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Dynamical convergence of polynomials to products of power maps and the exponential

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ABSTRACT. We study the dynamics of a family of polynomial functions and the relationship to the dynamics of a related entire transcendental family of functions. As the degree of the polynomial approaches infinity, the polynomial functions converge uniformly on compact sets to a function that is a product of a power map and the exponential function. The advantage to the approach in the paper is that we can use the relatively simple, although high degree, polynomial functions to aid our understanding of the dynamics of the related transcendental entire function. We study properties both in the dynamical plane as well as in the parameter plane.

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1. Introduction

The goal of this paper is to understand the dynamics of a family of polynomial functions and the relationship to the dynamics of a related entire transcendental family of functions. We study maps in the family

$$C_{a,m,d}(z) = az^m\left(1+\frac{z}{d}\right)^d, a \in \mathbb{C}^* = \mathbb{C}\setminus\{0\}, m \geq 2, d \geq 2.$$

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Each map $C_{a,m,d}$ has $\infty, 0, -d$, and $-md/(m+d)$ as critical points. For all $a \in \mathbb{C}^*$, ∞ and 0 are fixed points, and $C_{a,m,d}(-d) = 0$. Hence, to study the dynamics of $C_{a,m,d}$, we are interested in the orbit of the free critical point $c_{m,d} = -md/(m+d)$.

As $d \to \infty$, the maps $C_{a,m,d}$ converge uniformly on compact sets to

$$E_{a,m}(z) = az^m e^z, a \in \mathbb{C}^*, m \geq 2.$$

For $E_{a,m}$, the critical points are at $\infty, 0$, and $-m$. For all $a \in \mathbb{C}^*$, 0 and ∞ are fixed points for $E_{a,m}$. The free critical point of $E_{a,m}$ is $-m$, which is also the limit of the free critical points $c_{m,d}$ as $d \to \infty$. Núria Fagella and Antonio Garijo investigate both the dynamical and parameter planes of $E_{a,m}$ [11, 12]. In particular, they study the topology of the basin of attraction of the superattracting fixed point at 0, as well as properties of parameters of capture zones in parameter space (parameters for which the free critical point converges to 0 under iteration).

The advantage to the approach in the paper is that we can use the relatively simple, although high degree, polynomial functions to aid our understanding of the dynamics of the related transcendental entire function. The approach of understanding transcendental entire functions through polynomial or rational approximations began with the preprint [7], which was eventually published in [5]. The approach has also been previously used in papers such as [10, 13, 14, 15, 16, 17, 18, 19, 20, 21]. Here, we focus on how the dynamics of the polynomial functions change with respect to the parameter a as well as when d gets large, and how properties of the Julia and Fatou sets as well as the parameter spaces of $C_{a,m,d}$ compare to those of $E_{a,m}$. The paper is organized as follows. In Section 2, we relate the family $C_{a,m,d}$ to two other polynomial functions that have previously been studied. This section provides background and does not contain new results. In Section 3, we focus on properties of the dynamical plane for both $C_{a,m,d}$ and $E_{a,m}$. Finally, in Section 4, we discuss the parameter planes of $C_{a,m,d}$ and $E_{a,m}$, primarily by focusing on the capture zones.

2. Background

In [6], Bodil Branner and Núria Fagella use $C_{a,m,d}$ with $m = 1$ and $d \geq 2$ to give homeomorphisms between limbs of the Mandelbrot set with equal denominator. For $p < q$, $q \geq 3$, and $\gcd(p, q) = 1$, they construct a homeomorphism between the p/q limb of the Mandelbrot set and the 0 limb of the connectedness locus for $C_{a,m,q}$; that is, the set of parameters such that the Julia set is connected.

To aid in our computations and to relate the family $C_{a,m,d}$ to known results, we add symmetry to the parameter space and then to the dynamical space. For fixed $a \in \mathbb{C}^*$, $m \geq 2$, $d \geq 2$, we can choose a branch of the $m + d - 1$st root and take $\alpha = (d^d/a)^{1/(m+d-1)}$. Then $C_{a,m,d}$ is related to the monic map

$$M_{b,m,d}(z) = z^m(z + b)^d,$$

by $C_{a,m,d}(\alpha z) = \alpha M_{b,m,d}(z)$ and $b = d/\alpha$. Eliminating roots and solving for a, we can also write the relationship between the parameters as

$$a = b^{m+d-1}/d^{m-1}. \tag{2.1}$$

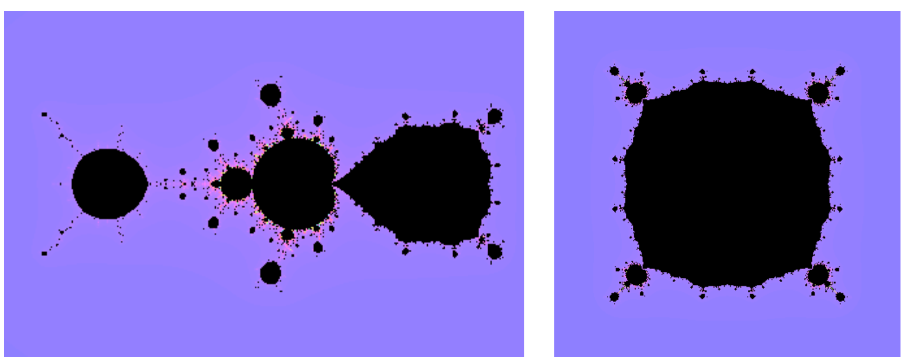
Each map $M_{b,m,d}$ has $\infty, 0, -b$, and $-mb/(m+d)$ as critical points. Similar to $C_{a,m,d}$, the critical points ∞ and 0 are fixed by $M_{b,m,d}$, and $M_{b,m,d}(-b) = 0$. The orbit of $-mb/(m+d)$ determines the parameter space for $M_{b,m,d}$ and is the image of the orbit of $-md/(m+d)$ under $C_{a,m,d}$ after conjugation. The conjugation and the parameter relationship in Equation (2.1) shows that the parameter space for $M_{b,m,d}$ is a $m+d-1$ cover of the parameter space for $C_{a,m,d}$.

To add symmetry to the dynamical space, we use $Pow_d(z) = z^d$ to show that $M_{b,m,d}$ is semiconjugate to the family

$$S_{b,m,d}(z) = z^m(z^d + b).$$

Specifically, $M_{b,m,d}(z^d) = (S_{b,m,d}(z))^d$. Since 0 is not in the Julia set, this semiconjugacy shows that the Julia set for $S_{b,m,d}$ is a d-fold cover of the Julia set for $M_{b,m,d}$, which itself is topologically isomorphic to the Julia set for $C_{a,m,d}$.

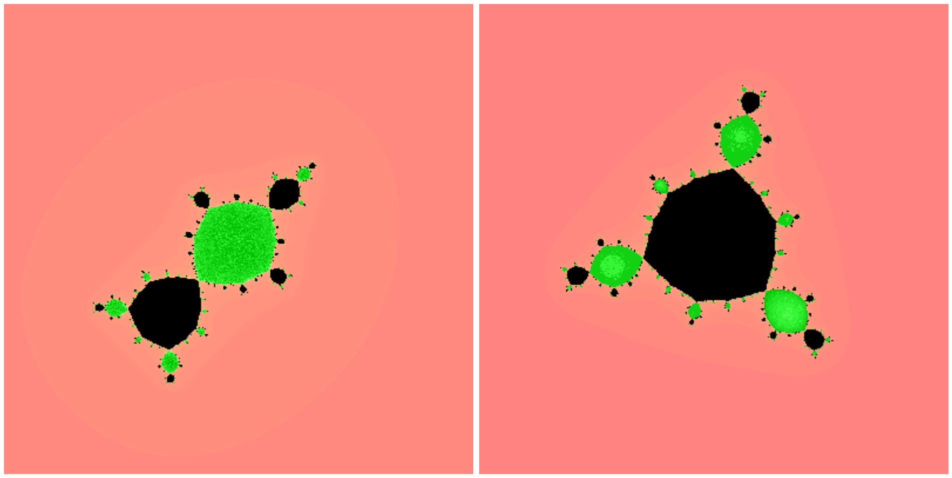
If ω is a dth root of unity and $z \in \mathbb{C}$, then we have the symmetry $S_{b,m,d}(\omega z) = \omega^m S_{b,m,d}(z)$. This symmetry implies that the Fatou and Julia sets of $S_{b,m,d}$ have d-fold rotational symmetry. In particular, if the critical points z_i and z_j are roots of $z^d = -mb/(m+d)$, then $|S^n_{b,m,d}(z_i)| = |S^n_{b,m,d}(z_j)|$ for all $n \in \mathbb{N}$. This implies that the nonzero, finite critical points have orbits that either all go to ∞, all go to 0, or all remain bounded away from ∞ and 0. Since ∞ and 0 are fixed points, the parameter space for $S_{b,m,d}$ is determined by the orbits of the roots of $z^d = -mb/(m+d)$. The symmetry and the semiconjugacy between $S_{b,m,d}$ and $M_{b,m,d}$ imply that the parameter spaces for $S_{b,m,d}$ and $M_{b,m,d}$ are the same. Figures 1 and 2 show examples of these symmetries in the parametric and dynamical planes.



(a) (b)

FIGURE 1. For $m = 2$ and $d = 3$, Figure 1a is the parameter space for $C_{a,m,d}$ for $-12.5 < \mathrm{Re}(a) < 2.5$ and $-5 < \mathrm{Im}(a) < 5$, and Figure 1b is the parameter space for $M_{b,m,d}$ and $S_{b,m,d}$ for $-2.5 < \mathrm{Re}(b) < 2.5$ and $-2.5 < \mathrm{Im}(b) < 2.5$.

