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Partial-isometric crossed products of dynamical systems by left LCM semigroups

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Dedicated to Sriwulan Adji

ABSTRACT. Let P be a left LCM semigroup, and α an action of P by endomorphisms of a C^*-algebra A. We study a semigroup crossed product C^*-algebra in which the action α is implemented by partial isometries. This crossed product gives a model for the Nica-Teoplitz algebras of product systems of Hilbert bimodules (associated with semigroup dynamical systems) studied first by Fowler, for which, we provide a structure theorem as it behaves well under short exact sequences and tensor products.

CONTENTS

1. Introduction

Let P be a unital semigroup whose unit element is denoted by e. Suppose that (A, P, α) is a dynamical system consisting of a C^*-algebra A, and an action $\alpha : P \to \mathrm{End}(A)$ of P by endomorphisms of A such that $\alpha_e = \mathrm{id}_A$. Note that, since the C^*-algebra A is not necessarily unital, we need to assume that each endomorphism α_x is extendible, which means that it extends to a strictly continuous endomorphism $\overline{\alpha}_x$ of the multiplier algebra $\mathcal{M}(A)$. Recall that an endomorphism α of A is extendible if and only if there exists an approximate identity $\{a_\lambda\}$ in A and a projection $p \in \mathcal{M}(A)$ such that $\alpha(a_\lambda)$ converges strictly to p in $\mathcal{M}(A)$. However, the extendibility of α does not necessarily imply $\overline{\alpha}(1_{\mathcal{M}(A)}) = 1_{\mathcal{M}(A)}$.

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There have been huge efforts on the study of C^*-algebras associated with semigroups and semigroup dynamical systems. In the line of those efforts in this regard, Fowler in [12], for the dynamical system (A, P, α), where P is the positive cone of a group G such that (G, P) is quasi-lattice ordered in the sense of Nica [24], defined a covariant representation called the *Nica-Toeplitz covariant representation* of the system, such that the endomorphisms α_x are implemented by partial isometries. He then showed that there exists a universal C^*-algebra $\mathcal{T}_{\text{cov}}(X)$ associated with the system generated by a universal Nica-Toeplitz covariant representation of the system such that there is a bijection between the Nica-Toeplitz covariant representations of the system and the nondegenerate representations of $\mathcal{T}_{\text{cov}}(X)$. To be more precise, X is actually the product system of Hilbert bimodules associated with the system (A, P, α) introduced by him, and the algebra $\mathcal{T}_{\text{cov}}(X)$ is universal for Toeplitz representations of X satisfying a covariance condition called *Nica covariance*. He called this universal algebra the *Nica-Toeplitz crossed product* (or *Nica-Toeplitz algebra*) of the system (A, P, α) and denoted it by $\mathcal{T}_{\text{cov}}(A \times_\alpha P)$. When the group G is totally ordered and abelian (with the positive cone $G^+ = P$), the Nica covariance condition holds automatically, and the Toeplitz algebra $\mathcal{T}(X)$ is the *partial-isometric crossed product* $A \times_\alpha^{\text{piso}} P$ of the system (A, P, α) introduced and studied by the authors of [23]. In other word, the semigroup crossed product $A \times_\alpha^{\text{piso}} P$ actually gives a model for the Teoplitz algebras $\mathcal{T}(X)$ of product systems X of Hilbert bimodules associated with the systems (A, P, α), where P is the positive cone of a totally ordered abelian group G. Further studies on the structure of the crossed product $A \times_\alpha^{\text{piso}} P$ have been done progressively in [3], [4], [5], [20], and [29] since then.

In the very recent years, mathematicians in [7, 15, 16], following the idea of Fowler, have extended and studied the notion of the Nica-Toeplitz algebra of a product system X over more general semigroups P, namely, right LCM semigroups (see also [11]). These are the semigroups that appear as a natural generalization of the well-known notion of quasi-latticed ordered groups introduced first by Nica in [24]. Recall that the notation $\mathcal{N}\mathcal{T}(X)$ is used for the Nica-Toeplitz algebra of X in [7, 15, 16], which are the works that brought this question to our attention that whether we could define a partial-isometric crossed product corresponding to the system (A, P, α), where the semigroup P goes beyond the positive cones of totally ordered abelian groups. Although, based on the work of Fowler in [12] (see also the effort in [2] in this direction), we were already aware that the answer to this question must be "yes" for the positive cones P of quasi-latticed ordered groups (G, P), [7, 15, 16] made us very enthusiastic to seek even more than that. Hence, the initial investigations in the present work indicated that the semigroup P must be left LCM (see §2). More precisely, in the dynamical system (A, P, α), we considered the semigroup P to be left LCM (so, the opposite semigroup P^{o} becomes right LCM). Then, following [12], we defined a covariant representation of the system satisfying a covariance condition called the *(right) Nica covariance*, in which the endomorphisms α_x are implemented by partial isometries. We called this representation the *covariant partial-isometric representation* of the system. More importantly, we showed that every system (A,P,α) admits a nontrivial covariant partial-isometric representation. Next, we proved that the Nica-Toeplitz algebra $\mathcal{NT}(X)$ of the product system X associated with the dynamical system (A,P,α) is generated by a covariant partial-isometric representation of the system which is universal for covariant partial-isometric representations of the system. We called this universal algebra the *partial-isometric crossed product* of the system (A,P,α) and denoted it by $A\times_{\alpha}^{\mathrm{piso}} P$ (following [23]), which is unique up to isomorphism. We then studied the behavior of crossed product $A\times_{\alpha}^{\mathrm{piso}} P$ under short exact sequences and tensor products, from which, a structure theorem followed. In addition, as an example, when P and P^{o} are both left LCM semigroups we studied the distinguished system (B_P,P,τ), where B_P is the C^*-subalgebra of $\ell^\infty(P)$ generated by the characteristic functions $\{1_y : y\in P\}$, and the action τ on B_P is induced by the shift on $\ell^\infty(P)$. It was shown that the algebra $B_P\times_{\tau}^{\mathrm{piso}} P$ is universal for *bicovariant partial-isometric representations* of P, which are the partial-isometric representations of P satisfying both right and left Nica covariance conditions.

Here, prior to talking about the organization of the present work, we would like to mention that, by [28], if P is the positive cone of an abelian lattice-ordered group G, then the Nica-Toeplitz algebra $\mathcal{T}_{\mathrm{cov}}(A\times_\alpha P)$ of the system (A,P,α) is a full corner in a classical crossed product by the group G. Thus, by the present work, since $A\times_{\alpha}^{\mathrm{piso}} P\simeq \mathcal{T}_{\mathrm{cov}}(A\times_\alpha P)$, the same corner realization holds for the partial-isometric crossed products of the systems (A,P,α) consisting of the positive cones P of abelian lattice-ordered groups (see also [29]).

Now, the present work as an extension of the idea in [23] follows the framework of [18] for partial-isometric crossed products. We begin with a preliminary section containing a summary on LCM semigroups and discrete product systems of Hilbert bimodules. In section 3 and 4, for the system (A,P,α) with a left LCM semigroup P, a covariant representation of the system is defined which satisfies a covariance condition called the *(right) Nica covariance*, where the endomorphisms α_x are implemented by partial isometries. This representation is called the *covariant partial-isometric representation* of the system. We also provide an example which shows that every system admits a nontrivial covariant partial-isometric representation. Then, we show that there is a C^*-algebra B associated with the system generated by a covariant partial-isometric representation of the system which is universal for covariant partial-isometric representations of the system, in the sense that there is a bijection between the covariant partial-isometric representations of the system and the nondegenerate representations of the C^*-algebra B. This universal algebra B is called the *partial-isometric crossed product* of the system (A,P,α) and denoted by $A\times_{\alpha}^{\mathrm{piso}} P$, which is unique up to isomorphism. We also show that this crossed product behaves well under short exact sequences. In section 5, we show that under some certain conditions the crossed product $(A \otimes_{\max} B) \times^{\text{piso}} (P \times S)$ can be decomposed as the maximal tensor product of the crossed products $A \times^{\text{piso}} P$ and $B \times^{\text{piso}} S$. Also, when P and the opposite semigroup P^{o} are both left LCM we consider the distinguished system (B_P, P, τ), where B_P is the C^*-subalgebra of $\ell^\infty(P)$ generated by the characteristic functions $\{1_y : y \in P\}$, and the action τ on B_P is induced by the shift on $\ell^\infty(P)$. Note that each 1_y is actually the characteristic function of the right ideal $yP = \{yx : x \in P\}$ in P. We then show that the crossed prodcut $B_P \times_\tau^{\text{piso}} P$ is universal for *bicovariant partial-isometric representations* of P, which are the partial-isometric representations of P satisfying both right and left Nica covariance conditions. In section 6, for the crossed product $(A \otimes_{\max} B) \times^{\text{piso}} P$ a composition series

$$0 \leq \mathcal{I}_1 \leq \mathcal{I}_2 \leq (A \otimes_{\max} B) \times^{\text{piso}} P$$

of ideals is obtained, for which we identify the subquotients

$$\mathcal{I}_1, \quad \mathcal{I}_2 / \mathcal{I}_1, \quad \text{and} \quad ((A \otimes_{\max} B) \times^{\text{piso}} P)/ \mathcal{I}_2$$

with familiar terms. Finally in section 7, as an application, we study the partial-isometric crossed product of the dynamical system considered in [19].

2. Preliminaries

2.1. LCM semigroups. Let P be a discrete semigroup. We assume that P is unital, which means that there is an element $e \in P$ such that $xe = ex = x$ for all $x \in P$. Recall that P is called *right cancellative* if $xz = yz$, then $x = y$ for every $x, y, z \in P$.

Definition 2.1. A unital semigroup P is called left LCM (least common multiple) if it is right cancellative and for every $x, y \in P$, we have either $Px \cap Py = \emptyset$ or $Px \cap Py = Pz$ for some $z \in P$.

Let P^* denote the set of all invertible elements of P, which is obviously not empty as $e \in P^*$. In fact, P^* is a group with the action inherited from P. Now, if $Px \cap Py = Pz$, since $z = ez \in Pz$, we have $sx = z = ty$ for some $s, t \in P$. So, z can be viewed as a least common left multiple of x, y. However, such a least common left multiple may not be unique. Actually one can see that if z and $\tilde{z}$ are both least common left multiples of x, y, then there is an invertible element u of P ($u \in P^*$) such that $\tilde{z} = uz$. Note that right LCM semigroups are defined similarly. A unital semigroup P is called *right LCM* if it is left cancellative and for every $x, y \in P$, we have either $xP \cap yP = \emptyset$ or $xP \cap yP = zP$ for some $z \in P$. Let P^{o} denote the opposite semigroup endowed with the action $\bullet$ such that $x \bullet y = yx$ for all $x, y \in P^{\text{o}}$. Clearly, P is a left LCM semigroup if and only if P^{o} is a right LCM semigroup.

LCM semigroups actually appear as a natural generalization of the well-known notion of quasi-latticed ordered groups introduced first by Nica in [24]. Let G be a group and P a unital subsemigroup such that $P \cap P^{-1} = \{e\}$. There is a partial order on G defined by P such that

$$x \leq_{\rm rt} y \Longleftrightarrow yx^{-1} \in P \Longleftrightarrow y \in Px \Longleftrightarrow Py \subseteq Px$$

for all $x, y \in G$. Moreover, we have

$$x \leq_{\rm rt} y \Longleftrightarrow xz \leq_{\rm rt} yz$$

for all $x, y, z \in G$, which means that this partial order is right-invariant. The partial order $\leq_{\rm rt}$ on G is said to be a *right quasi-lattice order* if every finite subset of G which has an upper bound in P has a least upper bound in P. In this case, the pair (G, P) is called a *right quasi-lattice ordered group*. Note that a left-invariant partial order on G is also defined by P such that $x \leq_{\rm lt} y$ iff $x^{-1}y \in P$. So, a *left quasi-lattice ordered group* (G, P) can be defined similarly. Now, it is not difficult to see that if (G, P) is a right quasi-lattice ordered group, then the semigroup P is a left LCM semigroup. Similarly, if (G, P) is a left quasi-lattice ordered group, then P is a right LCM semigroup. Note that, however, a LCM semigroup P is not necessarily embedded in a group G such that (G, P) is a quasi-lattice ordered group (see more in [21]).

