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Restriction of uniform crossnorms

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ABSTRACT. Consider the space $\mathcal{B}[\mathcal{X} \otimes_\alpha \mathcal{Y}]$ of operators on the tensor product $\mathcal{X} \otimes_\alpha \mathcal{Y}$ of normed spaces $\mathcal{X}$ and $\mathcal{Y}$ equipped with a uniform crossnorm $\|\cdot\|_\alpha$. Take the induced uniform norm on $\mathcal{B}[\mathcal{X} \otimes_\alpha \mathcal{Y}]$ and consider its restriction to the tensor product $\mathcal{B}[\mathcal{X}] \otimes \mathcal{B}[\mathcal{Y}]$ of the algebra of operators $\mathcal{B}[\mathcal{X}]$ and $\mathcal{B}[\mathcal{Y}]$. It is proved that such a restriction is a reasonable crossnorm on $\mathcal{B}[\mathcal{X}] \otimes \mathcal{B}[\mathcal{Y}]$.

CONTENTS

1. Introduction

The paper deals with a special tensor norm on the tensor product of a pair of spaces of bounded linear transformations; in particular, of a pair of spaces of operators. We avoid the term "operator space" in this note, since the term has already been consecrated to define a theory of certain subspaces of the algebra of Hilbert-space operators $\mathcal{B}[\mathcal{H}]$, which can be thought of as object of a category in the realm of C^*-algebras (see, e.g., [2], [9, Definition 1.2], [1], [10, Definition 1.1]). On the contrary, our aim in this note is much less ambitious. Let $\mathcal{B}[\mathcal{X}]$ be the normed algebra of all operators on a normed space $\mathcal{X}$, and consider the tensor product $\mathcal{B}[\mathcal{X}] \otimes \mathcal{B}[\mathcal{Y}]$. It is shown that this is included in the normed algebra $\mathcal{B}[\mathcal{X} \otimes_\alpha \mathcal{Y}]$, where $\mathcal{X} \otimes_\alpha \mathcal{Y}$ stands for the tensor product $\mathcal{X} \otimes \mathcal{Y}$ of normed spaces $\mathcal{X}$ and $\mathcal{Y}$ equipped with a uniform crossnorm $\|\cdot\|_\alpha$. We give an elementary proof that the induced uniform operator norm on $\mathcal{B}[\mathcal{X} \otimes_\alpha \mathcal{Y}]$, when restricted to $\mathcal{B}[\mathcal{X}] \otimes \mathcal{B}[\mathcal{Y}]$, acts as a reasonable crossnorm on the tensor product $\mathcal{B}[\mathcal{X}] \otimes \mathcal{B}[\mathcal{Y}]$.

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All terms and notation used above will be defined here in due course. The paper is organized into four more sections. Basic propositions, including notation and terminology, are summarized in Section 2. Supplementary results on uniform crossnorms required in the sequel are considered in Section 3. The main theorem is proved in Section 4. A discussion on equivalent uniform crossnorms, in light of the outcome of Section 4, closes the paper in Section 5.

2. Auxiliary results

All linear spaces in this paper are over the same scalar field $\mathbb{F}$, which is either $\mathbb{R}$ or $\mathbb{C}$. The algebraic *tensor product* of linear spaces $\mathcal{X}$ and $\mathcal{Y}$ is a linear space $\mathcal{X}\otimes\mathcal{Y}$ for which there is a bilinear map $\theta : \mathcal{X}\times\mathcal{Y}\to\mathcal{X}\otimes\mathcal{Y}$ (called the *natural bilinear map* associated with $\mathcal{X}\otimes\mathcal{Y}$) whose range spans $\mathcal{X}\otimes\mathcal{Y}$ with the following additional (universal) property: for every bilinear map $\phi : \mathcal{X}\times\mathcal{Y}\to\mathcal{Z}$ into any linear space $\mathcal{Z}$ there exists a (unique) linear transformation $\Phi : \mathcal{X}\otimes\mathcal{Y}\to\mathcal{Z}$ for which the diagram

$$\begin{array}{ccc} \mathcal{X}\times\mathcal{Y} & \xrightarrow{\phi} & \mathcal{Z} \\ & {\scriptstyle\theta}\searrow & \uparrow{\scriptstyle\Phi} \\ & & \mathcal{X}\otimes\mathcal{Y} \end{array}$$

commutes. A tensor product space $\mathcal{X}\otimes\mathcal{Y}$ exists for every pair of linear spaces $(\mathcal{X},\mathcal{Y})$ and is unique up to isomorphisms. Set $x\otimes y=\theta(x,y)\in\mathcal{X}\otimes\mathcal{Y}$ for each $(x,y)\in\mathcal{X}\times\mathcal{Y}$. These are the *single tensors*. An arbitrary element $\digamma$ in the linear space $\mathcal{X}\otimes\mathcal{Y}$ is a finite sum $\sum_i x_i\otimes y_i$ of single tensors, and the representation of $\digamma=\sum_i x_i\otimes y_i$ as a finite sum of single tensors is not unique. (For an exposition on algebraic tensor products see, e.g., [7].) If $\mathcal{X}$ and $\mathcal{Y}$ are linear spaces, then $\mathcal{L}[\mathcal{X},\mathcal{Y}]$ denotes the linear space of all linear transformations of $\mathcal{X}$ into $\mathcal{Y}$. Let $\mathcal{X},\mathcal{Y},\mathcal{V},\mathcal{W}$ be linear spaces and consider the tensor product spaces $\mathcal{X}\otimes\mathcal{Y}$ and $\mathcal{V}\otimes\mathcal{W}$. Take a pair of linear transformations $A\in\mathcal{L}[\mathcal{X},\mathcal{V}]$ and $B\in\mathcal{L}[\mathcal{Y},\mathcal{W}]$ and set

$$(A\otimes B)\sum_i x_i\otimes y_i=\sum_i Ax_i\otimes By_i$$

in $\mathcal{V}\otimes\mathcal{W}$ for every $\digamma=\sum_i x_i\otimes y_i$ in $\mathcal{X}\otimes\mathcal{Y}$. This defines a linear transformation $A\otimes B\in\mathcal{L}[\mathcal{X}\otimes\mathcal{Y},\mathcal{V}\otimes\mathcal{W}]$ of the linear space $\mathcal{X}\otimes\mathcal{Y}$ into the linear space $\mathcal{V}\otimes\mathcal{W}$, referred to as the *tensor product of the linear transformations* A and B, which is such that $(A\otimes B)(\digamma)$ does not depend on the representation $\sum_i x_i\otimes y_i$ of $\digamma\in\mathcal{X}\otimes\mathcal{Y}$ (see, e.g., [7, Proposition 3.6]). Consider the linear spaces $\mathcal{L}[\mathcal{X},\mathcal{V}]$ and $\mathcal{L}[\mathcal{Y},\mathcal{W}]$ and let $\mathbb{L}$ be an arbitrary element of $\mathcal{L}[\mathcal{X},\mathcal{V}]\otimes\mathcal{L}[\mathcal{Y},\mathcal{W}]$ so that $\mathbb{L}=\sum_j A_j\otimes B_j$ is a finite sum of single tensors $A_j\otimes B_j\in\mathcal{L}[\mathcal{X},\mathcal{V}]\otimes\mathcal{L}[\mathcal{Y},\mathcal{W}]$, and therefore

$$\mathcal{L}[\mathcal{X},\mathcal{V}]\otimes\mathcal{L}[\mathcal{Y},\mathcal{W}]\subseteq\mathcal{L}[\mathcal{X}\otimes\mathcal{Y},\mathcal{V}\otimes\mathcal{W}].$$

