma 3.1. *Let $k = 3$ and $A^2 = -n < 0$, $n \in \mathbb{N}$, be a primitive class. Then*

1. *If A is ordinary, then A is $D(M)$-equivalent to either $(\tau, \tau + 1, 0, 0)$ ($n = 2\tau + 1$ odd) or $(\tau - 1, \tau, 1, 0)$ ($n = 2\tau$ even).*
2. *If $n = 4\tau + 2$ and A is characteristic, then A is $D(M)$-equivalent to $(\tau - 1, \tau + 1, 1, 1)$*

Assuming this result holds, the proof of Theorem 1.1 can be completed for $k \geq 4$. Assume A is primitive ordinary with a simplified class in O_A. Assume further that $A = (a, b_1, ..., b_k)$ is simplified, then $3a \leq b_1 + b_2 + b_3$ and hence the class (a, b_1, b_2, b_3) satisfies $a^2 - b_1^2 - b_2^2 - b_3^2 < 0$. Thus by Lemma 3.1, this part can be reduced to $(\tilde{a}, \tilde{b}_1, \tilde{b}_2, 0)$ while keeping the remainder $(b_4, .., b_k)$ unchanged. Thus, A is equivalent to a class $(\tilde{a}, \tilde{b}_1, .., \tilde{b}_{k-1}, 0)$. The class $\tilde{A} = (\tilde{a}, \tilde{b}_1, .., \tilde{b}_{k-1})$ still satisfies $\tilde{A}^2 = -n$ and $O_{\tilde{A}}$, viewed as an orbit for $k-1$, still contains a simplified class in $k-1$ terms. If not then, by Lemma 2.5, it would contain a reduced class, and so too would O_A, in violation of Lemma 2.5. Hence, an inductive argument completes the proof of Theorem 1.1 for primitive ordinary classes under the assumption that Lemma 3.1 holds.

Similarly, assume that A is primitive characteristic. Then $n = 4\tau + k - 1$ and, if A is simplified, a brief calculation shows that $\tau \geq 0$ and even. Thus, $n \geq k-1$. As before, if O_A contains a characteristic class, then A is equivalent by Lemma 3.1 to a class $(\tilde{a}, \tilde{b}_1, .., \tilde{b}_{k-2}, 1, 1)$. The class $\tilde{A} = (\tilde{a}, \tilde{b}_1, .., \tilde{b}_{k-2})$ is characteristic, satisfies $\tilde{A}^2 = -n + 2 \leq -k + 3 < 0$ and $O_{\tilde{A}}$, viewed as an orbit for $k - 2$, still contains a simplified class in $k - 2$ terms. Hence, an inductive argument completes the proof of Theorem 1.1 for primitive characteristic classes under the assumption that Lemma 3.1 holds.

The inductive argument given above allows for an explicit description of a representative of each orbit, similar to Lemma 3.1.

Corollary 3.2. *Let $k \geq 3$, $A^2 = -n < 0$, $n \in \mathbb{N}$, be a primitive class and assume O_A contains a simplified class.*

(1) *If A is ordinary, then A is $D(M)$-equivalent to either $(\tau, \tau+1, 0, ..., 0)$ ($n = 2\tau+1$ odd) or $(\tau-1, \tau, 1, 0, .., 0)$ ($n = 2\tau$ even).*
(2) *If $n = 8\gamma + k - 1$ and A is characteristic, then A is $D(M)$-equivalent to $(2\gamma-1, 2\gamma+1, 1, ..., 1)$*

It is now possible to describe a fundamental domain for the $D(M)$-action on Z_n:

(1) If n is square free and $n \neq 8\gamma + k - 1$, then any class in Z_n is primitive ordinary. Thus, $Z_n/D(M)$ is in bijection with the set of reduced classes of square n and, if $n < 0$, additionally a single class from Cor. 3.2. These generate all of the orbits of the $D(M)$ action.
(2) If n is square free and $n = 8\gamma + k - 1$, then an additional simplified orbit appears, generated by the characteristic class in Cor. 3.2.
(3) If n is not square free, then in addition to the classes described previously, for each factor $n = p^2\tilde{n}$, all the $\tilde{n}$ orbits would also appear.

4. Reduction to a difference

This section will begin the study of negative orbits O_A for $k = 3$. Through an explicit family of operators in $D(M)$, the equivalence of two classes $A_1, A_2 \in Z_{-n}$ for $n > 0$ will be reduced to the study of the differences $|a - b_i|$. For ease of notation, this, and subsequent sections, will write $A = (a, b, c, d)$ and focus on the difference $|a-d|$. The elementary transformations suffice to reorganize any class so that the difference of interest is in the d-slot.

