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Regular ideals of locally-convex higher-rank graph algebras

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ABSTRACT. We give a vertex set description for basic, graded, regular ideals of locally-convex Kumjian-Pask Algebras. We also show that Condition (B) is preserved when taking the quotient by a basic, graded, regular ideal. We further show that when a locally-convex, row-finite k-graph satisfies Condition (B), all regular ideals are graded. We then show the same things hold for gauge-invariant, regular ideals in locally-convex k-graph C^*-algebras.

CONTENTS

1. Introduction

In this paper, we will study the regular ideals of higher-rank graph algebras. We study them first in the algebraic setting: Kumjian-Pask Algebras. We then look at them in the analytic setting of k-graph C^*-algebras.

Higher-rank graph C^*-algebras were first introduced in [6] in 2000. In Kumjian and Pask's work, they focus on row-finite k-graphs with no sources. They were able to show that the gauge-invariant uniqueness theorem could be generalized from graph C^*-algebras. Additionally, they were able to find conditions for simplicity. This gave evidence to support the hope that much of the theory from graph C^*-algebras might carry over. These findings were further generalized in [7], where Raeburn, Sims, and Yeend introduced the concept of local-convexity in this setting. The local-convexity condition allows for sources to appear in the

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graphs. They were able to prove a generalization of the Cuntz-Krieger Uniqueness Theorem and show that there is a lattice isomorphism from the saturated, hereditary sets of vertices to the gauge-invariant ideals of k-graph C^*-algebras.

Much like the creation of Leavitt path algebras as an algebraic analogue of graph C^*-algebras, Kumjian-Pask algebras were created as an algebraic analogue of k-graph C^*-algebras. They were first introduced in [1]. The authors were able to prove many algebraic analogues to theorems proved in the C^*-algebra setting, included proofs of the uniqueness theorems and the lattice isomorphism of saturated and hereditary sets of vertices to the basic, graded ideals. These works were again generalized to the locally-convex setting in [3].

In the recent work of [2] and [5], the regular ideals of row-finite, no-source graph C^*-algebras and Leavitt-Path algebras respectively were studied. In both of [2] and [5], a vertex description of the regular, gauge-invariant ideals and regular, graded ideals were found respectively. As directed graphs can be seen as 1-graphs, the current paper generalizes both into higher-rank and by allowing for sources. In [2] and [5], it was shown that Condition (L) (a graph satisfies Condition (L) if every cycle has an entry) is preserved when quotienting by regular ideals.

In this paper, we provide clear vertex set descriptions for the basic, graded, regular ideals of Kumjian-Pask algebras, in the form of Theorem 4.6, and the gauge-invariant regular ideals of k-graph C^*-algebras, Theorem 6.4. Condition (B) was introduced in [7] as a generalization of Condition (L) for locally-convex higher-rank-graphs. For this reason, it is a natural fit for replacing Condition (L) in theorems similar to those seen in [2] and [5]. We see in Theorem 4.8 and Theorem 6.5 that Condition (B) is preserved in the graph when quotienting by a regular ideal, which is not the case for arbitrary quotients. Condition (B) was introduced to extend the Cuntz-Krieger Uniqueness Theorem for k-graphs with sources [7]. We further show in Corollaries 4.12 and 6.9 that when a locally-convex, row-finite k-graph satisfies Condition (B) that all regular ideals of the Kumjian-Pask algebra are graded and all regular ideals of the k-graph C*-algebra are gauge-invariant respectively.

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2. Background: higher rank graphs

The information on k-graphs will pertain to both Kumjian-Pask algebras and k-graph C^*-algebras so we include it here in its own section. We use the definitions from [6], [8] and [3].

Definition 2.1. *Let $k \in \mathbb{N}\setminus\{0\}$. A k-graph is a pair (Λ, d) where Λ is a countable category and d is a functor from Λ to $\mathbb{N}^k$ which satisfies the factorization property: for all $\lambda \in Mor(\Lambda)$ and all $m, n \in \mathbb{N}^k$ such that $d(\lambda) = m+n$, there exists unique morphisms μ, ν in $Mor(\Lambda)$ such that $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$.*

We put a partial order on the elements of $\mathbb{N}^k$ in the following way: we say that $n \leq m$ if and only if $n_i \leq m_i$ for all i , where $m = (m_1, m_2, ..., m_k)$, and $n = (n_1, n_2, ..., n_k)$. We refer to elements of $Mor(\Lambda)$ as paths and write r and s for the domain and codomain maps respectively.

Our factorization property gives us that $d(\lambda) = 0$ if and only if $\lambda = id_v$ for some $v \in Obj(\Lambda)$. We often refer to the elements of $Obj(\Lambda)$ as vertices. Thus, we identify $Obj(\Lambda)$ with $\{\lambda \in Mor(\Lambda) : d(\lambda) = 0\}$, and write $\lambda \in \Lambda$ in place of $\lambda \in Mor(\Lambda)$. For any $\lambda \in \Lambda$ and $E \subset \Lambda$, we define

$$\lambda E := \{\lambda\mu : \mu \in E,\ r(\mu) = s(\lambda)\}$$

and

$$E\lambda := \{\mu\lambda : \mu \in E,\ s(\mu) = r(\lambda)\}.$$

By the factorization property, we know that for any $l \leq m \leq n \in \mathbb{N}^k$ if $d(\lambda) = n$ then there are unique elements denoted $\lambda(0, l),\ \lambda(l, m),\ \lambda(m, n)$ of Λ such that $d(\lambda(0, l)) = l,\ d(\lambda(l, m)) = m - l,\ d(\lambda(m, n)) = n - m$ and

$$\lambda = \lambda(0, l)\lambda(l, m)\lambda(m, n).$$

For $m \in \mathbb{N}^k$, we define $\Lambda^m = \{\lambda \in \Lambda : d(\lambda) = m\}$. Thus, $Obj(\Lambda) = \Lambda^0$. We say that a k-graph is row-finite if for each $v \in \Lambda^0$ and each $m \in \mathbb{N}^k$, the set $\Lambda^m(v)$ is finite. We say that $v \in \Lambda^0$ is a source if there exists $m \in \mathbb{N}^k$ such that $\Lambda^m(v) = \emptyset$.

As an example of a k-graph, we offer up a common one that is used in many descriptions in research and later in this paper. We will use the notation from [3].