From now on suppose $\mathcal{X}$ and $\mathcal{Y}$ are normed spaces. Let $\mathcal{B}[\mathcal{X},\mathcal{Y}]$ be the normed space of all bounded linear transformations of $\mathcal{X}$ into $\mathcal{Y}$ equipped with its standard induced uniform norm, and let $\mathcal{X}^* = \mathcal{B}[\mathcal{X},\mathbb{F}]$ denote the dual of $\mathcal{X}$. Let $x \otimes y$ and $f \otimes g$ be single tensors in the tensor product spaces $\mathcal{X} \otimes \mathcal{Y}$ and $\mathcal{X}^* \otimes \mathcal{Y}^*$. A norm $\|\cdot\|_\alpha$ on $\mathcal{X} \otimes \mathcal{Y}$ is a *reasonable crossnorm* if for every $x \in \mathcal{X}$, $y \in \mathcal{Y}$, $f \in \mathcal{X}^*$, $g \in \mathcal{Y}^*$,

(a) $\|x \otimes y\|_\alpha \le \|x\|\,\|y\|$,

(b) $f \otimes g$ lies in $(\mathcal{X} \otimes \mathcal{Y})^*$, and $\|f \otimes g\|_{*\alpha} \le \|f\|\,\|g\|$ (where $\|\cdot\|_{*\alpha}$ is the norm on the dual $(\mathcal{X} \otimes \mathcal{Y})^*$ when $\mathcal{X} \otimes \mathcal{Y}$ is equipped with the norm $\|\cdot\|_\alpha$), so that

$$\mathcal{X}^* \otimes \mathcal{Y}^* \subseteq (\mathcal{X} \otimes \mathcal{Y})^*.$$

Actually, $\|x \otimes y\|_\alpha = \|x\|\,\|y\|$ and $\|f \otimes g\|_{*\alpha} = \|f\|\,\|g\|$ whenever $\|\cdot\|_\alpha$ is a reasonable crossnorm (see, e.g., [4, Proposition 1.1.1]). Two special reasonable crossnorms on $\mathcal{X} \otimes \mathcal{Y}$ are the *injective* $\|\cdot\|_\vee$ and *projective* $\|\cdot\|_\wedge$ norms,

$$\begin{aligned}\|\digamma\|_\vee &= \sup_{\|f\|\le 1,\, \|g\|\le 1,\, f\in\mathcal{X}^*,\, g\in\mathcal{Y}^*} \Big|\sum\nolimits_i f(x_i)\, g(y_i)\Big|,\\ \|\digamma\|_\wedge &= \inf_{\{x_i\}_i,\{y_i\}_i,\, \digamma=\sum_i x_i\otimes y_i} \sum\nolimits_i \|x_i\|\,\|y_i\|,\end{aligned}$$

for every $\digamma = \sum_i x_i \otimes y_i$. Here the infimum is taken over all representations of $\digamma \in \mathcal{X} \otimes \mathcal{Y}$.

Proposition 2.1. *A norm $\|\cdot\|_\alpha$ on $\mathcal{X} \otimes \mathcal{Y}$ is a reasonable crossnorm if and only if*

$$\|\digamma\|_\vee \le \|\digamma\|_\alpha \le \|\digamma\|_\wedge \quad \textit{for every} \quad \digamma \in \mathcal{X} \otimes \mathcal{Y}.$$

Proof. See, e.g., [11, Proposition 6.1]. □

Let $\mathcal{X} \otimes_\alpha \mathcal{Y} = (\mathcal{X} \otimes \mathcal{Y}, \|\cdot\|_\alpha)$ be the tensor product space of normed spaces equipped with a norm $\|\cdot\|_\alpha$ on $\mathcal{X} \otimes \mathcal{Y}$, which is not necessarily complete even if $\|\cdot\|_\alpha$ is a reasonable crossnorm and $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces. Their completion is denoted by $\mathcal{X} \widehat{\otimes}_\alpha \mathcal{Y}$ (same notation $\|\cdot\|_\alpha$ for the extended norm on $\mathcal{X} \widehat{\otimes}_\alpha \mathcal{Y}$). In particular, $\mathcal{X} \widehat{\otimes}_\vee \mathcal{Y}$ and $\mathcal{X} \widehat{\otimes}_\wedge \mathcal{Y}$ are referred to as the *injective and projective tensor products*. For the theory of Banach space $\mathcal{X} \widehat{\otimes}_\alpha \mathcal{Y}$ (including $\mathcal{X} \widehat{\otimes}_\vee \mathcal{Y}$ and $\mathcal{X} \widehat{\otimes}_\wedge \mathcal{Y}$) see, e.g., [5, Chapters 15 and 16], [3, Section 12], [11, Section 6.1], [4, Sections 1.1 and 1.2].

Recall that

$$\mathcal{X}^* \otimes \mathcal{Y}^* \subseteq (\mathcal{X} \otimes_\alpha \mathcal{Y})^*.$$

When restricted to $\mathcal{X}^* \otimes \mathcal{Y}^*$ the norm $\|\cdot\|_{*\alpha}$ on $(\mathcal{X} \otimes_\alpha \mathcal{Y})^*$ is a reasonable crossnorm (with respect to $(\mathcal{X}^* \otimes \mathcal{Y}^*)^*$).

Proposition 2.2. *Let $\|\cdot\|_\alpha$ be a reasonable crossnorm on a tensor product space $\mathcal{X} \otimes \mathcal{Y}$ of normed spaces $\mathcal{X}$ and $\mathcal{Y}$, and take the dual $(\mathcal{X} \otimes_\alpha \mathcal{Y})^*$ of $\mathcal{X} \otimes_\alpha \mathcal{Y} = (\mathcal{X} \otimes \mathcal{Y}, \|\cdot\|_\alpha)$. When restricted to $\mathcal{X}^* \otimes \mathcal{Y}^*$ the norm $\|\cdot\|_{*\alpha}$ on $(\mathcal{X} \otimes_\alpha \mathcal{Y})^*$ is a reasonable crossnorm on $\mathcal{X}^* \otimes \mathcal{Y}^*$.*

Proof. See, e.g., [4, Proposition 1.1.2]. □

The purpose of this paper is to extend the above (nontrivial) result to the case where $\mathcal{X}^* = \mathcal{B}[\mathcal{X},\mathbb{F}]$, $\mathcal{Y}^* = \mathcal{B}[\mathcal{Y},\mathbb{F}]$, $\mathcal{X}^*\otimes\mathcal{Y}^* = \mathcal{B}[\mathcal{X},\mathbb{F}]\otimes\mathcal{B}[\mathcal{Y},\mathbb{F}]$ and $(\mathcal{X}\otimes_\alpha\mathcal{Y})^* = \mathcal{B}[\mathcal{X}\otimes_\alpha\mathcal{Y},\mathbb{F}]$ are replaced by (extended to) $\mathcal{B}[\mathcal{X},\mathcal{V}]$, $\mathcal{B}[\mathcal{Y},\mathcal{W}]$, $\mathcal{B}[\mathcal{X},\mathcal{V}]\otimes\mathcal{B}[\mathcal{Y},\mathcal{W}]$, and $\mathcal{B}[\mathcal{X}\otimes_\alpha\mathcal{Y},\mathcal{V}\otimes_\alpha\mathcal{W}]$, respectively, for arbitrary normed spaces $\mathcal{X},\mathcal{Y},\mathcal{V},\mathcal{W}$. This will be done in Section 4 (Theorem 4.1).