Example 2.2. For any positive integer k, the positive cone $\mathbb{N}^k$ of the abelian lattice-ordered group $\mathbb{Z}^k$ is a left LCM semigroup, such that

$$\mathbb{N}^k x \cap \mathbb{N}^k y = \mathbb{N}^k (x \vee y)$$

for all $x, y \in \mathbb{N}^k$, where $x \vee y$ denotes the supremum of x and y given by $(x \vee y)_i = \max\{x_i, y_i\}$ for $1 \leq i \leq k$.

Example 2.3. Let $\mathbb{N}^\times$ denote the set of positive integers, which is a unital (abelian) semigroup with the usual multiplication. It is indeed a left LCM semigroup such that

$$\mathbb{N}^\times r \cap \mathbb{N}^\times s = \mathbb{N}^\times (r \vee s)$$

for all $r, s \in \mathbb{N}^\times$, where

$$r \vee s = \text{the least common multiple of numbers } r \text{ and } s.$$

In fact, if $\mathbb{Q}_+^\times$ denotes the abelian multiplicative group of positive rationals, then it is a lattice-ordered group with

$$r \leq s \Longleftrightarrow (\frac{s}{r}) \in \mathbb{N}^\times \Longleftrightarrow sr^{-1} \in \mathbb{N}^\times$$

for all $r, s \in \mathbb{Q}_+^\times$. So, $\mathbb{N}^\times$ is actually the positive cone of $\mathbb{Q}_+^\times$, namely,

$$\mathbb{N}^\times = \{r \in \mathbb{Q}_+^\times : r \geq 1\}.$$

Example 2.4. Assume that n is an integer such that $n \geq 2$. Let $\mathbb{F}_n$ be the free group on n generators $\{a_1, a_2, ..., a_n\}$, and $\mathbb{F}_n^+$ the unital subsemigroup of $\mathbb{F}_n$ generated by the nonnegative powers of a_i's. Then, for the right-invariant partial order $\leq_{\rm rt}$ on $\mathbb{F}_n$ defined by $\mathbb{F}_n^+$, we have $x \leq_{\rm rt} y$ if and only if x is a final string on the right of y, whit which, $(\mathbb{F}_n, \mathbb{F}_n^+)$ is a right quasi-lattice ordered group. Thus, $\mathbb{F}_n^+$ is a left LCM semigroup. The left-invariant partial order $\leq_{\rm lt}$ on $\mathbb{F}_n$ is given such that $x \leq_{\text{lt}} y$ if and only if x is an initial string on the left of y, whit which, $(\mathbb{F}_n, \mathbb{F}_n^+)$ is a left quasi-lattice ordered group. So, in this case, $\mathbb{F}_n^+$ is a right LCM semigroup.

2.2. Discrete product systems of Hilbert bimodules. A *Hilbert bimodule* over a C^*-algebra A is a right Hilbert A-module X together with a homomorphism $\phi : A \to \mathcal{L}(X)$ which defines a left action of A on X by $a \cdot x = \phi(a)x$ for all $a \in A$ and $x \in X$. A *Toeplitz representation* of X in a C^*-algebra B is a pair (ψ, π) consisting of a linear map $\psi : X \to B$ and a homomorphism $\pi : A \to B$ such that

$$\psi(x \cdot a) = \psi(x)\pi(a), \;\; \psi(x)^*\psi(y) = \pi(\langle x, y\rangle_A), \;\; \text{and} \;\; \psi(a \cdot x) = \pi(a)\psi(x)$$

for all $a \in A$ and $x, y \in X$. Then, there is a homomorphism (Pimsner homomorphism) $\rho : \mathcal{K}(X) \to B$ such that

$$\rho(\Theta_{x,y}) = \psi(x)\psi(y)^* \quad \text{for all } x, y \in X. \tag{2.1}$$

The *Toeplitz algebra* of X is the C^*-algebra $\mathcal{T}(X)$ which is universal for Toeplitz representations of X (see [26, 13]).

Recall that every right Hilbert A-module X is essential, which means that we have

$$X = \overline{\text{span}}\{x \cdot a : x \in X, a \in A\}.$$

Moreover, a Hilbert bimodule X over a C^*-algebra A is called *essential* if

$$X = \overline{\text{span}}\{a \cdot x : a \in A, x \in X\} = \overline{\text{span}}\{\phi(a)x : a \in A, x \in X\},$$

which means that X is also essential as a left A-module.

Now, let A be a C^*-algebra and S a unital (countable) discrete semigroup. We recall from [12] that the disjoint union $X = \bigsqcup_{s \in S} X_s$ of Hilbert bimodules X_s over A is called a *discrete product system* over S if there is a multiplication

$$(x, y) \in X_s \times X_t \mapsto xy \in X_{st} \tag{2.2}$$

on X, with which, X is a semigroup, and the map (2.2) extends to an isomorphism of the Hilbert bimodules $X_s \otimes_A X_t$ and X_{xt} for all $s, t \in S$ with $s, t \neq e$. The bimodule X_e is ${}_A A_A$, and the multiplications $X_e \times X_s \mapsto X_s$ and $X_s \times X_e \mapsto X_s$ are just given by the module actions of A on X_s. Note that also we write $\phi_s : A \to \mathcal{L}(X_s)$ for the homomorphism which defines the left action of A on X_s.

Note that, for every $s, t \in S$ with $s \neq e$, there is a homomorphism $\iota_s^{st} : \mathcal{L}(X_s) \to \mathcal{L}(X_{st})$ characterized by

$$\iota_s^{st}(T)(xy) = (Tx)y$$

for all $x \in X_s$, $y \in X_t$ and $T \in \mathcal{L}(X_s)$. In fact, $\iota_s^{st}(T) = T \otimes \text{id}_{X_t}$.

A *Toeplitz representation* of the product system X in a C^*-algebra B is a map $\psi : X \to B$ such that

(1) $\psi_s(x)\psi_t(y) = \psi_{st}(xy)$ for all $s, t \in S$, $x \in X_s$, and $y \in X_t$; and
(2) the pair (ψ_s, ψ_e) is a Toeplitz representation of X_s in B for all $s \in S$,

where ψ_s denotes the restriction of ψ to X_s. For every $s \in S$, let $\psi^{(s)} : \mathcal{K}(X_s) \to B$ be the Pimsner homomorphism corresponding to the pair (ψ_s, ψ_e) defined by

$$\psi^{(s)}(\Theta_{x,y}) = \psi_s(x)\psi_s(y)^*$$

for all $x, y \in X_s$ (see (2.1)).

By [12, Proposition 2.8], for every product system X over S, there is a C^*-algebra $\mathcal{T}(X)$, called the *Toeplitz algebra* of X, which is generated by a universal Toeplitz representation $i_X : X \to \mathcal{T}(X)$ of X. The pair $(\mathcal{T}(X), i_X)$ is unique up to isomorphism, and i_X is isometric.

Next, we recall that for any quasi-lattice ordered group (G, S), the notions of compactly aligned product system over S and Nica covariant Toeplitz representation of it were introduced first by Fowler in [12]. Then, authors in [7] extended these notions to product systems over right LCM semigroups. Suppose that S is a unital right LCM semigroup. A product system X over S of Hilbert bimodules is called *compactly aligned* if for all $r, t \in S$ such that $rS \cap tS = sS$ for some $s \in S$ we have

$$\iota_r^s(R)\iota_t^s(T) \in \mathcal{K}(X_s)$$

for all $R \in \mathcal{K}(X_r)$ and $T \in \mathcal{K}(X_t)$. Let X be a compactly aligned product system over a right LCM semigroup S, and $\psi : X \to B$ a Toeplitz representation of X in a C^*-algebra B. Then, ψ is called *Nica covariant* if

$$\psi^{(r)}(R)\psi^{(t)}(T) = \begin{cases} \psi^{(s)}\big(\iota_r^s(R)\iota_t^s(T)\big) & \text{if } rS \cap tS = sS, \\ 0 & \text{if } rS \cap tS = \emptyset \end{cases} \tag{2.3}$$

for all $r, t \in S$, $R \in \mathcal{K}(X_r)$ and $T \in \mathcal{K}(X_t)$.

For a compactly aligned product system X over a right LCM semigroup S, the *Nica-Toeplitz algebra* $\mathcal{NT}(X)$ is the C^*-algebra generated by a Nica covariant Toeplitz representation $i_X : X \to \mathcal{NT}(X)$ which is universal for Nica covariant Toeplitz representations of X, which means that, for every Nica covariant Toeplitz representation of ψ of X, there is a representation ψ_* of $\mathcal{NT}(X)$ such that $\psi_* \circ i_X = \psi$ (see [12, 7]).

3. Nica partial-isometric representations

Let P be a left LCM semigroup. A *partial-isometric representation* of P on a Hilbert space H is a map $V : P \to B(H)$ such that each $V_x := V(x)$ is a partial isometry, and the map V is a unital semigroup homomorphism of P into the multiplicative semigroup $B(H)$. Moreover, if the representation V satisfies the equation

$$V_x^*V_xV_y^*V_y = \begin{cases} V_z^*V_z & \text{if } Px \cap Py = Pz, \\ 0 & \text{if } Px \cap Py = \emptyset, \end{cases} \tag{3.1}$$

then it is called a *Nica partial-isometric representation* of P on H. The equation (3.1) is called the *Nica covariance condition*. Of course, since the least common left multiple z may not be unique, we must check that whether the Nica covariance condition is well-defined. So, assume that $Pz = Px \cap Py = P\tilde{z}$. If follows that

$$V_x^*V_xV_y^*V_y = V_z^*V_z$$

and

$$V_x^*V_xV_y^*V_y = V_{\tilde{z}}^*V_{\tilde{z}}.$$

Now, since $\tilde{z} = uz$ for some invertible element u of P, we have

$$V_{\tilde{z}}^*V_{\tilde{z}} = V_{uz}^*V_{uz} = (V_uV_z)^*V_uV_z = V_z^*V_u^*V_uV_z.$$

But it is not difficult to see that V_u is actually a unitary, and therefore,

$$V_{\tilde{z}}^*V_{\tilde{z}} = V_z^*V_z.$$

This implies that the equation (3.1) is indeed well-defined.

The following example shows that given a left LCM semigroup P, a Nica partial-isometric representation of P exists.

