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On the convergence of multiple ergodic means

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ABSTRACT. Consider a sequence of measure preserving transformations $\mathfrak{U} = \{U_k : k = 1, 2, ...\}$ on a measurable space (X, μ). We prove a.e. convergence of the ergodic means

$$\frac{1}{s_1 \cdots s_n} \sum_{j_1=0}^{s_1-1} \cdots \sum_{j_n=0}^{s_n-1} f\left(U_1^{j_1} \cdots U_n^{j_n} x\right) \tag{0.1}$$

as $\min_j s_j \to \infty$, for any function $f \in L \log^{d-1}(X)$, where $d \leq n$ is the rank of the transformations $\mathfrak{U}$. The result gives a generalization of a theorem by N. Dunford and A. Zygmund, claiming the convergence of (0.1) in a narrower class of functions $L \log^{n-1}(X)$.

CONTENTS

1. Introduction

Birkhoff's ergodic theorem is one of the most important and beautiful result of probability theory. The study of ergodic theorems started in 1931 by von Neumann and Birkhoff, having its origins in statistical mechanics. Recall the definition of the measure-preserving transformation (see [4]).

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Definition 1.1. Let $(X, \mathcal{B}, \mu)$ be a probability space. A mapping $T : X \to X$ is said to be a measure-preserving transformation if for any measurable set $E \in \mathcal{B}$ the set $T^{-1}(E)$ is also measurable and $\mu(E) = \mu(T^{-1}(E))$. The combination $(X, \mathcal{B}, \mu, T)$ is called a measure-preserving system.

Theorem A (Birkhoff). *If $(X, \mathcal{B}, \mu, T)$ is a measure-preserving system, then for any function $f \in L^1(X)$ the averages*

$$\frac{1}{n}\sum_{j=0}^{n-1} f(T^j x)$$

converge almost everywhere to a T-invariant function $\bar{f}$ as $n \to \infty$.

There are different proofs and various generalizations of this classical theorem. Some of those clearly demonstrate strong link between the Lebesgue differentiation theory on $\mathbb{R}^n$ and pointwise convergence of different type of ergodic averages. The following multiple version of Birkhoff's theorem, proved by Zygmund [13] and Dunford [2] independently, is an example of such a resemblance. Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be non-decreasing function and $(X, \mathcal{B}, \mu)$ be a probability space. Denote by $L_\Phi(X)$ the class of $\mathcal{B}$-measurable functions f on X with $\Phi(|f|) \in L^1(\mathbb{T})$. The class $L_\Phi(X)$ corresponding to a function

$$\Phi(t) = t\,(1 + (\max\{0, \log t\})^n), \quad n \geq 1, \tag{1.1}$$

will be denoted by $L \log^n L(X)$. Clearly this class of function is strongly included in $L^1(X)$.

Theorem B (Dunford-Zygmund). *Let $U_1, \dots, U_n$ be measure-preserving one-to-one transformations of a probability space $(X, \mathcal{B}, \mu)$. Then for any function $f \in L \log^{n-1} L(X)$ the averages*

$$\frac{1}{s_1 \cdots s_n}\sum_{j_1=0}^{s_1-1} \cdots \sum_{j_n=0}^{s_n-1} f\left(U_1^{j_1} \cdots U_n^{j_n} x\right) \tag{1.2}$$

converge a.e. as $\min_j s_j \to \infty$.

This result has been generalized for general contraction operators on L^1, considering those instead of the operators $f \to f \circ U_k$ generated by the measure-preserving transformations U_k (Dunford-Schwartz [3], Fava [5]). Hagelstein and Stokolos in [10] proved the sharpness of the class of functions $L \log^{n-1} L(X)$ in the context of Theorem B. Namely,

Theorem C (Hagelstein-Stokolos). *Suppose a collection of invertible commuting measure-preserving transformations $\mathfrak{U} = \{U_k : k = 1, 2, \dots, n\}$ is non-periodic, that is for any non-trivial collection of integers $p_k \in \mathbb{Z}$, $k = 1, 2, \dots, n$ we have*

$$\mu\{U_1^{p_1} \circ \dots \circ U_n^{p_n}(x) = x\} = 0.$$

If $\Phi(t) = o(t \log^{n-1} t)$ as $t \to \infty$, then there exists a function $f \in L_\Phi(X)$ such that averages (1.2) unboundedly diverge a.e..

Definition 1.2. A set of invertible commuting measure-preserving (ICMP) transformations $\mathfrak{U} = \{U_k : k = 1, 2, \dots, n\}$ is said to be *dependent* if there is a non-trivial collection of integers $p_k \in \mathbb{Z}$, $k = 1, 2, \dots, n$, such that

$$(U_1^{p_1} \circ \dots \circ U_n^{p_n})(x) = x \tag{1.3}$$

almost everywhere on X. If there is no such a collection of integers p_k, then we say $\mathfrak{U}$ is *independent*. The *rank of* $\mathfrak{U}$ denoted by $\mathrm{rank}(\mathfrak{U})$ will be called the largest integer r for which there is an independent subset of cardinality r in $\mathfrak{U}$.

Remark 1.3. Note that according to our definition, the independence of $\mathfrak{U}$ requires the failure of (1.3) on a set of positive measure for any non-trivial collection of integers $\{p_k\}$, while the condition of non-periodicity in Theorem C is a stronger version of independence, since in this case the failure of (1.3) is required almost everywhere.

