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Foliations induced by metallic structures

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ABSTRACT. We give necessary and sufficient conditions for the real distributions defined by a metallic pseudo-Riemannian structure to be integrable and geodesically invariant, in terms of associated tensor fields to the metallic structures and of adapted connections. In the integrable case, we prove a Chen-type inequality for these distributions and provide conditions for a metallic map to preserve these distributions. If the structure is metallic Norden, we describe the complex metallic distributions in the same spirit.

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1. Introduction

Let M be a smooth manifold and let J be a $(1,1)$-tensor field on M. If $J^2 = pJ + qI$, for some p and q real numbers, then J is called *a metallic structure on M* and (M, J) is called *a metallic manifold*. If g is a pseudo-Riemannian metric on M such that J is g-symmetric, then (J, g) is called a *metallic pseudo-Riemannian structure* on M.

The aim of this paper is to consider the complementary distributions associated to a metallic pseudo-Riemannian structure and study their integrability and geodesically invariance in terms of associated tensor fields to the metallic structure and of adapted connections. In this sense, we consider the Schouten-van Kampen, Vrănceanu and Vidal connections, which seem to be the most important connections for the study of foliations of a pseudo-Riemannian manifold [1]. Moreover, for these distributions, we prove a Chen-type inequality giving a relation between the squared norm of the mean curvature and the Chen

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first invariant. We also prove a leaf correspondence theorem between the leaves of two metallic pseudo-Riemannian manifolds when there is given a metallic map between them with certain properties.

The sign of $p^2 + 4q$ is important in the study of foliations induced by metallic structures; if it is positive, then J has two real eigenvalues, if it is negative, J has two complex eigenvalues. In the real case, J can be related to almost product structures and in the complex case, to Norden structures. We remark that some properties of metallic distributions have also been studied in [11]. In this paper we consider both of these cases and we describe some similarities and differences between them. In particular, in the complex case, we compute the $\bar{\partial}$-operator in terms of J. Moreover, we construct the metallic complex cohomology and homology groups.

2. Preliminaries

2.1. Metallic pseudo-Riemannian structures.

Definition 2.1. [3] Let (M, g) be a pseudo-Riemannian manifold and let J be a metallic structure on M. We say that the pair (J, g) is *a metallic pseudo-Riemannian structure on M* if J is g-symmetric. In this case, (M, J, g) is called *a metallic pseudo-Riemannian manifold*. If $p^2 + 4q < 0$, then (J, g) is called *a metallic Norden structure* and (M, J, g) is called *a metallic Norden manifold*.

Remark 2.2. Let (M, g) be a pseudo-Riemannian manifold and let J be a metallic structure on M such that $J^2 = pJ + qI$. If we require that J is g-skew-symmetric, then we obtain that $p = 0$. Namely, if we assume $g(JX, Y) = -g(X, JY)$, for any X, $Y \in C^\infty(TM)$, then we get $g(JX, JY) = -g(X, J^2Y) = -pg(X, JY) - qg(X, Y) = pg(JX, Y) - qg(X, Y)$. On the other hand, $g(JX, JY) = -g(J^2X, Y) = -pg(JX, Y) - qg(X, Y)$, therefore $p = 0$. In particular, for $p \neq 0$, it is not possible to define the concept of metallic Hermitian structure.

Definition 2.3. [3] (i) A linear connection ∇ on M is called a *J-connection* if J is covariantly constant with respect to ∇, i.e. $\nabla J = 0$.

(ii) A metallic pseudo-Riemannian manifold (M, J, g) such that the Levi-Civita connection ∇ with respect to g is a J-connection is called a *locally metallic pseudo-Riemannian manifold*.

2.2. Associated tensors to a metallic pseudo-Riemannian structure. For a metallic pseudo-Riemannian structure (J, g) on the smooth manifold M with ∇ the Levi-Civita connection of g, we introduce some tensor fields [7] to characterize the properties of the metallic distributions defined by J:

(1) *the J-bracket*

$$[X, Y]_J := [JX, Y] + [X, JY] - J([X, Y]),$$

where $[\cdot, \cdot]$ is the Lie bracket, $[X, Y] = \nabla_X Y - \nabla_Y X$

(2) *the Nijenhuis tensor associated to J*

$$N_J(X, Y) := J([X, Y]_J) - [JX, JY]$$

(3) *the Jordan bracket associated to J*

$$\{X,Y\}_J := \{JX,Y\} + \{X,JY\} - J(\{X,Y\}),$$

where $\{\cdot,\cdot\}$ is the Jordan bracket, $\{X,Y\} = \nabla_X Y + \nabla_Y X$

(4) *the Jordan tensor associated to J*

$$M_J(X,Y) := J(\{X,Y\}_J) - \{JX,JY\}$$

(5) *the deformation tensor associated to J*

$$H_J(X,Y) := (J\circ\nabla_X J - \nabla_{JX}J)(Y),$$

which satisfies $2H_J = N_J + M_J$.

Remark 2.4. The J-bracket and the associated Nijenhuis tensor can be defined for any $(1,1)$-tensor field on a smooth manifold M, the Jordan bracket, the associated Jordan tensor and the deformation tensor can be defined for $(1,1)$-tensor fields on a pseudo-Riemannian manifold (M,g).

Assume that J satisfies $J^2 = pJ + qI$ with $p^2+4q > 0$. We denote by $\sigma_\pm := \frac{p\pm\sqrt{p^2+4q}}{2}$ and consider the projection operators $\mathcal{P}$ and $\mathcal{P}'$ [8]:

$$\mathcal{P} := -\frac{1}{\sqrt{p^2+4q}}J + \frac{\sigma_+}{\sqrt{p^2+4q}}I,\ \ \mathcal{P}' := \frac{1}{\sqrt{p^2+4q}}J - \frac{\sigma_-}{\sqrt{p^2+4q}}I$$

satisfying

$$\mathcal{P}^2 = \mathcal{P},\ \ \mathcal{P}'^2 = \mathcal{P}',\ \ \mathcal{P}+\mathcal{P}' = I,\ \ \mathcal{P}\circ\mathcal{P}' = 0,\ \ \mathcal{P}'\circ\mathcal{P} = 0.$$

By a direct computation, we get the following:

Proposition 2.5. *For the two projection operators $\mathcal{P}$ and $\mathcal{P}'$, we have:*

(1) $N_{\mathcal{P}} = N_{\mathcal{P}'} = \frac{1}{p^2+4q}N_J$;
(2) $M_{\mathcal{P}} = M_{\mathcal{P}'} = \frac{1}{p^2+4q}M_J$;
(3) $H_{\mathcal{P}} = H_{\mathcal{P}'} = \frac{1}{p^2+4q}H_J$.

Consider now *the deformation tensors* H and H':

$$H(X,Y) := \mathcal{P}'(\nabla_{\mathcal{P}X}\mathcal{P}Y) = \mathcal{P}'((\nabla_{\mathcal{P}X}\mathcal{P})Y),$$

$$H'(X,Y) := \mathcal{P}(\nabla_{\mathcal{P}'X}\mathcal{P}'Y) = \mathcal{P}((\nabla_{\mathcal{P}'X}\mathcal{P}')Y)$$

the twisting tensors L and L':

$$L(X,Y) := \frac{1}{2}[H(X,Y) - H(Y,X)],\ \ L'(X,Y) := \frac{1}{2}[H'(X,Y) - H'(Y,X)]$$

and *the extrinsic curvature tensors* K and K':

$$K(X,Y) := \frac{1}{2}[H(X,Y) + H(Y,X)],\ \ K'(X,Y) := \frac{1}{2}[H'(X,Y) + H'(Y,X)],$$

for any $X, Y \in C^\infty(TM)$.

By a direct computation we obtain:

$$H(X,Y) = \frac{1}{(p^2+4q)\sqrt{p^2+4q}}[J(\nabla_{JX}JY) - \sigma_+ J(\nabla_X JY) - \sigma_+ J(\nabla_{JX}Y) +$$

$$+\sigma_+^2 J(\nabla_X Y) - \sigma_- \nabla_{JX}JY - q\nabla_X JY - q\nabla_{JX}Y + q\sigma_+ \nabla_X Y] =$$

$$= \frac{1}{(p^2+4q)\sqrt{p^2+4q}}[J(\nabla_{JX}J) - \sigma_+ J(\nabla_X J) - \sigma_-(\nabla_{JX}J) - q(\nabla_X J)](Y)$$

$$H'(X,Y) = -\frac{1}{(p^2+4q)\sqrt{p^2+4q}}[J(\nabla_{JX}JY) - \sigma_- J(\nabla_X JY) - \sigma_- J(\nabla_{JX}Y) +$$

$$+\sigma_-^2 J(\nabla_X Y) - \sigma_+ \nabla_{JX}JY - q\nabla_X JY - q\nabla_{JX}Y + q\sigma_- \nabla_X Y] =$$

$$= -\frac{1}{(p^2+4q)\sqrt{p^2+4q}}[J(\nabla_{JX}J) - \sigma_- J(\nabla_X J) - \sigma_+(\nabla_{JX}J) - q(\nabla_X J)](Y).$$

In particular, we get:

$$H(X,Y) + H'(X,Y) = \frac{1}{(p^2+4q)\sqrt{p^2+4q}}(-\sigma_+ + \sigma_-)[J(\nabla_X J) - (\nabla_{JX}J)](Y) =$$

$$= \frac{1}{p^2+4q} H_J(X,Y).$$

Moreover:

$$L = \frac{1}{2(p^2+4q)\sqrt{p^2+4q}}(\sigma_- N_J - J \circ N_J),$$

$$L' = -\frac{1}{2(p^2+4q)\sqrt{p^2+4q}}(\sigma_+ N_J - J \circ N_J),$$

$$K = \frac{1}{2(p^2+4q)\sqrt{p^2+4q}}(\sigma_- M_J - J \circ M_J),$$

$$K' = -\frac{1}{2(p^2+4q)\sqrt{p^2+4q}}(\sigma_+ M_J - J \circ M_J).$$

3. Metallic distributions

Let (M, J, g) be a metallic pseudo-Riemannian manifold such that $J^2 = pJ + qI$ with $p^2 + 4q > 0$. Define the complementary distributions:

$$\mathcal{D} := \ker \mathcal{P}', \quad \mathcal{D}' := \ker \mathcal{P} \tag{1}$$

which we shall call *the metallic distributions* defined by the metallic structure J.

