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Solitons of the spacelike mean curvature flow in generalized Robertson-Walker spacetimes

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ABSTRACT. Our purpose in this paper is to study solitons of the spacelike mean curvature flow in a generalized Robertson-Walker (GRW) spacetime $-I \times_f M^n$. Under suitable constraints on the warping function f and on the curvatures of the Riemannian fiber M^n, we apply suitable maximum principles in order to obtain nonexistence and uniqueness results concerning these solitons. Applications to standard models of GRW spacetimes, namely, the Einstein-de Sitter spacetime, steady state type spacetimes, de Sitter and anti-de Sitter spaces, are given. Furthermore, we establish new Calabi-Bernstein type results related to entire spacelike mean curvature flow graphs constructed over the Riemannian fiber of the ambient spacetime.

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1. Introduction

Let $\mathbb{R}^{n+1}_1$ be the $(n+1)$-dimensional Minkowski space $(\mathbb{R}^{n+1}_1, \bar{g})$ with its standard Lorentzian metric

$$\bar{g} = -dx_1^2 + \sum_{i=2}^{n+1} dx_i^2.$$

Let $\psi : \Sigma^n \to \mathbb{R}^{n+1}_1$ be a spacelike immersion (which means that it has a Riemannian induced metric) in the Minkowski space. The *spacelike mean curvature flow* associated to ψ is a family of smooth spacelike immersions $\Psi_t = \Psi(t, \cdot) : \Sigma^n \to \mathbb{R}^{n+1}_1$ with corresponding images $\Sigma^n_t = \Psi_t(\Sigma^n)$ satisfying the following evolution equation

$$\begin{cases} \dfrac{\partial \Psi}{\partial t} = \vec{H} \\ \Psi(0, x) = \psi(x) \end{cases}$$

on some time interval, where $\vec{H}$ stands for the (non-normalized) mean curvature vector of the spacelike submanifold Σ^n_t in $\mathbb{R}^{n+1}_1$.

Mean curvature flow in the Minkowski space and, more generally, in a Lorentzian manifold has been extensively studied by several authors (see, for instance, [1, 18, 19, 20, 21, 22, 23, 28, 29, 31, 32, 34, 40]) and, according to [19], an important justification for this interest is the fact that spacelike translating solitons can be regarded as a natural way of foliating spacetimes by almost null like hypersurfaces. Particular examples may give insight into the structure of certain spacetimes at null infinity and have possible applications in General Relativity (for more details, see [19]).

More recently, Lambert and Lotay [33] proved long-time existence and convergence results for spacelike solutions to mean curvature flow in the n-dimensional pseudo-Euclidean space $\mathbb{R}^n_m$ of index m, which are entire or defined on bounded domains and satisfying Neumann or Dirichlet boundary conditions. In [24], Guilfoyle and Klingenberg proved the longtime existence for mean curvature flow of a smooth n-dimensional spacelike submanifold of an $(n+m)$-dimensional manifold whose metric satisfies the so-called timelike curvature condition. Meanwhile, Alías, de Lira and Rigoli [10] introduced the general definition of self-similar mean curvature flow in a Riemannian manifold $\overline{M}^{n+1}$ endowed with a vector field K and establishing the corresponding notion of mean curvature flow soliton. In particular, when $\overline{M}^{n+1}$ is a Riemannian warped product of the type $I \times_f M^n$ and $K = f(t)\partial_t$, they applied weak maximum principles to guarantee that a complete n-dimensional mean curvature flow soliton is a slice of $\overline{M}^{n+1}$. In [17], Colombo, Mari and Rigoli also studied some properties of mean curvature flow solitons in general Riemannian manifolds and in warped products, focusing on splitting and rigidity results under various geometric conditions, ranging from the stability of the soliton to the fact that the image of its

Gauss map be contained in suitable regions of the sphere. Moreover, they also investigated the case of entire mean curvature flow graphs. When the ambient space is a Lorentzian product space, the first and third authors [14] established nonexistence results for complete spacelike translating solitons under suitable curvature constraints on the curvatures of the Riemannian base of the ambient space. In particular, they obtained Calabi-Bernstein type results for entire translating graphs constructed over this Riemannian base. For this, they proved a version of the Omori-Yau's maximum principle for complete spacelike translating solitons. Besides, they also constructed new examples of rotationally symmetric spacelike translating solitons embedded in such an ambient space.

Proceeding with this picture, here we extend the techniques developed in [6, 10, 13, 14, 17] to study complete spacelike mean curvature flow solitons immersed in a generalized Robertson-Walker (GRW) spacetime, that is, a Lorentzian warped product $-I \times_f M^n$ with 1-dimensional negative definite base I and n-dimensional Riemannian fiber M^n. Under suitable constraints on the warping function f and on the curvatures of M^n, we apply suitable maximum principles in order to obtain nonexistence and uniqueness results concerning these solitons. Applications to standard GRW spacetimes as, for instance, the Einstein-de Sitter and steady state type spacetimes, are given. Furthermore, we establish new Calabi-Bernstein type results related to entire spacelike mean curvature flow graphs constructed over the Riemannian fiber of the ambient spacetime.

2. Spacelike hypersurfaces

Let (M^n, g_M) be a connected, n-dimensional, oriented Riemannian manifold, $I \subset \mathbb{R}$ an open interval and $f : I \to \mathbb{R}$ a positive smooth function. Also, in the product manifold $\overline{M}^{n+1} = I \times M^n$ let π_I and π_M denote the canonical projections onto the factors I and M^n, respectively.

The class of Lorentzian manifolds which will be of our concern here is the one obtained by furnishing $\overline{M}^{n+1}$ with the Lorentzian metric $\overline{g}$ given by

$$\overline{g}(u, v)_p = -g_I((\pi_I)_* v, (\pi_I)_* v)_{\pi_I(p)} + (f \circ (\pi_I))^2(p) g_M((\pi_M)_* v, (\pi_M)_* v)_{\pi_M(p)},$$

for all $p \in \overline{M}^{n+1}$ and $u,\ v \in T_p\overline{M}$, where g_I stands for the standard metric of $I \subset \mathbb{R}$. Along this work, we will simply write

$$\overline{M}^{n+1} = -I \times_f M^n. \tag{1}$$

According to the nomenclature established in [12], we say that $\overline{M}^{n+1}$ is a *generalized Robertson Walker* (GRW) spacetime with warping function f and Riemannian fiber M^n. When M^n has constant sectional curvature, (1) has been known in the mathematical literature as a Robertson-Walker (RW) spacetime, an allusion to the fact that, for $n = 3$, it is an exact solution of Einsteim's field equations (see, for instance, [37, Chapter 12]).

In this setting, we will consider the timelike conformal closed vector field $\mathcal{K} = f(\pi_I)\partial_t$ globally defined on $\overline{M}$, where $\partial_t = \frac{\partial}{\partial_t}$ stands for the coordinate timelike vector field tangent to I. From the relationship between the Levi-Civita connections of $\overline{M}$ and those ones of I and M^n (see [37, Proposition 7.35]), it follows that

$$\overline{\nabla}_V \mathcal{K} = f'(\pi_I)V, \tag{2}$$

for all $V \in \mathfrak{X}(\overline{M})$, where $\overline{\nabla}$ is the Levi-Civita connection of $\overline{g}$.

Let Σ^n be an n-dimensional connected manifold. A smooth immersion $\psi : \Sigma^n \to \overline{M}^{n+1}$ is said to be a *spacelike hypersurface* if Σ^n, furnished with the metric g induced from $\overline{g}$ via ψ, is a Riemannian manifold. We will denote by ∇ the Levi-Civita connection of g. Since $\overline{M}$ is time-orientable, it follows from the connectedness of Σ^n that one can uniquely choose a globally defined timelike unit vector field $N \in \mathfrak{X}^{\perp}(\Sigma)$, having the same time-orientation of ∂_t, that is, such that $\overline{g}(N, \partial_t) < 0$. In this case, one says that N is the *future-pointing Gauss map* of Σ^n and will always assume such a timelike orientation for Σ^n. From the inverse Cauchy-Schwarz inequality (see [37, Proposition 5.30]), we have that $\overline{g}(N, \partial_t) \leq -1$, with the equality holding at a point $p \in \Sigma^n$ if, and only if, $N = \partial_t$ at p.

We will denote by A and $H = -\mathrm{trace}(A)$ the shape operator and the (non-normalized) mean curvature function of the spacelike hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to its future-pointing Gauss map N. Throughout this paper, the mean curvature H taken with respect to such a choice of orientation N will be called the *future mean curvature* of Σ^n. In particular, for a fixed $t_* \in I$, from [7, Example 5.6] we have that the *slice* $\{t_*\}\times M^n$ has constant future mean curvature $H = n\frac{f'(t_*)}{f(t_*)}$ with respect to $N = \partial_t$.