Example 2.2. *For a fixed $m \in (\mathbb{N} \cup \{\infty\})^k$: we define*

$$\Omega_{k,m} := \{(p, q) \in \mathbb{N}^k \times \mathbb{N}^k : p \leq q \leq m\}.$$

The objects are $\Omega^0_{k,m} = \{p \in \mathbb{N}^k : p \leq m\}$, and range and source maps $r(p, q) = p$ and $s(p, q) = q$. The morphisms (p, q) and (r, s) are composable if and only if $q = r$. When they are composable, we have $(p, q)(q, s) = (p, s)$. The factorization property is fulfilled by $d : \Omega^{k,m} \mapsto \mathbb{N}^k$ defined by $d((p, q)) = q - p$. So, we have that the pair $(\Omega_{k,m}, d)$ is a k-graph.

We introduce the set $\Lambda^{\leq n}$ consisting of paths λ with $d(\lambda) \leq n$ which cannot be extended to paths $\lambda\mu$ with $d(\lambda) < d(\lambda\mu) \leq n$. Thus,

$$\Lambda^{\leq n} := \{\lambda \in \Lambda : d(\lambda) \leq n, \text{ and } d(\lambda)_i < n_i \text{ implies } s(\lambda)\Lambda^{e_i} = \emptyset\}.$$

We have that $v\Lambda^{\leq n} := v\Lambda \cap \Lambda^{\leq n}$ for $v \in \Lambda^0$ is always nonempty.

We will be interested in row-finite locally-convex k-graphs throughout the paper, which were introduced in [3] and [7]

Definition 2.3. *We say Λ is* locally-convex *if for every $v \in \Lambda^0, 1 \leq i, j \leq k$ with $i \neq j$, $\lambda \in v\Lambda^{e_i}$ and $\mu \in v\Lambda^{e_j}$, the sets $s(\lambda)\Lambda^{e_j}$ and $s(\mu)\Lambda^{e_i}$ are nonempty.*

It is worth noting that by [7, Remark 3.10] every row-finite 1-graph is locally-convex, as are all row-finite, higher-rank graphs with no sources.

As we hope to give a vertex description of the regular ideals, it is natural to introduce some vertex sets which will help us classify ideals in our respective k-graph algebras.

Definition 2.4. *Let (Λ, d) be a locally-convex row-finite k-graph. We say a subset H of Λ^0 is* hereditary *if $\lambda \in \Lambda$ and $r(\lambda) \in H$ imply $s(\lambda) \in H$. We say that H is* saturated *if for $v \in \Lambda^0$, $s(v\Lambda^{\leq e_i}) \subset H$ for some $i \in \{1, ..., k\}$ implies $v \in H$.*

As final definitions, we introduce boundary paths of locally-convex, row-finite k-graphs and Condition (B). Boundary paths are used to put certain conditions on our k-graphs including aperiodicity in [6] and Condition (C) of [8]. We use it here to define Condition (B). Aperiodicity was a condition that Kumjian and Pask used along with cofinality to find a simplicity condition of row-finte, no source k-graph C^*-algebras. It served as an analogue to Condition (L) of directed graphs. When switching to row-finite locally-convex k-graph C^*-algebras, Raeburn, Sims, and Yeend made Condition (B) as an analogue of aperiodicity to allow for sources. With this, they were able to prove the Cuntz-Krieger Uniqueness Theorem. By Theorem 8.4 of [3], when a row-finite k-graph has no sources, Condition (B) is equivalent to the aperiodicity condition.

Definition 2.5. *Let Λ be a locally convex k-graph. A* boundary path in Λ *is a graph morphism $x : \Omega_{k,m} \mapsto \Lambda$ for some $m \in (\mathbb{N} \cup \infty)^k$ such that, whenever $v \in Obj(\Omega_{k.m})$ satisfies $v(\Omega_{k.m})^{\leq e_i} = \{v\}$, we also have that $x(v)\Lambda^{\leq e_i} = \{x(v)\}$. We denote the collection of all boundary paths in Λ by $\Lambda^{\leq\infty}$. The range map of Λ extends naturally to $\Lambda^{\leq\infty}$ via $r(x) := x(0)$. For $v \in \Lambda^0$, we write $v\Lambda^{\leq\infty}$ for $\{x \in \Lambda^{\leq\infty} : r(x) = v\}$.*

If Λ has no sources, then $\Lambda^{\leq\infty} = \Lambda^{\infty}$. It should also be noted that a boundary path can be composed with finite paths. That is, if x is a boundary path and $\lambda \in \Lambda$ is a finite path with $s(\lambda) = r(x)$ then we define $\lambda x : \Omega_{k,m+d(\lambda)} \mapsto \Lambda$ such that

$$\lambda x(d(\lambda)) = r(x),\ \lambda x(0) = r(\lambda),$$
$$\lambda x(l, l + e_i) = \lambda(l, l + e_i) \text{ if } l + e_i \leq d(\lambda),$$
$$\lambda x(d(\lambda) + l, d(\lambda) + l + e_i) = x(l + l + e_i) \text{ if } l + e_i \leq m.$$

The rest of the graph morphism can be obtained by concatenating paths of length e_i. We have that λx is also a boundary path. For more information, the reader can refer to [3] and [7].

Definition 2.6. *We say that a vertex v in a k-graph, Λ, satisfies Condition (B) if;*

$$\text{there exists } x \in v\Lambda^{\leq\infty} \text{ such that } \alpha \neq \beta \in \Lambda \text{ implies } \alpha x \neq \beta x$$

We say that a k-graph, Λ, satisfies condition (B) if every vertex in Λ satisfies Condition (B).

3. Background: Kumjian-Pask algebras

As Kumjian-Pask algebras are an algebraic generalization of k-graph C^*-algebras, we introduce ghost paths to take the place of adjoints. We will again be using the definitions and ideas from [3].