3. Uniform crossnorms

If $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are normed spaces and if $T\in\mathcal{B}[\mathcal{X},\mathcal{Y}]$ and $S\in\mathcal{B}[\mathcal{Y},\mathcal{Z}]$, then $ST\in\mathcal{B}[\mathcal{X},\mathcal{Z}]$ and $\|ST\|\le\|S\|\,\|T\|$. This is a crucial property shared by the induced uniform norm of bounded linear transformations, referred to as the operator norm property (see, e.g.,[6, Proposition 4.16]). Its counterpart for the case of tensor products (rather than ordinary products) yields the notion of uniform crossnorm.

A *uniform crossnorm* $\|\cdot\|_\alpha$ is a reasonable crossnorm on every tensor product space (of arbitrary normed spaces $\mathcal{X},\mathcal{Y},\mathcal{V},\mathcal{W}$) such that for any bounded linear transformations $A\in\mathcal{B}[\mathcal{X},\mathcal{V}]$ and $B\in\mathcal{B}[\mathcal{Y},\mathcal{W}]$ the linear tensor product transformation $A\otimes B:\mathcal{X}\otimes_\alpha\mathcal{Y}\to\mathcal{V}\otimes_\alpha\mathcal{W}$ is bounded, i.e.,

$$A\otimes B\in\mathcal{B}[\mathcal{X}\otimes_\alpha\mathcal{Y},\mathcal{V}\otimes_\alpha\mathcal{W}],$$

and

$$\|A\otimes B\|\le\|A\|\,\|B\|,\quad\text{equivalently,}\quad\|A\otimes B\|=\|A\|\,\|B\|.$$

In fact, the above equivalence holds since

$$\begin{aligned}\|A\|\,\|B\| &= \sup_{\|x\|\le1}\|Ax\|\sup_{\|y\|\le1}\|By\|\le\sup_{\|x\|\|y\|\le1}\|Ax\|\,\|By\|=\sup_{\|x\otimes y\|_\alpha\le1}\|Ax\otimes By\|_\alpha\\ &=\sup_{\|x\otimes y\|_\alpha\le1}\|(A\otimes B)(x\otimes y)\|_\alpha\le\sup_{\|\mathcal{F}\|_\alpha\le1}\|(A\otimes B)\mathcal{F}\|_\alpha=\|A\otimes B\|,\end{aligned}$$

with $\|A\otimes B\|$, $\|A\|$, and $\|B\|$ standing for the induced uniform norms of $A\otimes B$ in $\mathcal{B}[\mathcal{X}\otimes_\alpha\mathcal{Y},\ \mathcal{V}\otimes_\alpha\mathcal{W}]$, A in $\mathcal{B}[\mathcal{X},\mathcal{V}]$, and B in $\mathcal{B}[\mathcal{Y},\mathcal{W}]$.

The projective $\|\cdot\|_\wedge$ and injective $\|\cdot\|_\vee$ norms are uniform crossnorms (see, e.g., [11, Propositions 2.3 and 3.2]).

Let $\|\cdot\|_{[\alpha,\alpha]}$ be the induced uniform norm on the normed space of bounded linear transformations $\mathcal{B}[\mathcal{X}\otimes_\alpha\mathcal{Y},\mathcal{V}\otimes_\alpha\mathcal{W}]$ (i.e., $\|\cdot\|_{[\alpha,\alpha]}$ stands for the induced uniform norm $\|\cdot\|$ on $\mathcal{B}[\mathcal{X}\otimes\mathcal{Y},\mathcal{V}\otimes\mathcal{W}]$ when $\mathcal{X}\otimes\mathcal{V}$ and $\mathcal{Y}\otimes\mathcal{W}$ are equipped with a uniform crossnorm $\|\cdot\|_\alpha$). The notation $\|\cdot\|_{[\alpha,\alpha]}$ highlights the dependence of $\|\cdot\|$ on the uniform crossnorm $\|\cdot\|_\alpha$ which equips both $\mathcal{X}\otimes\mathcal{Y}$ and $\mathcal{V}\otimes\mathcal{W}$. For instance,

$$\|A\otimes B\|_{[\alpha,\alpha]}=\sup_{\mathcal{F}\in\mathcal{X}\otimes\mathcal{Y},\ \|\mathcal{F}\|_\alpha\le1}\|(A\otimes B)\mathcal{F}\|_\alpha=\|A\|\,\|B\|.$$

Proposition 3.1. *If $\|\cdot\|_\alpha$ is a uniform crossnorm, then $\mathcal{B}[\mathcal{X},\mathcal{V}]\otimes\mathcal{B}[\mathcal{Y},\mathcal{W}]\subseteq\mathcal{B}[\mathcal{X}\otimes_\alpha\mathcal{Y},\ \mathcal{V}\otimes_\alpha\mathcal{W}]$ so that*

$$\mathcal{B}[\mathcal{X},\mathcal{V}]\otimes_{[\alpha,\alpha]}\mathcal{B}[\mathcal{Y},\mathcal{W}]\subseteq\mathcal{B}[\mathcal{X}\otimes_\alpha\mathcal{Y},\ \mathcal{V}\otimes_\alpha\mathcal{W}].$$

Proof. An arbitrary element $\mathbb{L}$ in the tensor product space $\mathcal{B}[\mathcal{X},\mathcal{V}]\otimes\mathcal{B}[\mathcal{Y},\mathcal{W}]$ is represented by a finite sum $\mathbb{L}=\sum_j A_j\otimes B_j$ of single tensors with

$$A_j\in\mathcal{B}[\mathcal{X},\mathcal{V}],\quad B_j\in\mathcal{B}[\mathcal{Y},\mathcal{W}].$$

If $\|\cdot\|_\alpha$ is a uniform crossnorm, then each linear transformation $A_j\otimes B_j$ lies in $\mathcal{B}[\mathcal{X}\otimes_\alpha\mathcal{Y},\mathcal{V}\otimes_\alpha\mathcal{W}]$ and so does $\mathbb{L}=\sum_j A_j\otimes B_j$. Therefore

$$\mathcal{B}[\mathcal{X},\mathcal{V}]\otimes\mathcal{B}[\mathcal{Y},\mathcal{W}]\subseteq\mathcal{B}[\mathcal{X}\otimes_\alpha\mathcal{Y},\ \mathcal{V}\otimes_\alpha\mathcal{W}].$$

Now, to finish the proof, equip $\mathcal{B}[\mathcal{X},\mathcal{V}]\otimes\mathcal{B}[\mathcal{Y},\mathcal{W}]$ with the norm $\|\cdot\|_{[\alpha,\alpha]}$ from $\mathcal{B}[\mathcal{X}\otimes_\alpha\mathcal{Y},\ \mathcal{V}\otimes_\alpha\mathcal{W}]$ and set

$$\mathcal{B}[\mathcal{X},\mathcal{V}]\otimes_{[\alpha,\alpha]}\mathcal{B}[\mathcal{Y},\mathcal{W}]=(\mathcal{B}[\mathcal{X},\mathcal{V}]\otimes\mathcal{B}[\mathcal{Y},\mathcal{W}],\ \|\cdot\|_{[\alpha,\alpha]}).\quad\square$$

Let $\|\cdot\|_{*[\alpha,\alpha]}$ be the induced uniform norm on $\mathcal{B}\big[\mathcal{B}[\mathcal{X},\mathcal{V}]\otimes_{[\alpha,\alpha]}\mathcal{B}[\mathcal{Y},\mathcal{W}],\mathbb{F}\big]$.