Lemma 4.1. *Let $\alpha \in \mathbb{Z}$, then*

$$M(\alpha) := T_{12}^{\alpha}(RT_1T_2)^{\alpha} = \begin{pmatrix} 1+\alpha^2 & \alpha & \alpha & -\alpha^2 \\ \alpha & 1 & 0 & -\alpha \\ \alpha & 0 & 1 & -\alpha \\ \alpha^2 & \alpha & \alpha & 1-\alpha^2 \end{pmatrix}$$

Proof. For $\alpha = 0$, this is the identity and for $\alpha = 1$, by matrix multiplication:

$$T_{12}RT_1T_2 = \begin{pmatrix} 2 & 1 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Now proceed by induction:

$$T_{12}^{\alpha+1}(RT_1T_2)^{\alpha+1} = T_{12}M(\alpha)RT_1T_2 =$$

$$= T_{12}\begin{pmatrix} \alpha^2+2\alpha+2 & \alpha+1 & \alpha+1 & -\alpha^2-2\alpha-1 \\ \alpha+1 & 0 & 1 & -\alpha-1 \\ \alpha+1 & 1 & 0 & -\alpha-1 \\ \alpha^2+2\alpha+1 & \alpha+1 & \alpha+1 & -\alpha^2-2\alpha \end{pmatrix} =$$

$$= \begin{pmatrix} 1+(\alpha+1)^2 & \alpha+1 & \alpha+1 & -(\alpha+1)^2 \\ \alpha+1 & 1 & 0 & -\alpha-1 \\ \alpha+1 & 0 & 1 & -\alpha-1 \\ (\alpha+1)^2 & \alpha+1 & \alpha+1 & 1-(\alpha+1)^2 \end{pmatrix}$$

If $\alpha < 0$, then observe that

$$(RT_1T_2)^{-1} = T_2T_1R \text{ and } T_{12}T_1T_2 = T_1T_2T_{12}$$

and hence,

$$T_{12}^{\alpha}(RT_1T_2)^{\alpha} = T_1T_2\left[T_{12}^{|\alpha|}(RT_1T_2)^{|\alpha|}\right]T_1T_2.$$

□

Remark: This family of maps can also be found in the proof of Lemma 3.3, [29].

Lemma 4.2. *Let $(\alpha,\beta)\in\mathbb{Z}^2$, then*

$$R(\alpha,\beta) := M(\alpha)T_1M(\beta) = \begin{pmatrix} 1+\alpha^2+\beta^2 & \beta-\alpha & \beta+\alpha & -\beta^2 \\ \alpha-\beta & -1 & 0 & \beta-\alpha \\ \alpha+\beta & 0 & 1 & -\alpha-\beta \\ \alpha^2+\beta^2 & \beta-\alpha & \beta+\alpha & 1-\alpha^2-\beta^2 \end{pmatrix}$$

Proof. This is obtained by matrix multiplication using the above lemma. □

For a given class (a,b,c,d), this map acts as

$$R(\alpha,\beta)(a,b,c,d) = (a(\alpha,\beta), b(\alpha,\beta), c(\alpha,\beta), d(\alpha,\beta))$$

where

$$\begin{aligned} a(\alpha,\beta) &= a + (\alpha^2+\beta^2)(a-d) + \beta(b+c) + \alpha(-b+c), \\ b(\alpha,\beta) &= -b + (\alpha-\beta)(a-d), \\ c(\alpha,\beta) &= c + (\alpha+\beta)(a-d), \\ d(\alpha,\beta) &= d + (\alpha^2+\beta^2)(a-d) + \beta(b+c) + \alpha(-b+c). \end{aligned}$$

The map $R(\alpha,\beta)$ preserves $a-d$.

The functions $a(\alpha,\beta)$ and $d(\alpha,\beta)$ describe paraboloids, hence admit a unique maximum or minimum. This extremal point is at

$$(\alpha_0,\beta_0) = \left(\frac{b-c}{2(a-d)}, -\frac{b+c}{2(a-d)}\right)$$

and takes values $b(\alpha_0,\beta_0) = c(\alpha_0,\beta_0) = 0$,

$$a(\alpha_0,\beta_0) = \frac{-n}{2(a-d)} + \frac{a-d}{2} \text{ and } d(\alpha_0,\beta_0) = a(\alpha_0,\beta_0) - (a-d)$$

These values only depend on n and $a-d$, hence for all classes which have the same $|a-d|$, the magnitude of these values are identical. In particular, note that if A has $a-d>0$, then, using elementary transformations, $(-a,b,c,-d)$ has $\tilde{a}-\tilde{d} = -a+d < 0$. Thus the sign of $a-d$ is not relevant, one can consider either $a-d$ or $-(a-d)$ as is convenient for the situation at hand.

Clearly there is no guarantee that $\alpha_0, \beta_0 \in \mathbb{Z}$, however there is an integer point nearby. The following theorem shows that this nearby point can be significant.

Theorem 4.3. *Let $k = 3$ and $A^2 = -n$.*

(1) *All classes with $|a - d| = 1$ are $D(M)$-equivalent.*
(2) *If $n \equiv_4 2$, all characteristic classes with $|a - d| = 2$ are $D(M)$-equivalent.*

Proof. Assume that by applying elementary transformations, the class (a, b, c, d) satisfies $a - d = -1$ or -2.