In [24], Pascale Roesch studies the case $S_{b,m,d}$ with $d = 1$, $m \geq 2$ and $b = (m+1)c/m$ for a parameter $c \in \mathbb{C}$. Roesch proves that the boundary is a Jordan curve for every component of the parameter space for which the orbit of the



(a) (b)

FIGURE 2. For $b = r + ri$ with $r = (625/54)^{1/4}/\sqrt{2}$, $m = 2$, and $d = 3$, Figure 2a is the Julia set for $M_{b,m,d}$ for $-3 < \mathrm{Re}(z) < 2$ and $-3 < \mathrm{Im}(z) < 2$, and Figure 2b is the Julia set for $S_{b,m,d}$ for $-2 < \mathrm{Re}(z) < 2$ and $-2 < \mathrm{Im}(z) < 2$.

free critical point ($-c$ in that parametrization) either approaches 0 or another finite attracting cycle and proves results on local connectedness. Additionally, Roesch describes the landscape of the parameter space by describing intersections between homeomorphic copies of the Mandelbrot set and components where the orbit of the free critical point approaches 0.

In [22, 23], Maryam Rabii studies $S_{b,m,d}$ with $b = -c$ for a parameter $c \in \mathbb{C}$, for the cases $m \geq 2$, $n \geq 1$ and $m = 1$, $n \geq 2$, with a focus on whether the Julia sets and connectedness locus are connected or locally connected.

Carlos Arteaga and Alexandre Alves have completed a number of investigations of the functions $S_{b,m,d}$ with $b = -c$: [2] studies $S_{b,m,d}$ for $m \geq 2$ and $d \geq 1$, [3] studies $S_{b,m,d}$ for $m = 1$ and $d \geq 2$, and [1] studies $S_{b,m,d}$ for $m \geq 1$ and $d \geq 2$. In these papers, they study the connectedness locus. For example, they show that the postcritically finite parameters are dense in the boundary of the connectedness locus. They also show that the filled Julia sets for c in the unit disk and the connectedness loci converge in the Hausdorff metric to the closed unit disk as $d \to \infty$.

3. Results in the dynamical plane

Many of the bounds in this section for the family $C_{a,m,d}$ are sharpened versions of bounds in [4].

3.1. Symmetry and Bounds. Although the function $C_{a,m,d}$ does not have the rotational symmetry of $S_{b,m,d}$ in the dynamical plane, there is a reflection symmetry between the Fatou and Julia sets of $C_{\overline{a},m,d}$ and $C_{a,m,d}$.

Theorem 3.1. *The Fatou and Julia sets of $C_{a,m,d}$ and $E_{a,m}$ are the reflections across the real axis of $C_{\overline{a},m,d}$ and $E_{\overline{a},m}$, respectively.*

Proof. We have

$$\begin{aligned} C_{\overline{a},m,d}(\overline{z}) &= \overline{a}\,\overline{z}^m\left(1+\frac{\overline{z}}{d}\right)^d \\ &= \overline{C_{a,m,d}(z)} \end{aligned}$$

and

$$\begin{aligned} E_{\overline{a},m}(\overline{z}) &= \overline{a}\,\overline{z}^m e^{\overline{z}} \\ &= \overline{E_{a,m}(z)}. \end{aligned}$$

Since these properties hold under iteration, the symmetries follow. □

All polynomials have a superattracting fixed point at infinity. We begin by determining an escape radius for functions in the family $C_{a,m,d}$.

Proposition 3.2. [4] *Let $r_\infty = \max\left\{2d, 2\left(\frac{d^{d+m}}{|a|}\right)^{1/(m-1)}\right\}$. Then for $|z| > r_\infty$, $\left|f^n_{a,m,d}(z)\right| \to \infty$ as $n \to \infty$*

We can use Proposition 3.2 to show that, for any m and d, there are always parameter values for which the Julia set is disconnected.

Theorem 3.3. *Given $m, d \geq 2$, if*

$$|a| > \max\left\{2d\left(\frac{md}{m+d}\right)^{-m}\left(\frac{d}{m+d}\right)^{-d},\right.$$
$$\left. 2^{\frac{m-1}{m}} d^{\frac{d+m}{m}}\left(\frac{md}{m+d}\right)^{1-m}\left(\frac{d}{m+d}\right)^{\frac{d(1-m)}{m}}\right\}$$

then $\left|C^n_{a,m,d}\left(c_{m,d}\right)\right| \to \infty$ as $n \to \infty$, and $J(C_{a,m,d})$ is disconnected.

Proof. Given $m, d \geq 2$, we have

$$\begin{aligned} \left|C_{a,m,d}\left(\frac{-md}{m+d}\right)\right| &= \left|a\left(\frac{-md}{m+d}\right)^m\left(1+\frac{-md}{m+d}\left(\frac{1}{d}\right)\right)^d\right| \\ &= |a|\left(\frac{md}{m+d}\right)^m\left(\frac{d}{m+d}\right)^d. \end{aligned} \tag{3.1}$$

From Proposition 3.2, suppose that $r_\infty = 2d$. By hypothesis, we know that $|a| > 2d\left(\frac{md}{m+d}\right)^{-m}\left(\frac{d}{m+d}\right)^{-d}$. Using Equation 3.1, we have

$$\begin{aligned}\left|C_{a,m,d}\left(\frac{-md}{m+d}\right)\right| &= |a|\left(\frac{md}{m+d}\right)^m\left(\frac{d}{m+d}\right)^d\\ &> 2d\left(\frac{md}{m+d}\right)^{-m}\left(\frac{d}{m+d}\right)^{-d}\left(\frac{md}{m+d}\right)^m\left(\frac{d}{m+d}\right)^d\\ &= 2d = r_\infty.\end{aligned}$$

Then by Proposition 3.2, $\left|C^n_{a,m,d}\left(c_{m,d}\right)\right| \to \infty$ as $n \to \infty$, and $J(C_{a,m,d})$ is disconnected.

Next, suppose that $r_\infty = 2\left(d^{d+m}/|a|\right)^{1/(m-1)}$ from Proposition 3.2. Using our hypothesis, we have

$$|a| > 2^{\frac{m-1}{m}} d^{\frac{d+m}{m}}\left(\frac{md}{m+d}\right)^{1-m}\left(\frac{d}{m+d}\right)^{\frac{d(1-m)}{m}},$$

which can be rewritten as

$$\left(\frac{md}{m+d}\right)\left(\frac{d}{m+d}\right)^{\frac{d}{m}} > 2^{\frac{1}{m}}\left(\frac{d^{\frac{d+m}{m}}}{|a|}\right)^{\frac{1}{m-1}}.$$

Using Equation 3.1, we have

$$\begin{aligned}\left|C_{a,m,d}\left(\frac{-md}{m+d}\right)\right| &= |a|\left(\frac{md}{m+d}\right)^m\left(\frac{d}{m+d}\right)^d\\ &> 2^{\frac{m-1}{m}} d^{\frac{d+m}{m}}\left(\frac{md}{m+d}\right)\left(\frac{d}{m+d}\right)^{\frac{d}{m}}\\ &> 2^{\frac{m-1}{m}} d^{\frac{d+m}{m}} 2^{\frac{1}{m}}\left(\frac{d^{\frac{d+m}{m}}}{|a|}\right)^{\frac{1}{m-1}}\\ &= 2\left(d^{d+m}/|a|\right)^{1/(m-1)} = r_\infty.\end{aligned}$$

In this case also, using Proposition 3.2, we have $\left|C^n_{a,m,d}\left(c_{m,d}\right)\right| \to \infty$ as $n \to \infty$, and $J(C_{a,m,d})$ is disconnected. □

All functions under study in this paper have a superattracting fixed point at the origin. When f is one of the functions $C_{a,m,d}, M_{b,m,d}, S_{b,m,d}$, or $E_{a,m}$, we use $A(f,0)$ to denote the basin of attraction of the origin, given by

$$A(f,0) = \{z \in \mathbb{C} : f^n(z) \to 0 \text{ as } n \to \infty\}.$$

The immediate basin of attraction of $z = 0$ is the connected component of $A(f,0)$ containing $z = 0$, and we denote it by $A^*(f,0)$. Since 0 is a superattracting fixed point, there is always a disk containing the origin that iterates to the origin. The next two results give estimates on the size of this disk for $C_{a,m,d}$ and $E_{a,m}$.

Fagella and Garijo showed that the immediate basin of attraction of the origin for $E_{a,m}$ contains a disk of radius δ_0, where δ_0 is defined in the following theorem. In addition, their paper showed that there is an unbounded region in the left half plane that maps to the disk of radius δ_0 under $E_{a,m}$. We state their result in the following theorem.

Theorem 3.4. [11]

(1) *There exists* $\delta_0 = \delta_0(|a|, m) > 0$, *defined as the unique positive solution of* $x^{m-1}e^x = 1/|a|$, *such that* $A^*(E_{a,m}, 0)$ *contains the disk* $D_{\delta_0} = \{z \in \mathbb{C}; |z| < \delta_0\}$.

(2) *There exist* $x_0 = x_0(|a|, m) < 0$ *and a function* $C(x) \geq 0$ *such that the open set*

$$H_{|a|,m} = \left\{ z = x + yi \mid \begin{array}{c} x \quad \in (-\infty, x_0) \\ y \in (-C(x), C(x)) \end{array} \right\}$$

satisfies $E_{a,m}\left(H_{|a|,m}\right) \subset D_{\delta_0}$.

The value of δ_0 *is always larger than or equal to* $\min\left\{1, (1/(|a|e))^{\frac{1}{m-1}}\right\}$. *In particular, if* $|a| \geq 1/e$ *then* $\delta_0 \geq (1/(|a|e))^{\frac{1}{m-1}}$ *and if* $|a| \leq 1/e$ *then* $\delta_0 \geq 1$.

We show that, in fact, the disk of radius δ_0 from Theorem 3.4 is contained in the immediate basin of attraction of the origin for all functions $C_{a,m,d}$.

Theorem 3.5. *For any function* $C_{a,m,d}$, *we have* $D_{\delta_0} \subset A^*(C_{a,m,d}, 0)$.