Example 3.1. Suppose that P is a left LCM semigroup and H a Hilbert space. Define a map $S : P \to B(\ell^2(P) \otimes H)$ by

$$(S_yf)(x) = \begin{cases} f(r) & \text{if } x = ry \text{ for some } r \in P, \\ 0 & \text{otherwise.} \end{cases}$$

for every $f \in \ell^2(P) \otimes H$. Note that $x = ry$ for some $r \in P$ is equivalent to saying that $x \in Py$. Moreover, if $sy = x = ry$ for some $r, s \in P$, then $s = r$ by the right cancellativity of P, and hence $f(r) = f(s)$. This implies that each S_y is well-defined. One can see that each S_y is a linear operator. We claim that each S_y is actually an isometry, and in particular, $S_e = 1$. We have

$$\|S_yf\|^2 = \sum_{x\in P} \|(S_yf)(x)\|^2 = \sum_{r\in P} \|(S_yf)(ry)\|^2 = \sum_{r\in P} \|f(r)\|^2 = \|f\|^2,$$

which implies that each S_y is an isometry. In particular,

$$(S_ef)(x) = (S_ef)(xe) = f(x),$$

which shows that $S_e = 1$. In addition, a simple calculation shows that

$$S_xS_y = S_{yx} = S_{x\bullet y} \text{ for all } x, y \in P. \tag{3.2}$$

Next, we want to show that the adjoint of each S_y is given by

$$(W_yf)(x) = f(xy)$$

for all $f \in \ell^2(P) \otimes H$. For every $f, g \in \ell^2(P) \otimes H$, we have

$$\begin{aligned} \langle S_yf|g\rangle &= \sum_{x\in P} \langle (S_yf)(x)|g(x)\rangle \\ &= \sum_{r\in P} \langle (S_yf)(ry)|g(ry)\rangle \\ &= \sum_{r\in P} \langle f(r)|g(ry)\rangle = \sum_{r\in P} \langle f(r)|(W_yg)(r)\rangle = \langle f|W_yg\rangle. \end{aligned}$$

So, $S_y^* = W_y$ for every $y \in P$. Also, for every $x, y \in P$, by applying (3.2), we get

$$W_x W_y = S_x^* S_y^* = [S_y S_x]^* = S_{xy}^* = W_{xy}.$$

Therefore, since each W_x is obviously a partial-isometry, it follows that the map $W : P \to B(\ell^2(P) \otimes H)$ defined by $(W_y f)(x) = f(xy)$ is a partial-isometric representation of P on $\ell^2(P) \otimes H$. We claim that the representation W satisfies the Nica covariance condition (3.1). Firstly,

$$W_x^* W_x W_y^* W_y = S_x S_x^* S_y S_y^*. \tag{3.3}$$

Then, for every $f \in \ell^2(P) \otimes H$, we have

$$(S_x^* S_y f)(r) = (S_x^*(S_y f))(r) = (S_y f)(rx) \text{ for all } r \in P. \tag{3.4}$$

Now, if $Px \cap Py = \emptyset$, since $rx \in Px$, it follows that $rx \notin Py$, and therefore,

$$(S_y f)(rx) = 0.$$

Thus, $(S_x^* S_y f)(r) = 0$, which implies that the equation (3.3) must be equal to zero when $Px \cap Py = \emptyset$. Suppose the otherwise, namely, $Px \cap Py = Pz$. Note that first, if $\{\varepsilon_s : s \in P\}$ is the usual orthonormal basis of $\ell^2(P)$, then each $S_y S_y^*$ is a projection onto the closed subspace $\ell^2(Py) \otimes H$ of $\ell^2(P) \otimes H$ spanned by the elements

$$\{\varepsilon_{sy} \otimes h : s \in P, h \in H\},$$

which is indeed equal to the $\ker(1 - S_y S_y^*)$. So, for every $f \in \ell^2(P) \otimes H$,

$$\big(S_x S_x^*(S_y S_y^* f)\big)(r) = \begin{cases} (S_y S_y^* f)(r) & \text{if } r \in Px, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for $(S_y S_y^* f)(r)$, where $r \in Px$, we have

$$(S_y S_y^* f)(r) = \begin{cases} f(r) & \text{if } r \in Py, \\ 0 & \text{otherwise.} \end{cases}$$

It thus follows that

$$\big(S_x S_x^*(S_y S_y^* f)\big)(r) = \begin{cases} f(r) & \text{if } r \in (Px \cap Py) = Pz, \\ 0 & \text{otherwise,} \end{cases}$$

which equals $(S_z S_z^* f)(r)$. Therefore, we have

$$S_x S_x^* S_y S_y^* = S_z S_z^*,$$

from which, for the equation (3.3), we get

$$W_x^* W_x W_y^* W_y = S_z S_z^* = W_z^* W_z.$$

Consequently, W is indeed a Nica partial-isometric representation.

Remark 3.2. If P is the positive cone of a totally ordered group G, then every partial-isometric representation V of P automatically satisfies the Nica covariance condition (3.1), such that

$$V_x^* V_x V_y^* V_y = V_{\max\{x,y\}}^* V_{\max\{x,y\}} \text{ for all } x, y \in P.$$

Lemma 3.3. *Consider the right quasi-lattice ordered group* $(\mathbb{F}_n, \mathbb{F}_n^+)$ *(see Example 2.4). A partial-isometric representation* V *of* $\mathbb{F}_n^+$ *satisfies the Nica covariance condition (3.1) if and only if the initial projections* $V_{a_i}^* V_{a_i}$ *and* $V_{a_j}^* V_{a_j}$ *have orthogonal ranges, where* $1 \leq i, j \leq n$ *such that* $i \neq j$.

Proof. Suppose that V is a partial-isometric representation of $\mathbb{F}_n^+$ on a Hilbert space H. If it satisfies the Nica covariance condition (3.1), then one can see that, for every $1 \leq i, j \leq n$ with $i \neq j$, we have

$$V_{a_i}^* V_{a_i} V_{a_j}^* V_{a_j} = 0$$

as $\mathbb{F}_n^+ a_i \cap \mathbb{F}_n^+ a_j = \emptyset$. So, it follows that the initial projections $V_{a_i}^* V_{a_i}$ and $V_{a_j}^* V_{a_j}$ have orthogonal ranges for every $1 \leq i, j \leq n$ with $i \neq j$.

Conversely, suppose that for every $1 \leq i, j \leq n$ with $i \neq j$, the initial projections $V_{a_i}^* V_{a_i}$ and $V_{a_j}^* V_{a_j}$ have orthogonal ranges. Therefore, we have

$$V_{a_i}^* V_{a_i} V_{a_j}^* V_{a_j} = 0 \tag{3.5}$$

for every $1 \leq i, j \leq n$ with $i \neq j$. Now, for every $x, y \in \mathbb{F}_n^+$, if $\mathbb{F}_n^+ x \cap \mathbb{F}_n^+ y \neq \emptyset$, then x is a final string on the right of y or y is a final string on the right of x. Suppose that x is a final string on the right of y, from which, it follows that $\mathbb{F}_n^+ x \cap \mathbb{F}_n^+ y = \mathbb{F}_n^+ y$, and $y = (yx^{-1})x$, where $yx^{-1} \in \mathbb{F}_n^+$. Therefore, we have

$$\begin{aligned} V_x^* V_x V_y^* V_y &= V_x^* V_x (V_{yx^{-1}x})^* V_y \\ &= V_x^* V_x (V_{yx^{-1}} V_x)^* V_y \\ &= V_x^* V_x V_x^* V_{yx^{-1}}^* V_y \\ &= V_x^* V_{yx^{-1}}^* V_y \\ &= (V_{yx^{-1}} V_x)^* V_y = (V_{yx^{-1}x})^* V_y = V_y^* V_y. \end{aligned}$$

If y is the final string on the right of x, a similar computation shows that $V_x^* V_x V_y^* V_y = V_x^* V_x$ as $\mathbb{F}_n^+ x \cap \mathbb{F}_n^+ y = \mathbb{F}_n^+ x$. If $\mathbb{F}_n^+ x \cap \mathbb{F}_n^+ y = \emptyset$, then $x \neq y$. Note that we can consider x and y as two strings of letters a_i's with equal lengths by adding, for example, a finite number of $(a_1)^0$ to the left of the shorter string. Therefore, since $x \neq y$, we can write

$$x = s a_i z \quad \text{and} \quad y = t a_j z,$$

where $s, t, z \in \mathbb{F}_n^+$, and $i \neq j$. It follows that

$$\begin{aligned} V_x^* V_x V_y^* V_y &= V_x^* V_{s a_i z} (V_{t a_j z})^* V_y \\ &= V_x^* V_s V_{a_i} V_z (V_t V_{a_j} V_z)^* V_y \\ &= V_x^* V_s (V_{a_i}) V_z V_z^* (V_{a_j}^*) V_t^* V_y \\ &= V_x^* V_s (V_{a_i} V_{a_i}^* V_{a_i}) V_z V_z^* (V_{a_j}^* V_{a_j} V_{a_j}^*) V_t^* V_y \\ &= V_x^* V_s V_{a_i} (V_{a_i}^* V_{a_i}) (V_z V_z^*) (V_{a_j}^* V_{a_j}) V_{a_j}^* V_t^* V_y. \end{aligned}$$

Now, in the bottom line, since the product $V_{a_i} V_z$ of the partial-isometries V_{a_i} and V_z is a partial-isometry, namely, $V_{a_i z}$, by [14, Lemma 2], $V_{a_i}^* V_{a_i}$ commutes with $V_zV_z^*$. So, we get

$$\begin{aligned} V_x^*V_xV_y^*V_y &= V_x^*V_sV_{a_i}(V_zV_z^*)(V_{a_i}^*V_{a_i}V_{a_j}^*V_{a_j})V_{a_j}^*V_t^*V_y \\ &= V_x^*V_sV_{a_i}(V_zV_z^*)(0)V_{a_j}^*V_t^*V_y = 0 \quad \text{(by (3.5))}. \end{aligned}$$

Thus, the representation V satisfies the Nica covariance condition (3.1). □

4. Partial-isometric crossed products

4.1. Covariant partial-isometric representations. Let P be a left LCM semigroup, and (A, P, α) a dynamical system consisting of a C^*-algebra A, and an action $\alpha : P \to \mathrm{End}(A)$ of P by extendible endomorphisms of A such that $\alpha_e = \mathrm{id}_A$.

Definition 4.1. A *covariant partial-isometric representation* of (A, P, α) on a Hilbert space H is a pair (π, V) consisting of a nondegenerate representation $\pi : A \to B(H)$ and a Nica partial-isometric representation $V : P \to B(H)$ of P such that

$$\pi(\alpha_x(a)) = V_x\pi(a)V_x^* \text{ and } V_x^*V_x\pi(a) = \pi(a)V_x^*V_x \tag{4.1}$$

for all $a \in A$ and $x \in P$.

Lemma 4.2. *Every covariant partial-isometric pair (π, V) extends to a covariant partial-isometric representation $(\overline{\pi}, V)$ of the system $(M(A), P, \overline{\alpha})$, and (4.1) is equivalent to*

$$\pi(\alpha_x(a))V_x = V_x\pi(a) \text{ and } V_xV_x^* = \overline{\pi}(\overline{\alpha}_x(1)) \tag{4.2}$$

for all $a \in A$ and $x \in P$.

Proof. We skip the proof as it follows by similar discussions to the first part of [23, §4]. □

The following example shows that every dynamical system (A, P, α) admits a nontrivial (nonzero) covariant partial-isometric representation.