Now, we consider two particular functions naturally attached to a spacelike hypersurface Σ^n immersed into a GRW spacetime $\overline{M}^{n+1} = -I \times_f M^n$, namely, the height function $h = (\pi_I)|_\Sigma$ and the hyperbolic angle function $\Theta = \overline{g}(N, \partial_t)$, where we recall that N denotes the future-pointing Gauss map of Σ^n and, consequently, $\Theta \leq -1$. A simple computation shows that

$$\overline{\nabla}\pi_I = -\overline{g}(\overline{\nabla}\pi_I, \partial_t)\partial_t = -\partial_t. \tag{3}$$

So, from (3) we have

$$\nabla h = (\overline{\nabla}\pi_I)^\top = -\partial_t^\top = -\partial_t - \Theta N. \tag{4}$$

Thus, (4) gives the following relation

$$|\nabla h|^2 = \Theta^2 - 1, \tag{5}$$

where $|\,.\,|$ stands for the norm of a tangent vector field on Σ^n in the metric g.

On the other hand, from (2) we have that

$$\overline{\nabla}_V \partial_t = \frac{f'(\pi_I)}{f(\pi_I)}\{V + \bar{g}(V, \partial_t)\partial_t\}, \tag{6}$$

for any tangent vector V on $\overline{M}^{n+1}$.

Hence, from (4) and (6) we deduce that, for any $X \in \mathfrak{X}(\Sigma)$, the Hessian of h in the metric g is given by

$$\begin{aligned} \nabla^2 h(X,X) &= g(\nabla_X \nabla h, X) \\ &= -\frac{f'(h)}{f(h)}\{|X|^2 + g(X,\nabla h)^2\} + g(AX,X)\Theta. \end{aligned} \tag{7}$$

Hence, from (7) we obtain that the Laplacian of h in the metric g is

$$\Delta h = -\frac{f'(h)}{f(h)}\{n + |\nabla h|^2\} - H\Theta. \tag{8}$$

3. Spacelike mean curvature flow solitons

We recall that the spacelike mean curvature flow $\Psi : [0,T) \times \Sigma^n \to \overline{M}^{n+1}$ of a spacelike hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ in a $(n+1)$-dimensional Lorentzian manifold $\overline{M}^{n+1}$, satisfying $\Psi(0,\cdot) = \psi(\cdot)$, looks for solutions of the equation

$$\frac{\partial \Psi}{\partial t} = \vec{H},$$

where $\vec{H}(t,\cdot)$ is the (non-normalized) mean curvature vector of $\Sigma_t^n = \Psi(t,\Sigma^n)$ (see, for instance, [33]). In our context, according to [10, Definition 1.1] and [17, Definition 1.1], a spacelike hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ immersed in a GRW spacetime $\overline{M}^{n+1} = -I \times_f M^n$ is said a *spacelike mean curvature flow soliton* with respect to $\mathcal{K} = f(t)\partial_t$ and with *soliton constant* $c \in \mathbb{R}$ if its (non-normalized) future mean curvature function satisfies

$$H = cf(h)\Theta. \tag{9}$$

In fact, considering that Ψ is a self-similar mean curvature flow with respect to some vector field X, we can reason as in [10, Proposition 2.1] to deduce that the corresponding mean curvature vector satisfies

$$\vec{H} = cX^{\perp},$$

for some constant $c \in \mathbb{R}$. In our setting, X is equal to $\mathcal{K} = f(t)\partial_t$ and, hence, assuming that $\psi : \Sigma^n \to \overline{M}^{n+1}$ satisfies equation (9) means that it is a solution of the mean curvature flow evolution equation.

Adopting the terminology introduced in [10] and [17], we will also consider the *soliton function*

$$\zeta_c(t) = nf'(t) + cf^2(t). \tag{10}$$

So, each slice $M_{t_*} = \{t_*\} \times M^n$ is a spacelike mean curvature flow soliton with respect to $\mathcal{K} = f(t)\partial_t$ and with soliton constant c given by

$$c = -n\frac{f'(t_*)}{f(t_*)^2}. \tag{11}$$

Moreover, t_* is implicitly given by the condition $\zeta_c(t_*) = 0$.

We close this section quoting important examples which will be addressed along the next sections.

Example 3.1. The 4-dimensional Einstein-de Sitter spacetime is modeled by the GRW spacetime $-\mathbb{R}^+ \times_{t^{\frac{2}{3}}} \mathbb{R}^3$, where $\mathbb{R}^3$ stands for the 3-dimensional Euclidean space endowed with its canonical metric. This spacetime is a classical exact solution to the Einstein field equation without cosmological constant. It is an open Friedmann-Robertson-Walker model, which incorporates homogeneity and isotropy (the cosmological principle) and permitted expansion (for more details, see [37, Chapter 12]). Here, we consider the $(n+1)$-dimensional Einstein-de Sitter spacetime $-\mathbb{R}^+ \times_{t^{\frac{2}{3}}} \mathbb{R}^n$. From (11) we conclude that the slice

$$\{(-\frac{2n}{3c})^{\frac{3}{5}}\} \times \mathbb{R}^n$$

is the only one that is a spacelike mean curvature flow soliton with respect to $\mathcal{K} = t^{\frac{2}{3}}\partial_t$ and with soliton constant $c < 0$.

Example 3.2. According to the terminology introduced by Albujer and Alías [4], a GRW spacetime $-\mathbb{R} \times_{e^t} M^n$ is called a *steady state type* spacetime. This terminology is due to the fact that the steady state model of the universe $\mathcal{H}^4$, proposed by Bondi-Gold [16] and Hoyle [26] when looking for a model of the universe which looks the same not only at all points and in all directions (that is, spatially isotropic and homogeneous) but also at all times, is isometric to the RW spacetime $-\mathbb{R} \times_{e^t} \mathbb{R}^3$ (for more details, see [25]). From (11) we conclude that the slice

$$\{\log(-\frac{n}{c})\} \times M^n$$

is the only one that is a spacelike mean curvature flow soliton with respect to $\mathcal{K} = e^t\partial_t$ and with soliton constant $c < 0$.

Example 3.3. From [35, Example 4.2], the $(n+1)$-dimensional de Sitter space $\mathbb{S}_1^{n+1}$ is isometric to the RW spacetime $-\mathbb{R} \times_{\cosh t} \mathbb{S}^n$, where $\mathbb{S}^n$ denotes the n-dimensional unit Euclidean sphere endowed with its standard metric. Taking into account the terminology introduced in [5], the open half-space $\mathbb{R}^+ \times \mathbb{S}^n \subset \mathbb{S}_1^{n+1}$ (respect. $\mathbb{R}^- \times \mathbb{S}^n \subset \mathbb{S}_1^{n+1}$) is called the *chronological future* (respect. *past*) of $\mathbb{S}_1^{n+1}$ with respect to the totally geodesic *equator* $\{0\} \times \mathbb{S}^n$. From (11) we see that the equator is a spacelike mean curvature flow soliton with respect to $\mathcal{K} = \cosh t\, \partial_t$ and constant soliton $c = 0$ and the slices

$$\{\sinh^{-1}(\frac{-n \pm \sqrt{n^2 - 4c^2}}{2c})\} \times \mathbb{S}^n$$

are spacelike mean curvature flow soliton with respect to $\mathcal{K} = \cosh t\, \partial_t$ and with soliton constant $0 < |c| \leq \frac{n}{2}$.

Example 3.4. Taking into account once more [35, Example 4.2], we consider the open region of $\mathbb{S}_1^{n+1}$ which is isometric to the RW spacetime $-\mathbb{R}^+ \times_{\sinh t}$

$\mathbb{H}^n$, where $\mathbb{H}^n$ denotes the n-dimensional hyperbolic space endowed with its standard metric. From (11) we have that the slices

$$\{\cosh^{-1}(\frac{-n-\sqrt{n^2+4c^2}}{2c})\}\times\mathbb{H}^n$$

are spacelike mean curvature flow soliton with respect to $\mathcal{K}=\sinh t\,\partial_t$ and with soliton constant $c<0$.

Example 3.5. Motivated by [35, Example 4.3], we will consider the open subset of the $(n+1)$-dimensional anti-de Sitter space $\mathbb{H}_1^{n+1}$ which is isometric to the RW spacetime $-(-\frac{\pi}{2},\frac{\pi}{2})\times_{\cos t}\mathbb{H}^n$. In analogy with the nomenclature of the de Sitter space, the open half-space $(0,\frac{\pi}{2})\times\mathbb{H}^n\subset\mathbb{H}_1^{n+1}$ (respect. $(-\frac{\pi}{2},0)\times\mathbb{H}^n\subset\mathbb{H}_1^{n+1}$) will be called the *chronological future* (respect. *past*) of $\mathbb{H}_1^{n+1}$ with respect to the totally geodesic *equator* $\{0\}\times\mathbb{H}^n$. From (11) we see that the equator is a spacelike mean curvature flow soliton with respect to $\mathcal{K}=\cos t\,\partial_t$ and constant soliton $c=0$ and the slices

$$\{\sin^{-1}(\frac{-n\pm\sqrt{n^2+4c^2}}{2c})\}\times\mathbb{H}^n$$

are spacelike mean curvature flow soliton with respect to $\mathcal{K}=\cos t\,\partial_t$ and with soliton constant $c\neq 0$.