Proof. Suppose $z \in D_{\delta_0}$. First, we establish the following inequality:

$$\begin{aligned}
|C_{a,m,d}(z)| &= |az^m(1 + z/d)^d| \\
&= |a||z^m||(1 + z/d)^d| \\
&= |a||z^m| \left| \sum_{k=0}^{d} \binom{d}{k} \frac{1}{d^k} z^k \right| \\
&\leq |a||z^m| \sum_{k=0}^{d} \frac{1}{k!} |z|^k \\
&\leq |a||z^m| \sum_{k=0}^{\infty} \frac{1}{k!} |z|^k \\
&= |a||z^m| e^{|z|} \\
&= |z||a||z|^{m-1} e^{|z|}.
\end{aligned}$$

Since $|z| < \delta_0$, we have that $|z|^{m-1}e^{|z|} < \delta_0^{m-1}e^{\delta_0}$. Using the definition of δ_0 from Theorem 3.4, we have that $\delta_0^{m-1}e^{\delta_0} = 1/|a|$. Then

$$|C_{a,m,d}(z)| \leq |z||a||z|^{m-1}e^{|z|} < |z|.$$

Since $|C_{a,m,d}(z)| < |z|$, $z \in A^*(C_{a,m,d}, 0)$. □

Since $C_{a,m,d}$ is a polynomial, it always has a superattracting Fatou component at infinity. Thus there isn't an unbounded region of the plane lying in the Fatou set for $C_{a,m,d}$ that corresponds with part (2) of Theorem 3.4. However, there is always a critical point at $-d$ that maps to the origin, so there is always a disk around $-d$ that maps inside D_{δ_0}. In fact, we can say more about the size of this disk. In particular, as d gets large, the radius of the disk around $-d$ grows with d.

Theorem 3.6. *Given $a \in \mathbb{C} - \{0\}$, $m \geq 2$, and real number $\epsilon > 0$, there exists an integer d_0 such that for all $d > d_0$, the image under $C_{a,m,d}$ of the disk of radius $d/(1+\epsilon)$ around $-d$ is contained in D_{δ_0}. That is, $C_{a,m,d}(D_{d/(1+\epsilon)}(-d)) \subset D_{\delta_0}$.*

Proof. Fix a, m, and ϵ. Let $f(x) = x^m/(1+\epsilon)^x$. Using calculus, for $x > 0$, f has one critical point at $c = m/\log(1+\epsilon)$. Further, f is increasing on $(0, c]$, is decreasing on $[c, \infty)$, is always positive, and $\lim_{x\to\infty} f(x) = 0$.

Let z be a point in the disk of radius $d/(1+\epsilon)$ around $-d$. We can write $z = -d + ud/(1+\epsilon)$ for a complex u with $|u| < 1$. Note that

$$\left|-d + u\frac{d}{1+\epsilon}\right| \leq d\frac{2+\epsilon}{1+\epsilon}.$$

We split our proof into two cases depending on the size of $|a|$.

First, suppose $|a| \geq 1/e$. Using Theorem 3.4, $\delta_0 > \left(\frac{1}{|a|e}\right)^{\frac{1}{m-1}}$. Let d_0 be the least integer larger than c such that

$$\frac{d_0^m}{(1+\epsilon)^{d_0}} < \left(\frac{1}{|a|e}\right)^{\frac{1}{m-1}} \frac{1}{|a|}\left(\frac{1+\epsilon}{2+\epsilon}\right)^m. \tag{3.2}$$

Notice that since f is decreasing, we have that $d^m/(1+\epsilon)^d < d_0^m/(1+\epsilon)^{d_0}$ for all integers $d > d_0$.

Then for $z \in D_{d/(1+\epsilon)}(-d)$,

$$\begin{aligned} |C_{a,m,d}(z)| &= \left| a\left(-d + u\frac{d}{1+\epsilon}\right)^m \left(1 + \frac{-d + u\frac{d}{1+\epsilon}}{d}\right)^d \right| \qquad (3.3)\\ &= |a| \left|-d + u\frac{d}{1+\epsilon}\right|^m \left|1 + \frac{-d + u\frac{d}{1+\epsilon}}{d}\right|^d \\ &\leq |a|\left(d\frac{2+\epsilon}{1+\epsilon}\right)^m \frac{1}{(1+\epsilon)^d} \\ &= \frac{d^m|a|}{(1+\epsilon)^d}\left(\frac{2+\epsilon}{1+\epsilon}\right)^m \end{aligned}$$

$$< \left(\frac{1}{|a|e}\right)^{\frac{1}{m-1}} < \delta_0,$$

using Equation 3.2 and Theorem 3.4 for the last two inequalities.

Next, suppose $|a| \leq 1/e$. Using Theorem 3.4, $\delta_0 \geq 1$. Let d_0 be the least integer larger than c such that

$$\frac{d_0^m}{(1+\epsilon)^{d_0}} < \frac{1}{|a|}\left(\frac{1+\epsilon}{2+\epsilon}\right)^m. \tag{3.4}$$

Using the computation from Equation 3.3, we have

$$|C_{a,m,d}(z)| \leq \frac{d^m|a|}{(1+\epsilon)^d}\left(\frac{2+\epsilon}{1+\epsilon}\right)^m < 1 \leq \delta_0.$$

□

We show some dynamical plane pictures in Figure 3 to illustrate Theorem 3.6. Figure 3 (a), (b), (c) shows a sequence of illustrations of the Julia and Fatou sets of $C_{a,m,d}$ for $a = -5$, $m = 2$, and $d = 2, 3$, and 6. The black regions are points with bounded orbits and thus show the bounded components of the Fatou set. The colored region is the set of points whose orbits are unbounded and represents the component of the Fatou set at infinity. The Julia set is the boundary between the two. In this sequence of figures, a bounded Fatou component on the left, which contains the critical point at $-d$, grows as d increases. The final picture shows the dynamical plane of $E_{-5,2}$ with a slightly different coloring scheme where each point is colored based on how many iterates it takes for the real part of the orbit to become larger than some bound. A point is colored black if the real part does not become large after some number of iterates. Some of the black regions contain numerical error from not taking a large enough number of iterates. Note the large Fatou component to the left extends to infinity. However, in this figure, the colored points, whose orbits are unbounded, are points belonging to the Julia set.

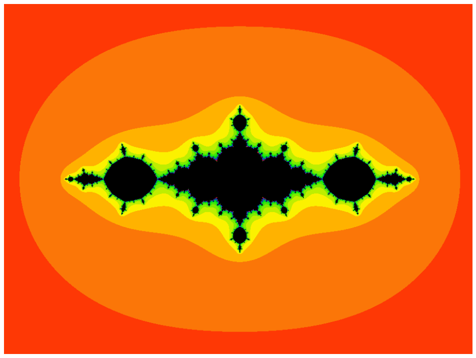
Note that the Julia set may change connectivity as d increases. Figure 4 (a) - (e) illustrates $C_{a,m,d}$ when $a = -729/64$ and $m = 2$ where the Julia set starts off disconnected but then becomes connected as d increases. Figure 4 (f) shows the Julia set for $E_{a,2}$.

3.2. Common Dynamics for All d. Next, we investigate the dynamics on the positive real axis when a is real. In contrast to Theorem 3.6, we can show that the dynamics on the positive real axis does not depend on d.

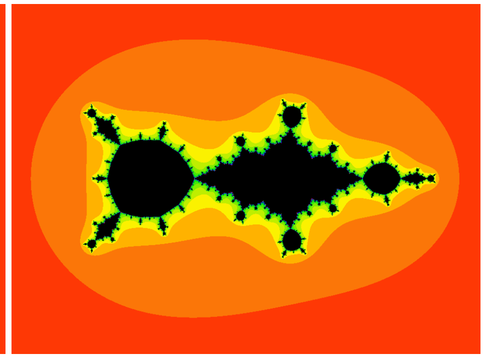
Theorem 3.7. *Let $m \geq 2$ and $d \geq 2$.*

(1) *Suppose $a \geq 1$. Then every point in real interval $[1, \infty)$ iterates to ∞ under $C_{a,m,d}$ and lies in the unbounded component of the Fatou set. The real interval $[1, \infty)$ iterates to ∞ under $E_{a,m}$ and lies in $J(E_{a,m})$.*

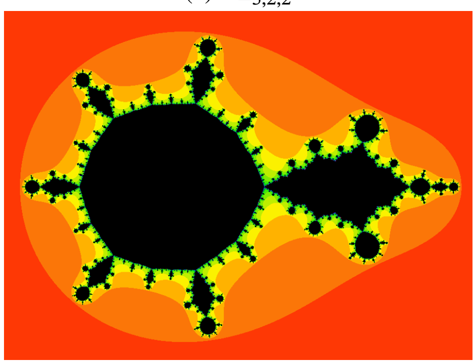
(2) *Suppose $0 < a < 1$. Then the real interval $[1/a, \infty)$ iterates to ∞ under $C_{a,m,d}$ and lies in the unbounded component of the Fatou set. The real interval $[1/a, \infty)$ iterates to ∞ under $E_{a,m}$ and lies in $J(E_{a,m})$.*



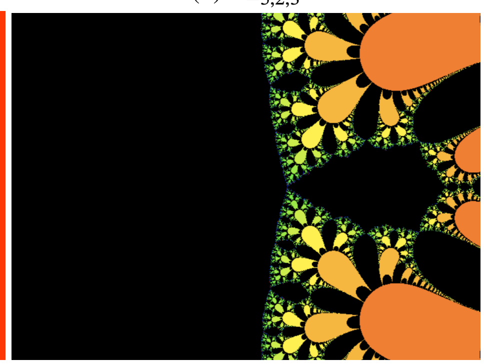
(a) $C_{-5,2,2}$



(b) $C_{-5,2,3}$



(c) $C_{-5,2,6}$



(d) $E_{-5,2}$

FIGURE 3. Julia sets for $C_{a,m,d}$ and $E_{a,m}$ for $a = -5$ and $m = 2$.

Proof. Suppose $m, d \geq 2$.