Definition 3.1. *Define $G(\Lambda) := \{\lambda^* : \lambda \in \Lambda\}$, and call each λ^* a* ghost path. *If $v \in \Lambda^0$, then we identify v and v^*. We extend the degree functor d and the range and source maps r and s to $G(\Lambda)$ by $d(\lambda^*) = -d(\lambda)$, $r(\lambda^*) = s(\lambda)$ and $s(\lambda^*) = r(\lambda)$. We extend the factorization property to the ghost paths by setting $(\mu\lambda)^* = \lambda^*\mu^*$. We denote by $\Lambda^{\neq 0}$ the set of paths which are not vertices and by $G(\Lambda^{\neq 0})$ the set of ghost paths that are not vertices.*

Definition 3.2. *Let Λ be a row-finite k-graph and let R be a commutative ring with 1. A Kumjian-Pask Λ-family (P, S) in an R-algebra A consists of two functions $P : \Lambda^0 \mapsto A$ and $S : \Lambda^{\neq 0} \cup G(\Lambda^{\neq 0}) \mapsto A$ such that:*

(KP1) *$\{P_v : v \in \Lambda^0\}$ is a family of mutually orthogonal idempotents;*
(KP2) *for all $\lambda, \mu \in \Lambda^{\neq 0}$ with $r(\mu) = s(\lambda)$, we have $S_\lambda S_\mu = S_{\lambda\mu}$, $S_{\mu^*} S_{\lambda^*} = S_{(\lambda\mu)^*}$, $P_{r(\lambda)} S_\lambda = S_\lambda = S_\lambda P_{s(\lambda)}$, and $P_{s(\lambda)} S_{\lambda^*} = S_{\lambda^*} = S_{\lambda^*} P_{r(\lambda)}$;*
(KP3) *for all $n \in \mathbb{N}^k \setminus \{0\}$ and $\lambda, \mu \in \Lambda^{\leq n}$, we have $S_{\lambda^*} S_\mu = \delta_{\lambda,\mu} P_{s(\lambda)}$;*
(KP4) *for all $v \in \Lambda^0$ and all $n \in \mathbb{N}^k \setminus \{0\}$, we have $P_v = \sum_{\lambda \in v\Lambda^{\leq n}} S_\lambda S_{\lambda^*}$.*

Theorem 3.3. [3, Proposition 3.3]. *Let Λ be a locally-convex, row-finite k-graph, (P, S) a Kumjian–Pask Λ-family in an R-algebra A, and $\lambda, \mu \in \Lambda$. If $n \in \mathbb{N}^k$ such that $d(\lambda)$, $d(\mu) \leq n$, then $S_{\lambda^*} S_\mu = \sum_{\lambda\alpha = \mu\beta, \lambda\alpha \in \Lambda^{\leq n}} S_\alpha S_{\beta^*}$.*

Definition 3.4. *We define $KP_R(\Lambda)$ to be the universal R-algebra generated by a Kumjian–Pask Λ-family (p, s), in the sense that if (Q, T) is a Kumjian–Pask Λ-family in an R-algebra A, then there exists a R-algebra homomorphism $\pi_{Q,T} : KP_R(\Lambda) \mapsto A$ such that $\pi_{Q,T} \circ p = Q$ and $\pi_{Q,T} \circ s = T$. For every $r \in R \setminus \{0\}$ and $v \in \Lambda^0$, we have $r p_v \neq 0$.*

Grading plays an important role in the lattice of ideals of Kumjian-Pask algebras. We take time here to define gradings, as it will pertain to later theorems.

Definition 3.5. *Let G be an additive abelian group. A ring A is G-*graded *if there are additive subgroups $\{A_g : g \in G\}$ of A such that $A_g A_h \subset A_{g+h}$ and every nonzero $a \in A$ can be written in exactly one way as a finite sum $\sum_{g \in F} a_g$ of nonzero elements $a_g \in A_g$. The elements of A_g are* homogeneous of degree g, *and $a = \sum_{g \in F} a_g$ is the* homogeneous decomposition of a.

Suppose that A is G-graded by $\{A_g : g \in G\}$. An ideal I in A is a *graded ideal* if $\{I \cap A_g : g \in G\}$ is a grading of I. Every ideal I which is generated by a set S of homogeneous elements is graded. The following theorem allows us to put a $\mathbb{Z}^k$ grading on $KP_R(\Lambda)$ as well as giving us a full description of the algebra.

Theorem 3.6. [3, Theorem 3.7]. *Let Λ be a locally-convex, row-finite k-graph.*

(i) *There is a unique R-algebra $KP_R(\Lambda)$, generated by a Kumjian–Pask Λ-family (p,s), such that if (Q,T) is a Kumjian–Pask Λ-family in an R-algebra A, then there exists a unique R-algebra homomorphism $\pi_{Q,T} : KP_R(\Lambda) \mapsto A$ such that $\pi_{Q,T}\circ p = Q$ and $\pi_{Q,T}\circ s = T$. For every $r \in R \setminus \{0\}$ and $v \in \Lambda^0$, we have $rp_v \neq 0$.*

(ii) *The subsets $KP_R(\Lambda)_n := span\{s_\alpha s_{\beta^*} : d(\alpha) - d(\beta) = n\}$ form a $\mathbb{Z}^k$-grading of $KP_R(\Lambda)$.*

By putting together Theorems 3.3 and 3.6, it can be seen that $KP_R(\Lambda) = span_R\{s_\lambda s_{\mu^*} : s(\lambda) = s(\mu)\}$.

Definition 3.7. *Let R be a ring. Let Λ be a k-graph. Let I be an ideal of the Kumjian-Pask Algebra $KP_R(\Lambda)$. We say that I is* basic *if it has the property such that if $rp_v \in I$ and $r \in R \setminus \{0\}$ then $p_v \in I$.*

Remark 3.8. *We note that if R is a field, all ideals of $KP_R(\Lambda)$ are basic. Indeed, let I be an ideal. Suppose that $rp_v \in I$ and $r \neq 0$. Then*

$$rp_v r^{-1} p_v = rr^{-1} p_v p_v = p_v \in I.$$

Definition 3.9. *For a subset H of Λ^0, define*

$$I(H) := span_R\{s_\alpha s_{\beta^*} : s(\alpha) = s(\beta) \in H\}.$$

$I(H)$ will be an ideal if H is saturated and hereditary.

In order to have a clear picture of the lattice isomorphism introduced in Definition 3.12, we give notations that will be concise to use throughout.