4. Nonexistence of spacelike mean curvature flow solitons

We initiate quoting the generalized maximum principle of Omori [36] and Yau [41] (see also [11] for a modern and accessible reference to the generalized maximum principle of Omori-Yau).

Lemma 4.1. *Let Σ^n be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below and let $u\in C^\infty(\Sigma)$ be a smooth function which is bounded from above on Σ^n. Then there exists a sequence of points $\{p_k\}_{k\geq 1}$ in Σ^n such that*

$$\lim_k u(p_k)=\sup_\Sigma u,\qquad \lim_k|\nabla u(p_k)|=0\qquad \textit{and}\qquad \limsup_k \Delta u(p_k)\leq 0.$$

It is not difficult to see that Lemma 4.1 is equivalent to the following one.

Lemma 4.2. *Let Σ^n be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below and let $u\in C^\infty(\Sigma)$ be a smooth function which is bounded from below on Σ^n. Then there exists a sequence of points $\{p_k\}_{k\geq 1}$ in Σ^n such that*

$$\lim_k u(p_k)=\inf_\Sigma u,\qquad \lim_k|\nabla u(p_k)|=0\qquad \textit{and}\qquad \liminf_k \Delta u(p_k)\geq 0.$$

In the results of this subsection, we will suppose that the GRW spacetime $\overline{M}^{n+1}=-I\times_f M^n$ obeys the *strong null convergence condition* (SNCC) which was introduced by Alías and Colares [8] and corresponds to the following suitable constraints on the sectional curvature K_M of the Riemannian fiber M^n of the GRW spacetime $-I \times_f M^n$

$$K_M \geq \sup_I (ff'' - f'^2). \tag{12}$$

We observe that the SNCC is a suitable change on the so-called *null convergence condition* (NCC), which means that the Ricci curvature of $\overline{M}^{n+1}$ being non-negative on null or lightlike directions (for more details concerning the NCC, see [35]).

Considering an immersed spacelike hypersurface in a GRW spacetime obeying (12), the next lemma gives sufficient conditions to its Ricci curvature with respect to a suitable conformal metric be bounded from below. For this, we will suppose that such a spacelike hypersurface lies in a *timelike bounded region*

$$\mathcal{B}_{t_1,t_2} := \{(t,p) \in -I \times_f M^n : t_1 \leq t \leq t_2 \quad \text{and} \quad p \in M^n\}.$$

Lemma 4.3. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime which obeys (12) and let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a spacelike hypersurface contained in a timelike bounded region $\mathcal{B}_{t_1,t_2} \subset \overline{M}^{n+1}$. If the second fundamental form and the hyperbolic angle function Θ are bounded, then the Ricci curvature $\widehat{\mathrm{Ric}}$ of Σ^n with respect to the conformal metric $\hat{g} := \dfrac{1}{f^2(h)} g$ is bounded from below.*

Proof. We recall that the curvature tensor R of Σ^n can be described in terms of its Weingarten operator A and the curvature tensor $\overline{R}$ of the ambient $-I \times_f M^n$ by the so-called Gauss' equation given by

$$\begin{aligned} g(R(X,Y)Z,W) &= \bar{g}(\overline{R}(X,Y)Z,W) - g(AX,Z)g(AY,W) \\ &\quad + g(AX,W)g(AY,Z), \end{aligned} \tag{13}$$

for every tangent vector fields $X, Y, Z \in \mathfrak{X}(\Sigma)$. Here, as in [37], the curvature tensor R is given by

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,$$

where $[\ ,\ ]$ denotes the Lie blanket and $X, Y, Z \in \mathfrak{X}(\Sigma)$.

Let us consider a smooth vector field $X \in \mathfrak{X}(\Sigma)$ and take a (local) orthonormal frame $\{E_1, \cdots, E_n\}$. It follows from Gauss equation (13) that the Ricci curvature Ric of Σ^n with respect to the induced metric g satisfies

$$\mathrm{Ric}(X,X) \geq \sum_i \bar{g}(\overline{R}(X,E_i)X,E_i) - \frac{H^2}{4}|X|^2. \tag{14}$$

To estimate the first summand on the right-hand side of (14), let us consider $X^* = (\pi_M)_*(X)$ and $E_i^* = (\pi_M)_*(E_i)$. So, from [37, Proposition 7.42] and (4) we have

$$\begin{aligned}\sum_i \bar{g}(\overline{R}(X,E_i)X,E_i) &= \sum_i g(R_M(X^*,E_i^*)X^*,E_i^*)_M \\ &\quad +(n-1)((\log f)'(h))^2|X|^2 \\ &\quad -(n-2)(\log f)''(h)g(X,\nabla h)^2 \\ &\quad -(\log f)''(h)|\nabla h|^2|X|^2,\end{aligned} \tag{15}$$

where R_M denotes the curvature tensor of M^n. But, writing

$$X^* = X + \bar{g}(X,\partial_t)\partial_t,$$

we can estimate the first summand on the right-hand side of (15) to get

$$\begin{aligned}\sum_i g(R_M(X^*,E_i^*)X^*,E_i^*)_M &= f^2(h)(|X^*|_M^2|E^*|_M^2 \\ &\quad -g(X^*,E^*)_M^2)K_M(X^*,E^*) \\ &\geq \frac{1}{f^2(h)}((n-1)|X|^2+|\nabla h^2||X|^2 \\ &\quad +(n-2)g(X,\nabla h)^2)\min_i K_M(X^*,E_i^*).\end{aligned} \tag{16}$$

Consequently, since our ambient space obeys (12), from (16) we have that

$$\begin{aligned}\sum_i g(R_M(X^*,E_i^*)X^*,E_i^*) &\geq ((n-1)|X|^2+|\nabla h|^2 \\ &\quad +(n-2)g(X,\nabla h)^2)(\log f)''(h).\end{aligned} \tag{17}$$

Substituting (17) into (15), we get

$$\begin{aligned}\sum_i \bar{g}(\overline{R}(X,E_i)X,E_i) &\geq ((n-1)|X|^2+|\nabla h|^2 \\ &\quad +(n-2)g(X,\nabla h)^2)(\log f)''(h) \\ &\quad +(n-1)((\log f)'(h))^2|X|^2 \\ &\quad -(n-2)(\log f)''(h)g(X,\nabla h)^2 \\ &\quad -(\log f)''(h)|\nabla h|^2|X|^2 \\ &= (n-1)\frac{f''(h)}{f(h)}|X|^2.\end{aligned} \tag{18}$$

Then, taking into account that $|A|^2 \geq \dfrac{H^2}{n}$, from (14) and (18) we arrive at the following lower estimate

$$\mathrm{Ric}(X,X) \geq -\left((n-1)\frac{|f''(h)|}{f(h)}+\frac{n|A|^2}{4}\right)|X|^2. \tag{19}$$

On the other hand, we have the following equation (see, for instance, [15, Section 1J], [30, Section A] or [39, page 168])

$$\begin{aligned}\widehat{\mathrm{Ric}}(X,X) &= \mathrm{Ric}(X,X) + \frac{1}{f^2(h)}\Big\{(n-2)f(h)\nabla^2 f(h)(X,X) \\ &\quad + (f(h)\Delta f(h) - (n-1)|\nabla f(h)|^2)|X|^2\Big\}.\end{aligned} \tag{20}$$

Consequently, from equation (20) we get

$$\begin{aligned}\widehat{\mathrm{Ric}}(X,X) &= \mathrm{Ric}(X,X) + \frac{1}{f^2(h)}\{(n-2)f(h)(f''(h)g(\nabla h,X)^2 \\ &\quad + f'(h)\nabla^2 h(X,X)) + (f(h)(f''(h)|\nabla h|^2 + f'(h)\Delta h) \\ &\quad -(n-1)(f'(h))^2|\nabla h|^2)|X|^2\}.\end{aligned} \tag{21}$$

Hence, considering (5), (7), (8) and (19) into (21), we obtain after a straightforward computation the following lower estimate

$$\begin{aligned}\widehat{\mathrm{Ric}}(X,X) &\geq \Big\{(n-1)\frac{(f'(h))^2}{f(h)} - (n-1)\Big(\frac{|f''(h)|}{f(h)} \\ &\quad + (n+1)\frac{(f'(h))^2}{f^2(h)}\Big)\Theta^2 - (n-\sqrt{n}-2)\frac{|f'(h)|}{f(h)}|A||\Theta| \\ &\quad - \frac{n|A|^2}{4}\Big\}|X|^2.\end{aligned} \tag{22}$$

Therefore, taking into account that $|A|$ and $|\Theta|$ are bounded and that Σ^n lies in a timelike bounded region of the ambient space, from (22) we conclude that $\widehat{\mathrm{Ric}}$ is bounded from below. □

Aiming to simplify the notation, along our main results we will consider the *modified soliton function* as being the function

$$\bar{\zeta}_c(t) := f'(t)\zeta_c(t), \tag{23}$$

where ζ_c is the soliton function defined in (10). So, we are in a position to state and prove our first nonexistence result concerning spacelike mean curvature flow solitons immersed in a GRW spacetime.