The main result of the present paper provides a generalization of Theorem B. Namely, it says that in fact a.e. convergence of averages (1.2) holds in a larger class of functions $L\log^{d-1} L \supset L\log^{n-1} L$, where $d = \mathrm{rank}(\mathfrak{U}) \le \mathrm{n}$. First we prove the following weak type maximal inequality, where $\mathrm{Log_n t}$ denotes the function in (1.1), i.e.

$$\mathrm{Log_n(t) = t\,(1 + (\max\{0, \log t\})^n)}\,.$$

Theorem 1.4. *Let $\mathfrak{U} = \{U_k : k = 1, 2, \dots, n\}$ be a set of ICMP transformations of rank d. Then, for any function $f \in L\log^{d-1} L(X)$ and $\lambda > 0$, we have*

$$\mu\left\{x \in X : \sup_{s_j \ge 0} \frac{1}{s_1 \dots s_n} \sum_{k_1=0}^{s_1-1} \cdots \sum_{k_n=0}^{s_n-1} \left| f\left((U_1^{k_1} \circ \cdots \circ U_n^{k_n})(x) \right) \right| > \lambda \right\}$$
$$\le C(\mathfrak{U}) \int_X \mathrm{Log_{d-1}}\left(\frac{\mathrm{|f|}}{\lambda}\right), \tag{1.4}$$

where $C(\mathfrak{U})$ is a constant depending only on $\mathfrak{U}$.

As a corollary of (1.4) we obtain the following.

Theorem 1.5. *Let $\mathfrak{U} = \{U_k : k = 1, 2, \dots, n\}$ be a set of ICMP transformations of rank d. Then, for any function $f \in L\log^{d-1} L(X)$ the averages (1.2) converge almost everywhere as $\min s_k \to \infty$.*

Remark 1.6. We will see in the last section that the class $L\log^{d-1} L(X)$ of the functions in Theorem 1.5 is optimal. More precisely, if the corresponding independent subset of cardinality $d = \mathrm{rank}(\mathfrak{U})$ in $\mathfrak{U}$ is "strongly independent" (i.e. non-periodic), then under the condition $\Phi(t) = o(t\log^{d-1} t)$ there exists a function $f \in L_\Phi(X)$ with a.e. diverging averages (1.2). In fact, the proof of this optimality immediately follows from Theorem C. We will just need to apply a simple lemma proved in Section 5 (Lemma 5.1).

The inequality (1.4) will be deduced from a maximal inequality on $\mathbb{R}^n$. Let $A : \mathbb{R}^n \to \mathbb{R}^d$ be a linear operator given by the matrix

$$A = \{a_{kj} : 1 \le j \le n, 1 \le k \le d\} \tag{1.5}$$

of size $d \times n$ (d-rows and n-columns). We consider the maximal function

$$M_A f(\mathbf{x}) = \sup_R \frac{1}{|R|} \int_R |f(\mathbf{x} + A \cdot \mathbf{t})| d\mathbf{t}, \quad \mathbf{x} \in \mathbb{R}^d, \tag{1.6}$$

where sup is taken over all n-dimensional symmetric intervals

$$R = \{\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}^n : t_j \in [-r_j, r_j], j = 1, 2, \dots, n\} \subset \mathbb{R}^n.$$

Denote by rankA the rank of the matrix A.

Theorem 1.7. *Let A be the matrix (1.5) and $r = \mathrm{rankA}$. Then for any function $f \in L(\log^+ L)^{r-1}(\mathbb{R}^d)$ the bound*

$$|\{\boldsymbol{x} \in \mathbb{R}^d : M_A f(\boldsymbol{x}) > \lambda)\}| \le C(A) \int_{\mathbb{R}^d} \mathrm{Log}_{\mathrm{r}-1}\left(\frac{|\mathrm{f}|}{\lambda}\right), \tag{1.7}$$

holds, where $C(A)$ is a constant, depending only the matrix A.

Remark 1.8. Observe that if $n = d = r$ and A is the identity matrix of size n, then (1.6) gives the well-known strong maximal function on $\mathbb{R}^n$, correspondingly, (1.7) becomes the weak type inequality due to M. de Guzman [6] (see also [7]). Moreover, inequality (1.7) holds even if A is a general invertible matrix and it follows from Guzman's inequality of [6], simply using the equivalence of rectangular and parallelepiped differentiation bases on $\mathbb{R}^n$. Our proof of the full version of inequality (1.7) is a reduction of the general case to the case of invertible A.

Remark 1.9. Note that papers [2] and [13] suggest different proofs of Theorem B. The proof of [2] is straightforward and the convergence of averages (1.2) was established only for the functions in L^p, $1 < p < \infty$, while Zygmund [13] provides an inequality, which is the analogue of a similar inequality for the strong maximal function, originally proved in [9]. The latter is the weaker version of Guzman's inequality of [6] .

Remark 1.10. The well known transfer principle of Calderón [1] enables to reduce certain ergodic maximal inequalities to maximal inequalities in harmonic analysis. A version of Calderón's principle in higher dimension was suggested in [11], where only non-periodic collections of measure-preserving transformations were considered. In fact, our proof of Theorem 1.4 is an extension of this higher dimensional principle to arbitrary collections of measure-preserving transformations.

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2. Proof of Theorem 1.7

We will use the following equivalent form of the maximal function (1.6)

$$M_{\mathfrak{U}}f(\mathbf{x})=\sup_{r_k>0}\frac{1}{2^n r_1\cdots r_n}\int_{-r_1}^{r_1}\cdots\int_{-r_n}^{r_n}|f(\mathbf{x}+t_1\mathbf{u}_1+\cdots+t_n\mathbf{u}_n)|dt_1\cdots dt_n, \quad (2.1)$$

where the vector set $\mathfrak{U}=\{\mathbf{u}_k,\ k=1,2,\dots,n\}$ is formed by the columns of the matrix (1.5). So the rank of vectors $\mathfrak{U}$ coincides with the rank of the matrix A. Once again note that that if the collection of vectors are independent, i.e. the matrix A is invertible, then inequality (1.7) is known, and we are going to reduce the general case to the case of invertible A. We need several lemmas, concerning parallelepipeds in $\mathbb{R}^d$ and associated measures.