Remark 3.1. The distributions $\mathcal{D}$ and $\mathcal{D}'$ are J-invariant and, if $q \neq 0$, then $\mathcal{D}$ and $\mathcal{D}'$ are also g-orthogonal.

Definition 3.2. A distribution $\mathcal{D} \subset TM$ on a smooth manifold M is called
(i) *involutive* if $X, Y \in \Gamma(\mathcal{D})$ implies $[X, Y] \in \Gamma(\mathcal{D})$;
(ii) *integrable* if for any $x \in M$, there exists a submanifold N_x which admits $\mathcal{D}|_{N_x}$ as tangent bundle.

According to the Frobenius theorem, a distribution $\mathcal{D}$ on M is involutive if and only if it is integrable. In this case, it defines a foliation whose leaves are the maximal connected submanifolds N_x of M which admit $\mathcal{D}|_{N_x}$ as tangent bundle.

Definition 3.3. We say that the metallic pseudo-Riemannian manifold (M, J, g) is *doubly foliated* if both of the distributions $\mathcal{D}$ and $\mathcal{D}'$ given by (1) are integrable and *singly foliated* if only one of them is integrable.

Remark 3.4. The distribution $\mathcal{D}$ (resp. $\mathcal{D}'$) given by (1) is integrable if and only if $(\nabla_X J)Y - (\nabla_Y J)X = 0$, for any $X, Y \in \Gamma(\mathcal{D})$ (resp. $X, Y \in \Gamma(\mathcal{D}')$), with ∇ a torsion-free linear connection on M. Indeed, for $X, Y \in \Gamma(\mathcal{D})$ we have $JX = \sigma_- X$, $JY = \sigma_- Y$ and $J(\nabla_X Y - \nabla_Y X) = -(\nabla_X J)Y + (\nabla_Y J)X + \sigma_-(\nabla_X Y - \nabla_Y X)$ which implies that $[X, Y] \in \Gamma(\mathcal{D})$ if and only if $(\nabla_X J)Y - (\nabla_Y J)X = 0$.

In particular, in a locally metallic pseudo-Riemannian manifold, the two distributions $\mathcal{D}$ and $\mathcal{D}'$ given by (1) are both integrable.

Proposition 3.5. *If (M, J, g) is a metallic pseudo-Riemannian manifold, then the distribution $\mathcal{D}$ is integrable if and only if:*

$$J \circ N_J(X, Y) = \sigma_- N_J(X, Y), \ \textit{for any} \ X, Y \in C^\infty(TM),$$

respectively, $\mathcal{D}'$ is integrable if and only if:

$$J \circ N_J(X, Y) = \sigma_+ N_J(X, Y), \ \textit{for any} \ X, Y \in C^\infty(TM).$$

In particular, both $\mathcal{D}$ and $\mathcal{D}'$ are integrable if and only if $N_J = 0$.

Proof. The distribution $\mathcal{D}$ is integrable if and only if

$$\mathcal{P}'([\mathcal{P}X, \mathcal{P}Y]) = 0,$$

for any $X, Y \in C^\infty(TM)$. Therefore, from a direct computation and using Proposition 2.5, we obtain that a necessary and sufficient condition for $\mathcal{D}$ to be integrable is:

$$0 = \mathcal{P}'([\mathcal{P}X, \mathcal{P}Y]) = -\mathcal{P}'(N_{\mathcal{P}}(X, Y)) = -\frac{1}{p^2 + 4q}\mathcal{P}'(N_J(X, Y)) =$$
$$= -\frac{1}{(p^2 + 4q)\sqrt{p^2 + 4q}}[J \circ N_J(X, Y) - \sigma_- N_J(X, Y)].$$

□

Definition 3.6. Given a linear connection ∇ on a smooth manifold M, we say that a distribution $\mathcal{D} \subset TM$ is ∇-*geodesically invariant* if $X, Y \in \Gamma(\mathcal{D})$ implies $\nabla_X Y + \nabla_Y X \in \Gamma(\mathcal{D})$.

In particular, if ∇ is the Levi-Civita of the pseudo-Riemannian manifold (M, g), then $\mathcal{D}$ is *geodesically invariant*.

We remark that the above condition is equivalent to the following: the distribution $\mathcal{D}$ is ∇-geodesically invariant if $X \in \Gamma(\mathcal{D})$ implies $\nabla_X X \in \Gamma(\mathcal{D})$.

Remark 3.7. For a linear connection ∇ on M, the distribution $\mathcal{D}$ (resp. $\mathcal{D}'$) given by (1) is ∇-geodesically invariant if and only if we have $(\nabla_X J)Y+(\nabla_Y J)X = 0$, for any X, $Y \in \Gamma(\mathcal{D})$ (resp. X, $Y \in \Gamma(\mathcal{D}')$). Indeed, for X, $Y \in \Gamma(\mathcal{D})$ we have $JX = \sigma_- X$, $JY = \sigma_- Y$ and $J(\nabla_X Y + \nabla_Y X) = -(\nabla_X J)Y - (\nabla_Y J)X + \sigma_-(\nabla_X Y + \nabla_Y X)$ which implies $\nabla_X Y + \nabla_Y X \in \Gamma(\mathcal{D})$ if and only if $(\nabla_X J)Y + (\nabla_Y J)X = 0$.

In particular, for any J-connection ∇, the distributions $\mathcal{D}$ and $\mathcal{D}'$ are ∇-geodesically invariant.

Proposition 3.8. *If (M, J, g) is a metallic pseudo-Riemannian manifold, then the distribution $\mathcal{D}$ is geodesically invariant if and only if:*

$$J \circ M_J(X,Y) = \sigma_- M_J(X,Y), \text{ for any } X, Y \in C^\infty(TM),$$

respectively, $\mathcal{D}'$ is geodesically invariant if and only if:

$$J \circ M_J(X,Y) = \sigma_+ M_J(X,Y), \text{ for any } X, Y \in C^\infty(TM).$$

In particular, both $\mathcal{D}$ and $\mathcal{D}'$ are geodesically invariant if and only if $M_J = 0$.

Proof. The distribution $\mathcal{D}$ is geodesically invariant if and only if

$$\mathcal{P}'(\{\mathcal{P}X, \mathcal{P}Y\}) = 0,$$

for any X, $Y \in C^\infty(TM)$. Therefore, from a direct computation and using Proposition 2.5, with a similar computation like in Proposition 3.5, we obtain the conclusion. □

Remark 3.9. $J_p := \mathcal{P} - \mathcal{P}'$ is an almost product structure on M and

$$J_p X = -\frac{1}{\sqrt{p^2 + 4q}}(2J - pI)X,$$

for any $X \in C^\infty(TM)$.

Direct computations provide the following relationship between J and J_p-brackets, J and J_p Nijenhuis tensors, Jordan bracket and Jordan tensors of the two structures. Precisely, we have the following:

Proposition 3.10.

$$[X,Y]_J = -\frac{\sqrt{p^2+4q}}{2}[X,Y]_{J_p} + \frac{p}{2}[X,Y]$$

$$N_J(X,Y) = \frac{p^2+4q}{4} N_{J_p}(X,Y)$$

$$\{X,Y\}_J = -\frac{\sqrt{p^2+4q}}{2}\{X,Y\}_{J_p} + \frac{p}{2}\{X,Y\}$$

$$M_J(X,Y) = \frac{p^2+4q}{4} M_{J_p}(X,Y).$$

In particular, the deformation tensors are related as follows:

$$H_J(X,Y) = \frac{p^2+4q}{4} H_{J_p}(X,Y).$$

The product conjugate connection of a linear connection ∇ is [2]:

$$\nabla_X^{(J_p)} Y = \mathcal{P}(\nabla_X \mathcal{P}Y) - \mathcal{P}(\nabla_X \mathcal{P}'Y) - \mathcal{P}'(\nabla_X \mathcal{P}Y) + \mathcal{P}'(\nabla_X \mathcal{P}'Y) \tag{2}$$

and we have:

Proposition 3.11. [2] *If $\nabla^{(J_p)}$ is torsion-free, then J_p is integrable, which means that $\mathcal{D}$ and $\mathcal{D}'$ are integrable distributions.*

Definition 3.12. We say that a linear connection ∇ restricts to a distribution $\mathcal{D} \subset TM$ on a metallic pseudo-Riemannian manifold (M,J,g) if $Y \in \Gamma(\mathcal{D})$ implies $\nabla_X Y \in \Gamma(\mathcal{D})$, for any $X \in C^\infty(TM)$.

We have:
1) ∇ restricts to $\mathcal{D}$ means $\mathcal{P}'(\nabla_X \mathcal{P}Y) = 0$ and $\mathcal{P}(\nabla_X \mathcal{P}Y) = \nabla_X \mathcal{P}Y$,
2) ∇ restricts to $\mathcal{D}'$ means $\mathcal{P}(\nabla_X \mathcal{P}'Y) = 0$ and $\mathcal{P}'(\nabla_X \mathcal{P}'Y) = \nabla_X \mathcal{P}'Y$.