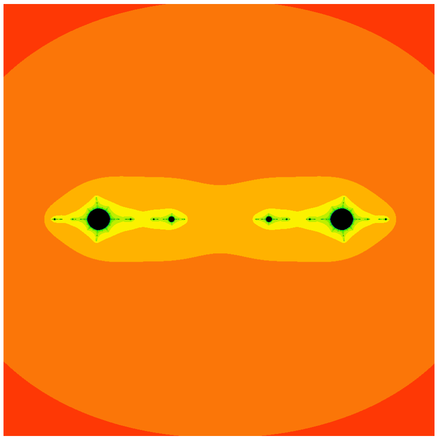
We start with the case $a \geq 1$. Observe that $0 = C_{a,m,d}(0) = C'_{a,m,d}(0) = C''_{a,m,d}(0)$ and $0 = E_{a,m}(0) = E'_{a,m}(0)$. Using calculus, we have that $C_{a,m,d}$ and $E_{a,m}$ are both increasing and concave up when $x > 0$. In addition, we have

$$1 < \frac{9a}{4} = C_{a,2,2}(1) < C_{a,m,d}(1) < E_{a,m}(1) = ae.$$

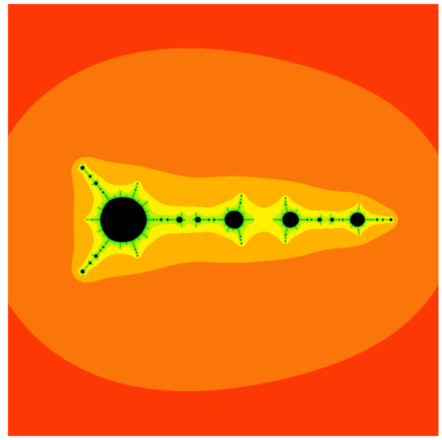
Thus, when $a \geq 1$, the functions $C_{a,m,d}$ and $E_{a,m}$ have a unique fixed point p on the positive axis and $0 < p < 1$. Since $C_{a,m,d}$ and $E_{a,m}$ are both increasing and concave up, this implies that $C^n_{a,m,d}(x) \to \infty$ and $E^n_{a,m}(x) \to \infty$ when $x > 1$.

Since $C_{a,m,d}$ is a polynomial, it has an attracting Fatou component containing infinity, so the interval $[1, \infty)$ lies in this Fatou component. Since $E_{a,m}$ is a critically finite entire function, the Julia set is the closure of the set of points whose orbits tend to infinity. Hence the interval $[1, \infty)$ lies in the Julia set.

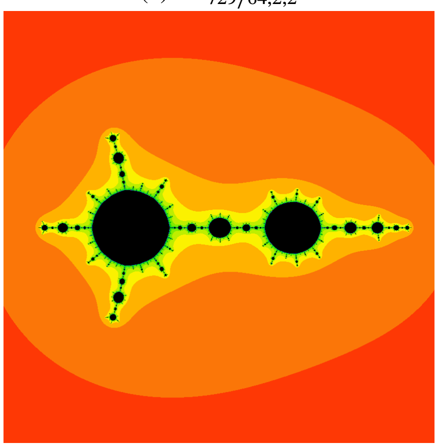
When $0 < a < 1$, we have that $C_{a,m,d}(1/a) > 1/a$ and $E_{a,m}(1/a) > 1/a$. Thus $C_{a,m,d}$ and $E_{a,m}$ have a unique fixed point p on the positive axis and $0 < p < 1/a$. A similar argument implies that $C^n_{a,m,d}(x) \to \infty$ and $E^n_{a,m}(x) \to \infty$ when $x > 1/a$. □



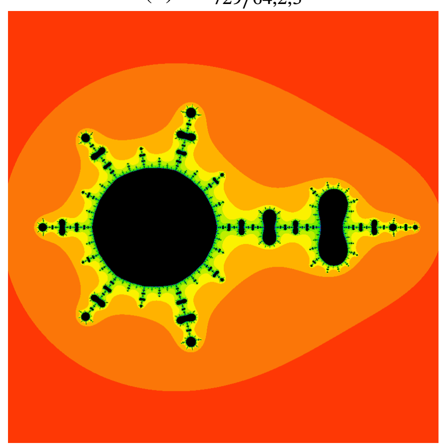
(a) $C_{-729/64,2,2}$



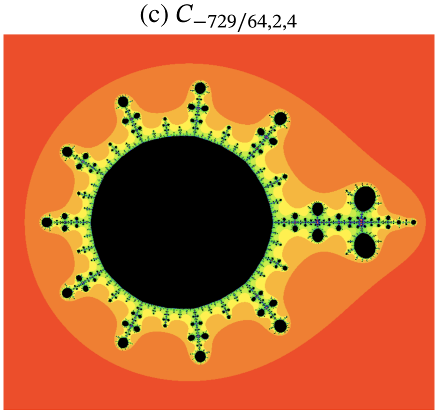
(b) $C_{-729/64,2,3}$



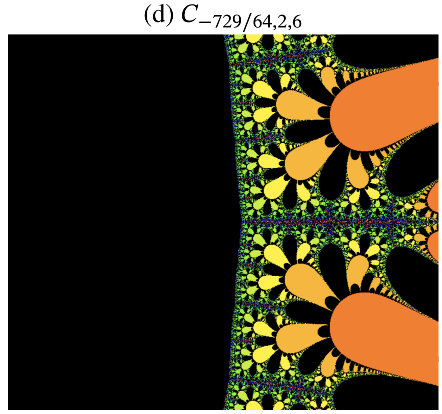
(c) $C_{-729/64,2,4}$



(d) $C_{-729/64,2,6}$



(e) $C_{-729/64,2,10}$



(f) $E_{-729/64,2}$

FIGURE 4. Julia sets for $C_{a,m,d}$ and $E_{a,m}$ when $a = -729/64$ and $m = 2$.

We can use Theorem 3.7 to improve our estimates from Theorem 3.3 when m is even to obtain parameters for which the Julia set is disconnected.

Theorem 3.8. *If m is even and $a > 4$, then $J(C_{a,m,d})$ is disconnected.*

Proof. We begin with the critical value

$$C_{a,m,d}\left(\frac{-md}{m+d}\right) = a\left(\frac{-md}{m+d}\right)^m\left(\frac{d}{m+d}\right)^d$$
$$= a\left(\frac{md}{m+d}\right)^m\left(\frac{d}{m+d}\right)^d,$$

since m is even. Note when $m = d = 2$, we have $C_{a,2,2}(c_{2,2}) = a/4$. We define a function

$$g(m,d) = a\left(\frac{md}{m+d}\right)^m\left(\frac{d}{m+d}\right)^d.$$

Using multivariable calculus, we can see that the minimum of g on the region $2 \leq m, n < \infty$ occurs at $(2,2)$. Thus $C_{a,m,d}(c_{m,d}) \geq C_{a,2,2}(c_{2,2})$. Using our assumption that $a > 4$, we have that

$$C_{a,m,d}(c_{m,d}) \geq C_{a,2,2}(c_{2,2}) = \frac{a}{4} > 1.$$

Using Theorem 3.7, $C_{a,m,d}(c_{m,d}) \to \infty$ and $J(C_{a,m,d})$ is disconnected. □

In contrast, we have the following result for $E_{a,m}$.

Theorem 3.9. *If $m = 2$ and $a > (e/2)^2$ or if $m = 2k$ for an integer $k \geq 2$ and $a > (e/m)^k$, then $F(E_{a,m}) = A(E_{a,m}, 0)$.*

Proof. First, suppose that $m = 2$ and $a > (e/2)^2$. Using the critical point $-m = -2$, we have

$$E_{a,m}(-2) = a(-2)^2e^{-2} > 1.$$

Using Theorem 3.7(1), the orbit of -2 tends to infinity.

Second, suppose that $m = 2k$ for an integer $k \geq 2$ and $a > (e/m)^k$. Again, using the critical point $-m$, we have

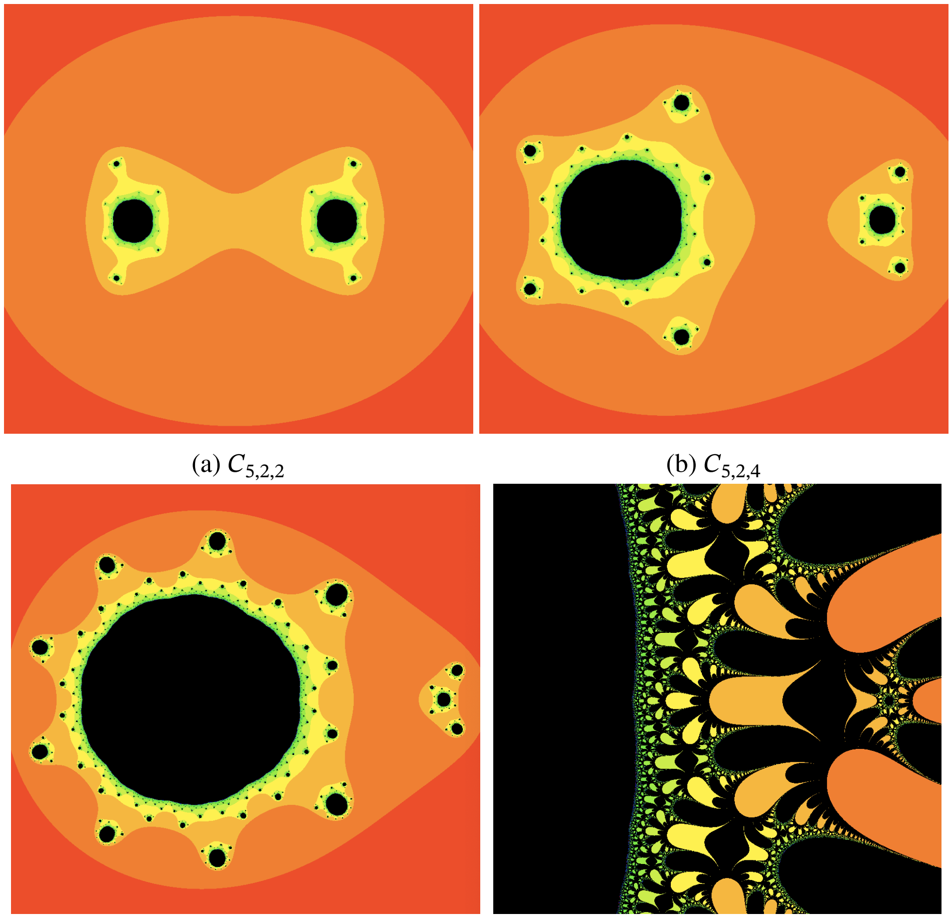
$$E_{a,m}(-m) = a(-m)^m e^{-m} > \left(\frac{m}{e}\right)^k = \frac{1}{a}.$$

Using Theorem 3.7(2), the orbit of $-m$ tends to infinity.