Definition 3.10. *For an ideal I in $KP_R(\Lambda)$, we define*
$H(I) := \{v \in \Lambda^0 : p_v \in I\}$

Lemma 3.11. [3, Lemma 9.2]. *Let H be a hereditary, saturated subset of Λ^0, and $I(H)$ be the ideal of $KP_R(\Lambda)$ generated by $\{p_v : v \in H\}$. Then*

$$I(H) = span\{s_\alpha s_{\beta^*} : \alpha, \beta \in \Lambda, s(\alpha) = s(\beta) \in H\}.$$

Theorem 3.12. [3, Theorem 9.4] *Let Λ be a row-finite, locally-convex k-graph. Let R be a commutative ring with 1. Then the map $H \to I(H)$ is a lattice isomorphism from the set of saturated hereditary subsets of Λ^0 onto the lattice of basic graded ideals of $KP_R(\Lambda)$.*

Remark 3.13. *The inverse of the lattice isomorphism described in the above theorem is the map $I \mapsto H(I)$.*

We now show that there is an isomorphism between the quotient algebra created by quotienting by a basic, graded ideal and the Kumjian-Pask algebra of the quotient graph. For I an ideal, we define $\Lambda \setminus H(I)$ to be the small category with objects $\Lambda^0 \setminus H(I)$, and morphisms $\{\lambda \in \Lambda : r(\lambda)$ and $s(\lambda) \in \Lambda^0 \setminus H(I)\}$, with the factorization property d inherited from (Λ, d).

Proposition 3.14. *Let Λ be a locally-convex row-finite k-graph and R a commutative ring with 1. Let I be a basic graded ideal of $KP_R(\Lambda)$, and let (q,t) and*

(p, m) be the universal Kumjian-Pask families in $KP_R(\Lambda \setminus H(I))$ and $KP_R(\Lambda)$, respectively. Then there exists an isomorphism $\pi : KP_R(\Lambda \setminus H(I)) \mapsto KP_R(\Lambda)/I$ such that

$$\pi(q_v) = p_v + I, \pi(t_\lambda) = m_\lambda + I, \text{ and } \pi(t_{\mu^*}) = m_{\mu^*} + I$$

for $v \in \Lambda^0/H(I)$ and $\lambda, \mu \in s^{-1}(\Lambda^0/H(I))$.

Proof. First note that $\Lambda\setminus H(I)$ is indeed a locally-convex k-graph. This is shown in the proof of [7, Theorem 5.2].

Now we show that $\{p_v+I, m_\lambda+I, m_{\mu^*+I}\}$ is a Kumjian-Pask $(\Lambda\setminus H(I))$ family. (KP 1) and (KP 2) hold as (p, m) is a Kumjian-Pask Λ family. To see (KP 3) and (KP 4), we show that $(\Lambda \setminus H(I))^{\leq n} = \Lambda^{\leq n} \setminus \{\lambda \in \Lambda : m_\lambda \in I\}$. Take $n \in \mathbb{N}^k$. Here we note that since I is basic and graded, it is generated by the set of idempotents of a hereditary and saturated set of vertices [3, Theorem 9.4]. Thus, $\{\lambda \in \Lambda : m_\lambda \in I\} = \{\lambda \in \Lambda : s(\lambda) \in H(I)\}$. To see that

$$(\Lambda \setminus H(I))^{\leq n} \subseteq \Lambda^{\leq n} \setminus \{\lambda \in \Lambda : s(\lambda) \in H(I)\},$$

take $\lambda \in (\Lambda \setminus H(I))^{\leq n}$. Note that $s(\lambda) \notin H(I)$, so we need only show that $\lambda \in \Lambda^{\leq n}$. Suppose that $d(\lambda) = n$. Then $\lambda \in \Lambda^{\leq n}$.

Now suppose $d(\lambda) < n$. For notation, assume that $s(\lambda) = v$. Then for every i such that $d(\lambda)_i < n_i$, $v(\Lambda \setminus H(I))^{e_i} = \emptyset$. Thus, we must show that for such an i, $v\Lambda^{e_i} = \emptyset$. To obtain a contradiction, suppose there exists an i so that $v\Lambda^{e_i} \neq \emptyset$. Then we have that $s(v\Lambda^{e_i}) \subseteq H(I)$ is nonempty. Thus, by definition of $v(\Lambda)^{\leq n}$, we get $s(v\Lambda^{e_i}) = s(v\Lambda^{\leq e_i})$. Which, since $H(I)$ is saturated, gives $v \in H(I)$ a contradiction. So it must be that

$$(\Lambda \setminus H(I))^{\leq n} \subseteq \Lambda^{\leq n} \setminus \{\lambda \in \Lambda : s(\Lambda) \in H(I)\}.$$

To see the other direction we simply note that by definition $(\Lambda\setminus H(I))^n \subseteq \Lambda^n$. So the inclusion is clear when $d(\lambda) = n$. When $d(\lambda) < n$, we note that since $(\Lambda \setminus H(I))^{e_i} \subseteq \Lambda^{e_i}$, it must be that $\lambda \in (\Lambda \setminus H(I))^{\leq n}$. Now it is simple to show that $\{p_v + I, m_\lambda + I, m_{\mu^*} + I\}$ satisfy (KP 3) and (KP 4).

Thus, by [3, Theorem 3.7], we know there is a homomorphism $\pi_{p+I,m+I}$ satisfying the equations stated. Since other generators of $KP_R(\Lambda)$ belong to I, the family $(p + I, m + I)$ generates $KP_R(\Lambda)/I$ and π is surjective. Suppose that $\pi(rq_v) = 0$ for some $r \in R \setminus \{0\}$ and $v \notin H(I)$. Then $rp_v + I = \pi(rq_v) = 0$, so that $rp_v \in I$ and since I is basic, $p_v \in I$ as well. And this implies that $v \in H(I)$, a contradiction. Thus, $\pi(rq_v) \neq 0$ for all $r \in R\setminus\{0\}$. Since I is graded, then $KP_R(\Lambda)/I$ is graded by $(KP_R(\Lambda)/I)_n = q(KP_R(\Lambda)_n)$, where $q : KP_R(\Lambda) \mapsto KP_R(\Lambda)/I$ is the quotient map. If $\alpha, \beta \in (\Lambda \setminus H(I))$ with $d(\alpha) - d(\beta) = n \in \mathbb{Z}^k$, then $\pi(t_\alpha t_{\beta^*}) = s_\alpha s_{\beta^*} + I = q(s_\alpha s_{\beta^*}) \in q(KP_R(\Lambda)_n) = (KP_R(\Lambda)/I)_n$. Thus, π is graded and thus, by the graded uniqueness theorem, [3, Theorem 4.1], π is injective. ∎

Our last two theorems will be helpful in classifying when a basic regular ideal must be graded.