Proposition 3.2. *If $\|\cdot\|_\alpha$ is a uniform crossnorm, then $\mathcal{B}[\mathcal{X},\mathcal{V}]^*\otimes\mathcal{B}[\mathcal{Y},\mathcal{W}]^*\subseteq\mathcal{L}\big[\mathcal{B}[\mathcal{X},\mathcal{Y}]\otimes_{[\alpha,\alpha]}\mathcal{B}[\mathcal{V},\mathcal{W}],\mathbb{F}\big]$. Moreover,*

$$\|\varphi\otimes\eta\|_{*[\alpha,\alpha]}\le\|\varphi\|\,\|\eta\|$$

for every $\varphi\in\mathcal{B}[\mathcal{X},\mathcal{V}]^*$ and $\eta\in\mathcal{B}[\mathcal{Y},\mathcal{W}]^*$. Consequently,

$$\mathcal{B}[\mathcal{X},\mathcal{V}]^*\otimes_{*[\alpha,\alpha]}\mathcal{B}[\mathcal{Y},\mathcal{W}]^*\subseteq(\mathcal{B}[\mathcal{X},\mathcal{Y}]\otimes_{[\alpha,\alpha]}\mathcal{B}[\mathcal{V},\mathcal{W}])^*.$$

Proof. Since $\mathbb{F}\otimes\mathbb{F}\cong\mathbb{F}$ (here $\cong$ stands for algebraic isomorphism), we get

$$\begin{aligned}\mathcal{B}[\mathcal{X},\mathcal{V}]^*\otimes\mathcal{B}[\mathcal{Y},\mathcal{W}]^*&=\mathcal{B}\big[\mathcal{B}[\mathcal{X},\mathcal{V}],\mathbb{F}]\big]\otimes\mathcal{B}\big[\mathcal{B}[\mathcal{Y},\mathcal{W}],\mathbb{F}]\big]\\&\subseteq\mathcal{L}\big[\mathcal{B}[\mathcal{X},\mathcal{V}],\mathbb{F}]\big]\otimes\mathcal{L}\big[\mathcal{B}[\mathcal{Y},\mathcal{W}],\mathbb{F}]\big]\\&\subseteq\mathcal{L}\big[\mathcal{B}[\mathcal{X},\mathcal{Y}]\otimes\mathcal{B}[\mathcal{V},\mathcal{W}],\mathbb{F}\otimes\mathbb{F}\big]\\&=\mathcal{L}\big[\mathcal{B}[\mathcal{X},\mathcal{Y}]\otimes_{[\alpha,\alpha]}\mathcal{B}[\mathcal{V},\mathcal{W}],\mathbb{F}\big],\end{aligned}$$

when $\mathcal{B}[\mathcal{X},\mathcal{Y}]\otimes\mathcal{B}[\mathcal{V},\mathcal{W}]$ is equipped with the uniform induced norm $\|\cdot\|_{\alpha,\alpha]}$ on $\mathcal{B}[\mathcal{X}\otimes_\alpha\mathcal{Y},\ \mathcal{V}\otimes_\alpha\mathcal{W}]$ by Proposition 3.1. Take arbitrary $\varphi\in\mathcal{B}[\mathcal{X},\mathcal{V}]^*$ and $\eta\in\mathcal{B}[\mathcal{Y},\mathcal{W}]^*$. According to the above inclusion,

$$\varphi\otimes\eta\in\mathcal{L}\big[\mathcal{B}[\mathcal{X},\mathcal{Y}]\otimes_{[\alpha,\alpha]}\mathcal{B}[\mathcal{V},\mathcal{W}],\mathbb{F}\big].$$

Set

$$\|\varphi\otimes\eta\|_{*[\alpha,\alpha]}=\sup_{\mathbb{L}\in\mathcal{B}[\mathcal{X},\mathcal{Y}]\otimes\mathcal{B}[\mathcal{V},\mathcal{W}],\ \|\mathbb{L}\|_{[\alpha,\alpha]}\le1}|(\varphi\otimes\eta)\mathbb{L}|.$$

It was shown in [12, Theorem 3] that the above supremum is not only finite but bounded by $\|\varphi\|\,\|\eta\|$ for the particular case of $\mathcal{V}=\mathcal{X}$ and $\mathcal{W}=\mathcal{Y}$ when these are Banach spaces, whose extension for arbitrary normed spaces $\mathcal{X},\mathcal{Y},\mathcal{V},\mathcal{W}$ follows the same argument. Thus

$$\|\varphi\otimes\eta\|_{*[\alpha,\alpha]}\le\|\varphi\|\,\|\eta\|.$$

An arbitrary element $\mathbb{k}$ in

$$\mathcal{B}[\mathcal{X},\mathcal{V}]^*\otimes\mathcal{B}[\mathcal{Y},\mathcal{W}]^*\subseteq\mathcal{L}\big[\mathcal{B}[\mathcal{X},\mathcal{Y}]\otimes_{[\alpha,\alpha]}\mathcal{B}[\mathcal{V},\mathcal{W}],\mathbb{F}\big]$$

is represented by a sum $\mathbb{k} = \sum_k \varphi_k \otimes \eta_k$ of finitely many single tensors with $\varphi_k \in \mathcal{B}[\mathcal{X},\mathcal{V}]^*$ and $\eta_k \in \mathcal{B}[\mathcal{Y},\mathcal{W}]^*$. Take the induced uniform norm on

$$\mathcal{B}\big[\mathcal{B}[\mathcal{X},\mathcal{V}] \otimes_{[\alpha,\alpha]} \mathcal{B}[\mathcal{Y},\mathcal{W}], \mathbb{F}\big],$$

say, $\|\cdot\|_{*[\alpha,\alpha]}$. The above displayed inequality ensures that

$$\begin{aligned}\|\mathbb{k}\|_{*[\alpha,\alpha]} &= \sup_{\|\mathbb{L}\|_{[\alpha,\alpha]}\leq 1} \Big|\Big(\sum\nolimits_k \varphi_k \otimes \eta_k\Big)\mathbb{L}\Big| \\ &\leq \sup_{\|\mathbb{L}\|_{[\alpha,\alpha]}\leq 1} \sum\nolimits_k \|\varphi_k \otimes \eta_k\|_{*[\alpha,\alpha]} \|\mathbb{L}\|_{[\alpha,\alpha]} \leq \sum\nolimits_k \|\varphi_k\|\,\|\eta_k\|,\end{aligned}$$

which is finite as the sum is finite. Thus $\mathbb{k} \in \mathcal{B}\big[\mathcal{B}[\mathcal{X},\mathcal{Y}] \otimes_{[\alpha,\alpha]} \mathcal{B}[\mathcal{V},\mathcal{W}], \mathbb{F}\big]$, and hence

$$\begin{aligned}\mathcal{B}[\mathcal{X},\mathcal{V}]^* \otimes \mathcal{B}[\mathcal{Y},\mathcal{W}]^* &\subseteq \mathcal{B}\big[\mathcal{B}[\mathcal{X},\mathcal{Y}] \otimes_{[\alpha,\alpha]} \mathcal{B}[\mathcal{V},\mathcal{W}], \mathbb{F}\big] \\ &= (\mathcal{B}[\mathcal{X},\mathcal{Y}] \otimes_{[\alpha,\alpha]} \mathcal{B}[\mathcal{V},\mathcal{W}])^*.\end{aligned}$$