In either case, since $E_{a,m}$ is entire and has a finite number of critical and asymptotic values, it does not have Herman rings, Baker domains, or wandering domains [9]. Since the orbit of $z = -m$ tends to ∞, $E_{a,m}$ cannot have Siegel disks, or parabolic domains, so the Fatou set must coincide with the basin of 0. □

We show a sample of the dynamics when $a = 5$ and $m = 2$ in Figure 5. Note that Theorems 3.6, 3.8 and 3.9 apply in this case. The Fatou component of $C_{5,2,d}$ containing $-d$ grows as d gets large, the Julia set of $C_{5,2,d}$ is disconnected, and the Fatou set of $E_{5,2}$ is the basin of 0.

Next, we consider the case when all finite critical points lie in the immediate attracting basin of the origin. We begin by focusing our attention on the real



(a) $C_{5,2,2}$ (b) $C_{5,2,4}$

(c) $C_{5,2,8}$ (d) $E_{5,2}$

FIGURE 5. Julia sets for $C_{a,m,d}$ and $E_{a,m}$ when $a = 5$ and $m = 2$.

function $C_{a,m,d}|_{\mathbb{R}}$ when $a = 1$. Figure 6 shows two examples of the real function $C_{1,m,d}|_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$, one example of an odd m in green, and one example of an even m in blue. If m is even, $C_{1,m,d}|_{\mathbb{R}}$ is strictly increasing from $y = 0$ at the critical point $x = -d$ to a maximum of $y = C_{1,m,d}(c_{m,d})$ at the critical point $x = c_{m,d}$, and then strictly decreasing from $y = C_{1,m,d}(c_{m,d})$ to $y = 0$ at $x = 0$. So $C_{1,m,d}|_{\mathbb{R}} : [-d, c_{m,d}] \to [0, C_{1,m,d}(c_{m,d})]$ is one-to-one and $C_{1,m,d}|_{\mathbb{R}} : [c_{m,d}, 0] \to [0, C_{1,m,d}(c_{m,d})]$ is one-to-one. If m is odd, $C_{1,m,d}|_{\mathbb{R}}$ is strictly decreasing from $y = 0$ at the critical point $x = -d$ to a minimum of $y = C_{1,m,d}(c_{m,d}) < 0$ at the critical point $c_{m,d}$, and then increases from $y = C_{1,m,d}(c_{m,d})$ to $y = 0$ at $x = 0$. We use these facts about $C_{1,m,d}$ in the following theorem to show the existence of parameter values for which all finite critical points lie in the immediate basin of 0.

Theorem 3.10. *For $C_{a,m,d}$, if $|a| \leq 1/m^m$, then all finite critical points lie in $A^*(C_{a,m,d}, 0)$.*

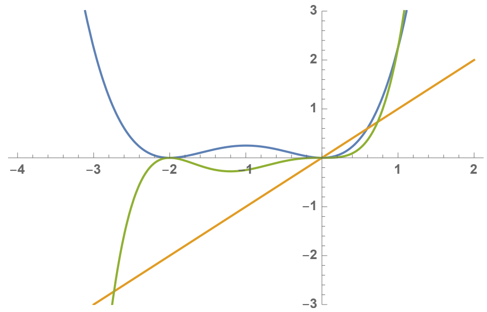


FIGURE 6. $C_{1,2,2}|_{\mathbb{R}}$ shown in blue, $C_{1,3,2}|_{\mathbb{R}}$ shown in green, and $y = x$ in orange.

Proof. Notice that $aC_{1,m,d}(z) = C_{a,m,d}(z)$ for any $a, z \in \mathbb{C}$. Then $C_{a,m,d}$ maps the real interval $[-d, 0]$ in a two-to-one fashion onto the line segment

$$\gamma = \{tC_{a,m,d}(c_{m,d}) :\ 0 \le t \le 1\}$$

from the origin to the complex number $C_{a,m,d}(c_{m,d})$.

Note that since $m \ge 2$, we have that $|a| < 1/e$. By Theorem 3.4, $\delta_0 = 1$.

Next, we show that the critical value $C_{a,m,d}(c_{m,d})$ lies in the disk of radius $\delta_0 = 1$. Notice that $md/(m+d) < m$, for all d. Then

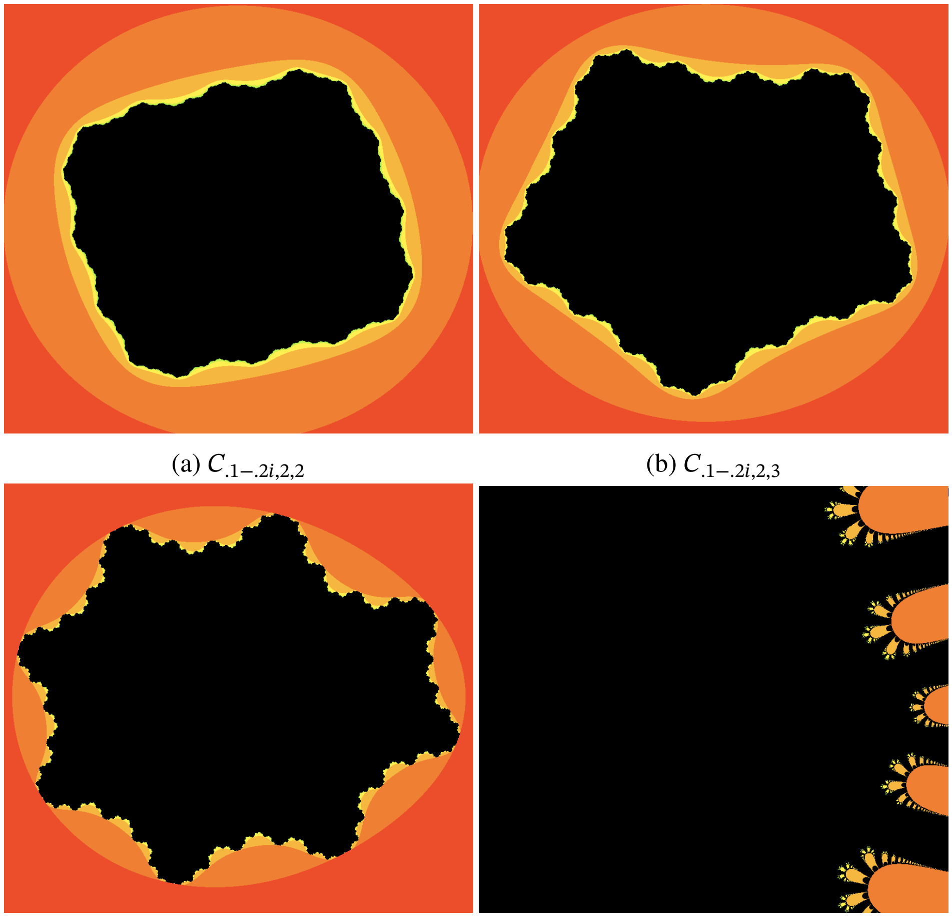
$$\begin{aligned}\left|C_{a,m,d}\left(c_{m,d}\right)\right| &= \left|a\left(\frac{-md}{m+d}\right)^m\left(1+\frac{-md}{m+d}\left(\frac{1}{d}\right)\right)^d\right|\\ &= |a|\left(\frac{md}{m+d}\right)^m\left(\frac{d}{m+d}\right)^d\\ &< |a|m^m \le 1 = \delta_0,\end{aligned}$$

since $|a| \le 1/m^m$.

Since $C_{a,m,d}(c_{m,d})$ lies within the zero-attracting radius, the entire ray γ lies within the zero attracting radius. This implies that the real interval $[-d, 0]$ containing all of the finite critical points lies in $A^*(C_{a,m,d}, 0)$. $\square$

We note Theorem 3.10 does not depend on d. Also, we observe that parameters a can satisfy both Theorems 3.6 and 3.10 simultaneously, and we show such examples in Figure 7 when $a = .1 - .2i$, so $|a| < 1/4$. These figures show $C_{.1-.2i,2,d}$ with $d = 2, 3, 6$ and $E_{.1-.2i,2}$. All critical points lie in the immediate basin of 0, but the black region to the left grows as d increases and is unbounded in the limit with $E_{.1-.2i,2}$. Theorem 3.8 implies that the interval $[1, \infty]$ iterates to infinity.

The following result follows immediately from the fact that $C_{a,m,d}$ is a polynomial. We state it as a theorem to compare with results from [11] about $E_{a,m}$ stated in Theorem 3.12.



(a) $C_{.1-.2i,2,2}$ (b) $C_{.1-.2i,2,3}$

(c) $C_{.1-.2i,2,6}$ (d) $E_{.1-.2i,2}$

FIGURE 7. Julia and Fatou sets for $C_{a,m,d}$ and $E_{a,m}$ when $a = 0.1 - 0.2i$ and $m = 2$.

Theorem 3.11. *Let* $C_{a,m,d}(z) = az^m(1 + z/d)^d$.

(1) *All bounded components of the Fatou set of* $C_{a,m,d}$ *are simply connected. The component of the Fatou set containing* ∞ *is simply or infinitely connected.*
(2) $A(C_{a,m,d}, 0)$ *has either one or infinitely many simply connected components.*
(3) *All the connected components of* $A(C_{a,m,d}, 0)$ *are bounded.*

Theorem 3.12. [11] *Let* $E_{a,m}(z) = az^m e^z$.