If $a - d = -1$, then consider two cases:

(1) If $b + c$ is even, then $\alpha_0, \beta_0 \in \mathbb{Z}$ and from above it shows that each class is equivalent to the class $\left(\frac{n-1}{2}, 0, 0, \frac{n+1}{2}\right)$.
(2) If $b + c$ is odd, then $\alpha_0, \beta_0 \in \frac{1}{2}\mathbb{Z}$, hence the closest integer points lie at $\pm\frac{1}{2}$ from (α_0, β_0). Due to the symmetry properties of the paraboloid, the value of $a(\alpha, \beta)$ and $d(\alpha, \beta)$ are the same at all four points and the values of $b(\alpha, \beta)$ and $c(\alpha, \beta)$ differ by signs and permutations. So the value only needs to be calculated at one point. Calculating the value shows each class in this case is equivalent to $\left(\frac{n-2}{2}, 1, 0, \frac{n}{2}\right)$.

Note that by Lemma 2.8, each case produces a vector in $\mathbb{Z}^4$.

Now consider the case $a - d = -2$. Then $\beta_0 = \frac{b+c}{4}$, $\alpha_0 = -\frac{b-c}{4}$. As A is characteristic, $n = 4\tau + 2$, $\tau \geq 0$ even, and each entry in A is odd. Moreover, one of $b + c$ and $b - c$ is divisible by 4, the other only by 2. Hence, the nearest integer point to (α_0, β_0) is either $(\alpha_0, \beta_0 + \frac{1}{2})$ or $(\alpha_0 + \frac{1}{2}, \beta_0)$. In either case, the values of $b(\alpha, \beta)$ and $c(\alpha, \beta)$ will be $1, 1$, the value for $a = \tau - 1$. Thus, each characteristic class with $|a - d| = 2$ is equivalent to $(\tau - 1, 1, 1, \tau + 1)$.

□

This result explains the origin of the classes which appear in Lemma 3.1.

Remark: Denote $k = |a - d|$. It can be shown, that if $k^2 \leq n$, any class is equivalent to a similar class as in the previous theorem. Such terminal classes (a, b, c, d) are characterized by the properties $a, b, c, d \geq 0$, $b \geq c$, $b + c \leq k$ and $d = a + k$. However, if $k \geq 4$, then these terminal classes may not be unique.

The goal in the following is to show that a given class is equivalent to one with $|a - d| = 1$ or, if A is characteristic, to $|a - d| = 2$. For this reason, define

$$\Delta = \begin{cases} 1 & A \text{ is ordinary,} \\ 2 & A \text{ is characteristic.} \end{cases}$$

5. Equivalence to Diophantine equation

Let $v^t = (1, 0, 0, -1)$, then

$$v^t A = a - d.$$

Thus, to test if a given class A has a difference that has value Δ, it is necessary to test in this fashion against all images of v under elementary transformations. In particular, to test whether O_A contains a class with $a - d = \Delta$, consider all numbers of the form

$$(Ev)^t LA$$

with E any product of elementary transformations and $L \in D(M)$. This can be written as

$$(L^t Ev)^t A$$

where $L^t E \in D(M)$ by the comments preceding Lemma 2.2. Thus, to determine if O_A contains a class with difference Δ, it suffices by Lemma 2.6 to find a class $\tilde{v}$ with square 0 such that

$$\tilde{v}^t A = \Delta.$$

Conversely, once such a class $\tilde{v}$ has been found, there exists an element $L \in D(M)$ such that $L\tilde{v} = v$, again by Lemma 2.6. Hence

$$\Delta = \tilde{v}^t A = (Lv)^t A = v^t L^t A$$

and thus the element $L^t A \in O_A$ has difference Δ.

If $\Delta = 1$, then $\tilde{v}$ is primitive. If $\Delta = 2$, then it is not possible for this equation to have a non-primitive solution, as any non-primitive solution would imply that there is a solution to the equation $v^t \cdot A = 1$, which is not possible in the characteristic case.

Theorem 5.1. *Let $M = \mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$ and $A = (a, b, c, d) \in H_2(M, \mathbb{Z})$. The following are equivalent:*

(1) *A is $D(M)$-equivalent to a class with difference Δ.*
(2) *The system of Diophantine equations*

$$z^2 = x^2 + y^2 + w^2, \quad az - bx - cy - dw = \Delta$$

has a solution $\tilde{v}^t = (z, x, y, w) \in \mathbb{Z}^4$.
(3) *The integral quaternary quadratic form $f(\epsilon, \sigma, \rho, \xi)$ given by*

$$F = \begin{pmatrix} a-b & 0 & -c & -d \\ 0 & a-b & -d & c \\ -c & -d & a+b & 0 \\ -d & c & 0 & a+b \end{pmatrix}$$

represents Δ or $-\Delta$ over the integers.