The proof is complete when we equip $\mathcal{B}[\mathcal{X},\mathcal{V}]^* \otimes \mathcal{B}[\mathcal{Y},\mathcal{W}]^*$ with the norm $\|\cdot\|_{*[\alpha,\alpha]}$ on $([\mathcal{B}[\mathcal{X},\mathcal{Y}] \otimes_{[\alpha,\alpha]} \mathcal{B}[\mathcal{V},\mathcal{W}])^*$ and set

$$\mathcal{B}[\mathcal{X},\mathcal{V}]^* \otimes_{*[\alpha,\alpha]} \mathcal{B}[\mathcal{Y},\mathcal{W}]^* = (\mathcal{B}[\mathcal{X},\mathcal{V}]^* \otimes \mathcal{B}[\mathcal{Y},\mathcal{W}]^*, \|\cdot\|_{*[\alpha,\alpha]}). \quad \square$$

4. Main result

Theorem 4.1 shows that $\|\cdot\|_{[\alpha,\alpha]}$ is a reasonable crossnorm on the space $\mathcal{B}[\mathcal{X},\mathcal{V}] \otimes \mathcal{B}[\mathcal{Y},\mathcal{W}]$, when inherited from $\mathcal{B}[\mathcal{X} \otimes_\alpha \mathcal{Y}, \mathcal{V} \otimes_\alpha \mathcal{W}]$, thus extending Proposition 2.2 from continuous linear functionals to arbitrary continuous linear transformations.

The proof of Theorem 4.1 is especially tailored to prompt the question that closes the paper, and also to support the statement of Corollary 4.4.

Theorem 4.1. *If $\|\cdot\|_\alpha$ is a uniform crossnorm, then*

(a) $\|\mathbb{L}\|_\vee \leq \|\mathbb{L}\|_{[\alpha,\alpha]} \leq \|\mathbb{L}\|_\wedge$ *for every $\mathbb{L} \in \mathcal{B}[\mathcal{X},\mathcal{V}] \otimes_{[\alpha,\alpha]} \mathcal{B}[\mathcal{Y},\mathcal{W}]$, where $\mathcal{X},\mathcal{Y},\mathcal{V},\mathcal{W}$ are arbitrary normed spaces and $\|\cdot\|_{[\alpha,\alpha]}$ is the associated induced uniform norm on $\mathcal{B}[\mathcal{X} \otimes_\alpha \mathcal{Y}, \mathcal{V} \otimes_\alpha \mathcal{W}]$. Hence*

(b) $\|\cdot\|_{[\alpha,\alpha]}$ *is a reasonable crossnorm on $\mathcal{B}[\mathcal{X},\mathcal{V}] \otimes \mathcal{B}[\mathcal{Y},\mathcal{W}]$.*

Proof. (a) Take an arbitrary element

$$\mathbb{L} \in \mathcal{B}[\mathcal{X},\mathcal{V}] \otimes \mathcal{B}[\mathcal{Y},\mathcal{W}] \subseteq \mathcal{B}[\mathcal{X} \otimes_\alpha \mathcal{Y}, \mathcal{V} \otimes_\alpha \mathcal{W}]$$

(according to Proposition 3.1) and let $\sum_j A_j \otimes B_j$ be any finite-sum representation of $\mathbb{L}$ in terms of single tensors in $\mathcal{B}[\mathcal{X},\mathcal{V}] \otimes \mathcal{B}[\mathcal{Y},\mathcal{W}]$. Regard $\mathbb{L}$ as a transformation in $\mathcal{B}[\mathcal{X} \otimes_\alpha \mathcal{Y}, \mathcal{V} \otimes_\alpha \mathcal{W}]$. For $\digamma$ in $\mathcal{X} \otimes_\alpha \mathcal{Y}$ set

$$\|\mathbb{L}(\digamma)\|_{\mathcal{V}\otimes_\alpha\mathcal{W}} = \|\mathbb{L}(\digamma)\|_\alpha, \quad \|\digamma\|_{\mathcal{X}\otimes_\alpha\mathcal{Y}} = \|\digamma\|_\alpha.$$

With $\|\cdot\|_{[\alpha,\alpha]}$ standing for the induced uniform norm on $\mathcal{B}[\mathcal{X}\otimes_\alpha \mathcal{Y}, \mathcal{V}\otimes_\alpha \mathcal{W}]$,

$$\begin{aligned}\|\mathbb{L}(\digamma)\|_\alpha &= \Big\|\Big(\sum\nolimits_j A_j\otimes B_j\Big)\digamma\Big\|_\alpha \le \sum\nolimits_j \|(A_j\otimes B_j)\digamma\|_\alpha \\ &\le \sum\nolimits_j \|A_j\otimes B_j\|_{[\alpha,\alpha]}\|\digamma\|_\alpha = \Big(\sum\nolimits_j \|A_j\|\,\|B_j\|\Big)\|\digamma\|_\alpha\end{aligned}$$

for every $\digamma\in\mathcal{X}\otimes\mathcal{Y}$. As $\|\mathbb{L}\|_{[\alpha,\alpha]} = \sup_{\|\digamma\|_\alpha\le 1}\|\mathbb{L}(\digamma)\|_\alpha$ we get

$$\|\mathbb{L}\|_{[\alpha,\alpha]} \le \sum\nolimits_j \|A_j\|\,\|B_j\|.$$

Since the above inequality holds for every representation $\sum_j A_j\otimes B_j$ of $\mathbb{L}$,

$$\|\mathbb{L}\|_{[\alpha,\alpha]} \le \inf_{\mathbb{L}=\sum_j A_j\otimes B_j} \sum\nolimits_j \|A_j\|\,\|B_j\| = \|\mathbb{L}\|_\wedge,$$

where $\|\cdot\|_\wedge$ is the projective norm on $\mathcal{B}[\mathcal{X},\mathcal{V}]\otimes\mathcal{B}[\mathcal{Y},\mathcal{W}]$. On the other hand, take an arbitrary $\mathbb{k}=\sum_k \varphi_k\otimes\eta_k$ in

$$\mathcal{B}[\mathcal{X},\mathcal{V}]^*\otimes\mathcal{B}[\mathcal{Y},\mathcal{W}]^* \subseteq (\mathcal{B}[\mathcal{X},\mathcal{Y}]\otimes_{[\alpha,\alpha]}\mathcal{B}[\mathcal{V},\mathcal{W}])^*$$

(according to Proposition 3.2). We will regard $\mathbb{k}$ as a functional in the dual space $(\mathcal{B}[\mathcal{X},\mathcal{Y}]\otimes_{[\alpha,\alpha]}\mathcal{B}[\mathcal{V},\mathcal{W}])^*$. With $\|\cdot\|_{*[\alpha,\alpha]}$ being the induced uniform norm in $\mathcal{B}\big[\mathcal{B}[\mathcal{X},\mathcal{Y}]\otimes_{[\alpha,\alpha]}\mathcal{B}[\mathcal{V},\mathcal{W}],\mathbb{F}\big]$,

$$\|\mathbb{k}\|_{*[\alpha,\alpha]} = \sup_{\|\mathbb{L}\|_{[\alpha,\alpha]}\le 1}|\mathbb{k}(\mathbb{L})|. \quad \text{Dually,} \quad \|\mathbb{L}\|_{[\alpha,\alpha]} = \sup_{\|\mathbb{k}\|_{*[\alpha,\alpha]}\le 1}|\mathbb{k}(\mathbb{L})|.$$