(1) *All connected components of the Fatou set of* $E_{a,m}$ *are simply connected.*
(2) $A(E_{a,m}, 0)$ *has either one or infinitely many connected components.*
(3) *All the connected components of* $A(E_{a,m}, 0)$ *except* $A^*(E_{a,m}, 0)$ *are unbounded.*

Next, we adapt a result for $S_{b,m,d}$ from [1] to our family $C_{a,m,d}$. We will use Theorem 3.14 in Section 4.

Theorem 3.13 ([1]). (1) *If a finite, nonzero critical point of $S_{b,m,d}$ belongs to $A^*(S_{b,m,d}, 0)$, then all other finite critical points also belong to $A^*(S_{b,m,d}, 0)$.*
(2) *Let $z_c \neq 0$ be a finite critical point of $S_{b,m,d}$. If $S_{b,m,d}(z_c)$ belongs to $A^*(S_{b,m,d}, 0)$, then z_c belongs to $A^*(S_{b,m,d}, 0)$.*

Theorem 3.14. *The critical point $c_{m,d} = -md/(m+d)$ belongs to the immediate basin $A^*(C_{a,m,d}, 0)$ if and only if the critical value $C_{a,m,d}(c_{m,d})$ belongs to $A^*(C_{a,m,d}, 0)$.*

Proof. Fix $a \neq 0$, $m \geq 2$, and $d \geq 2$. If

$$c_{m,d} = -md/(m+d) \in A^*(C_{a,m,d}, 0),$$

then clearly

$$C_{a,m,d}(c_{m,d}) \in A^*(C_{a,m,d}, 0),$$

since $A^*(C_{a,m,d}, 0)$ is forward invariant.

For the other direction, assume that $C_{a,m,d}(c_{m,d}) \in A^*(C_{a,m,d}, 0)$. Choose a parameter b such that $b^{m+d-1} = ad^{m-1}$. Let $z_1, \ldots, z_d$ be the finite, nonzero critical points of $S_{b,m,d}$. Since 0 is fixed by both the conjugacy and the semiconjugacy, we have $A^*(C_{a,m,d}, 0) = (d^d/a)^{1/(m+d-1)} A^*(M_{b,m,d}, 0)$ and $A^*(M_{b,m,d}, 0) = Pow_d(A^*(S_{b,m,d}, 0))$. It also follows that if $C_{a,m,d}(c_{m,d})$ belongs to $A^*(C_{a,m,d}, 0)$, then $S_{b,m,d}(z_c)$ belongs to $A^*(S_{b,m,d}, 0)$ for every finite, nonzero critical point z_c of $S_{b,m,d}$. By Theorem 3.13, since $S_{b,m,d}(z_c) \in A^*(S_{b,m,d}, 0)$, $z_c \in A^*(S_{b,m,d}, 0)$. Again, by the conjugacy and semiconjugacy, if $z_c \in A^*(S_{b,m,d}, 0)$, then $c_{m,d} \in A^*(C_{a,m,d}, 0)$. □

3.3. Absorbing and Superattracting Parameters. In Theorem 3.10, we showed that, for some parameters a, the free critical point belongs to $A^*(C_{a,m,d}, 0)$. In this section, we generalize to the concept of $-d$-absorbing parameters. It follows from the definition that $c_{m,d} \in A(C_{a,m,d}, 0)$ for these parameters a.

Definition 3.15. *We say that a parameter $a \in \mathbb{C}^*$ is $-d$-absorbing if the forward orbit of the critical point $c_{m,d}$ under the corresponding map $C_{a,m,d}$ contains the critical point $-d$.*

It is easy to solve for the parameter a such that $C_{a,m,d}(c_{m,d}) = -d$.

Theorem 3.16. *For each degree $d, m \geq 2$, the parameter*

$$a = -d\left(\frac{m+d}{-md}\right)^m \left(1 + \frac{m}{m+d}\right)^{-d}.$$

is a $-d$-absorbing parameter for $C_{a,m,d}$. Further, for m odd, we have

$$\lim_{d\to\infty} -d\left(\frac{m+d}{-md}\right)^m \left(1 + \frac{m}{m+d}\right)^{-d} = \infty,$$

and for m even, we have

$$\lim_{d\to\infty} -d\left(\frac{m+d}{-md}\right)^m\left(1+\frac{m}{m+d}\right)^{-d} = -\infty.$$

Proof. Fix m and d. Let

$$a = -d\left(\frac{m+d}{-md}\right)^m\left(1+\frac{m}{m+d}\right)^{-d}.$$

Then $C_{a,m,d}(c_{m,d}) = -d$. The critical orbit diagram is

$$c_{m,d} \longmapsto -d \longmapsto 0 \circlearrowleft .$$

To evaluate the limit, notice that

$$\begin{aligned}&\lim_{d\to\infty} -d\left(\frac{m+d}{-md}\right)^m\left(1+\frac{m}{m+d}\right)^{-d}\\ &= \lim_{d\to\infty}(-1)^{m+1}\frac{d}{m^m}\left[\left(1+\frac{m}{d}\right)^d\right]^m.\end{aligned}$$

Thus if m is odd, the limit is ∞, and if m is even, the limit is $-\infty$. □

To illustrate Theorem 3.16, the parameter $a = -729/64$ is -4-absorbing for the case $m = 2$ and $d = 4$. Figure 4c shows the dynamical plane for this example. The medium-sized black region in the middle of the picture is the Fatou component containing the critical point at $c_{2,4} = -4/3$. This component is mapped to the large black region on the left part of the figure, which is the Fatou component containing -4. That component is mapped to the large black Fatou component on the right, which contains the origin.

For $n > 1$, it is harder to solve $C^n_{a,m,d}(c_{m,d}) = -d$ for the parameter a. However, we can still show that these parameters escape to ∞ as $d \to \infty$.

Theorem 3.17. *Let $m \geq 2$ and $n \geq 1$ be integers. For all $M > 0$, there exists $D \in \mathbb{N}$ such that for all integers $d \geq D$, if $C^n_{a,m,d}(c_{m,d}) = -d$, then $|a| > M$.*

Proof. Fix an integer $m \geq 2$ and a real number $M > 0$. Aiming for the contrapositive, we use induction to first find $K_n > 0$ such that if $|a| \leq M$, then $|C^n_{a,m,d}(c_{m,d})| \leq K_n$, where K_n only depends on m and M (not on d).

For the base case, suppose $|a| \leq M$ and observe that

$$\begin{aligned}|C_{a,m,d}(c_{m,d})| &= |a|\left(\frac{md}{m+d}\right)^m\left(\frac{d}{m+d}\right)^d\\ &\leq Mm^m.\end{aligned}$$

Thus, we take $K_1 = Mm^m > 0$.

For the induction step, suppose that $|a| \leq M$ and for some $n \geq 1$ we have $K_n > 0$ (only depending on m and M) such that $|C^n_{a,m,d}(c_{m,d})| \leq K_n$. Then

$$\begin{aligned}|C^{n+1}_{a,m,d}(c_{m,d})| &\leq |a| \left|C^n_{a,m,d}(c_{m,d})\right|^m \left(1 + \frac{|C^n_{a,m,d}(c_{m,d})|}{d}\right)^d \\ &\leq M K_n^m e^{K_n}.\end{aligned}$$

Since $K_n > 0$ only depends on m and M, the bound $K_{n+1} = M K_n^m e^{K_n} > 0$ also only depends on m and M.

To finish the proof, fix $n \in \mathbb{N}$ and take $D > K_n$. Suppose $d \geq D$ and $|a| \leq M$. Then we have $|C^n_{a,m,d}(c_{m,d})| \leq K_n < d$, which implies that $C^n_{a,m,d}(c_{m,d}) \neq -d$. □

In the following theorem, we find parameters a for which the free critical point is superattracting.

Theorem 3.18. *For each $d, m \geq 2$, the parameter*

$$a = \left(\frac{-md}{m+d}\right)^{1-m} \left(1 + \frac{-md}{d(m+d)}\right)^{-d}.$$

is a superattracting parameter for $C_{a,m,d}$. The parameter $a = (-m)^{1-m} e^m$ is a superattracting parameter for $E_{a,m}$, and

$$\lim_{d\to\infty} \left(\frac{-md}{m+d}\right)^{1-m} \left(1 + \frac{-md}{d(m+d)}\right)^{-d} = (-m)^{1-m} e^m.$$

Proof. Fix m and d. Let

$$a = \left(\frac{-md}{m+d}\right)^{1-m} \left(1 + \frac{-md}{d(m+d)}\right)^{-d}.$$

Then

$$\begin{aligned}C_{a,m,d}(c_{m,d}) &= a \left(\frac{-md}{m+d}\right)^m \left(\frac{d}{m+d}\right)^d \\ &= \left(\frac{-md}{m+d}\right)^{1-m} \left(1 + \frac{-md}{d(m+d)}\right)^{-d} \left(\frac{-md}{m+d}\right)^m \left(\frac{d}{m+d}\right)^d \\ &= \left(\frac{-md}{m+d}\right) = c_{m,d}.\end{aligned}$$

In addition,

$$\begin{aligned}&\lim_{d\to\infty} \left(\frac{-md}{m+d}\right)^{1-m} \left(1 + \frac{-md}{d(m+d)}\right)^{-d} \\ &= \lim_{d\to\infty} \left(\frac{-md}{m+d}\right)^{1-m} \left(1 + \frac{m}{d}\right)^d \\ &= (-m)^{1-m} e^m.\end{aligned}$$

Finally, when $a = (-m)^{1-m} e^m$, the equation

$$E_{a,m}(-m) = (-m)^{1-m} e^m (-m)^m e^{-m} = -m$$

shows that $-m$ is a superattracting fixed point for $E_{a,m}$. □

4. Results in the parameter plane

Next, we move to a discussion of the parameter planes of $C_{a,m,d}$ and $E_{a,m}$. We begin by showing that both parameter spaces exhibit symmetry.