For a vector $\mathbf{x}=(x_1,\dots,x_d)\in\mathbb{R}^d$ we denote $|\mathbf{x}|=(x_1^2+\dots+x_d^2)^{1/2}$. Given a set of vectors $\mathfrak{B}\subset\mathbb{R}^d$ we denote by span($\mathfrak{B}$) the linear space generated by $\mathfrak{B}$ (sometimes this Euclidean space will be denoted by $\mathbb{R}_{\mathfrak{B}}$). The notation $|E|$ will stand for the Lebesgue measure of a set E in an Euclidean space.

Definition 2.1. Let $\mathfrak{U}=\{\mathbf{u}_k\ :\ k=1,2,\dots,n\}\subset\mathbb{R}^d$ be a set of unit vectors. Call a parallelepiped in $\mathbb{R}^d$ a set of the form

$$R=\left\{\mathbf{x}\in\mathbb{R}^d\ :\ \mathbf{x}=t_1\mathbf{u}_1+\dots+t_n\mathbf{u}_n,\ t_j\in[-r_j,r_j]\right\}. \quad (2.2)$$

The family of all parallelepipeds (2.2) generated by a fixed set of vectors $\mathfrak{U}$ will be denoted by $\mathcal{P}_{\mathfrak{U}}$.

Note that parallepipeds can have different representations (2.2). Clearly the arithmetic sum of two parallelepipeds R,Q

$$R+Q=\{\mathbf{x}+\mathbf{t}\ :\ \mathbf{x}\in R,\ t\in Q\}$$

is again a parallelepiped. For two parallelepipeds R and Q we write $Q\prec R$ if there is a parallelepiped R' such that $Q=R+R'$.

Lemma 2.2. *If $\mathfrak{U}=\{\boldsymbol{u}_k\ :\ k=1,2,\dots,n\}$ is a basis set of vectors in $\mathbb{R}^n$ and $R\in\mathcal{P}_{\mathfrak{U}}$ has a representation (2.2), then*

$$\{\boldsymbol{x}\in\mathbb{R}^n\ :\ |\boldsymbol{x}|\le 1\}\subset\frac{C(\mathfrak{U})}{\min_j r_j}\cdot R, \quad (2.3)$$

where $C(\mathfrak{U})$ is a constant, depending only on the set of vectors $\mathfrak{U}$.

Proof. For any $j=1,2,\dots,n$ we consider hyperplanes Γ_j^+ and Γ_j^- in $\mathbb{R}^n$ defined

$$\Gamma_j^{\pm}=\{\mathbf{x}=t_1\mathbf{u}_1+\dots+t_n\mathbf{u}_n\ :\ t_j=\pm r_j,\ t_i\in\mathbb{R},\ i\neq j\}$$

and let S_j be the closed strip domain lying between the hyperplanes $\Gamma_j^{\pm}$. We have $R=\cap_j S_j$. Denote by h_j the distance of the hyperplanes Γ_j^+ and Γ_j^- from the origin. It is clear that

$$\{\mathbf{x}\in\mathbb{R}^n\ :\ |\mathbf{x}|\le\min_j h_j\}\subset R. \quad (2.4)$$

One can also check that $c_j = h_j/r_j$ are constants, depending only on $\mathfrak{U}$. Denote $C(\mathfrak{U}) = (\min_j c_j)^{-1}$. From (2.4) we obtain

$$\{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| \le 1\} \subset \frac{1}{\min_j h_j} \cdot R \subset \frac{C(\mathfrak{U})}{\min_j r_j} \cdot R$$

and so (2.3). □

A version of the following lemma in the case of $d = 2$ was proved by Guzmán-Welland in [8] (see also [7], chap. 6, Lemma 2.1).

Lemma 2.3 (Guzmán-Welland). *Let $\mathfrak{U} = \{\boldsymbol{u}_k : k = 1, 2, \dots, n\}$ be a set of unit vectors in $\mathbb{R}^d$. Then for any parallelepiped $R \in \mathcal{P}_{\mathfrak{U}}$ there exist a subset $\mathfrak{B} \subset \mathfrak{U}$ of independent vectors and a parallelepiped $Q \in \mathcal{P}_{\mathfrak{B}}$ such that*

$$\operatorname{rank}(\mathfrak{B}) = \operatorname{rank}(\mathfrak{U}), \tag{2.5}$$

$$Q \prec R, \tag{2.6}$$

$$R \subset C(\mathfrak{U}) \cdot Q, \tag{2.7}$$

where $C(\mathfrak{U})$ is a constant depending only on the set of vectors $\mathfrak{U}$.