Example 4.3. Suppose that (A, P, α) is a dynamical system, and $\pi_0 : A \to B(H)$ a nondegenerate representation of A on a Hilbert space H. Define a map $\pi : A \to B(\ell^2(P) \otimes H)$ by

$$(\pi(a)f)(x) = \pi_0(\alpha_x(a))f(x)$$

for all $a \in A$ and $f \in \ell^2(P) \otimes H \simeq \ell^2(P, H)$. One can see that π is a representation of A on the Hilbert space $\ell^2(P) \otimes H$. Let $q : \ell^2(P) \otimes H \to \ell^2(P) \otimes H$ be a map defined by

$$(qf)(x) = \overline{\pi_0}(\overline{\alpha}_x(1))f(x)$$

for all $f \in \ell^2(P) \otimes H$. It is not difficult to see that $q \in B(\ell^2(P) \otimes H)$, which is actually a projection onto a closed subspace $\mathcal{H}$ of $\ell^2(P) \otimes H$. We claim that if $\{a_i\}$ is any approximate unit in A, then $\pi(a_i)$ converges strictly to q in $\mathcal{M}\big(\mathcal{K}(\ell^2(P) \otimes H)\big) = B(\ell^2(P) \otimes H)$. To prove our claim, since the net $\{\pi(a_i)\}$ is a norm bounded subset of $B(\ell^2(P) \otimes H)$, and $\pi(a_i)^* = \pi(a_i)$ for each i as well as $q^* = q$, by [27, Proposition C.7], we only need to show that $\pi(a_i) \to q$ strongly in $B(\ell^2(P) \otimes H)$. If $\{\varepsilon_x : x \in P\}$ is the usual orthonormal basis of $\ell^2(P)$, then it is enough to see that

$$\pi(a_i)(\varepsilon_x \otimes \pi_0(a)h) \to q(\varepsilon_x \otimes \pi_0(a)h)$$

for each spanning element $(\varepsilon_x \otimes \pi_0(a)h)$ of $\ell^2(P) \otimes H$ (recall that π_0 is nondegenerate). We have

$$\pi(a_i)(\varepsilon_x \otimes \pi_0(a)h) = \varepsilon_x \otimes \pi_0(\alpha_x(a_i))\pi_0(a)h = \varepsilon_x \otimes \pi_0(\alpha_x(a_i)a)h,$$

which is convergent to

$$\varepsilon_x \otimes \pi_0(\overline{\alpha}_x(1)a)h = \varepsilon_x \otimes \overline{\pi_0}(\overline{\alpha}_x(1))\pi_0(a)h = q(\varepsilon_x \otimes \pi_0(a)h)$$

in $\ell^2(P) \otimes H$. This is due to the extendibility of each α_x. Therefore, $\pi(a_i) \to q$ strictly in $B(\ell^2(P) \otimes H)$.

Next, let $W : P \to B(\ell^2(P) \otimes H)$ be the Nica partial-isometric representation introduced in Example 3.1. We aim at constructing a covariant partial-isometric representation (ρ, V) of (A, P, α) on the Hilbert space (closed subspace) $\mathcal{H}$ by using the pair (π, W). Note that, in general, π is not nondegenerate on $\ell^2(P) \otimes H$, unless $\overline{\alpha}_x(1) = 1$ for every $x \in P$. So, for our purpose, we first show that

$$W_x\pi(a) = \pi(\alpha_x(a))W_x \text{ and } W_x^*W_x\pi(a) = \pi(a)W_x^*W_x \tag{4.3}$$

for all $a \in A$ and $x \in P$. For every $f \in \ell^2(P) \otimes H$, we have

$$\begin{aligned}
(W_x\pi(a)f)(r) &= (W_x(\pi(a)f))(r) \\
&= (\pi(a)f)(rx) \\
&= \pi_0(\alpha_{rx}(a))f(rx) \\
&= \pi_0(\alpha_r(\alpha_x(a)))(W_xf)(r) \\
&= (\pi(\alpha_x(a))W_xf)(r)
\end{aligned}$$

for all $r \in P$. So, $W_x\pi(a) = \pi(\alpha_x(a))W_x$ is valid, from which, we get $\pi(a)W_x^* = W_x^*\pi(\alpha_x(a))$. One can apply these two equations to see that $W_x^*W_x\pi(a) = \pi(a)W_x^*W_x$ is also valid. Also, since $W_xW_x^* = S_x^*S_x = 1$ (see Example 3.1), each W_x is a coisometry, and hence, by applying the equation $W_x\pi(a) = \pi(\alpha_x(a))W_x$, we have

$$W_x\pi(a)W_x^* = \pi(\alpha_x(a))W_xW_x^* = \pi(\alpha_x(a)). \tag{4.4}$$

Now we claim that the pair $(\rho, V) = (q\pi q, qWq)$ is a covariant partial-isometric representation of of (A, P, α) on $\mathcal{H}$. More precisely, consider the maps

$$\rho : A \to qB(\ell^2(P) \otimes H)q \simeq B(\mathcal{H})$$

and

$$V : P \to qB(\ell^2(P) \otimes H)q \simeq B(\mathcal{H})$$

defined by

$$\rho(a) = q\pi(a)q = \pi(a) \quad \text{and} \quad V_x = qW_xq$$

for all $a \in A$ and $x \in P$, respectively. Since for any approximate unit $\{a_i\}$ in A, $\rho(a_i) = \pi(a_i) \to q$ strongly in $B(\mathcal{H})$, where $q = 1_{B(\mathcal{H})}$, it follows that the representation ρ is nondegenerate. Moreover, by applying (4.4), we have

$$V_x\rho(a)V_x^* = qW_xq\pi(a)qW_x^*q = qW_x\pi(a)W_x^*q = q\pi(\alpha_x(a))q = \rho(\alpha_x(a)).$$

Also, by applying the first equation of (4.3) and $\pi(a)W_x^* = W_x^*\pi(\alpha_x(a))$ along with the fact that $\rho(a) = q\pi(a)q = q\pi(a) = \pi(a)q = \pi(a)$, we get

$$\begin{aligned} V_x^*V_x\rho(a) &= qW_x^*qW_xq\pi(a)q \\ &= qW_x^*qW_x\pi(a)q \\ &= qW_x^*q\pi(\alpha_x(a))W_xq \\ &= qW_x^*\pi(\alpha_x(a))qW_xq \\ &= q\pi(a)W_x^*qW_xq \\ &= q\pi(a)qW_x^*qW_xq = \rho(a)V_x^*V_x. \end{aligned}$$

Thus, it is only left to show that the map V is a Nica partial-isometric representation. To see that each V_x is a partial-isometry, note that, for any approximate unit $\{a_i\}$ in A,

$$qW_x\pi(a_i)W_x^*qW_xq$$

converges strongly to

$$qW_xqW_x^*qW_xq = V_xV_x^*V_x$$

in $B(\ell^2(P) \otimes H)$. On the other hand, by applying the covariance equations of the pair (π, W), we have

$$\begin{aligned} q[W_x\pi(a_i)W_x^*]qW_xq &= q\pi(\alpha_x(a_i))qW_xq \\ &= q\pi(\alpha_x(a_i))W_xq = qW_x\pi(a_i)q, \end{aligned}$$

which converges strongly to $qW_xq = V_x$. So, we must have $V_xV_x^*V_x = V_x$, which means that each V_x is a partial-isometry. To see $V_xV_y = V_{xy}$ for every $x, y \in P$, we first need to compute V_xf for any $f \in \mathcal{H}$. So, knowing that $qf = f$, we have

$$\begin{aligned} [V_xf](r) = [qW_xf](r) &= [q(W_xf)](r) \\ &= \overline{\pi_0}(\overline{\alpha}_r(1))(W_xf)(r) \\ &= \overline{\pi_0}(\overline{\alpha}_r(1))f(rx) \\ &= \overline{\pi_0}(\overline{\alpha}_r(1))(qf)(rx) \\ &= \overline{\pi_0}(\overline{\alpha}_r(1))\overline{\pi_0}(\overline{\alpha}_{rx}(1))f(rx) \\ &= \overline{\pi_0}(\overline{\alpha}_r(1)\overline{\alpha}_{rx}(1))f(rx) \\ &= \overline{\pi_0}(\overline{\alpha}_{rx}(1))f(rx) \\ &= (qf)(rx) = f(rx) \end{aligned}$$

for all $r \in P$. Thus, by applying the above computation, we get

$$\begin{aligned} [V_xV_yf](r) &= [V_x(V_yf)](r) \\ &= (V_yf)(rx) \\ &= f((rx)y) \\ &= f(r(xy)) = [V_{xy}f](r). \end{aligned}$$

So, it follows that $V_xV_y = V_{xy}$ for all $x, y \in P$. Finally, we show that the partial-isometric representation V satisfies the Nica covariance condition (3.1). Let us first mention that the Hilbert space $\mathcal{H}$ is spanned by the elements

$$\{\varepsilon_r \otimes \overline{\pi_0}(\overline{\alpha}_r(1))h : r \in P, h \in H\}$$

as a closed subspace of $\ell^2(P) \otimes H$. Then, for every $y \in P$ and $f \in \mathcal{H}$, we have

$$(V_y^* f)(r) = (qW_y^* f)(r) = (q(S_y f))(r) = \overline{\pi_0}(\overline{\alpha}_r(1))(S_y f)(r).$$

Now, if $r = sy$ for some $s \in P$, which means that $r \in Py$, we get

$$(V_y^* f)(r) = \overline{\pi_0}(\overline{\alpha}_{sy}(1))(S_y f)(sy) = \overline{\pi_0}(\overline{\alpha}_{sy}(1))f(s).$$

Otherwise, $(V_y^* f)(r) = 0$. It therefore follows that, if $r = sy$ for some $s \in P$, then

$$\begin{aligned} [V_y^* V_y f](r) &= [V_y^*(V_y f)](r) \\ &= \overline{\pi_0}(\overline{\alpha}_{sy}(1))(V_y f)(s) \\ &= \overline{\pi_0}(\overline{\alpha}_{sy}(1))f(sy) \\ &= (qf)(sy) = f(sy) = f(r). \end{aligned}$$

Otherwise, $[V_y^* V_y f](r) = 0$. This implies that each $V_y^* V_y$ is the projection of $\mathcal{H}$ onto the closed subspace

$$\mathcal{H}_y := \{f \in \mathcal{H} : f(r) = 0 \text{ if } r \notin Py\} = \ker(1 - V_y^* V_y)$$

of $\mathcal{H}$, which is actually spanned by the elements

$$\{\varepsilon_{sy} \otimes \overline{\pi_0}(\overline{\alpha}_{sy}(1))h : s \in P, h \in H\}.$$

Now, if $Px \cap Py = \emptyset$, then for every $f \in \mathcal{H}$,

$$\begin{aligned} [V_x V_y^* f](r) &= [V_x(V_y^* f)](r) \\ &= (V_y^* f)(rx) = 0. \end{aligned}$$

This is due to the fact that, since $rx \in Px$, $rx \notin Py$. So, it follows that $V_x V_y^* = 0$, and hence,

$$V_x^* V_x V_y^* V_y = 0.$$

If $Px \cap Py = Pz$, for every $f \in \mathcal{H}$,

$$\big(V_x^* V_x (V_y^* V_y f)\big)(r) = \begin{cases} (V_y^* V_y f)(r) & \text{if } r \in Px, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for $(V_y^* V_y f)(r)$, where $r \in Px$, we have

$$(V_y^* V_y f)(r) = \begin{cases} f(r) & \text{if } r \in Py, \\ 0 & \text{otherwise.} \end{cases}$$

So, it follows that

$$\big(V_x^* V_x (V_y^* V_y f)\big)(r) = \begin{cases} f(r) & \text{if } r \in (Px \cap Py) = Pz, \\ 0 & \text{otherwise,} \end{cases}$$

which is equal to $(V_z^* V_z f)(r)$. Therefore,

$$V_x^* V_x V_y^* V_y = V_z^* V_z.$$

Consequently, the pair (ρ, V) is a (nontrivial) covariant partial-isometric representation of (A, P, α) on $\mathcal{H}$.

Note that, if π_0 is faithful, then it is not difficult to see that ρ becomes faithful. Hence, every system (A, P, α) has a (nontrivial) covariant pair (ρ, V) with ρ faithful.

4.2. Crossed products and Nica-Teoplitz algebras of Hilbert bimodules. Let P be a left LCM semigroup, and (A, P, α) a dynamical system consisting of a C^*-algebra A, and an action $\alpha : P \to \mathrm{End}(A)$ of P by extendible endomorphisms of A such that $\alpha_e = \mathrm{id}_A$.

Definition 4.4. A *partial-isometric crossed product* of (A, P, α) is a triple (B, i_A, i_P) consisting of a C^*-algebra B, a nondegenerate injective homomorphism $i_A : A \to B$, and a Nica partial-isometric representation $i_P : P \to \mathcal{M}(B)$ such that:

(i) the pair (i_A, i_P) is a covariant partial-isometric representation of (A, P, α) in B;
(ii) for every covariant partial-isometric representation (π, V) of (A, P, α) on a Hilbert space H, there exists a nondegenerate representation $\pi \times V : B \to B(H)$ such that $(\pi \times V) \circ i_A = \pi$ and $(\overline{\pi \times V}) \circ i_P = V$; and
(iii) the C^*-algebra B is generated by $\{i_A(a) i_P(x) : a \in A, x \in P\}$.