Proof. The equivalence of (1) and (2) has been argued above.

The first equation in the Diophantine system in (2) produces Pythagorean quadruples. Any solution to $z^2 = x^2 + y^2 + w^2$ with $\gcd(x, y, w) = 1$ and $z > 0$ can be written as

$$z = \epsilon^2 + \sigma^2 + \rho^2 + \xi^2, \; x = \epsilon^2 + \sigma^2 - \rho^2 - \xi^2,$$

$$y = 2(\epsilon\rho - \sigma\xi), \text{ and } w = 2(\epsilon\xi + \sigma\rho).$$

Note that the restriction $\gcd(x, y, w) = 1$ ensures that the class is not divisible, so is natural in this setting. Replacing (z, x, y, w) by these expressions in $az - bx - cy - dw$ leads to $f(\epsilon, \sigma, \rho, \xi)$.

Any solution to the Diophantine system in (2) for Δ with $z < 0$ also produces a solution to the corresponding equation for $-\Delta$ but with $z > 0$. Thus, a solution with $z > 0$ is always possible for $\pm\Delta$, if a solution exists at all.

Conversely, any integer representations of $\pm\Delta$ by the form f lead to classes (z, x, y, w) with $z > 0$ satisfying $az - bx - cy - dw = \pm\Delta$.

□

The goal is now to show that this form represents Δ in the integers. Note that $\det f = n^2$, hence is always a perfect square and thus does not fit into most general results on quadratic forms (see [10]).

Until now, it was not relevant if A^2 was positive or not. However, the following Lemma shows the distinction that now occurs with regards to the definiteness of f and the sign of A^2.

Lemma 5.2. *Let A be a primitive class with $A^2 \neq 0$. If O_A is a simplified orbit, then f is an indefinite form. If O_A is a reduced orbit, then f is definite.*

Proof. If O_A is a simplified orbit, then for a simplified class $a - b$ (corresponding to pairing with $(1, -1, 0, 0)$) must be negative while the pairing with $(1, 1, 0, 0)$ is positive. Hence, the form is indefinite.

If O_A is reduced, then $A^2 > 0$ when $k = 3$. The set of classes of positive square

$$\mathcal{P} = \{A \in H_2(M, \mathbb{Z}) \mid A^2 > 0\}$$

has two connected components. The Light Cone Lemma, essentially a corollary of the Cauchy-Schwartz inequality, states, that if A and B both lie in the closure of the same component of $\mathcal{P}$, then either $A \cdot B > 0$ or they are multiples of each other with $A^2 = 0$. If A and B lie in distinct components, then either $A \cdot B < 0$ or they are multiples of each other with $A^2 = 0$. Thus, if $A^2 > 0$, then the pairing with a class $v^t = (z, x, y, w)$ with $v^2 = 0$ and $z > 0$ will always have fixed sign. This then implies that f is definite.

□

6. Existence of solutions

The key result that will be applied in this section is the following:

Theorem 6.1 ([5], Ch. 9, Th. 1.5)**.** *Let f be a regular (i.e. $\det f \neq 0$) indefinite integral form in $k \geq 4$ variables and let $m \neq 0$ be an integer. Suppose that m is represented by f over all $\mathbb{Z}_p$, p prime. Then m is represented by f over $\mathbb{Z}$.*

Clearly, as f is indefinite and continuous over the reals, it has a real solution for $m = \Delta$ as there exist differences in O_A greater than Δ.

Assume that $p > 2$ is a prime dividing n. If $p \not| a - b$, then $a - b$ is a p-adic unit and through p-adic row and column operations F becomes

$$\begin{pmatrix} a-b & 0 & 0 & 0 \\ 0 & a-b & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

Hence, F diagonalizes over $\mathbb{Z}_p$ with a unimodular component of rank at least 2. Thus, if A is ordinary, it represents both 1 and -1 (see [5], Ch. 8, Lemma 3.4). If A is characteristic, then $2|a - b$, so after factoring out a common factor of 2 the matrix is still unimodular, hence f represents both 2 and -2.

If $p \not| a - b$ or $p \not| a + b$, then due to the symmetry of F, the argument above works. Assume that $p|a - b, a + b$. Then $p|a, b$. Assume that p also divides one of c or d. Then it in fact divides both and A is a divisible class.

Thus, $p \not| c$ or d. Assume $p \not| d$, then clearly the same holds for $a \pm d$. Diagonalizing F under the assumption that $p \not| d$, but $p|a \pm b$, as described in [7], Ch. 15, 4.4, means that $a - b$ is replaced by $a - b - 2d + a + b = 2(a - d)$, which is not p-divisible. Now replace $a - b$ by $a - d$ in the original argument above to see that f represents $\pm\Delta$.

Thus, f represents both Δ and $-\Delta$ in $\mathbb{Z}_p$ for $p \geq 3$. (p was assumed to divide n, clearly if $p \not| n$, then the f represents Δ and $-\Delta$ as it is unimodular over $\mathbb{Z}_p$.)