Theorem 4.4. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime satisfying (12). There does not exist a complete spacelike mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $\mathcal{K} = f(t)\partial_t$ with soliton constant c, whose second fundamental form and hyperbolic angle function are bounded, and lying in a timelike bounded region $\mathcal{B}_{t_1,t_2} \subset \overline{M}^{n+1}$ such that $\bar{\zeta}_c(t)$ has strict sign for all $t \in [t_1, t_2]$.*

Proof. By contradiction, let us assume the existence of a complete spacelike mean curvature flow soliton ψ satisfying the assumptions of Theorem 4.4. As before, consider on Σ^n the metric $\hat{g} = \frac{1}{f^2(h)}g$, which is conformal to its induced metric g. Denoting by $\hat{\Delta}$ the Laplacian with respect to the metric $\hat{g}$, from (5) and (8) we have

$$\hat{\Delta}h = -f(h)f'(h)\{n + (n-1)|\nabla h|^2\} - Hf^2(h)\Theta. \tag{24}$$

Thus, from (24) we get

$$\begin{aligned}\hat{\Delta}f(h) &= -nf(h)(f'(h))^2 - Hf'(h)f^2(h)\Theta \\ &\quad + f^3(h)\{(\log f)''(h) - (n-2)((\log f)'(h))^2\}|\nabla h|^2.\end{aligned} \tag{25}$$

For any positive real number α, with a straightforward computation from (25) we get

$$\begin{aligned}\hat{\Delta}f^{-\alpha}(h) &= -\alpha f^{-\alpha-1}(h)\Big\{-nf(h)(f'(h))^2 - Hf'(h)f^2(h)\Theta \\ &\quad + f^3(h)\left((\log f)''(h) - (n+\alpha-3)((\log f)'(h))^2\right)|\nabla h|^2\Big\} \\ &= -\alpha f^{-\alpha-1}(h)\Big\{-nf(h)(f'(h))^2\Theta^2 - Hf'(h)f^2(h)\Theta \\ &\quad + f^3(h)\left((\log f)''(h) - (\alpha-3)((\log f)'(h))^2\right)|\nabla h|^2\Big\}.\end{aligned} \tag{26}$$

Hence, from (9), (23) and (26) we obtain

$$\begin{aligned}\hat{\Delta}f^{-\alpha}(h) &= \alpha f^{-\alpha}(h)\Theta^2\bar{\zeta}_c(h) - \alpha f^{2-\alpha}(h)\Big\{(\log f)''(h) \\ &\quad -(\alpha-3)((\log f)'(h))^2\Big\}|\nabla h|^2.\end{aligned} \tag{27}$$

At this point, let us assume that $\bar{\zeta}_c(t) > 0$ for all $t_1 \le t \le t_2$. Since we are assuming that $|A|$ and Θ are bounded, we can apply Lemmas 4.1 and 4.3 to guarantee the existence of a sequence of points $\{p_k\}_{k\ge1}$ in Σ^n such that

$$\lim_k f^{-\alpha}(h)(p_k) = \sup_\Sigma f^{-\alpha}(h), \quad \lim_k |\hat{\nabla} f^{-\alpha}(h)(p_k)|_{\hat{g}} = 0,$$

$$\text{and} \quad \limsup_k \hat{\Delta} f^{-\alpha}(h)(p_k) \le 0, \tag{28}$$

where $|\cdot|_{\hat{g}}$ and $\hat{\nabla}$ denote, respectively, the norm and gradient with respect to the metric $\hat{g}$.

But, it is not difficult to verify that

$$|\hat{\nabla} f^{-\alpha}(h)|_{\hat{g}} = \alpha f^{-\alpha}(h)|f'(h)||\nabla h|. \tag{29}$$

So, since $\Sigma^n \subset \mathcal{B}_{t_1,t_2}$, with $|f'(t)| > 0$ for all $t_1 \le t \le t_2$, from (28) and (29) we get that

$$\lim_k |\nabla h(p_k)| = 0. \tag{30}$$

Consequently, from (5) and (30) we have that

$$\lim_k \Theta^2(p_k) = 1. \tag{31}$$

Moreover, from (27), (30) and (31) we obtain

$$
\begin{aligned}
0 \geq \limsup_k \hat{\Delta} f^{-\alpha}(h)(p_k) \geq \alpha \limsup_k & \left\{ f^{-\alpha}(h)\Theta^2 \bar{\zeta}_c(h) \right\}(p_k) \\
& -\alpha \limsup_k \Big\{ f^{2-\alpha}(h) \Big| (\log f)''(h) \\
& \quad -(\alpha - 3)((\log f)'(h))^2 \Big| |\nabla h|^2 \Big\}(p_k) \qquad (32) \\
& = \alpha \sup_\Sigma f^{-\alpha}(h) \limsup_k \bar{\zeta}_c(p_k) \geq 0.
\end{aligned}
$$

Hence, since $\sup_\Sigma f^{-\alpha}(h) > 0$ and $\bar{\zeta}_c(t) > 0$ for all $t_1 \leq t \leq t_2$, (32) gives us a contradiction.

Finally, in the case $\bar{\zeta}_c(t) < 0$ for all $t_1 \leq t \leq t_2$, using Lemma 4.2 instead of Lemma 4.1 we get

$$
\begin{aligned}
0 \leq \liminf_k \hat{\Delta} f^{-\alpha}(h)(p_k) \leq \alpha \liminf_k & \left\{ f^{-\alpha}(h)\Theta^2 \bar{\zeta}_c(h) \right\}(p_k) \\
& +\alpha \liminf_k \Big\{ f^{2-\alpha}(h) \Big| (\log f)''(h) \\
& \quad -(\alpha - 3)((\log f)'(h))^2 \Big| |\nabla h|^2 \Big\}(p_k) \qquad (33) \\
& = \alpha \inf_\Sigma f^{-\alpha}(h) \liminf_k \bar{\zeta}_c(p_k) \leq 0.
\end{aligned}
$$

Therefore, since $\inf_\Sigma f^{-\alpha}(h) > 0$ and $\bar{\zeta}_c(t) < 0$ for all $t_1 \leq t \leq t_2$, (33) also leads us to a contradiction. □

Taking into account that the $(n+1)$-dimensional Einstein-de Sitter spacetime $-\mathbb{R}^+ \times_{t^{\frac{2}{3}}} \mathbb{R}^n$ (see Example 3.1) satisfies (12), from Theorem 4.4 we obtain the following consequence.

Corollary 4.5. *Let $\overline{M}^{n+1} = -\mathbb{R}^+ \times_{t^{\frac{2}{3}}} \mathbb{R}^n$ be the $(n+1)$-dimensional Einstein-de Sitter spacetime. There does not exist a complete spacelike mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $\mathcal{K} = t^{\frac{2}{3}}\partial_t$ with soliton constant $c \geq 0$, whose second fundamental form and hyperbolic angle function are bounded, and lying in a timelike bounded region of $\overline{M}^{n+1}$.*

Since a steady state type spacetime (see Example 3.2) whose Riemannian fiber has nonnegative sectional curvature satisfies (12), from Theorem 4.4 we obtain the following application.

Corollary 4.6. *Let $\overline{M}^{n+1} = -\mathbb{R} \times_{e^t} M^n$ be a steady state type spacetime whose Riemannian fiber M^n has nonnegative sectional curvature. There does not exist a complete spacelike mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $\mathcal{K} = e^t\partial_t$ with soliton constant $c \geq 0$, whose second fundamental form and hyperbolic angle function are bounded, and lying in a timelike bounded region of $\overline{M}^{n+1}$.*

Considering the context of Example 3.3, from Theorem 4.4 we also get.