We call the algebra B the *partial-isometric crossed product* of the system (A, P, α) and denote it by $A \times_\alpha^{\mathrm{piso}} P$.

Remark 4.5. Note that in the definition above, for part (iii), we actually have

$$B = \overline{\mathrm{span}}\{i_P(x)^* i_A(a) i_P(y) : x, y \in P, a \in A\}. \tag{4.5}$$

To see this, we only need to show that the right hand side of (4.5) is closed under multiplication. To do so, we apply the Nica covariance condition to calculate each product

$$[i_P(x)^* i_A(a) i_P(y)][i_P(s)^* i_A(b) i_P(t)]. \tag{4.6}$$

We have

$$\begin{aligned}
&[i_P(x)^* i_A(a) i_P(y)][i_P(s)^* i_A(b) i_P(t)] \\
&= i_P(x)^* i_A(a) i_P(y)[i_P(y)^* i_P(y) i_P(s)^* i_P(s)] i_P(s)^* i_A(b) i_P(t),
\end{aligned}$$

which is zero if $Py \cap Ps = \emptyset$. But if $Py \cap Ps = Pz$ for some $z \in P$, then $ry = z = qs$ for some $r, q \in P$, and therefore by the covariance of the pair (i_A, i_P), we get

$$\begin{aligned}
&[i_P(x)^* i_A(a) i_P(y)][i_P(s)^* i_A(b) i_P(t)] \\
&= i_P(x)^* i_A(a) i_P(y) i_P(z)^* i_P(z) i_P(s)^* i_A(b) i_P(t) \\
&= i_P(x)^* i_A(a) i_P(y) i_P(ry)^* i_P(qs) i_P(s)^* i_A(b) i_P(t) \\
&= i_P(x)^* i_A(a)[i_P(y) i_P(y)^*] i_P(r)^* i_P(q)[i_P(s) i_P(s)^*] i_A(b) i_P(t) \\
&= i_P(x)^* i_A(a) \overline{i_A}(\overline{\alpha}_y(1)) i_P(r)^* i_P(q) \overline{i_A}(\overline{\alpha}_s(1)) i_A(b) i_P(t) \\
&= i_P(x)^* i_A(a \overline{\alpha}_y(1)) i_P(r)^* i_P(q) i_A(\overline{\alpha}_s(1) b) i_P(t) \\
&= i_P(x)^* i_P(r)^* i_A(\alpha_r(c)) i_A(\alpha_q(d)) i_P(q) i_P(t) \\
&= i_P(rx)^* i_A(\alpha_r(c) \alpha_q(d)) i_P(qt),
\end{aligned}$$

which is in the right hand side of (4.5), where $c = a\overline{\alpha}_y(1)$ and $d = \overline{\alpha}_s(1)b$. Thus, (4.5) is indeed true.

Next, we want to show that the partial-isometric crossed product of the system (A, P, α) always exists, and it is unique up to isomorphism. Firstly, since P is a left LCM semigroup, the opposite semigroup P^{o} is a right LCM semigroup. Therefore, one can easily see that (A, P^{o}, α) is a dynamical system in the sense of [15, Definition 3.1]. Then, following [12, §3] (see also [15, §3]), for every $s \in P$, let

$$X_s := \{s\} \times \overline{\alpha}_s(1)A,$$

where $\overline{\alpha}_s(1)A = \alpha_s(A)A = \overline{\operatorname{span}}\{\alpha_s(a)b : a, b \in A\}$ as each endomorphism α_s is extendible. Now, each X_s is given the structure of a Hilbert bimodule over A via

$$(s, x) \cdot a := (s, xa), \quad \langle (s, x), (s, y) \rangle_A := x^* y,$$

and

$$a \cdot (s, x) := (s, \alpha_s(a)x).$$

Let $X = \bigsqcup_{s \in P} X_s$, which is equipped with a multiplication

$$X_s \times X_t \to X_{s \bullet t}; \quad ((s, x), (t, y)) \mapsto (s, x)(t, y)$$

defined by

$$(s, x)(t, y) := (ts, \alpha_t(x)y) = (s \bullet t, \alpha_t(x)y)$$

for every $x \in \overline{\alpha}_s(1)A$ and $y \in \overline{\alpha}_t(1)A$. By [12, Lemma 3.2], X is a product system over the opposite semigroup P^{o} of essential Hilbert bimodules, and the left action of A on each fiber X_s is by compact operators. So, X is compactly aligned by [12, Proposition 5.8]. Let $(\mathcal{NT}(X), i_X)$ be the Nica-Toeplitz algebra corresponding to X (see [12], [7, §6], and [15]), which is generated by the universal Nica covariant Toeplitz representation $i_X : X \to \mathcal{NT}(X)$. We show that this algebra is the partial-isometric crossed product of the system (A, P, α). But we first need to recall that, for any approximate unit $\{a_i\}$ in A, by a similar discussion to [12, Lemma 3.3], one can see that $i_X(s, \alpha_s(a_i))$ converges strictly in the multiplier algebra $\mathcal{M}(\mathcal{NT}(X))$ for every $s \in P$. Now, we have:

Proposition 4.6. *Suppose that P is a left LCM semigroup, and (A, P, α) a dynamical system. Let $\{a_i\}$ be any approximate unit in A. Define the maps*

$$i_A : A \to \mathcal{NT}(X) \text{ and } i_P : P \to \mathcal{M}(\mathcal{NT}(X))$$

by

$$i_A(a) := i_X(e, a) \text{ and } i_P(s) := \lim_i i_X(s, \alpha_s(a_i))^* \text{ (strictly convergence)}$$

for all $a \in A$ and $s \in P$. Then the triple $(\mathcal{NT}(X), i_A, i_P)$ is a partial-isometric crossed product for (A, P, α), which is unique up to isomorphism.

Proof. For any approximate unit $\{a_i\}$ in A,

$$i_A(a_i) = i_X(e, a_i) = i_X(e, \alpha_e(a_i))$$

converges strictly to 1 in the multiplier algebra $\mathcal{M}(\mathcal{N}\mathcal{T}(X))$. One can see this again by [12, Lemma 3.3] similarly when $s = e$. It follows that i_A is a nondegenerate homomorphism. By a similar discussion to the first part of the proof of [12, Proposition 3.4], we can see that the map i_P is a partial-isometric representation such that together with the (nondegenerate) homomorphism i_A satisfy the covariance equations

$$i_A(\alpha_s(a)) = i_P(s)i_A(a)i_P(s)^* \text{ and } i_A(a)i_P(s)^*i_P(s) = i_P(s)^*i_P(s)i_A(a)$$

for all $a \in A$ and $s \in P$. So, we only need to show that the representation i_P satisfies the Nica covariance condition. By the same calculation as (3.7) in the proof of [12, Proposition 3.4], we have

$$i_A(ab^*)i_P(s)^*i_P(s) = i_X(s, \alpha_s(a))i_X(s, \alpha_s(b))^* \tag{4.7}$$

for all $a, b \in A$ and $s \in P$, and since

$$i_X(s, \alpha_s(a))i_X(s, \alpha_s(b))^* = i_X^{(s)}(\Theta_{(s,\alpha_s(a)),(s,\alpha_s(b))}),$$

it follows that

$$i_A(ab^*)i_P(s)^*i_P(s) = i_X^{(s)}(\Theta_{(s,\alpha_s(a)),(s,\alpha_s(b))}). \tag{4.8}$$

Therefore,

$$\begin{aligned}&i_A(ab^*)i_P(s)^*i_P(s)i_A(cd^*)i_P(t)^*i_P(t)\\&= i_X^{(s)}(\Theta_{(s,\alpha_s(a)),(s,\alpha_s(b))})i_X^{(t)}(\Theta_{(t,\alpha_t(c)),(t,\alpha_t(d))}),\end{aligned}$$

and since

$$\begin{aligned}&i_A(ab^*)i_P(s)^*i_P(s)i_A(cd^*)i_P(t)^*i_P(t)\\&= i_A(ab^*)i_A(cd^*)i_P(s)^*i_P(s)i_P(t)^*i_P(t)\\&= i_A(ab^*(cd^*))i_P(s)^*i_P(s)i_P(t)^*i_P(t),\end{aligned}$$

it follows that

$$\begin{aligned}&i_A(ab^*(cd^*))i_P(s)^*i_P(s)i_P(t)^*i_P(t)\\&\qquad= i_X^{(s)}(\Theta_{(s,\alpha_s(a)),(s,\alpha_s(b))})i_X^{(t)}(\Theta_{(t,\alpha_t(c)),(t,\alpha_t(d))}).\end{aligned} \tag{4.9}$$

Now, if $Ps \cap Pt = Pr$ for some $r \in P$, which is equivalent to saying that

$$s \bullet P^o \cap t \bullet P^o = r \bullet P^o,$$

since i_X is Nica covariant, we have

$$\begin{aligned}&i_A(ab^*(cd^*))i_P(s)^*i_P(s)i_P(t)^*i_P(t)\\&\qquad= i_X^{(r)}\Big(\iota_s^r\big(\Theta_{(s,\alpha_s(a)),(s,\alpha_s(b))}\big)\iota_t^r\big(\Theta_{(t,\alpha_t(c)),(t,\alpha_t(d))}\big)\Big).\end{aligned} \tag{4.10}$$

Next, we want to calculate the product

$$\iota_s^r\big(\Theta_{(s,\alpha_s(a)),(s,\alpha_s(b))}\big)\iota_t^r\big(\Theta_{(t,\alpha_t(c)),(t,\alpha_t(d))}\big)$$

of compact operators in $\mathcal{K}(X_r)$ to show that it is equal to

$$\Theta_{(r,\alpha_r(ab^*)),(r,\alpha_r(dc^*))}.$$

Since $s \bullet p = r = t \bullet q$ for some $p, q \in P$, and $X_t \otimes_A X_q \simeq X_{t \bullet q}$, it is enough to see this on the spanning elements $(t, \overline{\alpha}_t(1)f)(q, \overline{\alpha}_q(1)g)$ of $X_{t\bullet q} = X_r$, where $f, g \in A$. First, $(t, \overline{\alpha}_t(1)f)(q, \overline{\alpha}_q(1)g)$ by $\iota_t^r(\Theta_{(t,\alpha_t(c)),(t,\alpha_t(d))})$ is mapped to

$$\begin{aligned}
&\iota_t^{t\bullet q}\big(\Theta_{(t,\alpha_t(c)),(t,\alpha_t(d))}\big)\big((t, \overline{\alpha}_t(1)f)(q, \overline{\alpha}_q(1)g)\big)\\
&= \big(\Theta_{(t,\alpha_t(c)),(t,\alpha_t(d))}(t, \overline{\alpha}_t(1)f)\big)(q, \overline{\alpha}_q(1)g)\\
&= \big((t, \alpha_t(c)) \cdot \big\langle (t, \alpha_t(d)), (t, \overline{\alpha}_t(1)f)\big\rangle_A\big)(q, \overline{\alpha}_q(1)g)\\
&= \big((t, \alpha_t(c)) \cdot (\alpha_t(d^*)\overline{\alpha}_t(1)f)\big)(q, \overline{\alpha}_q(1)g)\\
&= \big((t, \alpha_t(c)) \cdot (\alpha_t(d^*)f)\big)(q, \overline{\alpha}_q(1)g)\\
&= \big((t, \alpha_t(c)\alpha_t(d^*)f)\big)(q, \overline{\alpha}_q(1)g)\\
&= (t, \alpha_t(cd^*)f)(q, \overline{\alpha}_q(1)g)\\
&= (qt, \alpha_q(\alpha_t(cd^*)f)\overline{\alpha}_q(1)g)\\
&= (qt, \alpha_q(\alpha_t(cd^*)f)g)\\
&= (qt, \alpha_{qt}(cd^*)\alpha_q(f)g)\\
&= (t \bullet q, \alpha_{t\bullet q}(cd^*)\alpha_q(f)g)\\
&= (r, \alpha_r(cd^*)\alpha_q(f)g)\\
&= (s \bullet p, \alpha_{s\bullet p}(cd^*)\alpha_q(f)g)\\
&= (ps, \alpha_{ps}(cd^*)\alpha_q(f)g) = (s, \alpha_s(cd^*))(p, \overline{\alpha}_p(1)\alpha_q(f)g).
\end{aligned}$$