A straightforward computation shows that the product conjugate connection $\nabla^{(J_p)}$ defined by (2) restricts to $\mathcal{D}$ and $\mathcal{D}'$. Moreover, if ∇ restricts to both $\mathcal{D}$ and $\mathcal{D}'$, then

$$\nabla_X^{(J_p)} Y = \nabla_X \mathcal{P}Y + \nabla_X \mathcal{P}'Y = \nabla_X Y \tag{3}$$

and so ∇ is an J_p-connection. Let us remark that the above connection (3) is exactly the Schouten-van Kampen connection of the pair $(\mathcal{D}, \mathcal{D}')$:

$$\nabla_X Y = \mathcal{P}(\nabla_X \mathcal{P}Y) + \mathcal{P}'(\nabla_X \mathcal{P}'Y)$$

which coincides with the metallic natural connection $\tilde{\nabla}$ [3] if ∇ is the Levi-Civita connection of g.

Now we can express the Kirichenko tensor fields [9] in terms of the projectors $\mathcal{P}, \mathcal{P}'$:

Proposition 3.13. [2] *The structural and virtual tensor fields of $J_p = \mathcal{P} - \mathcal{P}'$ are:*

$$\begin{cases} C_{\nabla}^{\mathcal{P}-\mathcal{P}'}(X,Y) = 2[\mathcal{P}(\nabla_{\mathcal{P}'X} \mathcal{P}'Y) + \mathcal{P}'(\nabla_{\mathcal{P}X} \mathcal{P}Y)] \\ B_{\nabla}^{\mathcal{P}-\mathcal{P}'}(X,Y) = -2[\mathcal{P}(\nabla_{\mathcal{P}X} \mathcal{P}'Y) + \mathcal{P}'(\nabla_{\mathcal{P}'X} \mathcal{P}Y)]. \end{cases}$$

Let us recall the well-known *fundamental tensor fields* of O'Neill-Gray:

$$\begin{cases} T(X,Y) = \mathcal{P}(\nabla_{\mathcal{P}'X} \mathcal{P}'Y) + \mathcal{P}'(\nabla_{\mathcal{P}'X} \mathcal{P}Y) \\ A(X,Y) = \mathcal{P}'(\nabla_{\mathcal{P}X} \mathcal{P}Y) + \mathcal{P}(\nabla_{\mathcal{P}X} \mathcal{P}'Y). \end{cases}$$

Then, a comparison of last two equations yields

$$\begin{cases} C_{\nabla}^{\mathcal{P}-\mathcal{P}'}(X,Y) = 2[T(X,\mathcal{P}'Y) + A(X,\mathcal{P}Y)] \\ B_{\nabla}^{\mathcal{P}-\mathcal{P}'}(X,Y) = -2[T(X,\mathcal{P}Y) + A(X,\mathcal{P}'Y)] \end{cases}$$

a fact which justifies the second name of T and A as *invariants* of the decomposition $TM = \mathcal{D} \oplus \mathcal{D}'$ [6].

On $\mathcal{D}$ with the induced metric $g_{\mathcal{D}}$, we consider the induced connection from the pseudo-Riemannian manifold (M, g, ∇) by [10]:

$$\nabla^{\mathcal{D}} : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D}), \ \ \nabla^{\mathcal{D}}_X Y := \mathcal{P}(\nabla_X Y)$$

which preserves the metric $g_{\mathcal{D}}$ and is torsion-free w.r.t. the bracket

$$[\cdot, \cdot]_{\mathcal{D}} : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D}), \ \ [X, Y]_{\mathcal{D}} := \mathcal{P}([X, Y]).$$

The bracket $[\cdot, \cdot]_{\mathcal{D}}$ has the usual properties of a Lie bracket excepting the Jacobi identity which is satisfied if and only if $\mathcal{D}$ is integrable.

The integrability of $\mathcal{D}$ can also be characterized in terms of second fundamental form of $\mathcal{D}$:

$$h : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D}'), \ \ h(X, Y) := \nabla_X Y - \nabla^{\mathcal{D}}_X Y,$$

and we can state:

Proposition 3.14. [10] *The distribution $\mathcal{D}$ is integrable if and only if one of the following assertions holds: (i) $\nabla^{\mathcal{D}}$ is torsion-free; (ii) h is symmetric.*

Similarly, on $(\mathcal{D}', g_{\mathcal{D}'})$ we define the induced connection from (M, g, ∇) by:

$$\nabla^{\mathcal{D}'} : \Gamma(\mathcal{D}') \times \Gamma(\mathcal{D}') \to \Gamma(\mathcal{D}'), \ \ \nabla^{\mathcal{D}'}_X Y := \mathcal{P}'(\nabla_X Y)$$

and consider the second fundamental form h' of $\mathcal{D}'$. Then the distribution $\mathcal{D}'$ is integrable if and only if one of the following assertions holds: (i) $\nabla^{\mathcal{D}'}$ is torsion-free; (ii) h' is symmetric.

We remark that the restrictions of the metallic natural connection $\tilde{\nabla}$, defined in [3], to $\mathcal{D}$ and respectively, to $\mathcal{D}'$, coincide with the two induced connections, respectively:

$$\tilde{\nabla}|_{\Gamma(\mathcal{D}) \times \Gamma(\mathcal{D})} = \nabla^{\mathcal{D}}, \ \ \tilde{\nabla}|_{\Gamma(\mathcal{D}') \times \Gamma(\mathcal{D}')} = \nabla^{\mathcal{D}'}.$$

Remark 3.15. For $p^2 + 4q = 0$, we get only one distribution, $\ker(J - \frac{p}{2}I)$, and $J_t := J - \frac{p}{2}I$ is an almost subtangent structure.

4. Adapted connections to $(\mathcal{D}, \mathcal{D}')$

Definition 4.1. We say that a linear connection ∇ on M is *adapted* to the decomposition $TM = \mathcal{D} \oplus \mathcal{D}'$ if $Y \in \Gamma(\mathcal{D})$ implies $\nabla_X Y \in \Gamma(\mathcal{D})$, for any $X \in C^\infty(TM)$ and $Y \in \Gamma(\mathcal{D}')$ implies $\nabla_X Y \in \Gamma(\mathcal{D}')$, for any $X \in C^\infty(TM)$.

Remark 4.2. If (M, J) is a metallic manifold such that $J^2 = pJ + qI$ with $p^2 + 4q > 0$, then a linear connection ∇ is adapted to $(\mathcal{D}, \mathcal{D}')$ given by (1) if and only if ∇ is a J-connection. Indeed, for $Y \in \Gamma(\mathcal{D})$ we have $JY = \sigma_- Y$ and $(\nabla_X J)Y = \sigma_- \nabla_X Y - J(\nabla_X Y)$, for any $X \in C^\infty(TM)$, which implies that $\nabla_X Y \in \Gamma(\mathcal{D})$ if and only if $\nabla J = 0$. Similarly we deduce the second implication.

In [1], A. Bejancu and H. R. Farran gave the expression of all adapted connections to $(\mathcal{D}, \mathcal{D}')$, namely:

$$\nabla^*_X Y = \mathcal{P}(\nabla_X \mathcal{P}Y) + \mathcal{P}'(\nabla_X \mathcal{P}'Y) + \mathcal{P}(S(X, \mathcal{P}Y)) + \mathcal{P}'(S(X, \mathcal{P}'Y)), \qquad (4)$$

for any $X, Y \in C^\infty(TM)$, where ∇ is a linear connection and S is a $(1,2)$-tensor field on M.

4.1. Schouten-van Kampen connection. An adapted connection to $(\mathcal{D}, \mathcal{D}')$ is *the Schouten-van Kampen connection* $\tilde{\nabla}$ of the linear connection ∇, obtained from (4) for $S := 0$:

$$\begin{aligned}\tilde{\nabla}_X Y &:= \mathcal{P}(\nabla_X \mathcal{P}Y) + \mathcal{P}'(\nabla_X \mathcal{P}'Y) = \qquad (5)\\ &= \nabla_X Y + \mathcal{P}((\nabla_X \mathcal{P})Y) + \mathcal{P}'((\nabla_X \mathcal{P}')Y).\end{aligned}$$

If (M, J, g) is a metallic pseudo-Riemannian manifold such that $J^2 = pJ + qI$ with $p^2 + 4q > 0$ and ∇ is torsion-free, then $\tilde{\nabla}$ is explicitly given by:

$$\tilde{\nabla}_X Y = \frac{1}{p^2 + 4q}[(2J - pI)(\nabla_X JY) - (pJ - (p^2 + 2q)I)(\nabla_X Y)], \qquad (6)$$

for any $X, Y \in C^\infty(TM)$. We remark that if ∇ is the Levi-Civita connection associated to g, then $\tilde{\nabla}$ is exactly the metallic natural connection defined in [3]. Moreover, $\tilde{\nabla}$ is a metric J-connection, i.e. $\tilde{\nabla} g = \tilde{\nabla} J = 0$, whose torsion is given by:

$$T^{\tilde{\nabla}}(X, Y) = \frac{1}{p^2 + 4q}[(2J - pI)(\nabla_X JY - \nabla_Y JX) - (pJ + 2qI)(\nabla_X Y - \nabla_Y X)],$$

for any $X, Y \in C^\infty(TM)$.