Corollary 4.7. *There does not exist a complete spacelike mean curvature flow soliton* $\psi : \Sigma^n \to \mathbb{S}_1^{n+1}$ *with respect to* $\mathcal{K} = \cosh t\,\partial_t$ *having soliton constant* $c \geq 0$ *(respect.* $c \leq 0$*), whose second fundamental form and hyperbolic angle function are bounded, and lying in a timelike bounded region contained in the chronological future (respect. past) of* $\mathbb{S}_1^{n+1}$ *with respect to the equator* $\{0\} \times \mathbb{S}^n$.

From Example 3.4 and Theorem 4.4 we obtain.

Corollary 4.8. *There does not exist a complete spacelike mean curvature flow soliton* $\psi : \Sigma^n \to -\mathbb{R}^+ \times_{\sinh t} \mathbb{H}^n \subset \mathbb{S}_1^{n+1}$ *with respect to* $\mathcal{K} = \sinh t\,\partial_t$ *having soliton constant* $c \geq 0$*, whose second fundamental form and hyperbolic angle function are bounded, and lying in a timelike bounded region of* $-\mathbb{R}^+ \times_{\sinh t} \mathbb{H}^n \subset \mathbb{S}_1^{n+1}$.

Finally, in the setting of Example 3.5, Theorem 4.4 reads as follows.

Corollary 4.9. *There does not exist a complete spacelike mean curvature flow soliton* $\psi : \Sigma^n \to -(-\frac{\pi}{2}, \frac{\pi}{2}) \times_{\cos t} \mathbb{H}^n \subset \mathbb{H}_1^{n+1}$ *with respect to* $\mathcal{K} = \cos t\,\partial_t$ *having soliton constant* $c \leq 0$ *(respect.* $c \geq 0$*), whose second fundamental form and hyperbolic angle function are bounded, and lying in a timelike bounded region contained in the chronological future (respect. past) of* $\mathbb{H}_1^{n+1}$ *with respect to the equator* $\{0\} \times \mathbb{H}^n$.

5. Uniqueness and further nonexistence results

We will also adopt the following notation

$$\mathcal{L}_g^p(\Sigma) := \{u : \Sigma^n \to \mathbb{R} : \int_\Sigma |u|^p d\Sigma < +\infty\},$$

where $d\Sigma$ stands for the measure related to the metric g. With this, we quote an extension of Hopf's theorem on a complete Riemannian manifold (Σ^n, g) due to Yau in [42].

Lemma 5.1. *Let* u *be a smooth function defined on a complete Riemannian manifold* (Σ^n, g)*, such that* Δu *does not change sign on* Σ^n*. If* $|\nabla u| \in \mathcal{L}_g^1(\Sigma)$*, then* Δu *vanishes identically on* Σ^n.

In what follows, we will assume that the warping function f of the ambient GRW spacetime $\overline{M}^{n+1} = -I \times_f M^n$ satisfies the following inequality

$$(\log f)'' \leq \gamma((\log f)')^2, \tag{34}$$

for some nonnegative constant γ. As it was observed in [6], the inequality (34) is a mild hypothesis due to the fact that, for instance, when $\overline{M}^{n+1}$ obeys the SNCC (respect. NCC) and its Riemannian fiber M^n is flat (respect. Ricci-flat), we have that (34) is automatically satisfied.

Returning to the study of spacelike mean curvature flow solitons, we get the following result.

Theorem 5.2. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime satisfying (34), occurring the equality only at isolated points of I, and whose Riemannian fiber M^n is complete. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike mean curvature flow soliton with respect to $\mathcal{K} = f(t)\partial_t$ and with soliton constant c, lying in a timelike bounded region $\mathcal{B}_{t_1,t_2} \subset \overline{M}^{n+1}$ such that $\bar{\zeta}_c(t) \geq 0$ for all $t_1 \leq t \leq t_2$. If the height function h is such that $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, then Σ^n is a slice M_{t_*} for some $t_* \in [t_1, t_2]$ which is implicitly given by the condition $\zeta_c(t_*) = 0$.*

Proof. We will consider again the conformal metric $\hat{g} := \dfrac{1}{f^2(h)} g$ and we will take $\alpha = \gamma + 3$. Since we are assuming that Σ^n lies in $\mathcal{B}_{t_1,t_2}$ and that $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, from (29) we get that $|\hat{\nabla} f^{-\alpha}(h)|_{\hat{g}} \in \mathcal{L}^1_{\hat{g}}(\Sigma)$.

Moreover, since $\bar{\zeta}_c(t) \geq 0$ for all $t_1 \leq t \leq t_2$, from (27) and (34) we obtain that $\hat{\Delta} f^{-\alpha}(h) \geq 0$. Consequently, we can apply Lemma 5.1 to infer that $\hat{\Delta} f^{-\alpha}(h) = 0$ on Σ^n.

Therefore, since we are assuming that the equality occurs in (34) just only at isolated points of I, returning to (27) we conclude that $|\nabla h|$ must vanishes identically on Σ^n. Therefore, Σ^n must be a slice M_{t_*} for some $t_* \in [t_1, t_2]$ which is implicitly given by the condition $\zeta_c(t_*) = 0$. □

Applying Theorem 5.2 to the Einstein-de Sitter spacetime, we obtain the following result.

Corollary 5.3. *Let $\overline{M}^{n+1} = -\mathbb{R}^+ \times_{t^{\frac{2}{3}}} \mathbb{R}^n$ be the $(n+1)$-dimensional Einstein-de Sitter spacetime. The only complete spacelike mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $\mathcal{K} = t^{\frac{2}{3}}\partial_t$ with soliton constant $c < 0$, lying in a timelike bounded region $\mathcal{B}_{t_1,t_2} \subset \overline{M}^{n+1}$ with $t_2 = (-\frac{2n}{3c})^{\frac{3}{5}}$, and such that its height function h satisfies $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, is the slice $\{(-\frac{2n}{3c})^{\frac{3}{5}}\} \times \mathbb{R}^n$.*

When the ambient space is a steady state type spacetime, Theorem 5.2 gives us the following consequence.

Corollary 5.4. *Let $\overline{M}^{n+1} = -\mathbb{R} \times_{e^t} M^n$ be a steady state type spacetime whose Riemannian fiber M^n is complete. The only complete spacelike mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $\mathcal{K} = e^t\partial_t$ with soliton constant $c < 0$, lying in a timelike bounded region $\mathcal{B}_{t_1,t_2} \subset \overline{M}^{n+1}$ with $t_2 = \log(-\frac{n}{c})$, and such that its height function h satisfies $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$, is the slice $\{\log(-\frac{n}{c})\} \times M^n$.*

From Theorem 5.2 we also get the following nonexistence result.

Corollary 5.5. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime satisfying (34), occurring the equality only at isolated points of I, and whose Riemannian fiber M^n*

is complete. There does not exist a complete spacelike mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ *with respect to* $\mathcal{K} = f(t)\partial_t$ *and with soliton constant c, lying in a timelike bounded region* $\mathcal{B}_{t_1,t_2} \subset \overline{M}^{n+1}$ *with* $\bar{\zeta}_c(t) > 0$ *for all* $t_1 \leq t \leq t_2$*, and such that its height function h satisfies* $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$*.*

When the ambient spacetime is the Einstein-de Sitter spacetime and a steady state type spacetime, respectively, Corollary 5.5 reads as follows.

Corollary 5.6. *Let* $\overline{M}^{n+1} = -\mathbb{R}^+ \times_{t^{\frac{2}{3}}} \mathbb{R}^n$ *be the* $(n+1)$*-dimensional Einstein-de Sitter spacetime. There does not exist a complete spacelike mean curvature flow soliton* $\psi : \Sigma^n \to \overline{M}^{n+1}$ *with respect to* $\mathcal{K} = t^{\frac{2}{3}}\partial_t$ *with soliton constant* $c \geq 0$*, lying in a timelike bounded region of* $\overline{M}^{n+1}$ *and such that its height function h satisfies* $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$*.*

Corollary 5.7. *Let* $\overline{M}^{n+1} = -\mathbb{R} \times_{e^t} M^n$ *be a steady state type spacetime whose Riemannian fiber* M^n *is complete. There does not exist a complete spacelike mean curvature flow soliton* $\psi : \Sigma^n \to \overline{M}^{n+1}$ *with respect to* $\mathcal{K} = e^t\partial_t$ *with soliton constant* $c \geq 0$*, lying in a timelike bounded region of* $\overline{M}^{n+1}$ *and such that its height function h satisfies* $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$*.*

We are also able to present a slight different version of Theorem 5.2.