Proof. Suppose that $R \in \mathcal{P}_{\mathfrak{U}}$ is the parallelepiped (2.2). Without loss of generality we can suppose that

$$r_1 \ge r_2 \ge \dots \ge r_n. \tag{2.8}$$

Denote

$$\mathfrak{B} = \{\mathbf{u}_k : \mathbf{u}_k \notin \operatorname{span}\{\mathbf{u}_1, \dots, \mathbf{u}_{k-1}\}\} \subset \mathfrak{U}.$$

One can easily check that the vectors of $\mathfrak{B}$ are independent and $\operatorname{rank}(\mathfrak{B}) = \operatorname{rank}(\mathfrak{U})$. One can split the set of vectors $\mathfrak{U}$ into groups

$$\mathfrak{U}_j = \{\mathbf{u}_k : k \in (k_{j-1}, k_j]\}, \quad j = 1, 2, \dots, s,$$
$$0 = k_0 < k_1 < \dots < k_s = n,$$

such that

$$\mathfrak{B} = \bigcup_{i \ge 0} \mathfrak{U}_{2i+1}, \quad \mathfrak{U}_{2j} \subset \operatorname{span}\left(\bigcup_{i=1}^{j} \mathfrak{U}_{2i-1}\right).$$

Considering the parallelepipeds

$$R_j = \left\{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} = \sum_{k=k_{j-1}+1}^{k_j} t_k \mathbf{u}_k,\ t_k \in [-r_k, r_k]\right\} \in \mathcal{P}_{\mathfrak{U}_j},$$

we can write

$$R = R_1 + R_2 + \dots + R_s.$$

Then the parallelepiped

$$Q = \sum_{j:\, 2j-1 \le s} R_{2j-1}$$

satisfies (2.5) and (2.6). If $\mathbf{x} \in R_{2j}$, then

$$|\mathbf{x}| \leq \sum_{i=k_{2j-1}+1}^{k_{2j}} r_j \leq n r_{k_{2j-1}}. \tag{2.9}$$

Let Y_j be the subspace of $\mathbb{R}^d$ generated by the independent vectors $\cup_{i \leq j} \mathfrak{U}_{2i-1}$. One can check

$$R_i \subset Y_j, \quad i = 1, 2, \dots, 2j.$$

Thus, applying Lemma 2.2 for the space Y_j, as well as (2.8), (2.9), we conclude

$$\begin{aligned}\frac{1}{n r_{k_{2j-1}}} R_{2j} \subset \{\mathbf{x} \in Y_j \,:\, |\mathbf{x}| \leq 1\} &\subset \frac{C(\mathfrak{U})}{r_{k_{2j-1}}}(R_1 + R_3 + \dots + R_{2j-1}) \\ &\subset \frac{C(\mathfrak{U})}{r_{k_{2j-1}}} \cdot Q\end{aligned}$$

Thus we get $R_{2j} \subset nC(\mathfrak{U}) \cdot Q$ and therefore

$$R \subset n^2 C(\mathfrak{U}) Q.$$

This gives us (2.7), completing the proof of lemma. □

Given a set of unit vectors $\mathfrak{U} = \{\mathbf{u}_k \,:\, k = 1, 2, \dots, n\} \subset \mathbb{R}^d$, let $\mathbb{R}_{\mathfrak{U}}$ be the subspace of $\mathbb{R}^d$ generated by the vectors $\mathfrak{U}$. We associate with a parallelepiped (2.2) a probability measure μ_R supported on R as follows. First, for each j we consider a probability measure μ_j uniformly distributed on the one dimensional parallelepiped $\{t\mathbf{u}_j \,:\, t \in [-r_j, r_j]\}$. The convolution of singular measures μ_j is the measure μ_R defined on the Lebesgue measurable sets of $E \subset \mathbb{R}_{\mathfrak{U}}$ by

$$\mu_R(E) = \int_{\mathbb{R}_{\mathfrak{U}}} \dots \int_{\mathbb{R}_{\mathfrak{U}}} \mathbf{1}_E(\mathbf{v}_1 + \dots + \mathbf{v}_n) d\mu_1(\mathbf{v}_1) \dots d\mu_n(\mathbf{v}_n). \tag{2.10}$$

One can check that μ_R is well-defined for any Lebesgue measurable set $E \subset \mathbb{R}_{\mathfrak{U}}$. Denote by f_R the density function of measure μ_R with respect to the Lebesgue measure on $\mathbb{R}_{\mathfrak{U}}$. Observe that if $\mathfrak{U}$ is independent, then

$$f_R(\mathbf{x}) = \begin{cases} |R|^{-1} & \text{if} \quad \mathbf{x} \in R, \\ 0 & \text{if} \quad \mathbf{x} \in \mathbb{R}_{\mathfrak{U}} \setminus R. \end{cases} \tag{2.11}$$

Lemma 2.4. *Let $\mathfrak{U} \subset \mathbb{R}^d$ be a set of arbitrary unit vectors and $R \in \mathcal{P}_{\mathfrak{U}}$. Then there exists a set of independent vectors $\mathfrak{B} \subset \mathfrak{U}$ such that* $\operatorname{rank}(\mathfrak{B}) = \operatorname{rank}(\mathfrak{U})$ *and there is a parallelepiped $R' \in \mathcal{P}_{\mathfrak{B}}$ such that*

$$\mu_R \leq C(\mathfrak{U}) \cdot \mu_{R'}. \tag{2.12}$$

Proof. Applying Lemma 2.3 in the Euclidean space $\mathbb{R}_{\mathfrak{U}}$, we find a set of independent vectors $\mathfrak{B} \subset \mathfrak{U}$, $\operatorname{rank}(\mathfrak{B}) = \operatorname{rank}(\mathfrak{U})$ and a parallelepiped $Q \in \mathcal{P}_{\mathfrak{B}}$ satisfying the conditions of lemma. Since $Q \prec R$, we have $R = Q + H$ for some parallelepiped H in $\mathbb{R}_{\mathfrak{U}}$. We can write

$$\mu_R(E) = \int_{\mathbb{R}_{\mathfrak{U}}} \int_{\mathbb{R}_{\mathfrak{U}}} \mathbf{1}_E(\mathbf{v} + \mathbf{v}') d\mu_Q(\mathbf{v}) d\mu_H(\mathbf{v}')$$

$$
\begin{aligned}
&= \int_{\mathbb{R}_{\mathfrak{U}}} \int_{\mathbb{R}_{\mathfrak{U}}} \mathbf{1}_E(\mathbf{v}+\mathbf{v}') f_Q(\mathbf{v}) d\mathbf{v} d\mu_H(\mathbf{v}') \\
&= \frac{1}{|Q|} \int_{\mathbb{R}_{\mathfrak{U}}} \int_{\mathbb{R}_{\mathfrak{U}}} \mathbf{1}_E(\mathbf{v}+\mathbf{v}') \mathbf{1}_Q(\mathbf{v}) d\mathbf{v} d\mu_H(\mathbf{v}') \\
&\le \frac{|E|}{|Q|}.
\end{aligned}
$$