We then let $\iota_s^r(\Theta_{(s,\alpha_s(a)),(s,\alpha_s(b))})$ act on $(s, \alpha_s(cd^*))(p, \overline{\alpha}_p(1)\alpha_q(f)g)$, and hence,

$$\begin{aligned}
&\iota_s^{s\bullet p}\big(\Theta_{(s,\alpha_s(a)),(s,\alpha_s(b))}\big)\big((s, \alpha_s(cd^*))(p, \overline{\alpha}_p(1)\alpha_q(f)g)\big)\\
&= \big(\Theta_{(s,\alpha_s(a)),(s,\alpha_s(b))}(s, \alpha_s(cd^*))\big)(p, \overline{\alpha}_p(1)\alpha_q(f)g)\\
&= \big((s, \alpha_s(a)) \cdot \big\langle (s, \alpha_s(b)), (s, \alpha_s(cd^*))\big\rangle_A\big)(p, \overline{\alpha}_p(1)\alpha_q(f)g)\\
&= \big((s, \alpha_s(a)) \cdot [\alpha_s(b^*)\alpha_s(cd^*)]\big)(p, \overline{\alpha}_p(1)\alpha_q(f)g)\\
&= \big((s, \alpha_s(a)) \cdot [\alpha_s(b^*cd^*)]\big)(p, \overline{\alpha}_p(1)\alpha_q(f)g)\\
&= (s, \alpha_s(a)\alpha_s(b^*cd^*))(p, \overline{\alpha}_p(1)\alpha_q(f)g)\\
&= (s, \alpha_s(ab^*cd^*))(p, \overline{\alpha}_p(1)\alpha_q(f)g)\\
&= (ps, \alpha_p(\alpha_s(ab^*cd^*))\overline{\alpha}_p(1)\alpha_q(f)g)\\
&= (ps, \alpha_p(\alpha_s(ab^*cd^*))\alpha_q(f)g)\\
&= (ps, \alpha_{ps}(ab^*cd^*)\alpha_q(f)g)\\
&= (s \bullet p, \alpha_{s\bullet p}(ab^*cd^*)\alpha_q(f)g) = (r, \alpha_r(ab^*cd^*)\alpha_q(f)g).
\end{aligned}$$

Thus, it follows that

$$\begin{aligned}
&\iota_s^r\big(\Theta_{(s,\alpha_s(a)),(s,\alpha_s(b))}\big)\iota_t^r\big(\Theta_{(t,\alpha_t(c)),(t,\alpha_t(d))}\big)\big((t, \overline{\alpha}_t(1)f)(q, \overline{\alpha}_q(1)g)\big)\\
&\qquad = (r, \alpha_r(ab^*cd^*)\alpha_q(f)g). \qquad (4.11)
\end{aligned}$$

On the other hand, since

$$\begin{aligned}
(t, \overline{\alpha}_t(1)f)(q, \overline{\alpha}_q(1)g) &= (qt, \alpha_q(\overline{\alpha}_t(1)f)\overline{\alpha}_q(1)g)\\
&= (qt, \alpha_q(\overline{\alpha}_t(1)f)g)\\
&= (qt, \overline{\alpha}_q(\overline{\alpha}_t(1))\alpha_q(f)g)\\
&= (qt, \overline{\alpha}_{qt}(1)\alpha_q(f)g)\\
&= (r, \overline{\alpha}_r(1)\alpha_q(f)g),
\end{aligned}$$

we have

$$\begin{aligned}
&\Theta_{(r,\alpha_r(ab^*)),(r,\alpha_r(dc^*))}\big((t,\overline{\alpha}_t(1)f)(q,\overline{\alpha}_q(1)g)\big)\\
&=\Theta_{(r,\alpha_r(ab^*)),(r,\alpha_r(dc^*))}(r,\overline{\alpha}_r(1)\alpha_q(f)g)\\
&=(r,\alpha_r(ab^*))\cdot\big\langle(r,\alpha_r(dc^*)),(r,\overline{\alpha}_r(1)\alpha_q(f)g)\big\rangle_A\\
&=(r,\alpha_r(ab^*))\cdot[\alpha_r(dc^*)^*\overline{\alpha}_r(1)\alpha_q(f)g]\\
&=(r,\alpha_r(ab^*))\cdot[\alpha_r(cd^*)\overline{\alpha}_r(1)\alpha_q(f)g]\\
&=(r,\alpha_r(ab^*))\cdot[\alpha_r(cd^*)\alpha_q(f)g]\\
&=(r,\alpha_r(ab^*)\alpha_r(cd^*)\alpha_q(f)g)\\
&=(r,\alpha_r(ab^*cd^*)\alpha_q(f)g).
\end{aligned}\tag{4.12}$$

So, we conclude by (4.11) and (4.12) that

$$\iota_s^r\big(\Theta_{(s,\alpha_s(a)),(s,\alpha_s(b))}\big)\iota_t^r\big(\Theta_{(t,\alpha_t(c)),(t,\alpha_t(d))}\big)=\Theta_{(r,\alpha_r(ab^*)),(r,\alpha_r(dc^*))}.\tag{4.13}$$

Consequently, if $Ps\cap Pt=Pr$, then by applying (4.13), (4.10), and (4.8), we get

$$\begin{aligned}
&i_A(ab^*(cd^*))i_P(s)^*i_P(s)i_P(t)^*i_P(t)\\
&=i_X^{(r)}\Big(\iota_s^r\big(\Theta_{(s,\alpha_s(a)),(s,\alpha_s(b))}\big)\iota_t^r\big(\Theta_{(t,\alpha_t(c)),(t,\alpha_t(d))}\big)\Big)\\
&=i_X^{(r)}\big(\Theta_{(r,\alpha_r(ab^*)),(r,\alpha_r(dc^*))}\big)\\
&=i_A(ab^*(dc^*)^*)i_P(r)^*i_P(r)\\
&=i_A(ab^*(cd^*))i_P(r)^*i_P(r).
\end{aligned}$$

We therefore have

$$i_A(ab^*cd^*)i_P(s)^*i_P(s)i_P(t)^*i_P(t)=i_A(ab^*cd^*)i_P(r)^*i_P(r)\tag{4.14}$$

for all $a,b,c,d\in A$. Since A contains an approximate unit, it follows by (4.14) that we must have

$$i_P(s)^*i_P(s)i_P(t)^*i_P(t)=i_P(r)^*i_P(r)$$

when $Ps\cap Pt=Pr$. If $Ps\cap Pt=\emptyset$, then again, since i_X is Nica covariant, the right hand side of (4.9) is zero, and therefore,

$$i_A(ab^*(cd^*))i_P(s)^*i_P(s)i_P(t)^*i_P(t)=0\tag{4.15}$$

for all $a,b,c,d\in A$. Thus, similar to the above, as A contains an approximate unit, we conclude that

$$i_P(s)^*i_P(s)i_P(t)^*i_P(t)=0.$$

So, the pair (i_A,i_P) is a covariant partial-isometric representation of (A,P,α) in the algebra $\mathcal{NT}(X)$, and therefore, condition (i) in Definition 4.4 is satisfied.

Next, suppose that (π,V) is a covariant partial-isometric representation of (A,P,α) on a Hilbert space H. Then, the pair (π,V^*) is a representation of the system $(A,P^{\mathrm{o}},\alpha)$ in the sense of [15, Definition 3.2], which is Nica covariant. Note that the semigroup homomorphism $V^*:P^{\mathrm{o}}\to B(H)$ is defined by $s\mapsto V_s^*$. Therefore, by [15, Proposition 3.11], the map $\psi:X\to B(H)$ defined by

$$\psi(s,x):=V_s^*\pi(x)$$

is a nondegenerate Nica covariant Toeplitz representation of X on H (see also [12, Proposition 9.2]). So, there is a homomorphism $\psi_* : \mathcal{NT}(X) \to B(H)$ such that $\psi_* \circ i_X = \psi$ (see [12, 7]), which is nondegenerate. Let $\pi \times V = \psi_*$. Then

$$(\pi \times V)(i_A(a)) = \psi_*(i_X(e,a)) = \psi(e,a) = V_e^* \pi(a) = \pi(a)$$

for all $a \in A$. Also, since $\pi \times V$ is nondegenerate, we have

$$\begin{aligned}
\overline{(\pi \times V)}(i_P(s)) &= \overline{(\pi \times V)}\big(\lim_i i_X(s, \alpha_s(a_i))^*\big) \\
&= \lim_i (\pi \times V)\big(i_X(s, \alpha_s(a_i))^*\big) \\
&= \lim_i \psi_*\big(i_X(s, \alpha_s(a_i))\big)^* \\
&= \lim_i \psi(s, \alpha_s(a_i))^* \\
&= \lim_i [V_s^* \pi(\alpha_s(a_i))]^* \\
&= \lim_i [\pi(a_i) V_s^*]^* \quad \text{(by the covariance of } (\pi, V)) \\
&= \lim_i V_s \pi(a_i) = V_s
\end{aligned}$$

for all $s \in P$. Thus, condition (ii) in Definition 4.4 is satisfied, too.

Finally, condition (iii) also holds as the elements of the form $i_X(s, \overline{\alpha}_s(1)a^*)^*$ generate $\mathcal{NT}(X)$, and

$$i_X(s, \overline{\alpha}_s(1)a^*)^* = i_A(a) i_P(s),$$

which follows by a simple computation.

To see that the homomorphism i_A is injective, we recall from Example 4.3 that the system (A, P, α) admits a (nontrivial) covariant partial-isometric representation (π, V) with π faithful. Therefore, it follows from the equation $(\pi \times V) \circ i_A = \pi$ that i_A must be injective.

For uniqueness, suppose that (C, j_A, j_P) is another triple which satisfies conditions (i)-(iii) in Definition 4.4. Then, by applying the universal properties (condition (ii)) of the algebras C and $\mathcal{NT}(X)$, once can see that there is an isomorphism of C onto $\mathcal{NT}(X)$ which maps the pair (j_A, j_P) to the pair (i_A, i_P). □

Remark 4.7. Recall that when P is the positive cone of an abelian lattice-ordered group G, by [12, Theorem 9.3], a covariant partial-isometric representation (π, V) of (A, P, α) on a Hilbert space H induces a faithful representation $\pi \times V$ of $A \times_\alpha^{\text{piso}} P$ if and only if, for every finite subset $F = \{x_1, x_2, ..., x_n\}$ of $P \backslash \{e\}$, π is faithful on the range of

$$\prod_{i=1}^{n} (1 - V_{x_i}^* V_{x_i}).$$

Also, note that, by [15, Theorem 3.13], a similar necessary and sufficient condition for the faithfulness of the representation $\pi \times V$ of $A \times_\alpha^{\text{piso}} P$ can be obtained for more general semigroups P, namely, LCM semigroups. (see also [11, Theorem 3.2]).