Theorem 5.8. *Let* $\overline{M}^{n+1} = -I \times_f M^n$ *be a GRW spacetime satisfying (34) whose Riemannian fiber* M^n *is complete. Let* $\psi : \Sigma^n \to \overline{M}^{n+1}$ *be a complete spacelike mean curvature flow soliton with respect to* $\mathcal{K} = f(t)\partial_t$ *with soliton constant c, lying in a timelike bounded region* $\mathcal{B}_{t_1,t_2} \subset \overline{M}^{n+1}$ *with* $\bar{\zeta}_c(t) \geq 0$ *and* $f'(t)$ *vanishing only in isolated points of* $[t_1, t_2]$*. If* $|\nabla h| \in \mathcal{L}^1_g(\Sigma)$*, then* Σ^n *is a slice* M_{t_*} *for some* $t_* \in [t_1, t_2]$ *which is implicitly given by the condition* $\zeta_c(t_*) = 0$*.*

Proof. As in the proof of Theorem 5.2, we get that $\hat{\Delta} f^{-\alpha}(h) = 0$ on Σ^n, for $\alpha = \gamma + 3$. Moreover, since Σ^n lies in $\mathcal{B}_{t_1,t_2}$, we can also verify that $|\widehat{\nabla} f^{-2\alpha}(h)|_{\hat{g}} \in \mathcal{L}^1_{\hat{g}}(\Sigma)$. But, we note that

$$\hat{\Delta} f^{-2\alpha}(h) = 2f^{-\alpha}(h)\hat{\Delta} f^{-\alpha}(h) + 2|\widehat{\nabla} f^{-\alpha}(h)|^2_{\hat{g}} = 2|\widehat{\nabla} f^{-\alpha}(h)|^2_{\hat{g}} \geq 0. \tag{35}$$

Thus, we can apply again Lemma 5.1 to obtain that $\hat{\Delta} f^{-2\alpha}(h) = 0$ on Σ^n. Hence, since we are assuming that $f'(t) > 0$ for $t_1 \leq t \leq t_2$, from (29) and (35) we obtain that $|\nabla h| = 0$ on Σ^n. Therefore, Σ^n must be a slice M_{t_*} for some $t_* \in [t_1, t_2]$ which is implicitly given by the condition $\zeta_c(t_*) = 0$. □

At this point, we need quoting two other auxiliary results also due to Yau in [42].

Lemma 5.9. *If u is a nonnegative smooth subharmonic function defined on* (Σ^n, g)*, with* $u \in \mathcal{L}^p_g(\Sigma)$ *for some* $p > 1$*, then u must be constant.*

Lemma 5.10. *All complete noncompact Riemannian manifolds with nonnegative Ricci curvature have at least linear volume growth.*

These previous lemmas enable us to prove the following nonexistence result.

Theorem 5.11. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime satisfying (34), occurring the equality only at isolated points of I, and whose Riemannian fiber M^n is complete noncompact with nonnegative Ricci curvature. There does not exist a complete spacelike mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $\mathcal{K} = f(t)\partial_t$ and soliton constant c, lying in a timelike bounded region $\mathcal{B}_{t_1,t_2} \subset \overline{M}^{n+1}$, with $\bar{\zeta}_c(t) \geq 0$ for all $t_1 \leq t \leq t_2$, and whose height function h is such that $(f(h))^{-1} \in \mathcal{L}^q_g(\Sigma)$ for some q with $q > \gamma + 3$.*

Proof. Supposing by contradiction the existence of such a spacelike mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ and taking once more $\alpha = \gamma + 3$, from (27) we obtain that $\hat{\Delta} f^{-\alpha}(h) \geq 0$ on Σ^n. Moreover, since Σ^n is contained in a timelike bounded region and $(f(h))^{-1} \in \mathcal{L}^q_g(\Sigma)$ for some q with $q > \alpha$, it is not difficult to verify that $f^{-\alpha}(h) \in \mathcal{L}^p_{\hat{g}}(\Sigma)$ for $p = \frac{q}{\alpha} > 1$. Thus, we can apply Lemma 5.9 to get that $f(h)$ is constant on Σ^n. Hence, since we are also supposing that the equality occurs in (34) just only at isolated points of I, returning to (27) we conclude that $|\nabla h|$ must vanish identically on Σ^n. Consequently, Σ^n is isometric (up to scaling) to M^n. So, since $f(h)$ is a positive constant, our assumption that $f(h) \in \mathcal{L}^q_g(\Sigma)$ also implies that M^n has finite volume. But, since M^n is a complete non-compact with nonnegative Ricci curvature, Lemma 5.10 leads us to a contradiction. □

From Theorem 5.11 we obtain the following consequences.

Corollary 5.12. *Let $\overline{M}^{n+1} = -\mathbb{R}^+ \times_{t^{\frac{2}{3}}} \mathbb{R}^n$ be the $(n+1)$-dimensional Einstein-de Sitter spacetime. There does not exist a complete spacelike mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $\mathcal{K} = t^{\frac{2}{3}}\partial_t$ with soliton constant $c \geq 0$, lying in a timelike bounded region of $\overline{M}^{n+1}$ and such that its height function h satisfies $h^{-\frac{2}{3}} \in \mathcal{L}^q_g(\Sigma)$ for some q with $q > 3$.*

Corollary 5.13. *Let $\overline{M}^{n+1} = -\mathbb{R} \times_{e^t} M^n$ be a steady state type spacetime whose Riemannian fiber M^n is complete noncompact with nonnegative Ricci curvature. There does not exist a complete spacelike mean curvature flow soliton $\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $\mathcal{K} = e^t\partial_t$ with soliton constant $c \geq 0$, lying in a timelike bounded region of $\overline{M}^{n+1}$ and such that its height function h satisfies $e^{-h} \in \mathcal{L}^q_g(\Sigma)$ for some q with $q > 3$.*

According to [38], a GRW spacetime $\overline{M}^{n+1} = I \times_f M^n$ is said to be *spatially parabolic* when its Riemannian fiber M^n is parabolic, that is, (M^n, g_M) is a non-compact complete Riemannian manifold such that the only superharmonic functions on it that are bounded from below are the constants. Analogously, $\overline{M}^{n+1}$ is said to be *spatially parabolic covered* when its universal Lorentzian covering is spatially parabolic. For our next uniqueness result, we need of the following parabolicity criterion due to Aledo, Rubio and Salamanca (see [6, Theorem 2.2])

Lemma 5.14. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike hypersurface immersed in a spatially parabolic covered GRW spacetime $\overline{M}^{n+1} = I \times_f M^n$. If $\sup_\Sigma f(h) < +\infty$ and the hyperbolic angle function Θ is bounded, then $(\Sigma^n, \hat{g})$, endowed with the conformal metric $\hat{g} = \frac{1}{f^2(h)} g$, is parabolic.*

Using Lemma 5.14 we obtain the following result.

Theorem 5.15. *Let $\overline{M}^{n+1} = I \times_f M^n$ be a spatially parabolic covered GRW spacetime satisfying (34), holding the equality only at isolated points of I. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete spacelike mean curvature flow soliton with respect to $\mathcal{K} = f(t)\partial_t$ with soliton constant c, lying in a timelike bounded region $\mathcal{B}_{t_1,t_2} \subset \overline{M}^{n+1}$, with $\bar{\zeta}_c(t) \geq 0$ for all $t_1 \leq t \leq t_2$. If the hyperbolic angle function Θ is bounded, then Σ^n is a slice M_{t_*} for some $t_* \in [t_1, t_2]$ which is implicitly given by the condition $\zeta_c(t_*) = 0$.*

Proof. First, we note that Lemma 5.14 guarantees that $(\Sigma^n, \hat{g})$ is parabolic. Moreover, it follows from (27) that $f(h)^{-\alpha}$ (where $\alpha = \gamma + 3$) is subharmonic on Σ^n. Thus, since the hypothesis that $\Sigma^n \subset \mathcal{B}_{t_1,t_2}$ implies in particular that $f(h)^{-\alpha}$ is bounded from above, it follows from the parabolicity of $(\Sigma^n, \hat{g})$ that $f(h)$ is constant on Σ^n. Consequently, since we are assuming that the equality holds in (34) only at isolated points of I, returning to (27) we conclude that $|\nabla h| = 0$ on Σ^n, which means that Σ^n is a slice. □

We close this section quoting the following applications of Theorem 5.15.