This clearly implies

$$\|f_R\|_\infty \le \|f_Q\|_\infty = |Q|^{-1}. \tag{2.13}$$

Denote $R' = C(\mathfrak{U})Q$, where $C(\mathfrak{U})$ is the constant in (2.7). From (2.7) and (2.11) we have

$$
\begin{aligned}
&R \subset R', \\
&\|f_{R'}\|_\infty = |R'|^{-1} = \left(C(\mathfrak{U})|Q|\right)^{-1}.
\end{aligned} \tag{2.14}
$$

Combining (2.13) and (2.14) we get the pointwise bound $f_R \le C(\mathfrak{U}) f_{R'}$, which implies (2.12). □

Proof of Theorem 1.7. Observe that the integral in (2.1) may be written as a convolution of measure (2.10) with the function f. Namely, we have

$$
\begin{aligned}
\frac{1}{2^n r_1 \dots r_n} \int_{-r_1}^{r_1} \dots \int_{-r_n}^{r_n} |f(\mathbf{x} + t_1\mathbf{u}_1 + \dots + t_n\mathbf{u}_n)| dt_1 \dots dt_n \\
= \int_{\mathbb{R}^d} |f(\mathbf{x}+\mathbf{v})| d\mu_R(\mathbf{v}).
\end{aligned} \tag{2.15}
$$

Applying Lemma 2.4, for any parallelepiped $R \in \mathcal{P}_{\mathfrak{U}}$ we find an independent vector set $\mathfrak{B} \subset \mathfrak{U}$ with $\mathrm{rank}(\mathfrak{B}) = \mathrm{rank}(\mathfrak{U})$ and a parallelepiped $R' \in \mathcal{P}_{\mathfrak{B}}$ such that (2.12) holds. Thus the last integral in (2.15) may be estimated as follows:

$$\int_{\mathbb{R}^d} f(\mathbf{x}+\mathbf{v}) d\mu_R(\mathbf{v}) \le C(\mathfrak{U}) \int_{\mathbb{R}^d} f(\mathbf{x}+\mathbf{v}) d\mu_{R'}(\mathbf{v}) \le C(\mathfrak{U}) M_{\mathfrak{B}} f(\mathbf{x}).$$

This implies

$$M_{\mathfrak{U}} f(\mathbf{x}) \le C(\mathfrak{U}) \max_{\mathfrak{B}} M_{\mathfrak{B}} f(\mathbf{x}),$$

where the maximum is taken over all the subsets $\mathfrak{B} \subset \mathfrak{U}$ of independent vectors such that $\mathrm{rank}(\mathfrak{B}) = \mathrm{rank}(\mathfrak{U})$. For each such $\mathfrak{B}$ the operator $M_{\mathfrak{B}}$ satisfies the bound (1.7) and the number of all collections $\mathfrak{B}$ is constant, depending only on n and so on $\mathfrak{U}$. Thus we get (1.7). □

3. A discrete maximal inequality

We will need a discrete version of inequality (1.7). Let $\phi : \mathbb{Z}^d \to \mathbb{R}$ be a d-dimensional sequence and let $A = \{a_{kj} : 1 \le j \le n, 1 \le k \le d\}$ be an integer matrix. Consider the maximal operator

$$\mathcal{D}_A\phi(\mathbf{n}) = \sup_{s_j\in\mathbb{N}} \frac{1}{s_1 \dots s_n} \sum_{k_1=0}^{s_1-1} \dots \sum_{k_n=0}^{s_n-1} \phi(\mathbf{n} + A\cdot\mathbf{k})$$
$$= \sup_{s_j\in\mathbb{N}} \frac{1}{s_1 \dots s_n} \sum_{\mathbf{k}=0}^{\mathbf{s}-1} \phi(\mathbf{n} + A\cdot\mathbf{k}), \quad \mathbf{n}\in\mathbb{N}^d.$$

From Theorem 1.7 we easily obtain the following.

Corollary 3.1. *For any integer matrix A of* rank(A) = r *we have the bound*

$$\#\{\boldsymbol{n}\in\mathbb{Z}^d \,:\, \mathcal{D}_A\phi(\boldsymbol{n}) > \lambda\} \le C(A) \sum_{\boldsymbol{n}\in\mathbb{Z}^d} \mathrm{Log}_{\mathrm{r}-1}\left(\frac{|\phi(\boldsymbol{n})|}{\lambda}\right).$$