Consider now the case $p = 2$ and let A be ordinary. By Lemma 2.8, it is always possible to choose $a - b$ to be odd. Assume this has been done. Then F can again be put in the form

$$\begin{pmatrix} a-b & 0 & 0 & 0 \\ 0 & a-b & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

If n is odd, then the argument given in [10] to prove the odd case of Theorem 1 applies here and f represents 1 and -1 in $\mathbb{Z}_2$.

If n is even, then nothing can be said about the lower right rank 2 component of the form above. However, the split off rank 2 part corresponds to $g(x_1, x_2) = u(x_1^2 + x_2^2)$ where u is a 2-adic unit. We now show that either $u(x_1^2 + x_2^2) = 1$ or $u(x_1^2 + x_2^2) = -1$ always has a solution. The unit can be written as $u = 1 + \alpha \cdot 2 + \beta \cdot 4 + u_8$, with $\alpha, \beta \in \{0, 1\}$ and $8|u_8$. For u, there are four possible combinations of these initial terms:

(1) 1: In this case, setting $x_1 = 1$ and $x_2 = 0$, $g(1, 0) \equiv_8 1$.
(2) $1 + 2$: Set $x_1 = 1$ and $x_2 = 2$, then $g(1, 2) \equiv_8 -1$.
(3) $1 + 2 + 4$: In this case, setting $x_1 = 1$ and $x_2 = 0$, $g(1, 0) \equiv_8 -1$.
(4) $1 + 4$: Set $x_1 = 1$ and $x_2 = 2$, then $g(1, 2) \equiv_8 1$.

Applying Hensel's Lemma (see [23], Ch. II, Cor 3) to g shows that f represents 1 or -1 in $\mathbb{Z}_2$. Theorem 6.1 now leads to the following.

Lemma 6.2. *Let A be primitive ordinary with $A^2 = -n < 0$ and $k = 3$. Then the quaternary quadratic form f of Theorem 5.1 represents either 1 or -1.*

Consider now the case $p = 2$ and let A be characteristic. Lemma 2.8 shows that $n = 4k + 2$, in particular $2|n$, but $4 \not| n$. Then $a \pm b$ is even, c, d odd. Thus, the reduction process for F leads to the block matrix

$$\begin{pmatrix} a-b & -d & 0 & 0 \\ -d & a+b & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}.$$

In particular, over $\mathbb{Z}_2$, F does not have an orthogonal basis. Thus, recalling that $\det f = n^2$, F splits as one of the following ([20], 93:11):

$$\epsilon H_i \oplus 2\eta H_j \text{ or } \epsilon H_i \oplus 2\eta \oplus 2\xi.$$

Here $\epsilon, \eta, \xi \in \mathbb{Z}_2$ are units, $i, j \in \{1, 2\}$,

$$H_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } H_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

In each case, the form represents 2 and -2. As before, Theorem 6.1 leads to the following.

Lemma 6.3. *Let A be primitive characteristic with $A^2 = -n < 0$ and $k = 3$. Then the quaternary quadratic form f of Theorem 5.1 represents either 2 or -2.*

This completes the proof of Lemma 3.1 and thus also Theorem 1.1.

7. The case $k = 2$

Thus far, the case with $k = 2$ has been largely ignored, mostly because it is known that Theorem 1.1 does not hold in this setting. Wall [25] showed that the classes $(1, 4, 1)$ and $(3, 4, 3)$ lie in distinct orbits. The argument looks at the orthogonal complement of the two vectors and finds that each contains a unique vector of self-intersection 1, one ordinary and one characteristic. Hence no map in $D(M)$ takes these into each other.

In the following, it is shown that whenever $n \geq 16$ is a perfect square there will be distinct ordinary and characteristic orbits.

As before, consider a family of transformations:

$$S(\beta)(a, b_j, b_k) = (ST_j)^\beta(a, b_j, b_k) =$$

$$= (a + 2\beta^2(a - b_k) + 2\beta b_j, b_k + 2\beta^2(a - b_k) + 2\beta b_j, b_j + 2\beta(a - b_k)).$$

Note that $R(\beta, \beta)$ is essentially this map, mapping b_i to $-b_i$. Thus this family of maps in $k = 2$ is present in $k = 3$ along the lines $b_i(\alpha, \beta) = const$ in the map $R(\alpha, \beta)$. This parabola has its vertex at $\beta_0 = -\frac{b_j}{2(a-b_k)}$. If $a - b_k = -\Delta$, then there is again a unique terminal class.

It follows, that once again there is an equivalence for square free m:

(1) $A = (a, b, c)$ is $D(M)$-equivalent to a class with difference m.

(2) The system of Diophantine equations

$$z^2 = x^2 + y^2, \quad az - bx - cy = m$$

has a solution $(z, x, y) \in \mathbb{Z}^3$.