4.2. Vrănceanu connection. Another adapted connection to $(\mathcal{D}, \mathcal{D}')$ is the *Vrănceanu connection* $\bar{\nabla}$ of the linear connection ∇, obtained from (4) for

$$S(X, Y) := -\mathcal{P}(\nabla_{\mathcal{P}'X} \mathcal{P}Y) - \mathcal{P}'(\nabla_{\mathcal{P}X} \mathcal{P}'Y) + \mathcal{P}([\mathcal{P}'X, \mathcal{P}Y]) + \mathcal{P}'([\mathcal{P}X, \mathcal{P}'Y]).$$

If (M, J, g) is a metallic pseudo-Riemannian manifold such that $J^2 = pJ + qI$ with $p^2 + 4q > 0$, then $\bar{\nabla}$ is explicitly given by:

$$\begin{aligned}\bar{\nabla}_X Y &= \tilde{\nabla}_{\mathcal{P}X} Y + \mathcal{P}([\mathcal{P}'X, \mathcal{P}Y]) + \mathcal{P}'([\mathcal{P}X, \mathcal{P}'Y]) = \qquad (7)\\ &= \nabla_X Y + \frac{1}{p^2 + 4q}[2J((\nabla_X J)Y) - p(\nabla_X J)Y + J((\nabla_Y J)X) +\\ &\quad + (\nabla_{JY} J)X - p(\nabla_Y J)X] +\\ &\quad + \frac{1}{p^2 + 4q}[T^\nabla(JX, JY) + J(T^\nabla(JX, Y)) - pT^\nabla(JX, Y) -\\ &\quad - J(T^\nabla(X, JY)) - qT^\nabla(X, Y)],\end{aligned}$$

for any $X, Y \in C^\infty(TM)$.

Moreover, $\bar{\nabla}$ is a J-connection, i.e. $\bar{\nabla} J = 0$, whose torsion is given by:

$$T^{\bar{\nabla}}(X, Y) = \frac{1}{p^2 + 4q} N_J(X, Y) + \mathcal{P}'(T^\nabla(\mathcal{P}'X, \mathcal{P}'Y)) - \mathcal{P}(T^\nabla(\mathcal{P}X, \mathcal{P}Y)),$$

for any $X, Y \in C^\infty(TM)$.

4.3. Vidal connection. Let (M, J, g) be a metallic pseudo-Riemannian manifold such that $J^2 = pJ + qI$ with $p^2 + 4q > 0$ and let ∇ be the Levi-Civita connection of g.

Another adapted connection to $(\mathcal{D}, \mathcal{D}')$ is *the Vidal connection* $\tilde{\tilde{\nabla}}$ associated to J, obtained from (4) for

$$S(X, Y) := -\mathcal{P}(\nabla_{\mathcal{P}Y}\mathcal{P}')X - \mathcal{P}'(\nabla_{\mathcal{P}'Y}\mathcal{P})X,$$

therefore:

$$\tilde{\tilde{\nabla}}_X Y = \tilde{\nabla}_X Y - \mathcal{P}(\nabla_{\mathcal{P}Y}\mathcal{P}')X - \mathcal{P}'(\nabla_{\mathcal{P}'Y}\mathcal{P})X = \tag{8}$$

$$= \tilde{\nabla}_X Y + \frac{1}{p^2+4q}[(\nabla_{JY}J)X + J((\nabla_Y J)X) - p(\nabla_Y J)X] =$$

$$= \nabla_X Y + \frac{1}{p^2+4q}[2J((\nabla_X J)Y) - p(\nabla_X J)Y + J((\nabla_Y J)X) +$$

$$+ (\nabla_{JY}J)X - p(\nabla_Y J)X],$$

for any $X, Y \in C^\infty(TM)$.

Moreover, $\tilde{\tilde{\nabla}}$ is a J-connection, i.e. $\tilde{\tilde{\nabla}}J = 0$, whose torsion is given by:

$$T^{\tilde{\tilde{\nabla}}}(X, Y) = \frac{1}{p^2+4q} N_J(X, Y),$$

for any $X, Y \in C^\infty(TM)$.

Remark 4.3. The Vrănceanu connection of the Levi-Civita connection coincides with the Vidal connection.

Moreover, we get:

$$(\tilde{\tilde{\nabla}}_X g)(Y, Z) = -\frac{1}{p^2+4q}[g((\nabla_{JY}J)X - (\nabla_Y J)JX, Z) +$$

$$+ g((\nabla_{JZ}J)X - (\nabla_Z J)JX, Y)] =$$

$$= \frac{1}{p^2+4q}[g(M_J(Y, X), Z) + g(M_J(Z, X), Y) + g((\nabla_{JX}J)Y + (\nabla_Y J)JX, Z) +$$

$$+ g((\nabla_{JX}J)Z + (\nabla_Z J)JX, Y)],$$

for any $X, Y, Z \in C^\infty(TM)$.

Since $\tilde{\nabla}J = \bar{\nabla}J = \tilde{\tilde{\nabla}}J = 0$, from Remark 3.7 we deduce:

Proposition 4.4. *The distributions $\mathcal{D}$ and $\mathcal{D}'$ are $\tilde{\nabla}$-geodesically invariant, $\bar{\nabla}$-geodesically invariant and $\tilde{\tilde{\nabla}}$-geodesically invariant.*

Using the Vidal connection $\tilde{\tilde{\nabla}}$, we characterize the integrability and the geodesic invariance of the metallic distributions defined by J in terms of the torsion and the covariant derivative of g w.r.t. this connection. From all the above considerations, we can state:

Theorem 4.5. *If (M, J, g) is a metallic pseudo-Riemannian manifold such that $J^2 = pJ + qI$ with $p^2 + 4q > 0$, then the following assertions are equivalent:*
(i) the distributions $\mathcal{D}$ and $\mathcal{D}'$ are integrable;
(ii) $N_J = 0$;
(iii) $L = 0$ and $L' = 0$;
(iv) the Vidal connection given by (8) is torsion-free.

Theorem 4.6. *If (M, J, g) is a metallic pseudo-Riemannian manifold such that $J^2 = pJ + qI$ with $p^2 + 4q > 0$, then the following assertions are equivalent:*
(i) the distributions $\mathcal{D}$ and $\mathcal{D}'$ are geodesically invariant;
(ii) $M_J = 0$;
(iii) $K = 0$ and $K' = 0$;
(iv) the Vidal connection given by (8) is metric with respect to g.

4.4. Leaves correspondence via metallic maps. We shall provide the condition for a metallic map between two metallic pseudo-Riemannian manifolds to preserve the metallic distributions. We recall the following:

Definition 4.7. A smooth map $\Phi : (M_1, J_1) \to (M_2, J_2)$ between two metallic manifolds is called a *metallic map* if:

$$\Phi_* \circ J_1 = J_2 \circ \Phi_*.$$

Remark 4.8. If $\Phi : (M_1, J_1) \to (M_2, J_2)$ is a metallic map and $J_i^2 = p_i J_i + q_i I$ with p_i and q_i real numbers, $i = 1, 2$, then:
(i) $\Phi_* \circ J_1^{2k+1} = J_2^{2k+1} \circ \Phi_*$, for any $k \in \mathbb{N}$;
(ii) $([(p_2^2 + q_2) - (p_1^2 + q_1)]J_1 + (p_2 q_2 - p_1 q_1)I)(TM_1) \subset \ker \Phi_*$;
(iii) in the particular case when one the structure is product and the other one is complex, then $ImJ_1 \subset \ker \Phi_*$.

Consider a metallic map $\Phi : (M_1, J_1) \to (M_2, J_2)$ between the metallic manifolds (M_i, J_i) such that $J_i^2 = p_i J_i + q_i I$ with $p_i^2 + 4q_i > 0$, $i = 1, 2$, and assume that the distributions $\mathcal{D}_i$ and $\mathcal{D}_i'$, $i = 1, 2$, are integrable. Then they define the foliations $\mathcal{F}_i$ and $\mathcal{F}_i'$, $i = 1, 2$, whose leaves are trivial metallic pseudo-Riemannian manifolds.

Denoting by $\Phi^* \mathcal{D}_2$ the pull-back distribution, i.e.:

$$(\Phi^* \mathcal{D}_2)_x := \{X_x \in T_x M : \Phi_{*x}(X_x) \in \mathcal{D}_{2\Phi(x)}\},$$

since Φ is a metallic map, we get:

$$(\Phi^* \mathcal{D}_2)_x = \{X_x \in T_x M : (J_1 - \sigma_{2_+} I)(X_x) \in \ker \Phi_{*x}\},$$

where $\sigma_{i_+} = \frac{p_i + \sqrt{p_i^2 + 4q_i}}{2}$, $i = 1, 2$ and

$$(\Phi^* \mathcal{D}_2')_x = \{X_x \in T_x M : (J_1 - \sigma_{2\_} I)(X_x) \in \ker \Phi_{*x}\},$$

where $\sigma_{i\_} = \frac{p_i - \sqrt{p_i^2 + 4q_i}}{2}$, $i = 1, 2$.

From the above considerations, we obtain a sufficient condition for the pull-back distribution $\Phi^* \mathcal{D}_2$ to coincide with one of the distributions $\mathcal{D}_1$ or $\mathcal{D}_1'$:

Proposition 4.9. *If* $\ker \Phi_* = (J_1 - \sigma_{2_+} I)(\ker(J_1 - \sigma_{1_+} I))$*, then* $\Phi^* \mathcal{D}_2 = \mathcal{D}_1$*. Moreover, if* Φ *is a surjective submersion with connected fibers, then a leaf of* $\mathcal{F}_2$ *corresponds to a leaf of* $\mathcal{F}_1$*.*

5. A Chen-type inequality for the metallic distributions

A fundamental problem in the theory of submanifolds is the problem posed by B. Y. Chen [4], namely, to find relations between the main intrinsic and extrinsic invariants of a submanifold. In this sense, the Chen's inequalities for submanifolds in real space forms was proved by B. Y. Chen [4], in complex space forms by Y. Doğru [5], in quaternionic space forms by G. E. Vîlcu [12] etc. In the same spirit, we shall prove a Chen-type inequality in the metallic case, for an integrable distribution defined by the metallic structure.