Suppose that (A, P, α) is a dynamical system, and I is an ideal of A such that $\alpha_s(I) \subset I$ for all $s \in P$. To define a crossed product $I \times_\alpha^{\text{piso}} P$ which we want it to sit naturally in $A \times_{\alpha}^{\text{piso}} P$ as an ideal, we need some extra condition. So, we need to recall a definition from [1]. Let α be an extendible endomorphism of a C^*-algebra A, and I an ideal of A. Suppose that $\psi : A \to \mathcal{M}(I)$ is the canonical nondegenerate homomorphism defined by $\psi(a)i = ai$ for all $a \in A$ and $i \in I$. Then, we say I is *extendible α-invariant* if it is α-invariant, which means that $\alpha(I) \subset I$, and the endomorphism $\alpha|_I$ is extendible, such that

$$\alpha(u_\lambda) \to \overline{\psi}(\overline{\alpha}(1_{\mathcal{M}(A)}))$$

strictly in $\mathcal{M}(I)$, where $\{u_\lambda\}$ is an approximate unit in I.

In addition, if (A, P, α) is a dynamical system and I is an ideal of A, then there is a dynamical system $(A/I, P, \tilde{\alpha})$ with extendible endomorphisms given by $\tilde{\alpha}_s(a+I) = \alpha_s(a) + I$ for every $a \in A$ and $s \in P$ (see again [1]).

The following theorem is actually a generalization of [5, Theorem 3.1]:

Theorem 4.8. *Let $(A \times_{\alpha}^{\text{piso}} P, i_A, V)$ be the partial-isometric crossed product of a dynamical system (A, P, α), and I an extendible α_x-invariant ideal of A for every $x \in P$. Then, there is a short exact sequence*

$$0 \longrightarrow I \times_{\alpha}^{\text{piso}} P \stackrel{\mu}{\longrightarrow} A \times_{\alpha}^{\text{piso}} P \stackrel{\varphi}{\longrightarrow} A/I \times_{\tilde{\alpha}}^{\text{piso}} P \longrightarrow 0 \tag{4.16}$$

of C^*-algebras, where μ is an isomorphism of $I \times_{\alpha}^{\text{piso}} P$ onto the ideal

$$\mathcal{E} := \overline{\text{span}}\{V_s^* i_A(i) V_t : i \in I, s, t \in P\}$$

of $A \times_{\alpha}^{\text{piso}} P$. If $q : A \to A/I$ is the quotient map, and the triples $(I \times_{\alpha}^{\text{piso}} P, i_I, W)$ and $(A/I \times_{\tilde{\alpha}}^{\text{piso}} P, i_{A/I}, U)$ are the crossed products of the systems (I, P, α) and $(A/I, P, \tilde{\alpha})$, respectively, then

$$\mu \circ i_I = i_A|_I,\ \overline{\mu} \circ W = V \ \text{ and } \ \varphi \circ i_A = i_{A/I} \circ q,\ \overline{\varphi} \circ V = U.$$

Proof. We first show that $\mathcal{E}$ is an ideal of $A \times_{\alpha}^{\text{piso}} P$. To do so, it suffices to see on the spanning elements of $\mathcal{E}$ that $V_r^* \mathcal{E}$, $i_A(a)\mathcal{E}$, and $V_r \mathcal{E}$ are all contained in $\mathcal{E}$ for every $a \in A$ and $r \in P$. This first one is obvious, and the second one follows easily by applying the covariance equation $i_A(a) V_s^* = V_s^* i_A(\alpha_s(a))$. For the third one, we have

$$V_r V_s^* i_A(i) V_t = V_r [V_r^* V_r V_s^* V_s] V_s^* i_A(i) V_t,$$

which is zero if $Pr \cap Ps = \emptyset$. But if $Pr \cap Ps = Pz$ for some $z \in P$, then there are $x, y \in P$ such that $xr = z = ys$, and therefore it follows that

$$\begin{aligned}
V_r V_s^* i_A(i) V_t &= V_r [V_z^* V_z] V_s^* i_A(i) V_t \\
&= V_r V_{xr}^* V_{ys} V_s^* i_A(i) V_t \\
&= V_r [V_x V_r]^* V_y V_s V_s^* i_A(i) V_t \\
&= [V_r V_r^*] V_x^* V_y [V_s V_s^*] i_A(i) V_t \\
&= \overline{i_A}(\overline{\alpha}_r(1)) V_x^* V_y \overline{i_A}(\overline{\alpha}_s(1)) i_A(i) V_t \quad \text{(by Lemma (4.2))} \\
&= V_x^* \overline{i_A}(\overline{\alpha}_x(\overline{\alpha}_r(1))) V_y i_A(\overline{\alpha}_s(1) i) V_t \quad \text{(by Lemma (4.2))} \\
&= V_x^* \overline{i_A}(\overline{\alpha}_{xr}(1)) i_A(\alpha_y(\overline{\alpha}_s(1) i)) V_y V_t \\
&= V_x^* \overline{i_A}(\overline{\alpha}_z(1)) i_A(\overline{\alpha}_y(\overline{\alpha}_s(1)) \alpha_y(i)) V_{yt} \\
&= V_x^* \overline{i_A}(\overline{\alpha}_z(1)) i_A(\overline{\alpha}_{ys}(1) \alpha_y(i)) V_{yt} \\
&= V_x^* \overline{i_A}(\overline{\alpha}_z(1)) i_A(\overline{\alpha}_z(1) \alpha_y(i)) V_{yt} \\
&= V_x^* i_A(\overline{\alpha}_z(1) \overline{\alpha}_z(1) \alpha_y(i)) V_{yt} \\
&= V_x^* i_A(\overline{\alpha}_z(1) \alpha_y(i)) V_{yt},
\end{aligned}$$

which belongs to $\mathcal{E}$. Thus, $\mathcal{E}$ is an ideal of $A \times_{\alpha}^{\text{piso}} P$. Let $\phi : A \times_{\alpha}^{\text{piso}} P \to \mathcal{M}(\mathcal{E})$ be the canonical nondegenerate homomorphism defined by $\phi(\xi)\eta = \xi\eta$ for all $\xi \in A \times_{\alpha}^{\text{piso}} P$ and $\eta \in \mathcal{E}$. Suppose that now the maps

$$k_I : I \to \mathcal{M}(\mathcal{E}) \text{ and } S : P \to \mathcal{M}(\mathcal{E})$$

are defined by the compositions

$$I \xrightarrow{i_A|_I} A \times_{\alpha}^{\text{piso}} P \xrightarrow{\phi} \mathcal{M}(\mathcal{E}) \text{ and } P \xrightarrow{V} \mathcal{M}(A \times_{\alpha}^{\text{piso}} P) \xrightarrow{\overline{\phi}} \mathcal{M}(\mathcal{E}),$$

respectively. We claim that the triple $(\mathcal{E}, k_I, S)$ is a partial-isometric crossed product of the system (I, P, α). First, exactly by the same discussion as in the proof of [5, Theorem 3.1] using the extendibility of the ideal I, it follows that the homomorphism k_I is nondegenerate. Also, it follows easily by the definition of the map S that it is indeed a Nica partial-isometric representation. Then, by some routine calculations, one can see that the pair (k_I, S) satisfies the covariance equations

$$k_I(\alpha_t(i)) = S_t k_I(i) S_t^* \text{ and } S_t^* S_t k_I(i) = k_I(i) S_t^* S_t$$

for all $i \in I$ and $t \in P$.

Next, suppose that the pair (π, T) is a covariant partial-isometric representation of (I, P, α) on a Hilbert space H. Let $\psi : A \to \mathcal{M}(I)$ be the canonical nondegenerate homomorphism which was mentioned about earlier. Let the map $\rho : A \to B(H)$ be defined by the composition

$$A \xrightarrow{\psi} \mathcal{M}(I) \xrightarrow{\overline{\pi}} B(H),$$

which is a nondegenerate representation of A on H. We claim that the pair (ρ, T) is a covariant partial-isometric representation of (A, P, α) on H. To prove our claim, we only need to show that the pair (ρ, T) satisfies the covariance equations (4.1). Since the ideal I is extendible, we have $\overline{\alpha_s|_I}\circ\psi = \psi\circ\alpha_s$ for all $s \in P$. It therefore follows that

$$\begin{aligned}\rho(\alpha_s(a)) &= (\overline{\pi}\circ\psi)(\alpha_s(a)) \\ &= \overline{\pi}(\psi\circ\alpha_s(a)) \\ &= \overline{\pi}(\overline{\alpha_s|_I}\circ\psi(a)) \\ &= (\overline{\pi}\circ\overline{\alpha_s|_I})(\psi(a)) \\ &= T_s\overline{\pi}(\psi(a))T_s^* = T_s\rho(a)T_s^*.\end{aligned}$$

Also, one can easily see that we have $T_s^*T_s\rho(a) = \rho(a)T_s^*T_s$. Thus, there is a nondegenerate representation $\rho\times T$ of $A\times_\alpha^{\mathrm{piso}}P$ on H, whose restriction $(\rho\times T)|_{\mathcal{E}}$ is a nondegenerate representation of $\mathcal{E}$ on H satisfying

$$(\rho\times T)|_{\mathcal{E}}\circ k_I = \pi \ \text{ and } \ \overline{(\rho\times T)|_{\mathcal{E}}}\circ S = T.$$

Finally, the elements of the form

$$\begin{aligned}S_s^*k_I(i)S_t &= \overline{\phi}(V_s)^*\overline{\phi}(i_A(i))\overline{\phi}(V_t) \\ &= \phi(V_s^*i_A(i)V_t) = V_s^*i_A(i)V_t\end{aligned}$$

obviously span the algebra $\mathcal{E}$. Thus, $(\mathcal{E}, k_I, S)$ is a partial-isometric crossed product of (I, P, α). So, by Proposition 4.6, there is an isomorphism $\mu : I\times_\alpha^{\mathrm{piso}}P \to \mathcal{E}$ such that

$$\mu(i_I(i)W_t) = k_I(i)S_t = \phi(i_A|_I(i))\overline{\phi}(V_t) = \phi(i_A|_I(i)V_t) = i_A|_I(i)V_t,$$

from which, it follows that

$$\mu\circ i_I = i_A|_I \ \text{ and } \overline{\mu}\circ W = V.$$

To get the desired homomorphism φ, let the homomorphism $j_A : A \to A/I\times_{\tilde{\alpha}}^{\mathrm{piso}}P$ be given by the composition

$$A \xrightarrow{\ q\ } A/I \xrightarrow{\ i_{A/I}\ } A/I\times_{\tilde{\alpha}}^{\mathrm{piso}}P,$$

which is nondegenerate. Then, it is not difficult to see that the pair (j_A, U) is a covariant partial-isometric representation of (A, P, α) in the algebra $A/I\times_{\tilde{\alpha}}^{\mathrm{piso}}P$. Thus, there is a nondegenerate homomorphism $\varphi := j_A\times U : A\times_\alpha^{\mathrm{piso}}P \to A/I\times_{\tilde{\alpha}}^{\mathrm{piso}}P$ such that

$$\varphi\circ i_A = j_A = i_{A/I}\circ q \ \text{ and } \overline{\varphi}\circ V = U,$$

which implies that φ is onto.

Finally, we show that $\mu(I\times_\alpha^{\mathrm{piso}}P) = \mathcal{E}$ is equal to $\ker\varphi$ which means that (4.16) is exact. The inclusion $\mathcal{E}\subset\ker\varphi$ is immediate. To see the other inclusion, take a nondegenerate representation Π of $A\times_\alpha^{\mathrm{piso}}P$ on a Hilbert space H with $\ker\Pi = \mathcal{E}$. Since $I \subset \ker(\Pi\circ i_A)$, the composition $\Pi\circ i_A$ gives a (well-defined) nodegenerate representation $\widetilde{\Pi}$ of A/I on H. Also, the composition $\overline{\Pi}\circ V$ defines a Nica partial-isometric representation P on H, such that together with $\widetilde{\Pi}$ forms a covariant partial-isometric representation of $(A/I, P, \tilde{\alpha})$ on H.