Proof. Given multiple sequence $\phi(\mathbf{m})$ consider the function

$$f(\mathbf{x}) = \sum_{\varepsilon_j=0,1,-1} \phi(m_1+\varepsilon_1,\dots,m_n+\varepsilon_n), \text{ if } [\mathbf{x}]=\mathbf{m}, \quad \mathbf{m}\in\mathbb{Z}_d, \tag{3.1}$$

on $\mathbb{R}^d$, where $[\mathbf{x}] = ([x_1],\dots,[x_d])$ denotes the coordinate wise integer part of the vector $\mathbf{x} = (x_1,\dots,x_d)$. Clearly there is a constant $\delta = \delta(A) < 1$ such that

$$A(\Delta) \subset (-1,1)^d, \text{ where } \Delta = [0,\delta)^n, \tag{3.2}$$

Using (3.1), (3.2), one can check that

$$\phi(\mathbf{n} + A\cdot\mathbf{k}) \le f(\mathbf{x} + A\cdot\mathbf{t}) \text{ if } \mathbf{t}\in\mathbf{k}+\Delta,\ [\mathbf{x}]=\mathbf{n}.$$

Thus we obtain

$$\sum_{\mathbf{k}=0}^{\mathbf{s}-1} \phi(\mathbf{n}+A\cdot\mathbf{k}) \le \sum_{\mathbf{k}=0}^{\mathbf{s}-1} \frac{1}{|\Delta|}\int_{\mathbf{k}+\Delta} |f(\mathbf{x}+A\cdot\mathbf{t})|d\mathbf{t}$$
$$\le \frac{1}{|\Delta|}\int_R |f(\mathbf{x}+A\cdot\mathbf{t})|d\mathbf{t},$$

for any $\mathbf{x}$ with $[\mathbf{x}] = \mathbf{n}$, where

$$R = \{\mathbf{t}\in\mathbb{R}^n \,:\, t_j\in[-1,s_j],\ j=1,\dots,n\}.$$

This implies

$$\mathcal{D}_A\phi(\mathbf{n}) \le C(A)M_Af(\mathbf{x}) \text{ if } [\mathbf{x}] = \mathbf{n}\in\mathbb{Z}^d$$

and so

$$\#\{\mathbf{n}\in\mathbb{Z}_+^d \,:\, \mathcal{D}_A\phi(\mathbf{n}) > \lambda\} \le |\{\mathbf{x}\in\mathbb{R}^d \,:\, M_Af(\mathbf{x}) > \lambda/C(A)\}|$$
$$\le C(A)\int_{\mathbb{R}^d} \mathrm{Log}_{\mathrm{r}-1}\left(\frac{|\mathrm{f}|}{\lambda}\right)$$
$$\le C(A)\sum_{\mathbf{n}\in\mathbb{Z}^d} \mathrm{Log}_{\mathrm{r}-1}\left(\frac{|\phi(\mathbf{n})|}{\lambda}\right).$$

This completes the proof. □

4. Proofs of Theorems 1.4 and 1.5

Proof of 1.4. Since rank($\mathfrak{U}$) = d, without loss of generality we can suppose that $U_1, \dots, U_d$ are independent and

$$U_k^{l_k} = U_1^{a_{1,k}} \circ \cdots \circ U_d^{a_{d,k}}, \quad d < k \le n, \tag{4.1}$$

where $l_k \ge 1$ and $a_{j,k}$ are some integers. First we suppose that $l_k = 1$. Thus we can write

$$\begin{aligned} f\left((U_1^{k_1} \circ \cdots \circ U_n^{k_n})(x)\right) \\ &= f\left((U_1^{k_1+a_{1,d+1}k_{d+1}+\cdots+a_{1,n}k_n} \circ \cdots \circ U_d^{k_d+a_{d,d+1}k_{d+1}+\cdots+a_{d,n}k_n})(x)\right) \\ &= \phi(x, A\cdot \mathbf{k}), \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} &\phi(x,\mathbf{n}) = f\left((U_1^{n_1} \circ \cdots \circ U_d^{n_d})(x)\right), \\ &\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}^d, \end{aligned}$$

and

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 & a_{1,d+1} & \cdots & a_{1,n} \\ 0 & 1 & \cdots & 0 & a_{2,d+1} & \cdots & a_{2,n} \\ \cdot & \cdot & \cdots & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & 1 & a_{d,d+1} & \dots & a_{d,n} \end{pmatrix}$$

is a matrix of size $d \times n$. Let

$$f_M^*(x,\mathbf{n}) = \max_{1\le s_j \le M} \frac{1}{s_1 \cdots s_n} \sum_{\mathbf{k}=0}^{\mathbf{s}-1} |\phi(x, \mathbf{n} + A\cdot \mathbf{k})|, \tag{4.3}$$

where $M \in \mathbb{N}$ and denote

$$\begin{aligned} &E_\lambda(x) = \{\mathbf{n} : 1 \le n_j \le N : f_M^*(x,\mathbf{n}) > \lambda\}, \\ &E_\lambda(\mathbf{n}) = \{x : f_M^*(x,\mathbf{n}) > \lambda\}, \quad \mathbf{n} \in \mathbb{Z}^d, \\ &E_\lambda = \{(x,\mathbf{n}) : 1 \le n_j \le N, f_M^*(x,\mathbf{n}) > \lambda\} = \cup_{x\in X} E_\lambda(x) \\ &\qquad = \cup_{1\le n_j \le N} E_\lambda(\mathbf{n}). \end{aligned} \tag{4.4}$$

Taking into account (4.2), observe that inequality (1.4) is the same as

$$\lim_{M\to\infty} \mu(E_\lambda(\mathbf{0})) \le C(U) \int_X \mathrm{Log}_{\mathrm{d}-1}\left(\frac{|\mathrm{f}|}{\lambda}\right). \tag{4.5}$$