Let (M, J, g) be an m-dimensional metallic Riemannian manifold and assume that the distribution $\mathcal{D}$ is integrable. In this case, the Riemann curvature tensors of $\mathcal{D}$ (computed with respect to the induced connection $\nabla^{\mathcal{D}}$ on $\mathcal{D}$ and the Lie bracket $[\cdot, \cdot]_{\mathcal{D}}$) and M satisfy [10]:

$$R^{\mathcal{D}}(X, Y, Z, W) = R^M(X, Y, Z, W) - \tag{9}$$

$$-g(h(X, Z), h(Y, W)) + g(h(X, W), h(Y, Z)),$$

for any $X, Y, Z, W \in \Gamma(\mathcal{D})$.

The relation between the mean curvature (the main extrinsic invariant) and the Chen first invariant (an intrinsic invariant), in a particular case of constant J-sectional curvature, is given in the following.

From a direct computation we obtain:

Proposition 5.1. *Let* (M, J, g) *be an* m*-dimensional metallic Riemannian manifold* $(m > 2)$ *such that* $J^2 = pJ + qI$ *with* $p^2 + 4q > 0$*, whose Riemann curvature tensor is given by*

$$R^M(X, Y, Z, W) = c[g(X, FW)g(Y, FZ) - g(X, FZ)g(Y, FW)], \tag{10}$$

for any $X, Y, Z, W \in C^\infty(TM)$*, where* $F := aJ + bI$ *with* a *and* b *real numbers satisfying* $qa^2 - pab - b^2 = 1$*. Then the* J*-sectional curvature of* M *is constant equal to* c*.*

Denote by $H := \frac{1}{n} tr(h)$ the *mean curvature* and by $\delta_{\mathcal{D}} := \tau^{\mathcal{D}} - \inf K^{\mathcal{D}}$ the *Chen first invariant* of $\mathcal{D}$, where $\tau^{\mathcal{D}}$ denotes the scalar curvature of $\mathcal{D}$ and $K^{\mathcal{D}}$ its sectional curvature.

Theorem 5.2. *Let* (M, J, g) *be an* m*-dimensional metallic Riemannian manifold* $(m > 2)$ *such that* $J^2 = pJ + qI$ *with* $p^2 + 4q > 0$*, whose Riemann curvature tensor is given by* (10) *and let* $\mathcal{D}$ *given by* (1) *be an* n*-dimensional integrable distribution. Then:*

$$\delta_{\mathcal{D}} \leq \frac{c(a\sigma_- + b)^2(n^2 - n + 2)}{2} + \frac{n^2(n-2)}{2(n-1)} ||H||^2.$$

Proof. Consider an orthonormal frame field $\{e_1, \dots, e_n\}$ for $\mathcal{D}$, $\{f_1, \dots, f_{m-n}\}$ an orthonormal frame field for $\mathcal{D}'$ and denote by

$$h_{ij}^k := g(h(e_i, e_j), f_k).$$

From (9) and (10) we get

$$2\tau^{\mathcal{D}} = c(a\sigma_- + b)^2 n(n-1) - ||h||^2 + n^2 ||H||^2.$$

Moreover

$$K^{\mathcal{D}}(e_1, e_2) = -c(a\sigma_- + b)^2 - \sum_{k=1}^{m-n} h_{11}^k h_{22}^k + \sum_{k=1}^{m-n} (h_{12}^k)^2$$

and

$$\tau^{\mathcal{D}} - K^{\mathcal{D}}(e_1, e_2) = \frac{c(a\sigma_- + b)^2(n^2 - n + 2)}{2} +$$

$$+ \sum_{k=1}^{m-n} [\sum_{3 \le i < j \le n} (h_{ii}^k h_{jj}^k - (h_{ij}^k)^2) + \sum_{j=3}^{n} (h_{11}^k + h_{22}^k) h_{jj}^k - \sum_{j=3}^{n} ((h_{1j}^k)^2 + (h_{2j}^k)^2)] \le$$

$$\le \frac{c(a\sigma_- + b)^2(n^2 - n + 2)}{2} + \frac{n-2}{2(n-1)} \sum_{k=1}^{m-n} (\sum_{j=1}^{n} h_{jj}^k)^2 - \sum_{k=1}^{m-n} \sum_{j=3}^{n} ((h_{1j}^k)^2 + (h_{2j}^k)^2) =$$

$$= \frac{c(a\sigma_- + b)^2(n^2 - n + 2)}{2} + \frac{n^2(n-2)}{2(n-1)} ||H||^2 - \sum_{k=1}^{m-n} \sum_{j=3}^{n} ((h_{1j}^k)^2 + (h_{2j}^k)^2) \le$$

$$\le \frac{c(a\sigma_- + b)^2(n^2 - n + 2)}{2} + \frac{n^2(n-2)}{2(n-1)} ||H||^2.$$

□

Remark 5.3. If $p = 0$ and $q = 1$, i.e. J is an almost product structure, then the inequality from Theorem 5.2 becomes

$$\delta_{\mathcal{D}} \le \frac{c(a-b)^2(n^2 - n + 2)}{2} + \frac{n^2(n-2)}{2(n-1)} ||H||^2.$$

In particular, if $a = 1$ and $b = 0$, i.e. $F = J$, we get

$$\delta_{\mathcal{D}} \le \frac{c(n^2 - n + 2)}{2} + \frac{n^2(n-2)}{2(n-1)} ||H||^2.$$

6. Metallic Norden structures

6.1. Complex metallic distributions. Let (M, J, g) be a metallic Norden manifold such that $J^2 = pJ + qI$ with $p^2 + 4q < 0$ and let $T^{\mathbb{C}}M := TM \otimes_{\mathbb{R}} \mathbb{C}$ be the complexified tangent bundle. Then we can define *the complexified metallic pseudo-Riemannian structure*:

$$J^{\mathbb{C}}(X + iY) := JX + iJY,$$

$$g^{\mathbb{C}}(X_1 + iY_1, X_2 + iY_2) := g(X_1, X_2) - g(Y_1, Y_2) + i[g(X_1, Y_2) + g(Y_1, X_2)],$$

for any $X, X_1, X_2, Y, Y_1, Y_2 \in C^\infty(TM)$.

Denote by $\sigma^{\mathbb{C}}_{\pm} := \frac{p\pm\sqrt{p^2+4q}}{2}$ and consider the projection operators $\mathcal{P}^{\mathbb{C}}$ and $\mathcal{P}^{\mathbb{C}'}$:

$$\mathcal{P}^{\mathbb{C}} := -\frac{1}{\sqrt{p^2+4q}}J^{\mathbb{C}} + \frac{\sigma^{\mathbb{C}}_{+}}{\sqrt{p^2+4q}}I^{\mathbb{C}}, \ \ \mathcal{P}^{\mathbb{C}'} := \frac{1}{\sqrt{p^2+4q}}J^{\mathbb{C}} - \frac{\sigma^{\mathbb{C}}_{-}}{\sqrt{p^2+4q}}I^{\mathbb{C}}$$

satisfying

$$\mathcal{P}^{\mathbb{C}^2} = \mathcal{P}^{\mathbb{C}}, \ \ \mathcal{P}^{\mathbb{C}'^2} = \mathcal{P}^{\mathbb{C}'}, \ \ \mathcal{P}^{\mathbb{C}} + \mathcal{P}^{\mathbb{C}'} = I^{\mathbb{C}}, \ \ \mathcal{P}^{\mathbb{C}} \circ \mathcal{P}^{\mathbb{C}'} = 0, \ \ \mathcal{P}^{\mathbb{C}'} \circ \mathcal{P}^{\mathbb{C}} = 0$$

and define the complementary distributions:

$$\mathcal{D}^{\mathbb{C}} := \ker \mathcal{P}^{\mathbb{C}'}, \mathcal{D}^{\mathbb{C}'} := \ker \mathcal{P}^{\mathbb{C}} \tag{11}$$

which we shall call *the complex metallic distributions* defined by J.

Remark 6.1. If (M, J, g) is a metallic Norden manifold such that $J^2 = pJ + qI$ with $p^2 + 4q < 0$, then $\mathcal{D}^{\mathbb{C}}$ and $\mathcal{D}^{\mathbb{C}'}$ are $J^{\mathbb{C}}$-invariant.

Lemma 6.2.

$$\mathcal{D}^{\mathbb{C}'} = \overline{\mathcal{D}^{\mathbb{C}}}$$

Proof. It follows from $\sigma^{\mathbb{C}}_{+} = \overline{\sigma^{\mathbb{C}}_{-}}$. □

In particular, if J is not trivial, that it admits two complex eigenvalues, or the two distributions are both different from 0, then the complexified tangent bundle splits as a direct sum of two conjugate subbundles:

$$T^{\mathbb{C}}M = \mathcal{D}^{\mathbb{C}} \oplus \overline{\mathcal{D}^{\mathbb{C}}}.$$

Extending the Lie bracket to:

$$[X_1 + iY_1, X_2 + iY_2]^{\mathbb{C}} := [X_1, X_2] - [Y_1, Y_2] + i([X_1, Y_2] + [Y_1, X_2]),$$

for any $X_1, X_2, Y_1, Y_2 \in C^\infty(TM)$, we say that:

Definition 6.3. A distribution $\mathcal{D}^{\mathbb{C}} \subset T^{\mathbb{C}}M$ is called *integrable* if $X, Y \in \Gamma(\mathcal{D}^{\mathbb{C}})$ implies $[X, Y]^{\mathbb{C}} \in \Gamma(\mathcal{D}^{\mathbb{C}})$.

Lemma 6.4. *The distribution $\mathcal{D}^{\mathbb{C}}$ is integrable if and only if*

$$\mathcal{P}^{\mathbb{C}'}([\mathcal{P}^{\mathbb{C}}X, \mathcal{P}^{\mathbb{C}}Y]^{\mathbb{C}}) = 0,$$

for any $X, Y \in C^\infty(T^{\mathbb{C}}M)$.