Corollary 5.16. *Let $\overline{M}^3 = -\mathbb{R}^+ \times_{t^{\frac{2}{3}}} \mathbb{R}^2$ be the 3-dimensional Einstein-de Sitter spacetime. The only complete spacelike mean curvature flow soliton $\psi : \Sigma^2 \to \overline{M}^3$ with respect to $\mathcal{K} = t^{\frac{2}{3}}\partial_t$ with soliton constant $c < 0$, lying in a timelike bounded region $\mathcal{B}_{t_1,t_2} \subset \overline{M}^3$ with $t_2 = (-\frac{4}{3c})^{\frac{3}{5}}$, and such that its hyperbolic angle function Θ is bounded, is the slice $\{(-\frac{4}{3c})^{\frac{3}{5}}\} \times \mathbb{R}^n$.*

Corollary 5.17. *Let $\overline{M}^{n+1} = -\mathbb{R} \times_{e^t} M^n$ be a spatially parabolic covered steady state type spacetime. The only complete spacelike mean curvature flow soliton*

$\psi : \Sigma^n \to \overline{M}^{n+1}$ with respect to $\mathcal{K} = e^t\partial_t$ with soliton constant $c < 0$, lying in a timelike bounded region $\mathcal{B}_{t_1,t_2} \subset \overline{M}^{n+1}$ with $t_2 = \log(-\frac{n}{c})$, and such that its hyperbolic angle function Θ is bounded, is the slice $\{\log(-\frac{n}{c})\} \times M^n$.

Remark 5.18. Related to Corollary 5.17 in the case $n = 2$, when the Riemannian fiber M^2 is a complete Riemannian surface having nonnegative Gaussian curvature, a classical result due to Ahlfors [2] and Blanc-Fiala-Huber [27] guarantees that M^2 has parabolic universal covering.

6. Entire spacelike mean curvature flow graphs

In this last section, we will use the theorems of the previous section in order to establish new Calabi-Bernstein type results concerning entire spacelike graphs constructed over the Riemannian fiber of a GRW spacetime. Before, we need to recall some basic facts related to these spacelike graphs.

Let $\Omega \subseteq M^n$ be a connected domain and let $u \in C^\infty(\Omega)$ be a smooth function such that $u(\Omega) \subseteq I$, then $\Sigma(u)$ will denote the (vertical) graph over Ω determined by u, that is,

$$\Sigma(u) = \{(u(x), x) : x \in \Omega\} \subset \overline{M}^{n+1} = -I \times_f M^n.$$

The graph is said to be entire if $\Omega = M^n$. Observe that $h(u(x), x) = u(x)$, for all $x \in \Omega$. Hence, h and u can be identified in a natural way. The metric induced on Ω from the Lorentzian metric of the ambient GRW spacetime via $\Sigma(u)$ is

$$g_u = -du^2 + f^2(u)g_M. \tag{36}$$

It can be easily seen from (36) that such a graph $\Sigma(u)$ is a spacelike hypersurface if and only if $|Du|_M < f(u)$, where Du stands for the gradient of u in M and $|Du|_M$ its norm, both with respect to the metric g_M. On the other hand, in the case where M^n is a simply connected manifold, from [12, Lemma 3.1] we have that every complete spacelike hypersurface $\psi : \Sigma^n \to -I \times_f M^n$ such that the warping function f is bounded on Σ^n is an entire spacelike graph over M^n. In particular, this happens for complete spacelike hypersurfaces lying in a timelike bounded region of $-I \times_f M^n$. It is also interesting to point out that, in contrast to the case of graphs in a Riemannian space, an entire spacelike graph $\Sigma(u)$ in a GRW spacetime is not necessarily complete, in the sense that the induced Riemannian metric (36) is not necessarily complete on M^n. For instance, Albujer constructed explicit examples of noncomplete entire maximal spacelike graphs (that is, whose mean curvature is identically zero) in the Lorentzian product space $-\mathbb{R} \times \mathbb{H}^2$ (see [3, Section 3]).

The future-pointing Gauss map of a spacelike graph $\Sigma(u)$ over Ω is given by the vector field

$$N = \frac{f(u)}{\sqrt{f^2(u) - |Du|_M^2}}\left(\partial_t + \frac{Du}{f^2(u)}\right). \tag{37}$$

The shape operator related to the future-pointing Gauss map (37) is given by

$$\begin{aligned} AX = &-\frac{1}{f(u)\sqrt{f^2(u)-|Du|_M^2}}D_X Du - \frac{f'(u)}{\sqrt{f^2(u)-|Du|_M^2}}X \\ &+\left(\frac{-g_M(D_X Du, Du)}{f(u)\left(f^2(u)-|Du|_M^2\right)^{3/2}} + \frac{f'(u)g_M(Du,X)}{\left(f^2(u)-|Du|_M^2\right)^{3/2}}\right)Du, \end{aligned} \tag{38}$$

for any vector field X tangent to Ω, where D denotes the Levi-Civita connection of (M^n, g_M). Consequently, if $\Sigma(u)$ is a spacelike graph defined over a domain $\Omega \subseteq M^n$, it is not difficult to verify from (38) that the future mean curvature function $H(u)$ of $\Sigma(u)$ is given by the following nonlinear differential equation:

$$\begin{aligned} H(u) \quad = \quad & \operatorname{div}_M\left(\frac{Du}{f(u)\sqrt{f^2(u)-|Du|_M^2}}\right) \\ & + \frac{f'(u)}{\sqrt{f^2(u)-|Du|_M^2}}\left(n + \frac{|Du|_M^2}{f^2(u)}\right), \end{aligned} \tag{39}$$

where div_M stands for the divergence operator computed in the metric g_M.

Hence, from (9) and (39) we have that $\Sigma(u)$ is a spacelike mean curvature flow soliton with respect to $K = f(t)\partial_t$ with soliton constant c if, and only if, $|Du|_M < f(u)$ and u is a solution of the following nonlinear differential equation:

$$\begin{aligned} \operatorname{div}_M\left(\frac{Du}{f(u)\sqrt{f(u)^2-|Du|_M^2}}\right) \quad = \quad & -\frac{1}{\sqrt{f(u)^2-|Du|_M^2}}\Big\{cf(u)^2 \\ & + f'(u)\Big(n + \frac{|Du|_M^2}{f(u)^2}\Big)\Big\}. \end{aligned} \tag{40}$$

We say that $u \in C^\infty(M)$ has finite C^2 norm when

$$||u||_{\mathcal{C}^2(M)} := \sup_{|k|\le 2} |D^k u|_{L^\infty(M)} < +\infty.$$

In this setting, we obtain a nonparametric version of Theorem 4.4.

Theorem 6.1. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime satisfying (12) and whose Riemannian fiber M^n is complete. Suppose that c is a constant such that the modified soliton function $\bar{\zeta}_c(t)$ has strict sign in I. There does not exist a smooth function $u : M^n \to I$ with finite C^2 norm which is solution of the spacelike mean curvature flow soliton equation (40) and such that $|Du|_M \le \beta f(u)$, for some constant $0 < \beta < 1$.*

PROOF. Let us assume the existence of such a smooth function $u : M^n \to I$. It follows from (38) that the shape operator A of an entire spacelike graph $\Sigma(u)$ is bounded provided that u has finite C^2 norm. Note also that the finiteness of the C^2 norm of u implies, in particular, that u is bounded, which, in turn, guarantees that $\Sigma(u)$ is contained in a bounded timelike region of $\overline{M}^{n+1}$.

On the other hand, under the assumptions of the theorem, $\Sigma(u)$ is a complete spacelike hypersurface. Indeed, proceeding as in [9, Corollary 5.1], from (36) and the Cauchy-Schwarz inequality we get

$$\begin{aligned} g_u(X,X) &= -g_M(Du,X^*)^2 + f^2(u)g_M(X^*,X^*) \\ &\geq (f^2(u) - |Du|_M^2)g_M(X^*,X^*), \end{aligned} \tag{41}$$

for every tangent vector field X on $\Sigma(u)$, where (as before) X^* denotes the projection of X onto the Riemannian fiber M^n. Thus, since $|Du|_M \leq \beta f(u)$, for some constant $0 < \beta < 1$, from (41) we get that

$$g_u(X,X) \geq \delta g_M(X^*,X^*), \tag{42}$$

where $\delta = (1-\beta^2)\inf_M f^2(u)$. So, (42) implies that $L = \sqrt{\delta} L_M$, where L and L_M denote the length of a curve on $\Sigma(u)$ with respect to the Riemannian metrics g_u and g_M, respectively. As a consequence, since we are always assuming that M^n is complete, the induced metric g_u must be also complete.

Moreover, from (37) we obtain that the hyperbolic angle function Θ of $\Sigma(u)$ is given by

$$\Theta = -\frac{f(u)}{\sqrt{f^2(u) - |Du|_M^2}}. \tag{43}$$

Hence, using once more that hypothesis that $|Du|_M \leq \beta f(u)$, for some constant $0 < \beta < 1$, from (43) we get that Θ is bounded. But, by applying Theorem 4.4 we have that $\Sigma(u)$ cannot exist. $\square$

From Theorem 6.1 we obtain the following applications.