Theorem 4.1. *The parameter spaces of $C_{a,m,d}$ and $E_{a,m}$ are symmetric with respect to reflection over the real axis.*

Proof. The critical point $c_{m,d}$ of the polynomial function is real and does not depend on a. The proof of Theorem 3.1 shows that the dynamical behavior of $c_{m,d}$ under $C_{a,m,d}$ is the same as under $C_{\overline{a},m,d}$ and the dynamical behavior of $-m$ under $E_{a,m}$ is the same as under $E_{\overline{a},m}$. □

For some parameter values, the orbit of the free critical point converges to the superattracting fixed point at the origin, and we say that the critical point has been *captured*. When f is one of the functions $C_{a,m,d}$ or $E_{a,m}$, and c_f is the free critical point, we define the *capture zones* by

$$\mathcal{H}^n(f) = \{\lambda \in \mathbb{C} \mid f^n(c_f) \in A^*(f,0) \text{ and } f^{n-1}(c_f) \notin A^*(f,0)\}.$$

The *main capture zone* is when the free critical point lies in the immediate basin of attraction of the origin, so

$$\mathcal{H}^0(f) = \left\{\lambda \in \mathbb{C} \mid c_f \in A^*(f,0)\right\}.$$

The following is an immediate result of Theorem 3.14.

Theorem 4.2. *For $C_{a,m,d}$, $\mathcal{H}^1(C_{a,m,d}) = \emptyset$.*

We begin by proving that the main capture zone contains a disk around the origin. Note that the radius ρ of the disk does not change with d and only depends on m.

Theorem 4.3. *For any integers $m \geq 2$ and $d \geq 2$, there exists $\rho = 1/m^m$ such that $D_\rho \subset \mathcal{H}^0(C_{a,m,d})$.*

Proof. Fix m and let $\rho = 1/m^m$. For $a \in D_\rho$, Theorem 3.10 implies that all finite critical points lie in $A^*(C_{a,m,d}, 0)$. Therefore, $D_\rho \subset \mathcal{H}^0(C_{a,m,d})$. □

Next, we show that the capture zones for $C_{a,m,d}$ are bounded for all $n \geq 0$.

Theorem 4.4. *For any integers $m \geq 2$ and $d \geq 2$, there exists ρ' such that $\mathcal{H}^n(C_{a,m,d}) \subset D_{\rho'}$ for all $n \geq 0$. Thus all the connected components of $\mathcal{H}^n(C_{a,m,d})$ are bounded.*

Proof. We fix m and d. Using Theorem 3.3, we define

$$\rho' = max\left\{2d\left(\frac{md}{m+d}\right)^{-m}\left(\frac{d}{m+d}\right)^{-d},\right.$$

$$2^{\frac{m-1}{m}} d^{\frac{d+m}{m}} \left(\frac{md}{m+d}\right)^{1-m} \left(\frac{d}{m+d}\right)^{\frac{d(1-m)}{m}} \Bigg\}.$$

Then for any a with $|a| > \rho'$, $C^n_{a,m,d}(c_{m,d}) \to \infty$. Thus the orbit of $c_{m,d}$ always lies outside of $A^*(C_{a,m,d}, 0)$, and $\mathcal{H}^n(C_{a,m,d})$ is bounded by ρ' for all $n \geq 0$. $\square$

Theorem 4.5 follows from applications of the maximum principle, as given in Lemma 1.9 and Lemma 1.12 of [24] for $S_{b,m,d}$ with $m \geq 2$ and $d = 1$. These proofs apply to $C_{a,m,d}$ with minor substitutions, and we include them here for completeness.

Theorem 4.5. (1) *The main capture zone $\mathcal{H}^0$ is connected.*
(2) *For integers $n = 0$ and $n \geq 2$, each connected component of $\mathcal{H}^n$ is simply connected.*

Proof. The proof of both parts is a proof by contradiction, using the maximum principle.

To prove part 1, suppose that $\mathcal{H}^0$ is not connected, and hence has a connected component U that does not contain 0. Take $a_1 \in \partial U$. Then the orbit of $c_{m,d}$ does not converge to 0. Since 0 is a superattracting fixed point, there exists some radius $r > 0$ such that $|C^k_{a_1,m,d}(c_{m,d})| > r$ for all integers $k \geq 0$. Take $a_2 \in U$. Since $c_{m,d}$ is in the attracting basin of 0, there exists an integer N such that for all $k \geq N$, $|C^k_{a_2,m,d}(c_{m,d})| < r/2$. For any $a \in U$, $c_{m,d} \in A^*(C_{a,m,d}, 0)$. The Böttcher coordinates around 0 extend to a neighborhood with $c_{m,d}$ on the boundary. The critical point $-d$ is contained in the interior of a preimage of this neighborhood under $C_{a,m,d}$. Hence, $C^k_{a,m,d}(c_{m,d}) \neq -d$ for all integers $k \geq 0$. Since the only preimages of 0 are 0 and $-d$, and since $a = 0$ is not in U, $C^k_{a,m,d}(c_{m,d}) \neq 0$ for all integers $k \geq 0$. In particular, this means that $a \mapsto 1/C^N_{a,m,d}(c_{m,d})$ is well-defined in a neighborhood of the closure of U, and the maximum principle is violated for this map on U.

To prove part 2, suppose that V is a connected component of $\mathcal{H}^n$, for some integer $n = 0$ or $n \geq 2$. For the sake of contradiction, suppose V is not simply connected, so there exists a bounded component B of $\mathbb{C}\backslash V$. Take $a \in \partial V \cap B$. As before, the orbit of $c_{m,d}$ does not converge to 0, so there exists some radius $r > 0$ such that $|C^k_{a,m,d}(c_{m,d})| > r$ for all integers $k \geq 0$. Since V is path connected, there exists a simple closed curve γ in V that surrounds B. Since every parameter b in the curve γ is in $\mathcal{H}^n$, the iterates $C^k_{b,m,d}(c_{m,d})$ converge to 0. Then, there exists an integer N such that for all $k \geq N$ and all parameters b in γ, $|C^k_{b,m,d}(c_{m,d})| < r/2$. This contradicts the maximum principle for the function $a \mapsto C^N_{a,m,d}(c_{m,d})$ on the bounded open set enclosed by γ. $\square$

We can compare Theorems 4.2, 4.3, and 4.4 with Theorem 4.6, noting that we obtain many similar results for the families $C_{a,m,d}$ and $E_{a,m}$. The main difference is that, when $n \geq 2$, $\mathcal{H}^n(C_{a,m,d})$ is bounded and $\mathcal{H}^n(E_{a,m})$ is unbounded.

Theorem 4.6. [12] *The following statements hold:*

(1) *The critical point* $-m$ *belongs to* $A^*(E_{a,m}, 0)$ *if and only if the critical value* $E_{a,m}(-m)$ *belongs to* $A^*(E_{a,m}, 0)$*. Hence* $\mathcal{H}^1(E_{a,m}) = \emptyset$*.*
(2) *Let* $\rho = \min(1/e, (e/m)^m)$ *and* $\rho' = (e/(m-1))^m$*. Then*

$$D_\rho \subset \mathcal{H}^0(E_{a,m}) \subset D_{\rho'}$$

where $D_r = \{z \in \mathbb{C} || z |< r\}$*.*
(3) *The main capture zone* $\mathcal{H}^0(E_{a,m})$ *is connected and simply connected.*
(4) *Let* $n \geq 2$*. All the connected components of* $\mathcal{H}^n(E_{a,m})$ *are simply connected and unbounded.*

In addition, when m is even, capture zones never intersect the ray $a > 4$ lying on the positive axis.

Theorem 4.7. *For even* m *and for all* $n \geq 0$*,* $\mathcal{H}^n(C_{a,m,d})$ *and* $\mathcal{H}^n(E_{a,m})$ *never intersect the ray* $[4, \infty)$*.*

Proof. Theorems 3.8 and 3.9 imply that when m is even, the real parameters with $a > 4$ have free critical orbit heading to infinity. □

A sequence of parameter spaces for $C_{a,2,d}$ with $d = 2, 3$, and 6 are shown in Figures 8a, 8b ,and 8c, and Figure 8d shows the parameter space for $E_{a,2}$. The capture zones are shown in orange, the black regions are parameters for which the free critical point is bounded but does not approach 0, and the white regions are parameters for which the free critical point heads to infinity and the Julia set is disconnected. First, note that Theorems 4.2 and 4.6 show that $\mathcal{H}^0(C_{a,m,d})$ is bounded and contains the origin for all d. Although we know that $\mathcal{H}^n(C_{a,m,d})$ is bounded for all d from Theorem 4.4, we can see in the sequence of parameter spaces in Figure 8 that the regions $\mathcal{H}^n(C_{a,m,d})$ grow as d increases. From Theorem 3.16, we can locate the regions $\mathcal{H}^2(C_{a,m,d})$ in each parameter space as the largest orange region intersecting the real axis on the left, and this absorbing parameter heads to $-\infty$ as d increases. We also notice that, in all parameter spaces shown in Figure 8, a ray along the positive axis is always colored white as indicated in Theorem 4.7.

Next, we discuss the properties of the attracting basins for parameters in the capture zones. Again, our goal is to compare results of $C_{a,m,d}$ with previously proven results from [11, 12]. Recall that a *quasicircle* is the image of a circle under a quasiconformal map.

Theorem 4.8. *Let* $\mathcal{H}^n(C_{a,m,d}), \mathcal{H}^0(C_{a,m,d})$ *be the capture zones for* $C_{a,m,d}$*. The following statements hold:*

(1) *If* $a \in \mathcal{H}^0(C_{a,m,d})$ *then* $A(C_{a,m,d}, 0) = A^*(C_{a,m,d}, 0)$*. However if* $a \notin \mathcal{H}^0(C_{a,m,d})$*, then* $A(C_{a,m,d}, 0)$ *has infinitely many connected components.*
(2) *If* $a \in \mathcal{H}^n(C_{a,m,d})$ *for any* $n \geq 0$ *the boundary of* $A^*(C_{a,m,d}, 0)$ *is a quasicircle.*
(3) *If* $a \in \mathcal{H}^0(C_{a,m,d})$ *then the Julia set is a quasicircle.*

Proof. Suppose $a \in \mathcal{H}^0(C_{a,m,d})$. Thus, $c_{m,d} = -md/(m+d) \in A^*(C_{a,m,d}, 0)$, which implies that $C_{a,m,d}(c_{m,d}) \in A^*(C_{a,m,d}, 0)$. Let γ be a simple curve in $A^*(C_{a,m,d}, 0)$ from $C_{a,m,d}(c_{m,d})$ to 0. Since $C_{a,m,d}(c_{m,d})$ and 0 are the only finite critical values, the preimage of γ under $C_{a,m,d}$ consists of $m+d$ simple curves that only intersect at endpoints. Since $c_{m,d}$ is a simple critical point, there are two curves with an endpoint at $c_{m,d}$. If either of these curves has its other endpoint at $-d$, then $-d \in A^*(C_{a,m,d}, 0)$.