Proposition 6.5. *The distribution $\mathcal{D}^{\mathbb{C}}$ (resp. $\mathcal{D}^{\mathbb{C}'}$) given by (11) is integrable if and only if $N_J = 0$.*

Extending the Levi-Civita connection ∇ of g to:

$$\nabla^{\mathbb{C}}_{X_1+iY_1}(X_2 + iY_2) := \nabla_{X_1}X_2 - \nabla_{Y_1}Y_2 + i(\nabla_{X_1}Y_2 + \nabla_{Y_1}X_2),$$

for any $X_1, X_2, Y_1, Y_2 \in C^\infty(TM)$, we pose the following:

Definition 6.6. Given a complex linear connection $\nabla^{\mathbb{C}}$ on a smooth manifold M, a distribution $\mathcal{D}^{\mathbb{C}} \subset T^{\mathbb{C}}M$ is called $\nabla^{\mathbb{C}}$-*geodesically invariant* if $X, Y \in \Gamma(\mathcal{D}^{\mathbb{C}})$ implies $\nabla^{\mathbb{C}}_X Y + \nabla^{\mathbb{C}}_Y X \in \Gamma(\mathcal{D}^{\mathbb{C}})$.

In particular, if $\nabla^{\mathbb{C}}$ is the Levi-Civita connection of the pseudo-Riemannian manifold $(M, g^{\mathbb{C}})$, then $\mathcal{D}^{\mathbb{C}}$ is called *geodesically invariant*.

Lemma 6.7. *The distribution $\mathcal{D}^{\mathbb{C}}$ is geodesically invariant if and only if*

$$\mathcal{P}^{\mathbb{C}'}(\{\mathcal{P}^{\mathbb{C}}X, \mathcal{P}^{\mathbb{C}}Y\}^{\mathbb{C}}) = 0,$$

for any $X, Y \in C^{\infty}(T^{\mathbb{C}}M)$, where $\{X, Y\}^{\mathbb{C}} := \nabla^{\mathbb{C}}_X Y + \nabla^{\mathbb{C}}_X Y$.

Proposition 6.8. *The distribution $\mathcal{D}^{\mathbb{C}}$ (resp. $\mathcal{D}^{\mathbb{C}'}$) given by (11) is geodesically invariant if and only if $M_J = 0$.*

Remark 6.9. For a complex linear connection $\nabla^{\mathbb{C}}$ on M, the distribution $\mathcal{D}^{\mathbb{C}}$ (resp. $\mathcal{D}^{\mathbb{C}'}$) given by (11) is $\nabla^{\mathbb{C}}$-geodesically invariant if and only if $(\nabla^{\mathbb{C}}_X J^{\mathbb{C}})Y + (\nabla^{\mathbb{C}}_Y J^{\mathbb{C}})X = 0$, for any $X, Y \in \Gamma(\mathcal{D}^{\mathbb{C}})$ (resp. $X, Y \in \Gamma(\mathcal{D}^{\mathbb{C}'})$). Indeed, for $X, Y \in \Gamma(\mathcal{D}^{\mathbb{C}})$ we have $J^{\mathbb{C}}X = \sigma^{\mathbb{C}}_{-}X$, $J^{\mathbb{C}}Y = \sigma^{\mathbb{C}}_{-}Y$ and $J^{\mathbb{C}}(\nabla^{\mathbb{C}}_X Y + \nabla^{\mathbb{C}}_Y X) = -(\nabla^{\mathbb{C}}_X J^{\mathbb{C}})Y - (\nabla^{\mathbb{C}}_Y J^{\mathbb{C}})X + \sigma^{\mathbb{C}}_{-}(\nabla^{\mathbb{C}}_X Y + \nabla^{\mathbb{C}}_Y X)$ which implies that $\nabla^{\mathbb{C}}_X Y + \nabla^{\mathbb{C}}_Y X \in \Gamma(\mathcal{D}^{\mathbb{C}})$ if and only if $(\nabla^{\mathbb{C}}_X J^{\mathbb{C}})Y + (\nabla^{\mathbb{C}}_Y J^{\mathbb{C}})X = 0$.

In particular, for any $J^{\mathbb{C}}$-connection $\nabla^{\mathbb{C}}$, the distributions $\mathcal{D}^{\mathbb{C}}$ and $\mathcal{D}^{\mathbb{C}'}$ are $\nabla^{\mathbb{C}}$-geodesically invariant.

Remark 6.10. $J_c := i(\mathcal{P}^{\mathbb{C}} - \mathcal{P}^{\mathbb{C}'})$ is a Norden structure on M and

$$J_c X = -\frac{1}{\sqrt{-p^2 - 4q}}(2J - pI)X,$$

for any $X \in C^{\infty}(TM)$.

By a direct computation we get:

Proposition 6.11. *The Nijenhuis tensors of J_c and J are related as follows:*

$$N_{J_c}(X, Y) = \frac{4}{-p^2 - 4q} N_J(X, Y),$$

for any $X, Y \in C^{\infty}(TM)$.

Moreover, if

$$T^{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M$$

is the decomposition of the complexified tangent bundle into $(1, 0)$ and $(0, 1)$ parts, with respect to the almost complex structure J_c, we have:

$$\mathcal{D}^{\mathbb{C}'} = T^{(1,0)}M, \ \ \mathcal{D}^{\mathbb{C}} = T^{(0,1)}M.$$

Definition 6.12. We say that a complex linear connection $\nabla^{\mathbb{C}}$ on M is *adapted* to the decomposition $T^{\mathbb{C}}M = \mathcal{D}^{\mathbb{C}} \oplus \mathcal{D}^{\mathbb{C}'}$ if $Y \in \Gamma(\mathcal{D}^{\mathbb{C}})$ implies $\nabla^{\mathbb{C}}_X Y \in \Gamma(\mathcal{D}^{\mathbb{C}})$, for any $X \in C^{\infty}(T^{\mathbb{C}}M)$ and $Y \in \Gamma(\mathcal{D}^{\mathbb{C}'})$ implies $\nabla_X Y \in \Gamma(\mathcal{D}^{\mathbb{C}'})$, for any $X \in C^{\infty}(T^{\mathbb{C}}M)$.

Remark 6.13. If (M, J, g) is a metallic Norden manifold such that $J^2 = pJ + qI$ with $p^2 + 4q < 0$, then a complex linear connection $\nabla^{\mathbb{C}}$ is adapted to $(\mathcal{D}^{\mathbb{C}}, \mathcal{D}^{\mathbb{C}'})$ given by (11) if and only if $\nabla^{\mathbb{C}}$ is a $J^{\mathbb{C}}$-connection. Indeed, for $Y \in \Gamma(\mathcal{D}^{\mathbb{C}})$ we have $J^{\mathbb{C}}Y = \sigma_-^{\mathbb{C}}Y$ and $(\nabla_X^{\mathbb{C}}J^{\mathbb{C}})Y = \sigma_-^{\mathbb{C}}\nabla_X^{\mathbb{C}}Y - J^{\mathbb{C}}(\nabla_X^{\mathbb{C}}Y)$, for any $X \in C^\infty(T^{\mathbb{C}}M)$, which implies that $\nabla_X^{\mathbb{C}}Y \in \Gamma(\mathcal{D}^{\mathbb{C}})$ if and only if $\nabla^{\mathbb{C}}J^{\mathbb{C}} = 0$. Similarly we deduce the second implication.

Proposition 6.14. *All adapted connections to $(\mathcal{D}^{\mathbb{C}}, \mathcal{D}^{\mathbb{C}'})$ are of the form:*

$$(\nabla^{\mathbb{C}})_X^*Y = \mathcal{P}^{\mathbb{C}}(\nabla_X^{\mathbb{C}}\mathcal{P}^{\mathbb{C}}Y) + \mathcal{P}^{\mathbb{C}'}(\nabla_X^{\mathbb{C}}\mathcal{P}^{\mathbb{C}'}Y) + \tag{12}$$
$$+\mathcal{P}^{\mathbb{C}}(S(X, \mathcal{P}^{\mathbb{C}}Y)) + \mathcal{P}^{\mathbb{C}'}(S(X, \mathcal{P}^{\mathbb{C}'}Y)),$$

for any X, $Y \in C^\infty(T^{\mathbb{C}}M)$, where $\nabla^{\mathbb{C}}$ is a complex linear connection and S is a complex $(1,2)$-tensor field on M.

Proof. We follow the same steps like in the real case [1]. □

Consider the following adapted connection to $(\mathcal{D}^{\mathbb{C}}, \mathcal{D}^{\mathbb{C}'})$:

1) *The complex Schouten-van Kampen connection* $\tilde{\nabla}^{\mathbb{C}}$ of the complex linear connection $\nabla^{\mathbb{C}}$, obtained from (12) for $S := 0$:

$$\tilde{\nabla}_X^{\mathbb{C}}Y := \mathcal{P}^{\mathbb{C}}(\nabla_X^{\mathbb{C}}\mathcal{P}^{\mathbb{C}}Y) + \mathcal{P}^{\mathbb{C}'}(\nabla_X^{\mathbb{C}}\mathcal{P}^{\mathbb{C}'}Y).$$

If (M, J, g) is a metallic Norden manifold such that $J^2 = pJ + qI$ with $p^2 + 4q < 0$ and $\nabla^{\mathbb{C}}$ is torsion-free, then $\tilde{\nabla}^{\mathbb{C}}$ is explicitly given by:

$$\tilde{\nabla}_X^{\mathbb{C}}Y = \frac{1}{p^2+4q}[(2J^{\mathbb{C}} - pI^{\mathbb{C}})(\nabla_X^{\mathbb{C}}JY) - (pJ^{\mathbb{C}} - (p^2+2q)I^{\mathbb{C}})(\nabla_X^{\mathbb{C}}Y)] = \tag{13}$$
$$= \nabla_X^{\mathbb{C}}Y + \frac{1}{p^2+4q}[2J^{\mathbb{C}}(\nabla_X^{\mathbb{C}}J^{\mathbb{C}}) - p(\nabla_X^{\mathbb{C}}J^{\mathbb{C}})]Y,$$

for any X, $Y \in C^\infty(T^{\mathbb{C}}M)$.