Theorem 3.15. *Let Λ be a row-finite, locally-convex k-graph. Let J be a basic ideal in $KP_R(\Lambda)$, then $I(H(J))$ is the largest basic, graded ideal contained in J.*

Proof. As J is an ideal and $I(H(J))$ is the ideal generated by $\{p_v : p_v \in J\}$ it is clear that $I(H(J)) \subseteq J$. Further, as J is an ideal, $H(J)$ is hereditary and saturated by $(KP2-4)$. Thus, $I(H(J))$ is basic and graded by Theorem 3.12. It remains to show that $I(H(J))$ is the largest. For this, we note that by Theorem 3.12 all basic, graded ideals are generated by the vertex idempotents of a saturated and hereditary set of vertices. As $I(H(J))$ contains all such idempotents in J, there can be no larger basic, graded ideal in J. ∎

Theorem 3.16. *Let Λ be a row-finite, locally-convex k-graph. Let J be a basic ideal in $KP_R(\Lambda)$, if $\Lambda \setminus H(J)$ satisfies Condition (B) then J is basic and graded.*

Proof. By Proposition 3.14 we can identify $KP_R(\Lambda)/I(H(J))$ with $KP_R(\Lambda \setminus H(J))$. We have that $I(H(J)) \subseteq J$. Let N be the image of J under the quotient map $KP_R(\Lambda) \mapsto KP_R(\Lambda)/I(H(J)) \cong KP_R(\Lambda \setminus H(J))$. Then

$$KP_R(\Lambda)/J \cong KP_R(\Lambda \setminus H(J))/N.$$

Consider our quotient mapping $q : KP_R(\Lambda \setminus H(J)) \mapsto KP_R(\Lambda \setminus H(J))/N$. Note that $H(N) \subseteq (\Lambda \setminus H(J))^0$ is empty. Thus, $q(p_v) \neq 0$ for all $v \in \Lambda \setminus H(J)^0$. Since $\Lambda \setminus H(J)$ satisfies Condition (B), we have by [3, Theorem 4.2] that the quotient map is injective. Hence, N is trivial and $J = I(H(J))$. ∎

4. Regular ideals of Kumjian-Pask algebras

We start off this section by giving a series of observations which will help us to our goal of giving a vertex description of the basic, regular, graded ideals of $KP_R(\Lambda)$.

Definition 4.1. *An ideal J in an algebra A is called regular if $J^{\perp\perp} = J$ where $J^\perp = \{a \in A : ax = xa = 0 \ \forall\, x \in J\}$.*

We note that if J is an ideal then $J^\perp$ is a regular ideal. The proof of the following lemma is largely the same as Lemma 3.2 of [5] but we include it here for completeness.

Lemma 4.2. *Let Λ be a row-finite locally-convex k-graph. Let R be a commutative ring with 1. If J is a graded ideal of $KP_R(\Lambda)$ then $J^\perp$ is a graded ideal.*

Proof. Let $z = \sum_{n\in\mathbb{Z}^k} r_n z_n \in J^\perp$. We need to prove that each $r_n z_n \in J^\perp$. Let $x \in J$. Since J is graded, then $x = \sum_{n\in\mathbb{Z}^k} t_n x_n$, where each $t_n x_n$ is homogeneous and $t_n x_n \in J$. So it is enough to check that $r_n z_n t_i x_i = t_i x_i r_n z_n = 0$ for each i, n. Since $z \in J^\perp$, and $t_i x_i \in J$ for each fixed $i \in \mathbb{Z}^k$, we have that $\sum_{n\in\mathbb{Z}^k} t_i x_i r_n z_n = t_i x_i z = 0 = z t_i x_i = \sum_{n\in\mathbb{Z}^k} r_n z_n t_i x_i$, and hence the grading (and the fact that $t_i x_i$ is homogeneous) implies that $r_n z_n t_i x_i = t_i x_i r_n z_n = 0$ as desired. ∎

Lemma 4.3. *Let Λ be a row-finite, locally-convex k-graph. Let R be a commutative ring with 1. If J is a basic graded ideal of $KP_R(\Lambda)$ then $J^\perp$ is a basic graded ideal.*

Proof. We have that $J^\perp$ is graded by Lemma 4.2. It remains to show that it is basic. As J is a graded basic ideal it must be that $J = I(H)$ for some saturated hereditary set H by Theorem 3.12. Suppose that $rp_v \in H(J^\perp)$ then we have that $rp_v s_\alpha s_{\beta^*} = 0$ and $s_\alpha s_{\beta^*} rp_v = rs_\alpha s_{\beta^*} p_v = 0$ for all $s_\alpha s_{\beta^*} \in I(H)$. But this is true if and only if $v \neq r(\alpha)$ and $v \neq r(\beta)$ for all $s_\alpha s_{\beta^*} \in I(H)$. Thus, we also get that $p_v s_\alpha s_{\beta^*} = s_\alpha s_{\beta^*} p_v = 0$ for all $s_\alpha s_{\beta^*} \in I(H)$. Thus, $p_v \in J^\perp$. ∎

The following notation will be useful for the remainder of the paper as they reoccur.

Definition 4.4. (i) *For $w \in \Lambda^0$, put $T(w) = \{s(\lambda) : \lambda \in \Lambda, r(\lambda) = w\}$.*
(ii) *If $I \subseteq KP_R(\Lambda)$ an ideal, let $\overline{H}(I) \subseteq \Lambda^0$ be the set*

$$\overline{H}(I) = \{r(\lambda) : \lambda \in \Lambda \text{ and } s(\lambda) \in H(I)\}.$$

We are now ready to describe the vertex set of $J^\perp$ and in turn give a vertex description of $J^\perp$.

Lemma 4.5. *Let Λ be a row finite, locally-convex k-graph. Let H be a hereditary and saturated subset of Λ^0. Let J be the ideal generated by H. Then*

$$H(J^\perp) = \{v \in \Lambda^0 : v\Lambda H = \emptyset\} = \Lambda^0 \setminus \bar{H}(J).$$

Proof. Since we define $H(J^\perp)$ to be $\{v \in \Lambda^0 : p_v \in J^\perp\}$,

$$H(J^\perp) = \{v \in \Lambda : p_v s_\alpha s_\beta^* = s_\alpha s_\beta^* p_v = 0 \text{ for all } s_\alpha s_\beta^* \in J\}$$

So $v \in H(J^\perp)$ if and only if for all $s_\alpha s_\beta^* \in J$:

(i) $p_v s_\alpha s_\beta^* = 0$; and
(ii) $s_\alpha s_\beta^* p_v = 0$.