Suppose neither of the curves has its other endpoint at $-d$, then both curves have their other endpoint at 0. Hence, the union of the two curves is a simple closed curve that encloses an open region R. By the Open Mapping Theorem, $C_{a,m,d}(R)$ is an open set, which implies that it contains no boundary points. Hence, the boundary of $C_{a,m,d}(R)$ must be contained in γ. However, γ is a simple, not closed, curve. The only open set that has γ as its boundary is $\hat{\mathbb{C}}\backslash\gamma$. The set $C_{a,m,d}(R)$ cannot contain $\hat{\mathbb{C}}\backslash\gamma$ because R is contained in $A^*(C_{a,m,d}, 0)$, which does not contain ∞. By contradiction, the preimage of γ under $C_{a,m,d}$ must contain a curve in $A^*(C_{a,m,d}, 0)$ that joins $c_{m,d}$ to $-d$, so $-d$ is in $A^*(C_{a,m,d}, 0)$. Since $-d \in A^*(C_{a,m,d}, 0)$, the immediate basin is completely invariant, so $A(C_{a,m,d}, 0) = A^*(C_{a,m,d}, 0)$.

Suppose $a \notin \mathcal{H}^0(C_{a,m,d})$. Then $c_{m,d}$ and 0 are in separate components of $A(C_{a,m,d}, 0)$. In particular, $A^*(C_{a,m,d}, 0)$ is not completely invariant. Therefore, $A(C_{a,m,d}, 0)$ has infinitely many components.

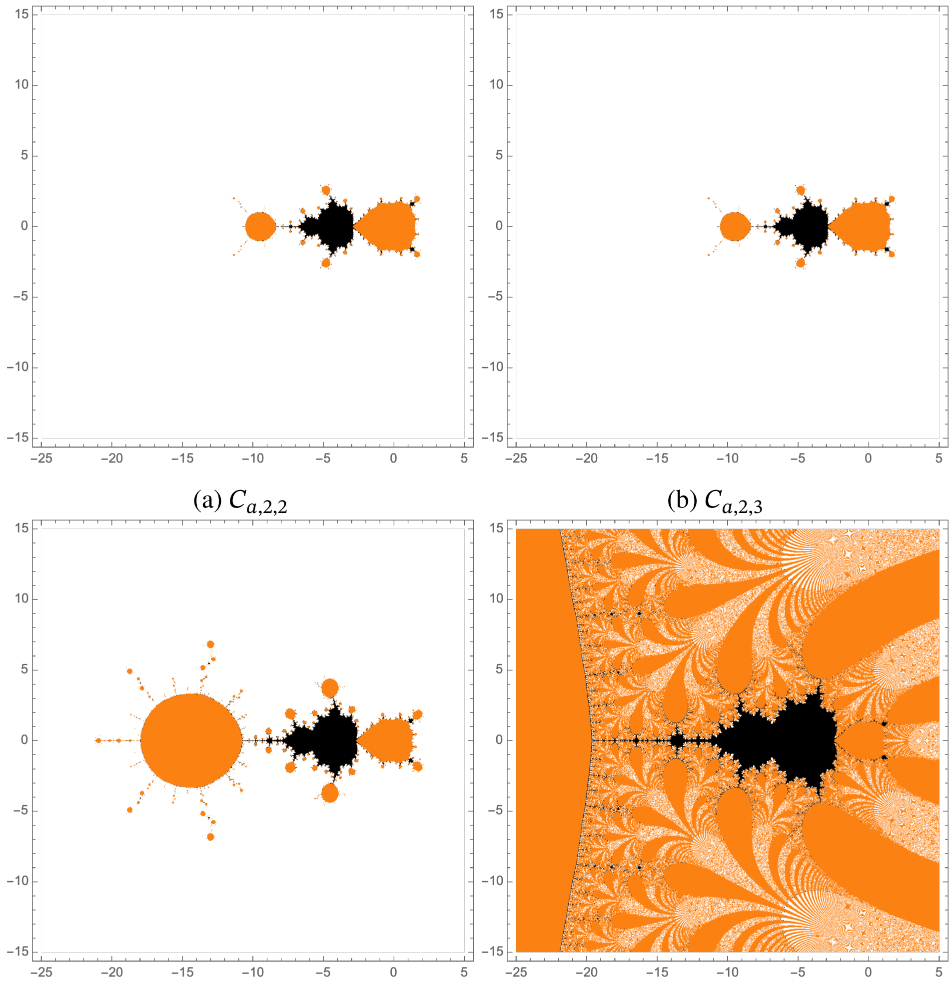
Part (2) follows immediately from [25] since $C_{a,m,d}$ is a polynomial and is hyperbolic since all critical points lie in superattracting components. For (3), we note that part (1) implies that $F(C_{a,m,d})$ consists of two completely invariant components so $J(C_{a,m,d})$ is the boundary between the two. Then part (2) gives that $J(C_{a,m,d})$ is a quasicircle. □

Note that sets that are quasicircles are necessarily locally connected. Theorem 4.8(2) only implies that the boundary of $A^*(C_{a,m,d})$ is locally connected. To discuss the local connectivity of the Julia set in this case, we begin by stating some definitions and results regarding renormalizable polynomials. For more information on this topic, see [8].

Definition 4.9. *A polynomial-like map $f : U \to V$ is a map between topological disks such that $\overline{U}$ is a compact subset of V and for every compact $K \subset V$, $f^{-1}(K)$ is compact.*

If $f : U \to V$ is a polynomial-like map, then the filled Julia set of f is $K(f) = \bigcap_{n\geq 0} f^{-n}(U)$ and the Julia set of f, $J(f)$, is the boundary of $K(f)$. Just as in the case of polynomials, $J(f)$ is connected if, and only if, all critical points of f are in $K(f)$.

Definition 4.10. *A map f is renormalizable at a distinguished critical point ω if and only if there exist $n \geq 1$ and topological disks U and V around ω such that $f^n : U \to V$ is a polynomial-like map with connected Julia set and ω is the only critical point of f in U.*



(a) $C_{a,2,2}$ (b) $C_{a,2,3}$

(c) $C_{a,2,6}$ (d) $E_{a,2}$

FIGURE 8. Parameter spaces for $C_{a,m,d}$ and $E_{a,m}$ when $m = 2$.

Theorem 4.11 ([22]). *Let $S_{-b,m,d}(z) = z^m(z^d - b)$, $m \geq 2$, $d \geq 1$, satisfy the following conditions:*

(1) *the Julia set of $S_{-b,m,d}$ is connected;*
(2) *$S_{-b,m,d}$ is not renormalizable away from zero.*

Then the Julia set of $S_{-b,m,d}$ is locally connected.

After using Theorem 4.11 to find parameters b such that $S_{-b,m,d}$ is locally connected, we can use Theorem 4.12 to find parameters a such that $C_{a,m,d}$ is locally connected.

Theorem 4.12. *Let $a, b \in \mathbb{C}^*$ such that $a = \dfrac{(-b)^{m+d-1}}{d^{m-1}}$. If the Julia set of $S_{-b,m,d}$ is locally connected, then the Julia set of $C_{a,m,d}$ is locally connected.*

Proof. Let $a, b \in \mathbb{C}^*$ such that $a = \frac{(-b)^{m+d-1}}{d^{m-1}}$. Suppose the Julia set of $S_{-b,m,d}$ is locally connected. Since the Julia set of $S_{-b,m,d}$ is a d-fold cover of the Julia set of $M_{b,m,d}$ under the map $Pow_d(z) = z^d$ and the Julia set of $M_{b,m,d}$ is topologically isomorphic to the Julia set of $C_{a,m,d}$, the Julia set of $C_{a,m,d}$ is locally connected. □

Finally, we present results on $E_{a,m}$ to compare to Theorem 4.8.

Theorem 4.13. [11, 12] *Let $\mathcal{H}^n(E_{a,m}), \mathcal{H}^0(E_{a,m})$ be the capture zones for $E_{a,m}$. The following statements hold:*

(1) *If $a \in \mathcal{H}^0(E_{a,m})$, then*

$$A(E_{a,m}, 0) = A^*(E_{a,m}, 0).$$

However, if $a \notin \mathcal{H}^0(E_{a,m})$ then $A(E_{a,m}, 0)$ has infinitely many connected components.

(2) *If $a \in \mathcal{H}^n(E_{a,m})$ for any $n \geq 2$ the boundary of $A^*(E_{a,m}, 0)$ is a quasicircle.*

(3) *If $a \in \mathcal{H}^0(E_{a,m})$, then the boundary of $A^*(E_{a,m}, 0)$ (which is equal to the Julia set) is a Cantor bouquet and thus is disconnected and not locally connected.*

The primary difference between Theorems 4.8 and 4.13 lie in part (3) of the statements. We note that Figure 7 (a),(b),(c) illustrates Theorem 4.8 (3). By Theorem 3.10, all critical values are contained in the immediate attracting basin, so $a = .1 - .2i \in \mathcal{H}^0(C_{.1-.2i,2,d})$ and the Julia sets of $C_{.1-.2i,2,d}$ are locally connected for all d. Figure 7 (d) illustrates Theorem 4.13 (3). By Theorem 4.6(2), $a = .1 - .2i \in \mathcal{H}^0(E_{.1-.2i,2})$, so the Julia set of $E_{.1-.2i,2}$ is a Cantor bouquet and not locally connected.

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