We remark that if $\nabla^{\mathbb{C}}$ is the Levi-Civita connection associated to $g^{\mathbb{C}}$, then $\tilde{\nabla}^{\mathbb{C}}$ is a metric $J^{\mathbb{C}}$-connection, i.e. $\tilde{\nabla}^{\mathbb{C}}g^{\mathbb{C}} = \tilde{\nabla}^{\mathbb{C}}J^{\mathbb{C}} = 0$, whose torsion is given by:

$$T^{\tilde{\nabla}^{\mathbb{C}}}(X,Y) = \frac{1}{p^2+4q}[(2J^{\mathbb{C}} - pI^{\mathbb{C}})(\nabla_X^{\mathbb{C}}JY - \nabla_Y^{\mathbb{C}}J^{\mathbb{C}}X) -$$
$$-(pJ^{\mathbb{C}} + 2qI^{\mathbb{C}})(\nabla_X^{\mathbb{C}}Y - \nabla_Y^{\mathbb{C}}X)],$$

for any X, $Y \in C^\infty(T^{\mathbb{C}}M)$.

2) *The complex Vrănceanu connection* $\bar{\nabla}^{\mathbb{C}}$ of the complex linear connection $\nabla^{\mathbb{C}}$, obtained from (12) for

$$S(X,Y) := -\mathcal{P}^{\mathbb{C}}(\nabla_{\mathcal{P}^{\mathbb{C}'}X}^{\mathbb{C}}\mathcal{P}^{\mathbb{C}}Y) - \mathcal{P}^{\mathbb{C}'}(\nabla_{\mathcal{P}^{\mathbb{C}'}X}^{\mathbb{C}}\mathcal{P}^{\mathbb{C}'}Y) +$$
$$+\mathcal{P}^{\mathbb{C}}([\mathcal{P}^{\mathbb{C}'}X, \mathcal{P}^{\mathbb{C}}Y]^{\mathbb{C}}) + \mathcal{P}^{\mathbb{C}'}([\mathcal{P}^{\mathbb{C}}X, \mathcal{P}^{\mathbb{C}'}Y]^{\mathbb{C}}).$$

If (M, J, g) is a metallic Norden manifold such that $J^2 = pJ + qI$ with $p^2 + 4q < 0$, then $\bar{\nabla}^{\mathbb{C}}$ is explicitly given by:

$$\bar{\nabla}_X^{\mathbb{C}}Y = \tilde{\nabla}_{\mathcal{P}^{\mathbb{C}}X}^{\mathbb{C}}Y + \mathcal{P}^{\mathbb{C}}([\mathcal{P}^{\mathbb{C}'}X, \mathcal{P}^{\mathbb{C}}Y]^{\mathbb{C}}) + \mathcal{P}^{\mathbb{C}'}([\mathcal{P}^{\mathbb{C}}X, \mathcal{P}^{\mathbb{C}'}Y]^{\mathbb{C}}), \tag{14}$$

for any $X, Y \in C^\infty(T^{\mathbb{C}}M)$.

Moreover, $\bar{\nabla}^{\mathbb{C}}$ is a $J^{\mathbb{C}}$-connection, i.e. $\bar{\nabla}^{\mathbb{C}} J^{\mathbb{C}} = 0$, whose torsion is given by:

$$T^{\bar{\nabla}^{\mathbb{C}}}(X,Y) = \frac{1}{p^2+4q} N_{J^{\mathbb{C}}}(X,Y) + \mathcal{P}^{\mathbb{C}'}(T^{\nabla^{\mathbb{C}}}(\mathcal{P}^{\mathbb{C}'}X, \mathcal{P}^{\mathbb{C}'}Y)) -$$
$$-\mathcal{P}^{\mathbb{C}}(T^{\nabla^{\mathbb{C}}}(\mathcal{P}^{\mathbb{C}}X, \mathcal{P}^{\mathbb{C}}Y)),$$

for any $X, Y \in C^\infty(T^{\mathbb{C}}M)$.

3) *The complex Vidal connection* $\tilde{\tilde{\nabla}}^{\mathbb{C}}$ associated to the metallic Norden structure (J, g), obtained from (12) for

$$S(X,Y) := -\mathcal{P}^{\mathbb{C}}(\nabla_{\mathcal{P}^{\mathbb{C}}Y}\mathcal{P}^{\mathbb{C}'})X - \mathcal{P}^{\mathbb{C}'}(\nabla_{\mathcal{P}^{\mathbb{C}'}Y}\mathcal{P}^{\mathbb{C}})X,$$

therefore:

$$\tilde{\tilde{\nabla}}^{\mathbb{C}}_X Y = \tilde{\nabla}^{\mathbb{C}}_X Y - \mathcal{P}^{\mathbb{C}}(\nabla_{\mathcal{P}^{\mathbb{C}}Y}\mathcal{P}^{\mathbb{C}'})X - \mathcal{P}^{\mathbb{C}'}(\nabla_{\mathcal{P}^{\mathbb{C}'}Y}\mathcal{P}^{\mathbb{C}})X = \tag{15}$$
$$= \tilde{\nabla}^{\mathbb{C}}_X Y + \frac{1}{p^2+4q}[(\nabla_{J^{\mathbb{C}}Y}J^{\mathbb{C}})X + J^{\mathbb{C}}((\nabla_Y J^{\mathbb{C}})X) - p(\nabla_Y J^{\mathbb{C}})X],$$

for any $X, Y \in C^\infty(T^{\mathbb{C}}M)$, where $\nabla^{\mathbb{C}}$ is the Levi-Civita connection of $g^{\mathbb{C}}$.

Moreover, $\tilde{\tilde{\nabla}}^{\mathbb{C}}$ is a $J^{\mathbb{C}}$-connection, i.e. $\tilde{\tilde{\nabla}}^{\mathbb{C}} J^{\mathbb{C}} = 0$, whose torsion is given by:

$$T^{\tilde{\tilde{\nabla}}^{\mathbb{C}}}(X,Y) = \frac{1}{p^2+4q} N_{J^{\mathbb{C}}}(X,Y),$$

for any $X, Y \in C^\infty(T^{\mathbb{C}}M)$.

Moreover, we get:

$$(\tilde{\tilde{\nabla}}^{\mathbb{C}}_X g^{\mathbb{C}})(Y,Z) = -\frac{1}{p^2+4q}[g^{\mathbb{C}}((\nabla^{\mathbb{C}}_{J^{\mathbb{C}}Y}J^{\mathbb{C}})X - (\nabla^{\mathbb{C}}_Y J^{\mathbb{C}})J^{\mathbb{C}}X, Z) +$$
$$+ g^{\mathbb{C}}((\nabla_{J^{\mathbb{C}}Z}J^{\mathbb{C}})X - (\nabla^{\mathbb{C}}_Z J^{\mathbb{C}})J^{\mathbb{C}}X, Y)] =$$
$$= \frac{1}{p^2+4q}[g^{\mathbb{C}}(M_{J^{\mathbb{C}}}(Y,X), Z) + g^{\mathbb{C}}(M_{J^{\mathbb{C}}}(Z,X), Y) +$$
$$+ g^{\mathbb{C}}((\nabla^{\mathbb{C}}_{J^{\mathbb{C}}X}J^{\mathbb{C}})Y + (\nabla^{\mathbb{C}}_Y J^{\mathbb{C}})J^{\mathbb{C}}X, Z) + g^{\mathbb{C}}((\nabla^{\mathbb{C}}_{J^{\mathbb{C}}X}J^{\mathbb{C}})Z + (\nabla^{\mathbb{C}}_Z J)J^{\mathbb{C}}X, Y)],$$

for any $X, Y, Z \in C^\infty(T^{\mathbb{C}}M)$.

Since $\tilde{\nabla}^{\mathbb{C}} J^{\mathbb{C}} = \bar{\nabla}^{\mathbb{C}} J^{\mathbb{C}} = \tilde{\tilde{\nabla}}^{\mathbb{C}} J^{\mathbb{C}} = 0$, from Remark 6.9 we deduce:

Proposition 6.15. *The distributions $\mathcal{D}^{\mathbb{C}}$ and $\mathcal{D}^{\mathbb{C}'}$ are $\tilde{\nabla}^{\mathbb{C}}$-geodesically invariant, $\bar{\nabla}^{\mathbb{C}}$-geodesically invariant and $\tilde{\tilde{\nabla}}^{\mathbb{C}}$-geodesically invariant.*

From all the above considerations, we can state:

Theorem 6.16. *If (M, J, g) is a metallic Norden manifold such that $J^2 = pJ + qI$ with $p^2 + 4q < 0$, then the following assertions are equivalent:*
(i) the distributions $\mathcal{D}^{\mathbb{C}}$ and $\mathcal{D}^{\mathbb{C}'}$ are integrable;
(ii) (M, J_c) is a complex manifold;
(iii) the complex Vidal connection given by (15) is torsion-free.

Theorem 6.17. *If (M, J, g) is a metallic Norden manifold such that $J^2 = pJ + qI$ with $p^2 + 4q < 0$, then the following assertions are equivalent:*
(i) the distributions $\mathcal{D}^{\mathbb{C}}$ and $\mathcal{D}^{\mathbb{C}'}$ are geodesically invariant;
(ii) the complex Vidal connection given by (15) is metric with respect to $g^{\mathbb{C}}$.