For (i) to be true, we require that $v \neq r(\alpha)$. So, we require that $v \neq r(\alpha)$ for all α with $s_\alpha s_\beta^* \in J$. So, for $v \in H(J^\perp)$ there can be no path from a vertex in H to v (as $s_\alpha = s_\alpha p_{s(\alpha)} \in J$).
For (ii) to be true we require that $v \neq r(\beta)$. So, we require that $v \neq r(\beta)$ for all β with $s_\alpha s_\beta^* \in J$. So for $v \in H(J^\perp)$ there can be no path from a vertex in H to v.

So, we have the following description:

$$H(J^\perp) = \{v \in \Lambda^0 : \text{ there is no path from a vertex in } H \text{ to } v\}$$

as desired. ∎

In [5], a vertex set description was given for the regular ideals of row-finite, no source Leavitt Path Algebras. As Leavitt Path Algebras are isomorphic to Kumjian-Pask Algebras generated by a 1-graph, the following result generalizes [5] both in moving to higher-rank and by allowing for sources.

Theorem 4.6. *Let Λ be a locally-convex, row-finite k-graph. Let R be a commutative ring with 1. Let $J \subseteq KP_R(\Lambda)$ be a basic graded ideal. Then:*

(i) $J^\perp = I(\Lambda^0 \setminus \bar{H}(J))$;
(ii) $J^{\perp\perp} = I(\{w \in \Lambda^0 : T(w) \subseteq \bar{H}(J)\})$; *and*
(iii) *J is regular if and only if $H(J) = \{w \in \Lambda^0 : T(w) \subseteq \bar{H}(J)\}$*

Proof. We know that $J = I(H)$ for some saturated and hereditary set H by Theorem 3.12. Thus, from Lemma 4.5, we know $H(J^\perp) = \Lambda^0 \setminus \bar{H}(J)$. We also know that $J^\perp$ is basic and graded since J is basic, and graded by Lemma 4.3. So it must be generated by $\{p_v : v \in H(J^\perp)\}$. This proves (i). The rest follows. ∎

Corollary 4.7. *Let Λ be a locally-convex, row-finite k-graph. Let R be a field. Let $J \subseteq KP_R(\Lambda)$ be a graded ideal. Then:*

(i) $J^\perp = I(\Lambda^0 \setminus \bar{H}(J))$;
(ii) $J^{\perp\perp} = I(\{w \in \Lambda^0 : T(w) \subseteq \bar{H}(J)\})$; *and*
(iii) *J is regular if and only if $H(J) = \{w \in \Lambda^0 : T(w) \subseteq \bar{H}(J)\}$*

Proof. All ideals are basic as R is a field. The rest follows from above. ∎

We now show that quotienting by a basic, graded, regular ideal of a Kumjian-Pask Algebra preserves Condition (B). It was shown in [5] that Condition (L) is preserved when quotienting by a basic, graded, regular ideal of a Leavitt Path Algebra.

Theorem 4.8. *Suppose that Λ is a row-finite, locally-convex k-graph which satisfies Condition (B). If J is a regular, basic, graded ideal, then $\Lambda \setminus J$ satisfies Condition (B).*

Proof. First note that using Lemma 4.5 and replacing J with $J^\perp$, and since J is regular, basic and graded, we have that $(\Lambda/J)^0 = \Lambda^0 \setminus H(J) = \overline{H}(J^\perp)$. For a vertex $v \in H(J^\perp)$ we know there exists an $x \in v\Lambda^{\leq\infty}$ such that if $\alpha \neq \beta$ then $\alpha x \neq \beta x$. As $H(J^\perp)$ is saturated and hereditary (since $J^\perp$ is an ideal) and $r(x) = v$ we know that $x(i,i) \in H(J^\perp)$ for all i. Thus, since $J^\perp$ is an ideal we conclude that $x \in (\Lambda/J)^{\leq\infty}$. Hence all vertices in $H(J^\perp)$ satisfy Condition (B). For a vertex w in $\overline{H}(J^\perp)$ we know there exists a finite path γ with $r(\gamma) = w$ and $s(\gamma) \in H(J^\perp)$. Therefore $\gamma x_{s(\gamma)} \in (\Lambda/J)^{\leq\infty}$ satisfies Condition (B) at w. To see this note that if $\alpha \neq \beta$ then $\alpha\gamma \neq \beta\gamma$ and $x_{s(\gamma)}$ satisfies Condition (B) at $s(\gamma)$. ∎

We remind the reader that for a row-finite k-graph with no sources that Condition (B) is equivalent to the aperiodicity condition [3, Lemma 8.4]. The corollary follows immediately.

Corollary 4.9. *Let Λ be a row finite k-graph with no sources which is aperiodic. Let J be a basic, graded regular ideal of $KP_R(\Lambda)$ then Λ/J is aperiodic.*

We finish the section with some theorems that allow us to show sufficient conditions for when a basic regular ideal must be graded.

Lemma 4.10. *Let Λ be a locally-convex, row-finite k-graph. Let J be a basic, regular ideal of $KP_R(\Lambda)$. Then $I(H(J)) \subseteq J$ is a regular basic, graded ideal.*

Proof. We know that $I(H(J)) \subseteq J$ and that $I(H(J)) \subseteq I(H(J))^{\perp\perp} \subseteq J^{\perp\perp} = J$. As $I(H(J))$ is basic and graded, by Lemma 4.3 we have $I(H(J))^{\perp}$ and $I(H(J))^{\perp\perp}$ are basic and graded. By Theorem 3.15, $I(H(J))$ is the largest gauge-invariant ideal in J. Thus, $I(H(J)) = I(H(J))^{\perp\perp}$. ∎

Proposition 4.11. *If Λ is a locally-convex, row-finite k-graph satisfying Condition (B), and J is a basic, regular ideal in $KP_R(\Lambda)$, then J is graded.*

Proof. As J is regular, we have that $I(H(J))$ is regular by Lemma 4.10. Thus, by Theorem 4.8, $\Lambda \setminus H(J)$ satisfies Condition (B). It follows that J is graded by Theorem 3.16. ∎

Putting together Theorem 4.3, Proposition 4.11 and Theorem 4.8, we get the following corollary.

Corollary 4.12. *Let Λ be a locally-convex, row-finite k-graph satisfying Condition (B). Let J be a basic, regular ideal in $KP_R(\Lambda)$. Then $\Lambda \setminus J$ satisfies Condition (B) and $KP_R(\Lambda/J) \cong KP_R(\Lambda \setminus H(J))$.*

5. Background: k-graph C^*-algebras

In the following section we will be giving background information and theorems to help us establish similar classification to the regular ideals in k-graph C^*-algebras. We begin by defining the Cuntz-Kreiger Λ family for a C^*-algebra.