Corollary 6.2. *For any constants $c \geq 0$ and $0 < \beta < 1$, there does not exist a smooth function $u : \mathbb{R}^n \to \mathbb{R}^+$ with finite C^2 norm which is a solution of the following system*

$$\begin{cases} \operatorname{div}_{\mathbb{R}^n}\left(\dfrac{Du}{u^{\frac{2}{3}}\sqrt{u^{\frac{4}{3}} - |Du|_{\mathbb{R}^n}^2}}\right) = -\dfrac{1}{\sqrt{u^{\frac{4}{3}} - |Du|_{\mathbb{R}^n}^2}}\left(cu^{\frac{4}{3}} + \dfrac{2n}{3u^{\frac{1}{3}}} + \dfrac{2|Du|_{\mathbb{R}^n}^2}{3u^{\frac{5}{3}}}\right) \\ \\ |Du|_{\mathbb{R}^n} \leq \beta u^{\frac{2}{3}} \end{cases} \tag{44}$$

Corollary 6.3. *Let M^n be a complete Riemannian manifold with nonnegative sectional curvature. For any constants $c \geq 0$ and $0 < \beta < 1$, there does not exist* *a smooth function $u : M^n \to I$ with finite C^2 norm which is a solution of the following system*

$$\begin{cases} \operatorname{div}_M\left(\frac{Du}{e^u\sqrt{e^{2u}-|Du|_M^2}}\right) = -\frac{1}{\sqrt{e^{2u}-|Du|_M^2}}\left(ce^{2u}+ne^u+\frac{|Du|_M^2}{e^u}\right) \\ |Du|_M \le \beta e^u \end{cases} \tag{45}$$

Remark 6.4. From Examples 3.3, 3.4 and 3.5, it is not difficult to see that we can also obtain applications of Theorem 6.1 to the de Sitter and anti-de Sitter spaces similar to Corollaries 6.2 and 6.3.

Proceeding, from Theorem 5.2 we obtain the following Calabi-Bernstein type result.

Theorem 6.5. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime satisfying (34), occurring the equality only at isolated points of I, and whose Riemannian fiber M^n is complete. Suppose that c is a constant such that $\bar{\zeta}_c(t) \ge 0$ for all $t \in I$. If $\Sigma(u) \subset \overline{M}^{n+1}$ is an entire spacelike graph determined by a bounded function $u \in C^\infty(M)$ which is solution of the spacelike mean curvature flow soliton equation (40) with $|Du|_M \le \beta f(u)$, for some constant $0 < \beta < 1$, and $|Du|_M \in \mathcal{L}^1_{g_M}(M)$, then $u \equiv t_*$ for some $t_* \in I$ which is implicitly given by the condition $\zeta_c(t_*) = 0$.*

Proof. Since we are supposing that $|Du|_M \le \beta f(u)$, for some constant $0 < \beta < 1$, it follows from the proof of Theorem 6.1 that $\Sigma(u)$ is a complete spacelike hypersurface.

On the other hand, reasoning once more as in [9, Corollary 5.1], we deduce from the induced metric (36) that $d\Sigma = \sqrt{|G|}dM$, where dM and $d\Sigma$ stand for the Riemannian volume elements of (M^n, g_M) and $(\Sigma(u), g_u)$, respectively, and $G = det(g_{ij})$ with

$$g_{ij} = g_u(E_i, E_j) = f^2(u)\delta_{ij} - E_i(u)E_j(u). \tag{46}$$

Here, $\{E_1, \dots, E^n\}$ denotes a local orthonormal frame with respect to the metric g_M. So, it is not difficult to verify that

$$|G| = f^{2(n-1)}(u)(f^2(u) - |Du|_M^2). \tag{47}$$

Hence, from (46) and (47) we obtain

$$d\Sigma = f^{n-1}(u)\sqrt{f^2(u) - |Du|_M^2}dM. \tag{48}$$

Moreover, since we have that $N = N^* - \Theta\partial_t$, from (4) we get

$$|\nabla h|^2 = f^2(u)|N^*|_M^2. \tag{49}$$

Thus, from (37) and (49) we obtain

$$|\nabla h|^2 = \frac{|Du|_M^2}{f^2(u) - |Du|_M^2}. \tag{50}$$

Consequently, from (50) and (48) we get

$$|\nabla h| d\Sigma = f^{n-1}(u)|Du|_M dM. \tag{51}$$

Hence, since we are assuming that u is bounded with $|Du|_M \in \mathcal{L}^1_{g_M}(M)$, relation (51) guarantees that $|\nabla h| \in \mathcal{L}^1_g(\Sigma(u))$. Therefore, the result follows by applying Theorem 5.2. □

From Theorem 6.5 we obtain the following applications.

Corollary 6.6. *For any constants $c < 0$ and $0 < \beta < 1$, the only bounded smooth function $u : \mathbb{R}^n \to \mathbb{R}^+$, with $u(x) \le (-\frac{2n}{3c})^{\frac{3}{5}}$ for all $x \in \mathbb{R}^n$, $|Du|_{\mathbb{R}^n} \in \mathcal{L}^1_{g_{\mathbb{R}^n}}(\mathbb{R}^n)$ and which is solution of the system (44), is the constant $u = (-\frac{2n}{3c})^{\frac{3}{5}}$.*

Corollary 6.7. *Let M^n be a complete Riemannian manifold. For any constants $c < 0$ and $0 < \beta < 1$, the only bounded smooth function $u : M^n \to \mathbb{R}$, with $u(x) \le \log(-\frac{n}{c})$ for all $x \in M^n$, $|Du|_M \in \mathcal{L}^1_{g_M}(M)$ and which is solution of the system (45), is the constant $u = \log(-\frac{n}{c})$.*

Taking into account once more relation (48), it is not difficult to see that from Theorem 5.11 we obtain the following nonexistence result.

Theorem 6.8. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a GRW spacetime satisfying (34), occurring the equality only at isolated points of I, and whose Riemannian fiber M^n is complete noncompact with nonnegative Ricci curvature. Suppose that c is a constant such that $\bar{\zeta}_c(t) \ge 0$ for all $t \in I$. There does not exist a bounded entire solution $u \in C^\infty(M)$ of the spacelike mean curvature flow soliton equation (39), with $|Du|_M \le \beta f(u)$, for some constant $0 < \beta < 1$, and such that $(f(u))^{-1} \in \mathcal{L}^q_{g_M}(M)$ for some q with $q > \gamma + 3$.*

We have the following applications of Theorem 6.8.

Corollary 6.9. *For any constants $c \ge 0$ and $0 < \beta < 1$, there does not exist a bounded smooth function $u : \mathbb{R}^n \to \mathbb{R}^+$ such that $u^{-\frac{2}{3}} \in \mathcal{L}^q_{g_{\mathbb{R}^n}}(\mathbb{R}^n)$, for some $q > 3$, and which is a solution of the system (44).*

Corollary 6.10. *Let M^n be a complete noncompact Riemannian manifold with nonnegative Ricci curvature. For any constants $c \ge 0$ and $0 < \beta < 1$, there does not exist a bounded smooth function $u : M^n \to I$ such that $e^{-u} \in \mathcal{L}^q_{g_M}(M)$, for some $q > 3$, and which is a solution of the system (45).*

Observing once more that the assumption $|Du|_M \le \beta f(u)$, for some constant $0 < \beta < 1$, implies that the hyperbolic angle function Θ given by (43) is bounded, Theorem 5.15 allows us to obtain the following result.

Theorem 6.11. *Let $\overline{M}^{n+1} = -I \times_f M^n$ be a spatially parabolic covered GRW spacetime satisfying (34), holding the equality only at isolated points of I. Suppose* *that c is a constant such that $\bar{\zeta}_c(t) \geq 0$ for all $t \in I$. If $\Sigma(u) \subset \overline{M}^{n+1}$ is an entire spacelike graph determined by a bounded function $u \in C^\infty(M)$ which is solution of the spacelike mean curvature flow soliton equation (40) with $|Du|_M \leq \beta f(u)$, for some constant $0 < \beta < 1$, then $u \equiv t_*$ for some $t_* \in I$ which is implicitly given by the condition $\zeta_c(t_*) = 0$.*

We finish this manuscript with the following applications of Theorem 6.11.

Corollary 6.12. *For any constants $c < 0$ and $0 < \beta < 1$, the only bounded smooth function $u : \mathbb{R}^2 \to \mathbb{R}^+$, with $u(x) \leq (-\frac{4}{3c})^{\frac{3}{5}}$ for all $x \in \mathbb{R}^2$, and which is solution of the system (44) for $n = 2$, is the constant $u = (-\frac{4}{3c})^{\frac{3}{5}}$.*

Corollary 6.13. *Let M^n be a complete Riemannian manifold with parabolic universal covering. For any constants $c < 0$ and $0 < \beta < 1$, the only bounded smooth function $u : M^n \to \mathbb{R}$, with $u(x) \leq \log(-\frac{n}{c})$ for all $x \in M^n$, and which is solution of the system (45), is the constant $u = \log(-\frac{n}{c})$.*

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