6.2. The $\bar{\partial}$-operator of a metallic complex structure.

Definition 6.18. A metallic manifold (M, J) such that $J^2 = pJ + qI$ with $p^2 + 4q < 0$ and J integrable is called *metallic complex manifold*.

Let (M, J) be a metallic complex manifold and let $J_c = -\frac{1}{\sqrt{-p^2-4q}}(2J - pI)$ be the associated complex structure. Consider its dual map $J_c^* : T^*M \to T^*M$, defined by $(J_c^*\alpha)(X) := \alpha(J_cX)$, for any $\alpha \in C^\infty(T^*M)$ and for any $X \in C^\infty(TM)$.

We shall define the real differential operator d^c acting on forms:

$$d^c := J_c^* \circ d \circ J_c^*,$$

where d is the real differential operator.

If (M, J, g) is an integrable metallic Norden manifold, we can consider the real codifferential operator δ^c acting on forms:

$$\delta^c := \star \circ d^c \circ \star,$$

where $\star$ is the Hodge-star operator with respect to the metric g.

We obtain

$$d^c \circ d^c = 0, \ \ d \circ d^c + d^c \circ d = 0,$$
$$\delta^c \circ \delta^c = 0, \ \ \delta \circ \delta^c + \delta^c \circ \delta = 0,$$

where δ is the codifferential operator, and with respect to the scalar product $\langle \cdot, \cdot \rangle$ induced by g, the operators d^c and δ^c are adjoint, i.e.

$$\langle d^c \alpha, \beta \rangle = \langle \alpha, \delta^c \beta \rangle,$$

for any $\alpha, \beta \in C^\infty(T^*M)$.

Remark that $J^* \circ \star = \star \circ J^*$ (and $J_c^* \circ \star = \star \circ J_c^*$) implies $\delta^c = J_c^* \circ \delta \circ J_c^*$ and

$$d^c \circ J_c^* = -J_c^* \circ d, \ \ J_c^* \circ d^c = -d \circ J_c^*,$$
$$\delta^c \circ J_c^* = -J_c^* \circ \delta, \ \ J_c^* \circ \delta^c = -\delta \circ J_c^*.$$

From the above relations, we can state:

Proposition 6.19. *Let α be a real form on M.*
(i) If α is d^c-closed (resp. δ^c-coclosed), then $J_c^*\alpha$ is closed (resp. coclosed).
(ii) If α is closed (resp. coclosed), then $J_c^*\alpha$ is d^c-closed (resp. δ^c-coclosed).
(iii) If α is J_c^*-invariant, i.e. $J_c^*\alpha = \alpha$, then α is d^c-closed (resp. δ^c-coclosed) if and only if it is closed (resp. coclosed).

Therefore, the d^c-closed (resp. δ^c-coclosed) forms are the J_c^*-invariant closed (resp. coclosed) forms. Then

$$\ker(d^c) = \ker(d) \cap \{J_c^* - invariant\, forms\}, \ \ Im(d^c) = J_c^*(Im(d)),$$
$$\ker(\delta^c) = \ker(\delta) \cap \{J_c^* - invariant\, forms\}, \ \ Im(\delta^c) = J_c^*(Im(\delta)).$$

Then we can consider *the metallic cohomology groups*

$$H^r(M) := \ker(d_r^c)/Im(d_{r-1}^c),$$

where

$$d_r^c : C^\infty(\Lambda^r(M)) \to C^\infty(\Lambda^{r+1}(M))$$

and *the metallic homology groups*

$$H_r(M) := \ker(\delta_r^c)/Im(\delta_{r+1}^c),$$

where

$$\delta_r^c : C^\infty(\Lambda^r(M)) \to C^\infty(\Lambda^{r-1}(M)).$$

Now we can introduce *the metallic Hodge-Laplace operator*

$$\Delta^c : C^\infty(\Lambda^r(M)) \to C^\infty(\Lambda^r(M)),\ \ \Delta^c := d^c \circ \delta^c + \delta^c \circ d^c,$$

which is symmetric and self-adjoint w.r.t. $\langle \cdot, \cdot \rangle$. Remark that

$$\Delta^c = -J_c^* \circ \Delta \circ J_c^*,$$

where $\Delta = d \circ \delta + \delta \circ d$ is the Hodge-Laplace operator, and Δ^c satisfies

$$\Delta^c \circ J_c^* = J_c^* \circ \Delta,\ \ J_c^* \circ \Delta^c = \Delta \circ J_c^*.$$

Definition 6.20. A real form α is called *J-harmonic* if it belongs to the kernel of the metallic Hodge-Laplace operator, i.e. $\Delta^c \alpha = 0$.

From the above relations, we get:

Proposition 6.21. *Let α be a real form on M.*

(i) If α is J-harmonic, then $J_c^* \alpha$ is harmonic.

(ii) If α is harmonic, then $J_c^* \alpha$ is J-harmonic.

(iii) If α is J_c^*-invariant, i.e. $J_c^* \alpha = \alpha$, then α is J-harmonic if and only if it is harmonic.

(iv) α is J-harmonic if and only if it is d^c-closed and δ^c-coclosed.

Therefore, the J-harmonic forms are the J_c^*-invariant harmonic forms. Then

$$\ker(\Delta^c) = \ker(\Delta) \cap \{J_c^* - \mathit{invariant\ forms}\},\ \ Im(\Delta^c) = J_c^*(Im(\Delta)).$$

Let

$$T^{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M = \mathcal{D}^{\mathbb{C}'} \oplus \mathcal{D}^{\mathbb{C}}$$

be the decomposition of the complexified tangent bundle into $(1,0)$ and $(0,1)$ parts, with respect to the complex structure J_c or, equivalently, with respect to the distributions defined by J.

The $\bar{\partial}$-operator and $\bar{\bar{\partial}}$-operator acting on (r,s)-forms on M are defined as follows:

$$\bar{\partial} : C^\infty(\Lambda^{(r,s)}(M)) \to C^\infty(\Lambda^{(r,s+1)}(M)),\ \ \bar{\partial} := \frac{1}{2}(d - id^c),$$

$$\bar{\bar{\partial}} : C^\infty(\Lambda^{(r,s+1)}(M)) \to C^\infty(\Lambda^{(r,s)}(M)),\ \ \bar{\bar{\partial}} := \frac{1}{2}(\delta - i\delta^c).$$

Remark that the integrability of J (which is equivalent to the integrability of J_c) implies

$$\bar{\partial}\circ\bar{\partial} = 0, \quad \bar{\bar{\partial}}\circ\bar{\bar{\partial}} = 0,$$

therefore we can consider *the metallic complex cohomology groups*

$$H^{(r,s)}(M) := \ker(\bar{\partial}_{(r,s)})/Im(\bar{\partial}_{(r,s-1)}),$$

where

$$\bar{\partial}_{(r,s)} : C^{\infty}(\Lambda^{(r,s)}(M)) \to C^{\infty}(\Lambda^{(r,s+1)}(M))$$

and *the metallic complex homology groups*

$$H_{(r,s)}(M) := \ker(\bar{\bar{\partial}}_{(r,s)})/Im(\bar{\bar{\partial}}_{(r,s+1)}),$$

where

$$\bar{\bar{\partial}}_{(r,s)} : C^{\infty}(\Lambda^{(r,s)}(M)) \to C^{\infty}(\Lambda^{(r,s-1)}(M)).$$

Now, if

$$T^{*^{\mathbb{C}}}M = \mathcal{D}^{*^{\mathbb{C}}} \oplus \overline{\mathcal{D}^{*^{\mathbb{C}}}}$$

is the decomposition of the complexified cotangent bundle defined by J^*, then we get the following:

Proposition 6.22. *Let (M, J) be a metallic complex manifold such that $J^2 = pJ + qI$ with $p^2 + 4q < 0$. Then the $\bar{\partial}$-operator:*

$$\bar{\partial} = \frac{1}{2(p^2+4q)}[(p^2+4q)d + i(4J^*\circ d\circ J^* - 2pd\circ J^* - 2pJ^*\circ d + p^2 d)]$$

is acting on $C^{\infty}(\Lambda^r(\mathcal{D}^*)) \otimes C^{\infty}(\Lambda^s(\overline{\mathcal{D}^{*^{\mathbb{C}}}}))$.

Proof. We have:

$$d^c = \left[-\frac{1}{\sqrt{-p^2-4q}}(2J^* - pI)\right]\circ d\circ\left[-\frac{1}{\sqrt{-p^2-4q}}(2J^* - pI)\right] =$$

$$= -\frac{1}{p^2+4q}(4J^*\circ d\circ J^* - 2pd\circ J^* - 2pJ^*\circ d + p^2 d).$$

Then the statement follows. □

Similarly, we prove that:

Proposition 6.23. *Let (M, J, g) be a metallic Norden manifold such that $J^2 = pJ + qI$ with $p^2 + 4q < 0$. Then the $\bar{\bar{\partial}}$-operator:*

$$\bar{\bar{\partial}} = \frac{1}{2(p^2+4q)}[(p^2+4q)\delta + i(4J^*\circ\delta\circ J^* - 2p\delta\circ J^* - 2pJ^*\circ\delta + p^2\delta)]$$

is acting on $C^{\infty}(\Lambda^r(\mathcal{D}^*)) \otimes C^{\infty}(\Lambda^s(\overline{\mathcal{D}^{*^{\mathbb{C}}}}))$.

Remark 6.24. The operators d^c and $\bar{\partial}$ can be defined on metallic complex manifolds and δ^c, Δ^c and $\bar{\bar{\partial}}$ only on metallic Norden manifolds.

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