Definition 5.1. *Let Λ be a row-finite k-graph. A* Cuntz–Krieger Λ-family *in a C^*-algebra B consists of a family of partial isometries $\{s_\lambda : \lambda \in \Lambda\}$ satisfying the Cuntz–Krieger relations:*

(KP1) *$\{s_v : v \in \Lambda^0\}$ is a family of mutually orthogonal projections;*
(KP2) *$s_{\lambda\mu} = s_\lambda s_\mu$ for all $\lambda, \mu \in \Lambda$ with $s(\lambda) = r(\mu)$;*
(KP3) *$s_\lambda^* s_\lambda = s_{s(\lambda)}$;*
(KP4) *$s_v = \sum_{\lambda \in \Lambda^{\leq m(v)}} s_\lambda s_\lambda^*$ for all $v \in \Lambda^0$ and $m \in \mathbb{N}^k$*

Theorem 5.2. [7, Theorem 3.15.] *Let (Λ, d) be a row-finite k-graph. Then there is a Cuntz–Krieger Λ-family $\{s_\lambda : \lambda \in \Lambda\}$ with each s_λ non-zero if and only if Λ is locally-convex.*

Given a row-finite k-graph (Λ, d), there is a C^*-algebra $C^*(\Lambda)$ generated by a universal Cuntz–Krieger Λ-family $\{s_\lambda : \lambda \in \Lambda\}$ [7]. We call this algebra the k-graph C^*-algebra for Λ and denote it $C^*(\Lambda)$.

Theorem 5.3. [7, Proposition 3.5]*Let (Λ, d) be a row-finite k-graph and let $\{s_\lambda : \lambda \in \Lambda\}$ be a Cuntz–Krieger Λ-family. Then for $\lambda, \mu \in \Lambda$ and $q \in \mathbb{N}^k$ with $d(\lambda), d(\mu) \leq q$ we have*

$$s_\lambda^* s_\mu = \sum_{\lambda\alpha = \mu\beta, \lambda\alpha \in \Lambda^{\leq q}} s_\alpha s_{\beta^*}$$

Hence, Theorem 5.3 gives us that $C^*(\Lambda) = \overline{span}\{s_\alpha s_{\beta^*} : s(\beta) = s(\alpha)\}$.

Similar to the graph C^*-algebra, the universality of $C^*(\Lambda)$ gives us an action of $\mathbb{T}^k$ on $C^*(\Lambda)$ known as the gauge action.

Definition 5.4. *Let (Λ, d) be a row-finite k-graph. For $z \in \mathbb{T}^k$ and $n \in \mathbb{Z}^k$, let $z^n := z_1^{n_1} \dots z_k^{n_k}$. Then $\{z^{d(\lambda)} s_\lambda : \lambda \in \Lambda\}$ is a Cuntz–Krieger Λ-family which generates $C^*(\Lambda)$, and the universal property of $C^*(\Lambda)$ gives a homomorphism $\gamma_z : C^*(\Lambda) \mapsto C^*(\Lambda)$ such that $\gamma_z(s_\lambda) = z^{d(\lambda)} s_\lambda$ for $\lambda \in \Lambda$; $\gamma_{\overline{z}}$ is an inverse for γ_z, so γ_z is an automorphism. This action is strongly continuous and known as the* gauge action.

For an ideal I in $C^*(\Lambda)$ we denote $H(I) := \{v \in \Lambda^0 : p_v \in I\}$.

Theorem 5.5. [7, Theorem 5.2] *Let (Λ, d) be a locally-convex row-finite k-graph. For each subset H of Λ^0, let $I(H)$ be the closed ideal in $C^*(\Lambda)$ generated by $\{s_v : v \in H\}$.*

(i) *The map $H \mapsto I(H)$ is an isomorphism of the lattice of saturated hereditary subsets of Λ^0 onto the lattice of closed gauge-invariant ideals of $C^*(\Lambda)$.*
(ii) *Suppose H is saturated and hereditary. Then $\Lambda \setminus H$, the small category with objects $\Lambda^0 \setminus H$, and morphisms $\{\lambda \in \Lambda : r(\lambda)$ and $s(\lambda) \in \Lambda^0 \setminus H(I)\}$, with the factorization property d inherited from (Λ, d). is a locally-convex row-finite k-graph, and $C^*(\Lambda)/I(H)$ is canonically isomorphic to $C^*(\Lambda \setminus H)$.*
(iii) *If H is any hereditary subset of Λ^0, then $\Lambda(H)$, the small category with objects H and morphisms $\{\lambda \in \Lambda : r(\lambda) \in H\}$ and the factorization property d inherited from Λ, is a locally-convex row-finite k-graph, $C^*(\Lambda(H))$ is canonically isomorphic to the subalgebra $C^*(s_\lambda : r(\lambda) \in H)$ of $C^*(\Lambda)$, and this subalgebra is a full corner in $I(H)$.*

Remark 5.6. *The inverse of the lattice isomorphism in (i) is $I \mapsto H(I)$.*

Remark 5.7. *By putting together Theorems* 5.5 *and* 5.3 *we get the following for J an ideal, generated by a saturated and hereditary H of a locally-convex, row-finite k-graph.*

$$C^*(\Lambda) = \overline{span}\{s_\alpha s_\beta^* : s(\alpha) = s(\beta)\},$$

Hence,

$$J = \overline{span}\{p_v s_\alpha s_\beta^*, s_\alpha s_\beta^* p_v : v \in H,\}$$

$$= \overline{span}\{s_\alpha s_\beta^* : r(\beta) \in H \text{ or } r(\alpha) \in H\}.$$

Since we need that $s(\alpha) = s(\beta)$ for $s_\alpha s_\beta^* \neq 0$, we can conclude that for $s_\alpha s_\beta^* \in J$ if $r(\beta) \in H$ then $s(\alpha) \in H$ as H is hereditary. Similarly, we get if $r(\alpha) \in H$ then $s(\beta) \in H$. So we have:

$$J = \overline{span}\{s_\alpha s_\beta^* : r(\beta) \in H \text{ or } r(\alpha) \in H, \text{ and } s(\alpha) = s(\beta) \in H\}$$

We finish this section with two additional applications of Theorem 5.5.