In (4.3) the coordinates of $A \cdot \mathbf{k}$ may vary in the interval $[-R, R]$, where $R = R(A, M)$ is a constant depending only on the matrix A and the integer M. From Corollary 3.1 it follows that

$$\#(E_\lambda(x)) \le C(A) \sum_{1\le n_j \le N+R} \mathrm{Log}_{\mathrm{r}-1}\left(\frac{|\phi(\mathrm{x},\mathbf{n})|}{\lambda}\right) \text{ for all x} \in \mathrm{X}.$$

Then, since U_k are measure-preserving, the sets $E_\lambda(\mathbf{n})$ have equal measures for different $\mathbf{n} \in \mathbb{Z}^d$. Thus from (4.4) we obtain

$$\begin{aligned}\mu(E_\lambda(\mathbf{0})) &= \frac{1}{N^d}\sum_{1\le n_j\le N}\mu(E_\lambda(\mathbf{n})) = \frac{1}{N^d}\int_X \#\,(E_\lambda(x))\\ &\le \frac{C(A)}{N^d}\sum_{1\le n_j\le N+R}\int_X \mathrm{Log}_{\mathrm{r}-1}\left(\frac{|\phi(\mathrm{x},\mathbf{n})|}{\lambda}\right)\\ &= \frac{C(A)(N+R)^d}{N^d}\int_X \mathrm{Log}_{\mathrm{r}-1}\left(\frac{|\mathrm{f}|}{\lambda}\right).\end{aligned}$$

Fixing M and letting $N\to\infty$, we get

$$|E_\lambda(\mathbf{0})| \le C(A)\int_X \mathrm{Log}_{\mathrm{r}-1}\left(\frac{|\mathrm{f}|}{\lambda}\right),$$

which implies (4.5). The general case $l_k \ge 1$ can be easily deduced from the case of $l_k = 1$. Fix an integer vector $\mathbf{r} = (r_{d+1},\dots,r_n)$, $0 \le r_j < l_j$, and denote by $Q^{\mathbf{r}}_{s_1,\dots,s_d} f(x)$ the sum of functions

$$\left|f\left(U_1^{k_1}\dots U_n^{k_n}x\right)\right|,$$

over the integer vectors $\mathbf{k} = (k_1,\dots,k_n)$, satisfying

$$1 \le k_j < s_j, \quad 1 < j \le n, \tag{4.6}$$
$$k_j = \bar{k}_j l_j + r_j, \quad \bar{k}_j \in \mathbb{Z}, \quad d < j \le n. \tag{4.7}$$

Under the conditions (4.7) we can write

$$f\left(U_1^{k_1}\dots U_n^{k_n}x\right) = \bar{f}\left(U_1^{k_1}\dots U_d^{k_d}\bar{U}_{d+1}^{\bar{k}_{d+1}}\dots \bar{U}_n^{\bar{k}_n}x\right) \tag{4.8}$$

where

$$\begin{aligned}\bar{f}(x) &= f\left(U_{d+1}^{r_{d+1}}\dots U_n^{r_n}x\right),\\ \bar{U}_j &= U_j^{l_j}, \quad d < j \le n.\end{aligned}$$

From (4.1) it follows that

$$\bar{U}_k = U_1^{a_{1,k}}\circ\dots\circ U_d^{a_{d,k}}, \quad d < k \le n, \tag{4.9}$$

Denote by $\alpha(\mathbf{s},\mathbf{r})$ the number of integer vectors $\mathbf{k} = (k_1,\dots,k_n)$, satisfying (4.6) and (4.7). According to (4.8) and (4.9) we can say that

$$\frac{Q^{\mathbf{r}}_{s_1,\dots,s_n} f(x)}{\alpha(\mathbf{s},\mathbf{r})} \tag{4.10}$$

are certain ergodic averages, obeying the case of $l_k = 1$ in (4.1). Thus we conclude that the averages (4.10) satisfy the weak estimate (1.4) for all vectors $\mathbf{r}$.

On the other hand, taking into account $\alpha(\mathbf{s},\mathbf{r}) \le s_1 \dots s_n$, we have

$$\begin{aligned}\frac{1}{s_1 \dots s_n} \sum_{k_1=0}^{s_1-1} \dots \sum_{k_n=0}^{s_n-1} \left| f\left((U_1^{k_1} \circ \dots \circ U_n^{k_n})(x)\right)\right| &= \frac{1}{s_1 \dots s_n} \sum_{\mathbf{r}} Q^{\mathbf{r}}_{s_1,\dots,s_n} f(x)\\ &= \sum_{\mathbf{r}} \frac{\alpha(\mathbf{s},\mathbf{r})}{s_1 \dots s_n} \frac{Q^{\mathbf{r}}_{s_1,\dots,s_n} f(x)}{\alpha(\mathbf{s},\mathbf{r})}\\ &\le \sum_{\mathbf{r}} \frac{Q^{\mathbf{r}}_{s_1,\dots,s_n} f(x)}{\alpha(\mathbf{s},\mathbf{r})}.\end{aligned}$$

Thus, since the averages (4.10) satisfy the weak estimate (1.4) and the number of different vectors $\mathbf{r} = l_{d+1} \dots l_n$ is a constant depending on $\mathfrak{U}$ only, we obtain (1.4) in full generality. The theorem is proved. □

Proof of Theorem 1.5. According to Theorem B the averages (1.2) converge a.e. for any function from $L\log^{n-1} L$ and so for any $f \in L^\infty(X)$. To prove convergence for any $f \in L\log^{d-1} L(\mathbb{T})$, fix $\varepsilon > 0$ and choose a function $g \in L^\infty$ such that

$$\int_X \mathrm{Log}_{d-1}\left(\frac{|f-g|}{\varepsilon}\right) < \varepsilon.$$