Since for every $\mathbb{k}=\sum_k \varphi_k\otimes\eta_k\in\mathcal{B}[\mathcal{X},\mathcal{V}]^*\otimes\mathcal{B}[\mathcal{Y},\mathcal{W}]^*$ the value of $\mathbb{k}(\mathbb{L})\in\mathbb{F}$ is

$$\mathbb{k}(\mathbb{L}) = \Big(\sum\nolimits_k \varphi_k\otimes\eta_k\Big)\sum\nolimits_j A_j\otimes B_j = \sum\nolimits_{k,j}\varphi_k(A_j)\otimes\eta_k(B_j) = \sum\nolimits_{k,j}\varphi_k(A_j)\,\eta_k(B_j)$$

for every $\mathbb{L}=\sum_j A_j\otimes B_j\in\mathcal{B}[\mathcal{X},\mathcal{Y}]\otimes\mathcal{B}[\mathcal{V},\mathcal{W}]$, we get

$$\begin{aligned}\|\mathbb{L}\|_{[\alpha,\alpha]} &= \sup_{\|\mathbb{k}\|_{*[\alpha,\alpha]}\le 1}|\mathbb{k}(\mathbb{L})| = \sup_{\|\sum_k\varphi_k\otimes\eta_k\|_{*[\alpha,\alpha]}\le 1}\Big|\sum\nolimits_{k,j}\varphi_k(A_j)\,\eta_k(B_j)\Big| \\ &\ge \sup_{\|\varphi\otimes\eta\|_{*[\alpha,\alpha]}\le 1}\Big|\sum\nolimits_j\varphi(A_j)\,\eta(B_j)\Big| \ge \sup_{\|\varphi\|\,\|\eta\|\le 1}\Big|\sum\nolimits_j\varphi(A_j)\,\eta(B_j)\Big|,\end{aligned}$$

because $\|\varphi\otimes\eta\|_{*[\alpha,\alpha]}\le\|\varphi\|\,\|\eta\|$ for every $\varphi\in\mathcal{B}[\mathcal{X},\mathcal{V}]^*$ and $\eta\in\mathcal{B}[\mathcal{Y},\mathcal{W}]^*$ according to Proposition 3.2. Hence

$$\|\mathbb{L}\|_{[\alpha,\alpha]} \ge \sup_{\|\varphi\|\,\|\eta\|\le 1}\Big|\sum\nolimits_j\varphi(A_j)\,\eta(B_j)\Big| \ge \sup_{\|\varphi\|\le 1,\,\|\eta\|\le 1}\Big|\sum\nolimits_j\varphi(A_j)\,\eta(B_j)\Big| = \|\mathbb{L}\|_\vee,$$

where $\|\cdot\|_\wedge$ is the injective norm on $\mathcal{B}[\mathcal{X},\mathcal{V}]\otimes\mathcal{B}[\mathcal{Y},\mathcal{W}]$. Consequently,

$$\|\mathbb{L}\|_\vee \le \|\mathbb{L}\|_{[\alpha,\alpha]} \le \|\mathbb{L}\|_\wedge$$

for every $\mathbb{L}\in\mathcal{B}[\mathcal{X},\mathcal{V}]\otimes_{[\alpha,\alpha]}\mathcal{B}[\mathcal{Y},\mathcal{W}]$.

(b) Thus, according to Proposition 2.1, the induced uniform norm $\|\cdot\|_{[\alpha,\alpha]}$ becomes a *reasonable crossnorm* on the tensor product space $\mathcal{B}[\mathcal{X},\mathcal{V}]\otimes\mathcal{B}[\mathcal{Y},\mathcal{W}]$. $\square$

Particular case. Set $\mathcal{V}=\mathcal{X}$, $\mathcal{W}=\mathcal{Y}$ and write $\mathcal{B}[\mathcal{X}]$, $\mathcal{B}[\mathcal{Y}]$, and $\mathcal{B}[\mathcal{X}\otimes_\alpha\mathcal{Y}]$ for $\mathcal{B}[\mathcal{X},\mathcal{X}]$, $\mathcal{B}[\mathcal{Y},\mathcal{Y}]$, and $\mathcal{B}[\mathcal{X}\otimes_\alpha\mathcal{Y},\mathcal{X}\otimes_\alpha\mathcal{Y}]$, respectively. Thus Theorem 4.1 yields the result stated in the Abstract on tensor products of algebra of operators.

It is worth noticing that a first application of the inequalities in part (a) of Theorem 4.1 is the assignment of a reasonable crossnorm to the tensor product space $\mathcal{B}[\mathcal{X},\mathcal{V}]\otimes\mathcal{B}[\mathcal{Y},\mathcal{W}]$ associated with a single uniform crossnorm as in part (b).

Remark 4.2. If $\|\cdot\|_\alpha$ is a uniform crossnorm, then (by definition)

$$\mathcal{B}[\mathcal{X},\mathcal{V}]\otimes\mathcal{B}[\mathcal{Y},\mathcal{W}]\subseteq\mathcal{B}[\mathcal{X}\otimes_\alpha\mathcal{Y},\mathcal{V}\otimes_\alpha\mathcal{W}].$$

As before, let $\|\cdot\|_{[\alpha,\alpha]}$ be the induced uniform norm on $\mathcal{B}[\mathcal{X}\otimes_\alpha\mathcal{Y},\mathcal{V}\otimes_\alpha\mathcal{W}]$. Take an arbitrary $\mathbb{L}=\sum_j A_j\otimes B_j\in\mathcal{B}[\mathcal{X},\mathcal{V}]\otimes\mathcal{B}[\mathcal{Y},\mathcal{W}]\subseteq\mathcal{L}[\mathcal{X}\otimes\mathcal{Y},\mathcal{V}\otimes\mathcal{W}]$. Since $\|\cdot\|_\vee\le\|\cdot\|_\alpha\le\|\cdot\|_\wedge$ (cf. Proposition 2.1),

$$\sup_{\mathcal{F}\neq0}\frac{\|\mathbb{L}(\mathcal{F})\|_\vee}{\|\mathcal{F}\|_\wedge}\le\sup_{\mathcal{F}\neq0}\frac{\|\mathbb{L}(\mathcal{F})\|_\alpha}{\|\mathcal{F}\|_\alpha}\le\sup_{\mathcal{F}\neq0}\frac{\|\mathbb{L}(\mathcal{F})\|_\wedge}{\|\mathcal{F}\|_\vee}. \qquad (*)$$

Thus set $\|\mathbb{L}\|_{[\wedge,\vee]}=\sup_{\mathcal{F}\neq0}\frac{\|\mathbb{L}(\mathcal{F})\|_\vee}{\|\mathcal{F}\|_\wedge}$ which is finite as $\|\mathbb{L}\|_{[\alpha,\alpha]}=\sup_{\mathcal{F}\neq0}\frac{\|\mathbb{L}(\mathcal{F})\|_\alpha}{\|\mathcal{F}\|_\alpha}$ is a norm, which is enough to ensure that

$$\|\cdot\|_{[\wedge,\vee]}\text{ is the induced uniform a norm on }\mathcal{B}[\mathcal{X}\otimes_\wedge\mathcal{Y},\mathcal{V}\otimes_\vee\mathcal{W}],$$

and that $\mathbb{L}\in\mathcal{B}[\mathcal{X}\otimes_\wedge\mathcal{Y},\mathcal{V}\otimes_\vee\mathcal{W}]$. (Note: $\sup_{\mathcal{F}\neq0}\frac{\|\mathbb{L}(\mathcal{F})\|_\wedge}{\|\mathcal{F}\|_\vee}$ may not be finite.) So

$$\mathcal{B}[\mathcal{X}\otimes_\wedge\mathcal{Y},\mathcal{V}\otimes_\vee\mathcal{W}]\subseteq\mathcal{B}[\mathcal{X}\otimes_\alpha\mathcal{Y},\mathcal{V}\otimes_\alpha\mathcal{W}]\quad\text{where}\quad\|\cdot\|_{[\wedge,\vee]}\le\|\cdot\|_{[\alpha,\alpha]}$$

by $(*)$, and since $\mathbb{L}\in\mathcal{B}[\mathcal{X}\otimes_\wedge\mathcal{Y},\mathcal{V}\otimes_\vee\mathcal{W}]$ we also get

$$\mathcal{B}[\mathcal{X},\mathcal{V}]\otimes\mathcal{B}[\mathcal{Y},\mathcal{W}]\subseteq\mathcal{B}[\mathcal{X}\otimes_\wedge\mathcal{Y},\mathcal{V}\otimes_\vee\mathcal{W}].$$