For Pythagorean triples one has a general description, plugging this in here leads to the equivalent binary quadratic form for $A = (a, b, c)$

$$f(\epsilon, \sigma) = (a - c)\epsilon^2 - 2b\epsilon\sigma + (a + c)\sigma^2.$$

The discriminant of this form is $D = 4b^2 - 4(a - c)(a + c) = -4A^2$. Thus the form is indefinite if $A^2 < 0$, and definite if $A^2 > 0$. If $n = \tau^2$, then the following is true:

Theorem 7.1 ([4], Theorem 1.3.1)**.** *Let f be an integral binary quadratic form. Then the following statements are equivalent:*

(1) *The discriminant of f is a perfect square.*
(2) *The form f is a product of two integral binary forms.*
(3) *There is $(\epsilon, \sigma) \in \mathbb{Z}^2$ with $(\epsilon, \sigma) \neq (0, 0)$ and $f(\epsilon, \sigma) = 0$.*

Thus, the last point means that there exists a class in the orbit of O_A with difference 0. This has the form (a, τ, a). Using the map $S(t)$, one can ensure that $a \leq \tau$.

Note that f is dependent on the order of b, c, yet the discriminant is not. Thus consider the forms associated to (a, τ, a) and (a, a, τ) separately. In each case, the Theorem implies that f factors over $\mathbb{Z}$. The results are

$$(a, \tau, a) \to f_1(\epsilon, \sigma) = 2y(-\tau\epsilon + a\sigma)$$

and

$$(a, a, \tau) \to f_2(\epsilon, \sigma) = (\epsilon - \sigma)[(a - \tau)\epsilon - (a + \tau)\sigma]$$

If A is ordinary, then $f_1 = 1$ has no solution in the integers. For f_2, $\epsilon - \sigma = \pm 1$, which, after considering all permutations, only leads to the solution $a = \tau - 1$. Thus only the class $(\tau - 1, \tau, \tau - 1)$ has an orbit with difference 1. Note that this is precisely the class $(3, 4, 3)$ of Wall.

Assuming that A is characteristic, τ must be odd. Then $f_1 = \pm 2$ implies that $y = \pm 1$ and $a = \tau - 1$, which is even. From $f_2 = \pm 2$ it follows that $a = \tau - 2$. So only one of the characteristic orbits contains a class with difference of 2.

Lemma 7.2. *Assume $n \geq 16$ is a perfect square and A is primitive. Then there exist at least two distinct $D(M)$-orbits of primitive ordinary classes and, if they exist, at least two distinct $D(M)$-orbits of primitive characteristic classes.*

If n is not a perfect square, the situation becomes more muddled. On the one side there is the $D(M)$ action. On the other, binary quadratic forms also admit a group action and a classification up to equivalence, see [4] for details. In fact, writing (a, b, c) for the homology class and $[a - c, -2b, a + c]$ for the binary quadratic form, $S(\alpha)(a, b, c)$ leads to a family of forms

$$[a - c, -2(b + 2\alpha(a - c)), a + c + 4\alpha^2(a - c) + 4\alpha b].$$

On the other side, the group generated by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

acts on forms. This action on the form $[a - c, -2b, a + c]$ produces

$$[a - c, -2b + 2\beta(a - c), a + c - 2b\beta + \beta^2(a - c)]$$

These are remarkably similar, in fact if $\beta = -2\alpha$ they coincide. However, for odd β there does not appear to be a corresponding $D(M)$ action.

Under the action of elementary transformations, (a, b, c) and (a, c, b) are equivalent. However, the equivalence of forms has no action that appears to take $[a - c, -2b, a + c]$ to $[a - b, -2c, a + b]$.

Due to this non-alignment of equivalences and the work needed to check if a form represents Δ over the integers (see Ch. 6, [4]), there does not appear to be any appreciable advantage of this method over simply calculating all simplified classes and manually checking for equivalence whenever n is not a perfect square.

8. Minimal genus

The minimal genus is the smallest possible genus g_A of a smooth representative of the class $A \in H_2(M, \mathbb{Z})$. If A has minimal genus g_A, then every class $D(M)$-equivalent to A does as well. Therefore, it is possible to determine the minimal genus for certain simplified orbits using Theorem 1.1 and Cor. 3.2,.

8.1. Primitive ordinary classes. In the primitive ordinary case, the classes $(\tau, \tau+1)$ and $(\tau-1, \tau)$ are both represented by complex or symplectic curves for some choice of complex or symplectic structure on $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$. Note that these structures can be chosen to be compatible with the orientation of $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$.

Now perform blow-ups (in the correct category) at appropriate points on or away from these curves. By Corollary 1.2, [22], these curves minimize the genus in their class. Hence the classes $(\tau, \tau + 1, 0, ..., 0)$ ($n = 2\tau + 1$ odd) or $(\tau - 1, \tau, 1, 0, .., 0)$ ($n = 2\tau$ even) both have minimal genus $g_A = 0$. This completes the proof of Lemma 1.2.