Applying (1.4), for the averages

$$\mathrm{A}_{\mathbf{m}}(f) = \frac{1}{m_1 \dots m_n} \sum_{j_1=0}^{m_1-1} \dots \sum_{j_n=0}^{m_n-1} f\left(U_1^{j_1} \dots U_n^{j_n} x\right)$$

we obtain

$$\begin{aligned}&\mu\left\{x : \limsup_{\min n_j \to \infty} |\mathrm{A}_{\mathbf{n}}(f) - \mathrm{A}_{\mathbf{m}}(f)| > 2\varepsilon\right\}\\ &= \mu\left\{x : \limsup_{\min n_j \to \infty} |\mathrm{A}_{\mathbf{n}}(f-g) - \mathrm{A}_{\mathbf{m}}(f-g)| > 2\varepsilon\right\}\\ &\le \mu\left\{x : \sup_{\mathbf{n}} |\mathrm{A}_{\mathbf{n}}(f-g)| > \varepsilon\right\}\\ &\le C(\mathfrak{U}) \int_X \mathrm{Log}_{d-1}\left(\frac{|f-g|}{\varepsilon}\right) < C(\mathfrak{U})\varepsilon.\end{aligned}$$

This implies a.e. convergence of $A_{\mathbf{n}}(f)$, completing the proof of the theorem. □

5. Sharpness in Theorem 1.5 and an extension

Let us show that the class of functions $L\log^{d-1}L(X)$ in Theorem 1.5 is optimal. Suppose the rank of $\mathfrak{U}=\{U_k\,:\,k=1,2,\dots,n\}$ is d and $\{U_1,\dots,U_d\}$ is the corresponding independent subset $\mathfrak{U}$, which is moreover non-periodic. According to Theorem C for $\Phi(t)=o(t\log^{d-1}t)$ there exists a function $f\in L_\Phi(X)$ with a.e. diverging averages

$$\frac{1}{s_1\dots s_d}\sum_{j_1=0}^{s_1-1}\dots\sum_{j_d=0}^{s_d-1} f\left(U_1^{j_1}\dots U_d^{j_d}x\right). \tag{5.1}$$

It turns out that for the same function f we have a.e. divergence of the averages

$$\frac{1}{s_1\dots s_n}\sum_{j_1=0}^{s_1-1}\dots\sum_{j_n=0}^{s_n-1} f\left(U_1^{j_1}\dots U_n^{j_n}x\right). \tag{5.2}$$

This immediately follows from the following lemma.

Lemma 5.1. *Let $\mathfrak{U}=\{U_k\,:\,k=1,2,\dots,n\}$ be a set of measure-preserving transformations and $d\le n$. If averages (5.1) diverge unboundedly a.e, then extended averages (5.2) also diverge unboundedly a.e.*

Proof. Denote by $A_{\mathbf{s}}(f)$ and $\bar{A}_{\mathbf{s}}(f)$ the averages (5.1) and (5.2) respectively and consider the functions

$$M_p(f)=\max_{\mathbf{s}\in\mathbb{Z}_+^d,\,s_j\ge p}A_{\mathbf{s}}(f),\quad \bar{M}_p(f)=\max_{\mathbf{s}\in\mathbb{Z}_+^n,\,s_j\ge p}\bar{A}_{\mathbf{s}}(f).$$

The unbounded divergence of averages (5.1) implies $M_p(f)=\infty$ a.e. for any $p>0$. If $\mathbf{s}=(s_1,\dots,s_d)$ and $\bar{\mathbf{s}}=(s_1,\dots,s_d,\dots,s_n)$, then we have

$$\bar{A}_{\bar{\mathbf{s}}}(f)\ge\frac{A_{\mathbf{s}}(f)}{s_{d+1}\dots s_n},$$

and thus, for any $p>0$

$$\bar{M}_p(f)\ge\frac{1}{p^{n-d}}M_p(f)=\infty\text{ a.e..}$$

□

A set of real numbers

$$\Theta=\{\theta_1,\theta_2,\dots,\theta_n\} \tag{5.3}$$

is said to be *dependent* (with respect to the rational numbers) if there is a non-trivial collection of integers r_k, $k=1,2,\dots,n$, such that

$$r_1\theta_1+r_2\theta_2+\dots+r_n\theta_n=0\mod 1,$$

If there are no such integers, then we say that Θ is *independent*. The rank of a collection $\Theta=\{\theta_1,\theta_2,\dots,\theta_n\}$ will be called the largest integer d, for which there is an independent subset of cardinality d in $\mathfrak{U}$. Consider the probability space of Lebesgue measure on $\mathbb{T}=\mathbb{R}/\mathbb{Z}$ with modulo one addition. Applying Theorem

1.7 and the ergodicity of the rotation mapping $x \to x + \theta$ for an irrational θ, we obtain

Corollary 5.2. *If* (5.3) *is a sequence of rank d, then*

1) for any $f \in L\log^{d-1} L(\mathbb{T})$ *the limit below holds a.e.*

$$\lim_{\min\{s_k\}\to\infty} \frac{1}{s_1 \cdots s_n} \sum_{k_1=0}^{s_1-1} \cdots \sum_{k_n=0}^{s_n-1} f(x + k_1\theta_1 + \cdots + k_n\theta_n) = \int_{\mathbb{T}} f(x)dx, \tag{5.4}$$

2) for any increasing function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$, *satisfying* $\Phi(t) = o(t(\log t)^{d-1})$, *there exists a function* $f \in L_\Phi(\mathbb{T})$ *such that the averages in* (5.4) *are a.e. divergent as* $\min\{s_k\} \to \infty$.

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