Therefore we may equip $\mathcal{B}[\mathcal{X},\mathcal{V}]\otimes\mathcal{B}[\mathcal{Y},\mathcal{W}]$ with both induced uniform norms, namely, $\|\cdot\|_{[\vee,\wedge]}$ and $\|\cdot\|_{[\alpha,\alpha]}$, and so we may consider the normed spaces

$$\mathcal{B}[\mathcal{X},\mathcal{V}]\otimes_{[\wedge,\vee]}\mathcal{B}[\mathcal{Y},\mathcal{W}]\subseteq\mathcal{B}[\mathcal{X},\mathcal{V}]\otimes_{[\alpha,\alpha]}\mathcal{B}[\mathcal{Y},\mathcal{W}]\quad\text{with}\quad\|\cdot\|_{[\wedge,\vee]}\le\|\cdot\|_{[\alpha,\alpha]}.$$

Remark 4.3. For the induced uniform norm $\|\cdot\|_{[\wedge,\vee]}$ on $\mathcal{B}[\mathcal{X}\otimes_\wedge\mathcal{Y},\mathcal{V}\otimes_\vee\mathcal{W}]$,

$$\|\mathbb{L}\|_\vee\le\|\mathbb{L}\|_{[\wedge,\vee]}\le\|\mathbb{L}\|_\wedge$$

for every $\mathbb{L}\in\mathcal{B}[\mathcal{X},\mathcal{V}]\otimes_{[\wedge,\vee]}\mathcal{B}[\mathcal{Y},\mathcal{W}]$, and hence

$$\|\cdot\|_{[\wedge,\vee]}\text{ is a reasonable crossnorm on }\mathcal{B}[\mathcal{X},\mathcal{V}]\otimes\mathcal{B}[\mathcal{Y},\mathcal{W}].$$

Indeed, it can be verified that $\|\cdot\|_\vee\le\|\cdot\|_{[\wedge,\vee]}$. Also, $\|\cdot\|_{[\wedge,\vee]}\le\|\cdot\|_{[\alpha,\alpha]}$ for an arbitrary uniform crossnorm $\|\cdot\|_\alpha$ according to Remark 4.2. Moreover, Theorem 4.1(a) says that $\|\cdot\|_{[\alpha,\alpha]}\le\|\cdot\|_\wedge$. Summing up,

$$\|\cdot\|_\vee\le\|\cdot\|_{[\wedge,\vee]}\le\|\cdot\|_{[\alpha,\alpha]}\le\|\cdot\|_\wedge.$$

Again, Proposition 2.1 ensures that the induced uniform norm $\|\cdot\|_{[\wedge,\vee]}$ is a reasonable crossnorm on the $\mathcal{B}[\mathcal{X},\mathcal{V}]\otimes\mathcal{B}[\mathcal{Y},\mathcal{W}]$.

The injective and projective norms $\|\cdot\|_\vee$ and $\|\cdot\|_\wedge$, being uniform crossnorms, act on every tensor product space and are the least and the greatest uniform crossnorm on every tensor product space (cf. Proposition 2.1). In particular, on $\mathcal{B}[\mathcal{X},\mathcal{V}]\otimes\mathcal{B}[\mathcal{Y},\mathcal{W}]$ (as in the above displayed inequalities). Also, $\|\cdot\|_{[\vee,\vee]}$ and $\|\cdot\|_{[\wedge,\wedge]}$ are reasonable crossnorms on $\mathcal{B}[\mathcal{X},\mathcal{V}]\otimes\mathcal{B}[\mathcal{Y},\mathcal{W}]$ by Theorem 4.1 because $\|\cdot\|_\vee$ and $\|\cdot\|_\wedge$ are uniform crossnorm. Therefore

$$\|\mathbb{L}\|_\vee\le\|\mathbb{L}\|_{[\vee,\vee]}\quad\text{and}\quad\|\mathbb{L}\|_{[\wedge,\wedge]}\le\|\mathbb{L}\|_\wedge.$$

Question. Does $\|\mathbb{L}\|_{[\alpha,\alpha]}$ lie in between, so that $\|\mathbb{L}\|_{[\vee,\vee]}$ is the least and $\|\mathbb{L}\|_{[\wedge,\wedge]}$ is the greatest reasonable crossnorm on $\mathcal{B}[\mathcal{X},\mathcal{V}]\otimes\mathcal{B}[\mathcal{Y},\mathcal{W}]$ that are inherited from $\mathcal{B}[\mathcal{X}\otimes_\alpha\mathcal{V},\mathcal{Y}\otimes_\alpha\mathcal{W}]$ for arbitrary uniform crossnorms $\|\cdot\|_\alpha$? (Note that there are reasonable crossnorms on the space $\mathcal{B}[\mathcal{X},\mathcal{V}]\otimes\mathcal{B}[\mathcal{Y},\mathcal{W}]$ that are not restrictions of the uniform induced norm on $\mathcal{B}[\mathcal{X}\otimes_\alpha\mathcal{V},\mathcal{Y}\otimes_\alpha\mathcal{W}]$ for any uniform crossnorm $\|\cdot\|_\alpha$, as is the case of $\|\mathbb{L}\|_\vee$ and $\|\mathbb{L}\|_\wedge$.)

Corollary 4.4 shows how $\|\mathbb{L}\|_{[\alpha,\alpha]}$ naturally fits between $\|\mathbb{L}\|_{[\vee,\vee]}$ and $\|\mathbb{L}\|_{[\wedge,\wedge]}$, giving a first estimate to the above question. (Compare the result in Corollary 4.4 below with $(*)$ in Remark 4.2.) Consider the setup in the proof of Theorem 4.1.