Observe that these classes are equivalent to the class $(-a, -(a + 1), 1, ..., 1)$, compare with the results in Section 4, [28] and Section 3, [6].

Remark: Using results on symplectic genus in [17], it is easy to see that no divisible ordinary class is represented by a symplectic surface.

8.2. Primitive characteristic classes. Cor. 3.2 shows that every primitive characteristic class with a simplified class in O_A is equivalent to $(2\gamma - 1, 2\gamma + 1, 1, ..., 1)$ for $\gamma \geq 0$. The minimal genus in two cases is well known: If $\gamma = 0$, the minimal genus is 0; when $\gamma = 1$, the minimal genus is 1. Hence, assume in the following that $\gamma \geq 2$.

Lemma 8.1. *Let* $A = (2\gamma - 1, 2\gamma + 1, 1, ..., 1) \in \mathbb{C}P^2\#k\overline{\mathbb{C}P^2}$.

(1) *If* $1 \leq k \leq 2\gamma - 1$, *then the minimal genus of any embedded surface representing* A *is bounded above by* $2\gamma - k$.
(2) *If* γ *is odd and* $k \geq 2\gamma - 1$, *then the minimal genus of any embedded surface representing* A *is 1.*
(3) *If* γ *is even and* $k \geq 2\gamma$, *then the minimal genus of any embedded surface representing* A *is 0.*

Proof. Consider first the class $(2\gamma+\alpha, 2\gamma+\alpha+2) \in H_2(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \mathbb{Z})$. Applying Theorem 2, [16], see also [22], the minimal genus of this class is

$$\frac{1}{2}((2\gamma+\alpha+1)(2\gamma+\alpha) - (2\gamma+\alpha)(2\gamma+\alpha-1)) = 2\gamma+\alpha$$

The minimal genus is realized by a non-singular holomorphic curve for some complex structure on $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ which is not orientation-compatible with the standard orientation.

Hence, if $k = 1$ and $\alpha = -1$, the class $A = (2\gamma-1, 2\gamma+1)$ has minimal genus $2\gamma - 1$.

Assume that $2 \leq k \leq 2\gamma - 1$. For $k = 2$, the class $(2\gamma-1, 2\gamma+1, 1)$ is D(M)-equivalent to $(2\gamma-3, 2\gamma-1, 3)$ via the map $S(1)(2\gamma-1, 1, 2\gamma+1)$. This class is representable by an embedded curve with genus $2\gamma - 2$ obtained by connected sum of a torus in the class $3E_2$ and the curve of genus $2\gamma - 3$ in $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$.

The map $S(1) = R(1,1)$, as remarked in Section 7, hence repeated application of this map to replace the 1's by 3's while reducing the leading term as much as possible shows that

$$(2\gamma-1, 2\gamma+1, \underbrace{1, ..., 1}_{k-1}) \simeq (2\gamma-3, 2\gamma-1, 3, \underbrace{1, ..., 1}_{k-2}) \simeq$$
$$(2\gamma-5, 2\gamma-3, 3, 3, \underbrace{1, ..., 1}_{k-3}) \simeq ... \simeq (2(\gamma-k+1)-1, 2(\gamma-k+1)+1, \underbrace{3, ..., 3}_{k-1}) =$$
$$= (2(\gamma-k)+1, 2(\gamma-k)+3, \underbrace{3, ..., 3}_{k-1}),$$

which has an embedded representative of genus $2\gamma - 2k + 1 + k - 1 = 2\gamma - k$ generated by connected sums of $k-1$ tori with the representative of $(2(\gamma-k)+1, 2(\gamma-k)+3)$. Thus, for $k \leq \gamma$ the result is proven.

Note that if $k = \gamma$, the class thus obtained is $(1, 3, ..., 3)$, hence the class A is representable by k-tubed tori and the minimal genus is bounded above by γ.

In what follows, Theorem 1.1 will be applied repeatedly without explicit reference, it allows us to determine an equivalent class to A simply by numerical calculation. In order to produce examples for $k > \gamma$, a construction by Acosta [1] (see also [12], [2]) is repeated and expanded.

- Consider the class $(1,3) \in H_2(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \mathbb{Z})$. Represent the class $H \in \mathbb{C}P^2$ by two lines and one line with opposite orientation. These three lines can be chosen to meet transitively in three distinct points, two with negative intersection, one with positive intersection. Smooth the negative points to produce an immersed sphere with a single positive self-intersection. The class $3E \in \overline{\mathbb{C}P^2}$ is represented by three lines, all meeting transitively in three distinct negative points. Smoothing one of these and connect summing the resulting curve with the curve in $\mathbb{C}P^2$ at the two singular points with opposite orientation produces an immersed sphere with a single negative self-intersection representing the class $(1, 3) \in H_2(\mathbb{C}P^2\#\overline{\mathbb{C}P^2}, \mathbb{Z})$.