Theorem 5.8. *Let Λ be a locally-convex k-graph. Let J be an ideal in $C^*(\Lambda)$, then $I(H(J))$ is the largest gauge-invariant ideal contained in J.*

Theorem 5.9. *Let Λ be a locally-convex row-finite k-graph. Let J be an ideal in $C^*(\Lambda)$ if $\Lambda \setminus H(J)$ satisfies Condition (B) then J is gauge invariant.*

The proofs follow the same reasoning as in the Kumjian-Pask algebra case.

6. Regular ideals of k-graph C^*-algebras

In this section, we give analagous proofs of those in Section 4 for the k-graph C^*-Algebras. As many of the proofs follow the same reasoning as the Kumjian-Pask Algebra case, we omit them here when logical. We refer the reader to Section 4 for the full details.

We intend to give a vertex description of the gauge-invariant regular ideals in $C^*(\Lambda)$. For this purpose, we refer the reader to Definition 4.1 for the definition of regular ideals and recall that if J is a regular ideal, then so is $J^\perp$. The next lemma shows that $J^\perp$ must be gauge-invariant, if J is. It is a good starting point for obtaining a vertex description of $J^\perp$ and, in turn, J.

Lemma 6.1. *Let J be a gauge-invariant ideal in a k-graph C^*-algebra $C^*(\Lambda)$. Then $J^\perp$ is a gauge-invariant regular ideal.*

Proof. For an ideal J we know that $J^\perp$ is always a regular ideal. It remains to show that it is gauge-invariant. Suppose that $a \in J^\perp$ then for any $z \in \mathbb{T}^k$ we have that:

$$\{\gamma_z(a)b : b \in J\} = \{\gamma_z(a)\gamma_z(\gamma_{\overline{z}}(b)) : b \in J\} = \{\gamma_z(a\gamma_{\overline{z}}(b)) : b \in J\} = \{0\}$$

as $\gamma_{\overline{z}}(b) \in J$ since J gauge-invariant. Similarly:

$$\{b\gamma_z(a) : b \in J\} = \{\gamma_z(\gamma_{\overline{z}}(b))\gamma_z(a) : b \in J\} = \{\gamma_z(\gamma_{\overline{z}}(b)a) : b \in J\} = \{0\}.$$

So we have $\gamma_z(a) \in J$.

■

We give now the C^*-algebra definitions that are analogues of the ones used in the regular ideal section for Kumjian-Pask algebras:

Definition 6.2. (i) *For $w \in \Lambda^0$, put $T(w) = \{s(\lambda) : \lambda \in \Lambda, r(\lambda) = w\}$.*

(ii) *If $I \subseteq C^*(\Lambda)$ an ideal, let $\overline{H}(I) \subseteq \Lambda^0$ be the set $\overline{H}(I) = \{r(\lambda) : \lambda \in \Lambda$ and $s(\lambda) \in H(I)\}$.*

Lemma 6.3. *Let Λ be a row finite, locally-convex k-graph. Let H be a hereditary and saturated subset of Λ^0. Let J be the ideal generated by H. Then*

$$H(J^\perp) = \{v \in \Lambda^0 : v\Lambda H = \emptyset\} = \Lambda^0 \setminus \bar{H}(J).$$

Proof. Since we define $H(J^\perp)$ to be $\{v \in \Lambda^0 : p_v \in J^\perp\}$,

$$H(J^\perp) = \{v \in C^*(\Lambda) : p_v s_\alpha s_\beta^* = s_\alpha s_\beta^* p_v = 0 \text{ for all } s_\alpha s_\beta^* \in J\}$$

The rest of the proof follows a similar reasoning to the Kumjian-Pask case. ■

Theorem 6.4. *Let Λ be a locally-convex, row-finite k-graph. Let $J \subseteq C^*(\Lambda)$ be a gauge-invariant ideal. Then:*

(i) $J^\perp = I(\Lambda^0 \setminus \bar{H}(J))$;
(ii) $J^{\perp\perp} = I(\{w \in \Lambda^0 : T(w) \subseteq \bar{H}(J)\})$; *and*
(iii) *J is regular if and only if* $H(J) = \{w \in \Lambda^0 : T(w) \subseteq \bar{H}(J)\}$.

Proof. From [7] [Theorem 5.2] we know that J must be generated by $\{p_v : v \in H\}$ for some saturated and hereditary set H. Thus, by Lemma 6.3, We know that $H(J^\perp) = \Lambda^0 \setminus \bar{H}(J)$. But from Lemma 6.1, we know that $J^\perp$ is gauge-invariant. So, using [7] [Th. 5.2] it must be that $J^\perp = I(\Lambda^0 \setminus \bar{H}(J))$. Proving i. The rest follow. ∎

Theorem 6.5. *Suppose that Λ is a row-finite, locally-convex k-graph which satisfies Condition (B). If J is a regular gauge-invariant ideal, then $\Lambda \setminus J$ satisfies Condition (B).*

The proof follows a similar reasoning as the Kumjian-Pask algebra case since the regular ideals of both have the same vertex description and satisfying Condition (B) is a property of the graph. We further remind the reader that for a row-finite k-graph without sources, satisfying Condition (B) is equivalent to being aperiodic [3, Lemma 8.4], so the corollary follows.

Corollary 6.6. *Let Λ be a row finite k-graph with no sources which is aperiodic. Let J be a gauge-invariant regular ideal of $C^*(\Lambda)$ then Λ/J is aperiodic.*

Lemma 6.7. *Let Λ be a locally-convex, row-finite k-graph. Let J be a regular ideal of $C^*(\Lambda)$. Then $I(H(J)) \subseteq J$ is a regular gauge-invariant ideal.*

The proof is again similar to the Kumjian-Pask algebra case.

Proposition 6.8. *If Λ is a locally-convex, row-finite k-graph satisfying Condition (B), and J is a regular ideal in $C^*(\Lambda)$, then J is gauge-invariant.*

Proof. As J is regular, we have that $I(H(J))$ is regular by lemma 6.7. Thus, by Theorem 6.5 $\Lambda \setminus H(J)$ satisfies Condition (B). It follows that J is gauge invariant by Theorem 5.9. ∎

Putting together Theorem 5.5, Proposition 6.8 and Theorem 6.5, we get the following Corollary.

Corollary 6.9. *Let Λ be a row-finite, locally-convex k-graph satisfying Condition (B). Let J be a regular ideal in $C^*(\Lambda)$. Then $\Lambda \setminus J$ satisfies Condition (B) and $C^*(\Lambda)/H(J) \cong C^*(\Lambda \setminus H(J))$.*

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