Corollary 4.4. *If $\|\cdot\|_\alpha$ is a uniform crossnorm, then*

$$\inf_{0\ne\digamma\in\mathcal{X}\otimes\mathcal{Y}}\frac{\|\digamma\|_\vee}{\|\digamma\|_\alpha}\|\mathbb{L}\|_{[\vee,\vee]}\le\|\mathbb{L}\|_{[\alpha,\alpha]}\le\sup_{0\ne\digamma\in\mathcal{X}\otimes\mathcal{Y}}\frac{\|\digamma\|_\wedge}{\|\digamma\|_\alpha}\|\mathbb{L}\|_{[\wedge,\wedge]}.$$

Proof. Consider the uniform crossnorms $\|\cdot\|_\vee$, $\|\cdot\|_\alpha$, and $\|\cdot\|_\wedge$ on $\mathcal{X}\otimes\mathcal{Y}$ and on $\mathcal{V}\otimes\mathcal{W}$. According to Proposition 2.1 $\|\cdot\|_\vee\le\|\cdot\|_\alpha\le\|\cdot\|_\wedge$. Regarding the setup in the proof of Theorem 4.1, take $\mathbb{L}(\digamma)\in\mathcal{V}\otimes\mathcal{W}$ for an arbitrary $\digamma\in\mathcal{X}\otimes\mathcal{Y}$. First consider the inequality $\|\mathbb{L}(\digamma)\|_\alpha\le\|\mathbb{L}(\digamma)\|_\wedge$ so that

$$\|\mathbb{L}\|_{[\alpha,\alpha]}=\sup_{\digamma\ne0}\frac{\|\mathbb{L}(\digamma)\|_\alpha}{\|\digamma\|_\alpha}\le\sup_{\digamma\ne0}\frac{\|\mathbb{L}(\digamma)\|_\wedge}{\|\digamma\|_\alpha}\le\|\mathbb{L}\|_{[\wedge,\wedge]}\sup_{\digamma\ne0}\frac{\|\digamma\|_\wedge}{\|\digamma\|_\alpha}.$$

Next take the inequality $\|\mathbb{L}(\digamma)\|_\vee\le\|\mathbb{L}(\digamma)\|_\alpha$. Similarly,

$$\|\mathbb{L}\|_{[\vee,\vee]}\le\|\mathbb{L}\|_{[\alpha,\alpha]}\sup_{\digamma\ne0}\frac{\|\digamma\|_\alpha}{\|\digamma\|_\vee};\quad\text{equivalently,}\quad\inf_{\digamma\ne0}\frac{\|\digamma\|_\vee}{\|\digamma\|_\alpha}\|\mathbb{L}\|_{[\vee,\vee]}\le\|\mathbb{L}\|_{[\alpha,\alpha]},$$

with $\inf_{\digamma\ne0}\frac{\|\digamma\|_\vee}{\|\digamma\|_\alpha}=\big(\sup_{\digamma\ne0}\frac{\|\digamma\|_\alpha}{\|\digamma\|_\vee}\big)^{-1}$. □

Remark 4.5. It is not usual to consider more than one uniform crossnorm, one equipping the domain and the other equipping the codomain, of a tensor product $A\otimes B$ of bounded linear transformations — see, e.g., [11, Section 6.1] and [4, Section 1.2]. In fact, a uniform crossnorm $\|\cdot\|_\alpha$ is a reasonable crossnorm which is supposed to be assigned to every tensor product for arbitrary normed spaces making $A\otimes B$ continuous when acting from $\mathcal{X}\otimes_\alpha\mathcal{Y}$ to $\mathcal{V}\otimes_\alpha\mathcal{W}$; both tensor product spaces equipped with the same norm (cf. Section 3). Therefore, if one takes another reasonable crossnorm $\|\cdot\|_\beta$, also supposed to be assigned to every tensor product space of arbitrary normed spaces, and requires that

$A \in \mathcal{B}[\mathcal{X}, \mathcal{V}]$ and $B \in \mathcal{B}[\mathcal{Y}, \mathcal{W}]$ imply $A \otimes B \in \mathcal{B}[\mathcal{X} \otimes_\alpha \mathcal{Y}, \mathcal{V} \otimes_\beta \mathcal{W}]$, then it is readily verified that $\|A\|\,\|B\| \leq \|A \otimes B\| = \|A \otimes B\|_{[\alpha,\beta]}$. A new concept of a *jointly uniform crossnorm* would require a new definition to make the above inequality an identity. Also, under a new definition of a jointly uniform crossnorm, Propositions 3.1 and 3.2 would require a restatement. (Extending Proposition 3.1 to such a new setup seems to be a simple task, but Proposition 3.2 would perhaps require some additional, possibly nontrivial, arguments). Although we will not proceed along this line — it goes beyond the purpose of the present paper — it seems that a possible version of the first part of Theorem 4.1 involving a pair of distinct *jointly uniform crossnorms* might lead to promising further research.

5. Final remark

Suppose a pair of normed spaces $(\mathcal{X}, \mathcal{Y})$ is such that $\sup_{0\neq\mathcal{F}\in\mathcal{X}\otimes\mathcal{Y}} \frac{\|\mathcal{F}\|_\wedge}{\|\mathcal{F}\|_\vee} < \infty$ when their tensor product space $\mathcal{X} \otimes \mathcal{Y}$ is equipped with the injective norm $\|\cdot\|_\vee$ and with the projective norm $\|\cdot\|_\wedge$. (Trivial example: if $\mathcal{X}$ and $\mathcal{Y}$ are finite-dimensional, where all norms are equivalent, so that $\mathcal{X} \otimes_\vee \mathcal{Y} \cong \mathcal{X} \otimes_\wedge \mathcal{Y}$ — here $\cong$ means topological isomorphism). In such a case (when the above supremum is finite), set $\sup_{0\neq\mathcal{F}\in\mathcal{X}\otimes\mathcal{Y}} \frac{\|\mathcal{F}\|_\wedge}{\|\mathcal{F}\|_\vee} = \gamma$, and the injective and projective norms become *equivalent uniform crossnorms* on $\mathcal{X} \otimes \mathcal{Y}$ and so is any uniform crossnorm $\|\cdot\|_\alpha$ since

$$\|\cdot\|_\vee \leq \|\cdot\|_\alpha \leq \|\cdot\|_\wedge \leq \gamma\|\cdot\|_\vee,$$

and so $\sup_{\mathcal{F}\neq0} \frac{\|\mathcal{F}\|_\wedge}{\|\mathcal{F}\|_\alpha} < \infty$ and $\sup_{\mathcal{F}\neq0} \frac{\|\mathcal{F}\|_\alpha}{\|\mathcal{F}\|_\vee} < \infty$. It is attributed to Grothendieck the origin of the question whether $\mathcal{X} \widehat{\otimes}_\vee \mathcal{Y} \cong \mathcal{X} \widehat{\otimes}_\wedge \mathcal{Y}$ holds for some pair of infinite-dimensional Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ (see [8, p.181]). A solution was given by Pisier in [8, Theorem 3.2(b)] where it was exhibited a separable infinite-dimensional Banach space $\mathcal{P}$, now called *Pisier space*, such that $\mathcal{P} \widehat{\otimes}_\vee \mathcal{P} \cong \mathcal{P} \widehat{\otimes}_\wedge \mathcal{P}$ (here $\cong$ means isometric isomorphism), which shows in addition that all reasonable crossnorms (and all uniform crossnorms) on $\mathcal{P} \widehat{\otimes} \mathcal{P}$ (and so on $\mathcal{P} \otimes \mathcal{P}$) are isomorphically equivalent, where in this case $\gamma = 1$ and, consequently, for $\mathbb{L} \in \mathcal{B}[\mathcal{P}] \otimes \mathcal{B}[\mathcal{P}]$,

$$\|\mathbb{L}\|_{[\vee,\vee]} = \|\mathbb{L}\|_{[\alpha,\alpha]} = \|\mathbb{L}\|_{[\wedge,\wedge]},$$

with respect to the setup in the proof of Theorem 4.1. Corollary 4.4 gives just a first estimate to the question of whether the equal signs "=" in the above equation can be replaced by "≤" (perhaps weighted with positive constants) for an arbitrary transformation $\mathbb{L} \in \mathcal{B}[\mathcal{X}, \mathcal{V}] \otimes \mathcal{B}[\mathcal{Y}, \mathcal{W}]$ acting on arbitrary (or on specific classes of) normed spaces $\mathcal{X}, \mathcal{Y}, \mathcal{V}, \mathcal{W}$.

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