- Acosta performs a similar construction to produce a representative of $(1, 3) \in H_2(\mathbb{C}P^2\#\mathbb{C}P^2, \mathbb{Z})$ consisting of two spheres, intersecting in a positive and a negative point. Either by changing orientation on the ambient manifold or by mimicking this construction, it can be shown that the class $(1, 3) \in H_2(\overline{\mathbb{C}P^2}\#\overline{\mathbb{C}P^2}, \mathbb{Z})$ is also represented by two spheres meeting in two points, one positive and one negative.

These two examples will form the building blocks of the constructions for $k > \gamma$.

If $\gamma < k \leq 2\gamma - 1$, then $A \simeq (1, \underbrace{3, \dots, 3}_{\gamma}, \underbrace{1, \dots, 1}_{k-\gamma})$. Reorganize this as

$$(1, 3, \underbrace{1, 3, \dots, 1, 3}_{k-\gamma-1 \text{ pairs}}, \underbrace{3, .., 3}_{2\gamma-k-1})$$

and produce an embedded representative as follows: The first $(1, 3)$-pair is represented by an immersed sphere with a single negative intersection point. Form the connected sum at this intersection point with the positive intersection point of the representative of the second pair $(1, 3)$ which is formed out of two spheres. This again produces an immersed sphere with a single negative self-intersection point, but now representing the class $(1, 3, 1, 3) \in H_2(\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}, \mathbb{Z})$. Continue this procedure until an immersed sphere with a single negative intersection point using all of the $k - \gamma - 1$ pairs above has been produced. Smoothing the intersection point produces an embedded torus. The remaining $(3, \dots, 3)$ have minimal genus (see [21]) $2\gamma - k - 1$. Thus forming the connected sum of these two curves produces an embedded curve representing A of genus $2\gamma - k$. When $k = 2\gamma - 1$, this produces a torus. This completes the proof for $k \leq 2\gamma - 1$.

Assume now that $k > 2\gamma - 1$. A classical result of Kervaire and Milnor ([12], [2]) states that for a characteristic class A to be representable by a sphere, it must hold that

$$A^2 \equiv_{16} \sigma.$$

This implies, that if γ is odd, then the minimal genus is at least 1. Thus, in this case, when $k \geq 2\gamma - 1$ this minimal genus is achieved.

Finally, assume that γ is even and $k > 2\gamma - 1$. Then the minimal genus is either 0 or 1. The class $A \simeq (1, \underbrace{3, ..., 3}_{\gamma}, \underbrace{1, ..., 1}_{k-\gamma})$ can now be reorganized as

$$(1, 3, \underbrace{1, 3, ..., 1, 3}_{\gamma-1 \text{ pairs}}, \underbrace{1, .., 1}_{k-2\gamma+1}).$$

The class $(3, 1, 3, ..., 1, 3) \in H_2(\#_{2\gamma-1}\overline{\mathbb{C}P^2}, \mathbb{Z})$ can be represented by an embedded torus by connect summing along opposite oriented intersection points of the two sphere representatives of $(1, 3)$, ultimately producing a representative consisting of two spheres intersecting in two points, one positive and one negative, but now representing the class $(1, 3, ..., 1, 3)$. This can be connect summed with a representative of the class $3E_1$ which is an immersed sphere with a single negative self-intersection point to produce a curve which can be smoothed to make an embedded torus.

This class has self-intersection $-9\gamma - \gamma + 1 = -8\gamma - 2\gamma + 1 \equiv_{16} -2\gamma + 1 = \sigma$. Thus a result of Yasuhara (Connecting Lemma III, [27]) states that the class $(1, 3, 1, 3, ..., 1, 3, 1) \in H_2(\mathbb{C}P^2\#2\gamma\overline{\mathbb{C}P^2}, \mathbb{Z})$ is represented by an embedded sphere.

$\square$

This result can be viewed as a stabilization result, eventually every characteristic class will have the smallest possible minimal genus.

Remark:

(1) It seems plausible that the minimal genus is in fact $2\gamma - k$ for $k \le 2\gamma - 1$.

(2) The minimal genus of characteristic classes can be estimated from below using results in [1],[3] and [9]: If $n = 8\gamma + k - 1$, then $g_A \ge \gamma - k + 2$. In particular, if A is to be represented by a sphere, then γ must be even and thus

$$k \ge \gamma + 2 = 2(\nu + 1).$$

For example, as $\gamma \ge 2$, this gives the following estimates

$$\gamma = 2 : k \ge 4; \ \ \gamma = 4 : k \ge 6; \text{ and } \gamma = 6 : k \ge 8$$

under the restriction that $k \le 9$. Applying Lemma 8.1, this leaves a gap of classes for which no determination can be made:

$$\gamma = 2 : k = 4; \ \ \gamma = 4 : 6 \le k \le 8; \text{ and } \gamma = 6 : k \ge 8$$

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