Then the corresponding (nondegenerate) representation $\widetilde{\Pi} \times (\overline{\Pi} \circ V)$ lifts to Π, which means that $[\widetilde{\Pi} \times (\overline{\Pi} \circ V)] \circ \varphi = \Pi$, from which the inclusion $\ker \varphi \subset \mathcal{E}$ follows. This completes the proof. □

Example 4.9. Suppose that S is a unital right LCM semigroup. See in [6, 25] that associated to S there is a universal C^*-algebra

$$C^*(S) = \overline{\operatorname{span}}\{W_s W_t^* : s, t \in S\}$$

generated by a universal isometric representation $W : S \to C^*(S)$, which is *Nica-covariant*, which means that it satisfies

$$W_r W_r^* W_s W_s^* = \begin{cases} W_t W_t^* & \text{if } rS \cap sS = tS, \\ 0 & \text{if } rS \cap sS = \emptyset. \end{cases} \tag{4.17}$$

In addition, by [7, Corollary 7.11], $C^*(S)$ is isomorphic to the Nica-Toeplitz algebra $\mathcal{N}\mathcal{T}(X)$ of the compactly aligned product system X over S with fibers $X_s = \mathbb{C}$ for all $s \in S$. Now, consider the trivial dynamical system $(\mathbb{C}, P, \mathrm{id})$, where P is a left LCM semigroup. So, the opposite semigroup P^o is right LCM. Then, it follows by Proposition 4.6 that there is an isomorphism

$$i_p(x) \in (\mathbb{C} \times_{\mathrm{id}}^{\mathrm{piso}} P) \mapsto W_x^* \in C^*(P^o)$$

for all $x \in P$, where W is the universal Nica-covariant isometric representation of P^o which generates $C^*(P^o)$.

For the C^*-algebra $C^*(S)$ associated to any arbitrary left cancellative semigroup S, readers may refer to [22].

Lemma 4.10. *For the dynamical system* (A, P, id) *in which* P *is a left LCM semigroup and* id *denotes the trivial action, we have*

$$(A \times_{\mathrm{id}}^{\mathrm{piso}} P, i) \simeq A \otimes_{\max} C^*(P^o).$$

The isomorphism maps each (spanning) element $i_p(x)^* i_A(a) i_p(y)$ *of* $A \times_{\mathrm{id}}^{\mathrm{piso}} P$ *to* $a \otimes W_x W_y^*$*, where* W *is the universal Nica-covariant isometric representation of* P^o *which generates* $C^*(P^o)$.

Proof. We skip the proof as it follows easily by some routine calculations. □

5. Tensor products of crossed products

Let (A, P, α) and (B, S, β) be dynamical systems in which P and S are left LCM semigroups. Then, $P \times S$ is a unital semigroup with the unit element (e_P, e_S), where e_P and e_S are the unit elements of P and S, respectively. In addition, since

$$\begin{aligned} (P \times S)(x, r) \cap (P \times S)(y, s) &= (Px \times Sr) \cap (Py \times Ss) \\ &= (Px \cap Py) \times (Sr \cap Ss), \end{aligned} \tag{5.1}$$

it follows that $P \times S$ is a left LCM semigroup. More precisely, if $Px \cap Py = Pz$ and $Sr \cap Ss = St$ for some $z \in P$ and $t \in S$, then it follows by (5.1) that

$$(P \times S)(x, r) \cap (P \times S)(y, s) = Pz \times St = (P \times S)(z, t),$$

which means that (z, t) is a least common left multiple of (x, r) and (y, s) in $P \times S$. Otherwise, $(P \times S)(x, r) \cap (P \times S)(y, s) = \emptyset$. Thus, $P \times S$ is actually a left LCM semigroup (note that, the similar fact holds if P and S are right LCM semigroups).

Next, for every $x \in P$ and $r \in S$, as α_x and β_r are endomorphisms of the algebras A and B, respectively, it follows by [27, Lemma B. 31] that there is an endomorphism $\alpha_x \otimes \beta_r$ of the maximal tensor product $A \otimes_{\max} B$ such that $(\alpha_x \otimes \beta_r)(a \otimes b) = \alpha_x(a) \otimes \beta_r(b)$ for all $a \in A$ and $b \in B$. We therefore have an action

$$\alpha \otimes \beta : P \times S \to \mathrm{End}(A \otimes_{\max} B)$$

of $P \times S$ on $A \otimes_{\max} B$ by endomorphisms such that

$$(\alpha \otimes \beta)_{(x,r)} = \alpha_x \otimes \beta_r \text{ for all } (x, r) \in P \times S.$$

Moreover, it follows by the extendibility of the actions α and β that the action $\alpha \otimes \beta$ on $A \otimes_{\max} B$ is actually given by extendible endomorphisms (see [18, Lemma 2.3]). Thus, we have a dynamical system $(A \otimes_{\max} B, P \times S, \alpha \otimes \beta)$, for which, we want to study the corresponding partial-isometric crossed product. We actually aim to show that under some certain conditions we have the following isomorphism:

$$(A \otimes_{\max} B) \times^{\mathrm{piso}}_{\alpha\otimes\beta} (P \times S) \simeq (A \times^{\mathrm{piso}}_{\alpha} P) \otimes_{\max} (B \times^{\mathrm{piso}}_{\beta} S).$$

In fact, those conditions are to ensure that the Nica partial-isometric representations of P and S are $*$-commuting. Hence, we first need to assume that the unital semigroups P, P^{o}, S, and S^{o} are all left LCM. It thus turns out that all of them must actually be both left and right LCM semigroups. The other condition comes from the following definition:

Definition 5.1. Suppose that P and P^{o} are both left LCM semigroups. A *bicovariant partial-isometric representation* of P on a Hilbert space H is a Nica partial-isometric representation $V : P \to B(H)$ which satisfies

$$V_r V_r^* V_s V_s^* = \begin{cases} V_t V_t^* & \text{if } rP \cap sP = tP, \\ 0 & \text{if } rP \cap sP = \emptyset. \end{cases} \tag{5.2}$$

Note the equation (5.2) is a kind of Nica covariance condition, too. So, to distinguish it from the covariance equation (3.1), we view (3.1) as the *right Nica covariance condition* and (5.2) as the *left Nica covariance condition*.

Note that similar to (3.1), we can see that the equation (5.2) is also well-defined.

Lemma 5.2. *Suppose that the unital semigroups P, P^{o}, S, and S^{o} are all left LCM. Let V and W be bicovariant partial-isometric representations of P and S on a Hilbert space H, respectively, such that each V_p $*$-commutes with each W_s for all $p \in P$ and $s \in S$. Then, there exits a bicovariant partial-isometric representation U of $P \times S$ on H such that $U_{(p,s)} = V_p W_s$. Moreover, every bicovariant partial-isometric representation of $P \times S$ arises this way.*

Proof. Define a map $U : P \times S \to B(H)$ by

$$U_{(p,s)} = V_p W_s$$

for all $(p,s) \in P \times S$. Since each V_p $*$-commutes with each W_s, it follows that each $U_{(p,s)}$ is a partial isometry, as

$$\begin{aligned} U_{(p,s)}U^*_{(p,s)}U_{(p,s)} &= V_p W_s [V_p W_s]^* V_p W_s \\ &= V_p W_s [W_s V_p]^* V_p W_s \\ &= V_p W_s V_p^* W_s^* V_p W_s \\ &= V_p V_p^* W_s V_p W_s^* W_s \\ &= V_p V_p^* V_p W_s W_s^* W_s = V_p W_s = U_{(p,s)}. \end{aligned}$$

Also, a simple computation shows that

$$U_{(p,s)}U_{(q,t)} = U_{(p,s)(q,t)}$$

for every (p,s) and (q,t) in $P \times S$. Thus, the map U is a (unital) semigroup homomorphism (with partial-isometric values). Next, we want to show that it satisfies the Nica covariance conditions (3.1) and (5.2), and hence, it is bicovariant. To see (3.1), we first have

$$\begin{aligned} U^*_{(p,s)}U_{(p,s)}U^*_{(q,t)}U_{(q,t)} &= [V_p W_s]^* V_p W_s [V_q W_t]^* V_q W_t \\ &= [W_s V_p]^* V_p W_s [W_t V_q]^* V_q W_t \\ &= V_p^* W_s^* V_p W_s V_q^* W_t^* V_q W_t \\ &= V_p^* V_p W_s^* V_q^* W_s V_q W_t^* W_t \\ &= V_p^* V_p V_q^* W_s^* V_q W_s W_t^* W_t \\ &= [V_p^* V_p V_q^* V_q][W_s^* W_s W_t^* W_t]. \end{aligned} \tag{5.3}$$

If

$$(P \times S)(p,s) \cap (P \times S)(q,t) = (P \times S)(z,r) = Pz \times Sr$$

for some $(z,r) \in P \times S$, then it follows by (5.1) (for the left hand side in above) that

$$(Pp \cap Pq) \times (Ss \cap St) = Pz \times Sr.$$

Thus, we must have

$$Pp \cap Pq = Pz \text{ and } Ss \cap St = Sr,$$

and hence, for (5.3), we get

$$\begin{aligned} U^*_{(p,s)}U_{(p,s)}U^*_{(q,t)}U_{(q,t)} &= [V_p^* V_p V_q^* V_q][W_s^* W_s W_t^* W_t] \\ &= V_z^* V_z W_r^* W_r \\ &= V_z^* W_r^* V_z W_r \\ &= [W_r V_z]^* U_{(z,r)} \\ &= [V_z W_r]^* U_{(z,r)} = U^*_{(z,r)}U_{(z,r)}. \end{aligned}$$

If $(P \times S)(p,s) \cap (P \times S)(q,t) = \emptyset$, then again, by (5.1), we get

$$(Pp \cap Pq) \times (Ss \cap St) = \emptyset.$$

It follows that

$$Pp \cap Pq = \emptyset \ \vee \ Ss \cap St = \emptyset,$$

which implies that,

$$V_p^*V_pV_q^*V_q = 0 \ \vee \ W_s^*W_sW_t^*W_t = 0.$$

Thus, for (5.3), we have

$$U_{(p,s)}^*U_{(p,s)}U_{(q,t)}^*U_{(q,t)} = [V_p^*V_pV_q^*V_q][W_s^*W_sW_t^*W_t] = 0.$$

A similar discussion shows that the representation U satisfies the Nica covariance condition (5.2), too. Therefore, U is a bicovariant partial-isometric representation of $P\times S$ on H satisfying $U_{(p,s)} = V_pW_s$.

Conversely, suppose that U is any bicovariant partial-isometric representation of $P\times S$ on a Hilbert space H. Define the maps

$$V : P \to B(H) \ \text{ and } \ W : S \to B(H)$$

by

$$V_p := U_{(p,e_S)} \ \text{ and } \ W_s := U_{(e_P,s)}$$

for all $p\in P$ and $s\in S$, respectively. It is easy to see that each V_p is a partial isometry as well as each W_s, and the maps V and W are (unital) semigroup homomorphisms. Next, we show that the presentation V is bicovariant, and we skip the proof for the presentation W as it follows similarly. To see that the presentation V satisfies the Nica covariance condition (3.1), firstly,

$$V_p^*V_pV_q^*V_q = U_{(p,e_S)}^*U_{(p,e_S)}U_{(q,e_S)}^*U_{(q,e_S)}. \tag{5.4}$$

Now, if $Pp\cap Pq = Pz$ for some $z\in P$, then it follows by (5.1) that

$$\begin{aligned}(P\times S)(p,e_S)\cap(P\times S)(q,e_S) &= (Pp\cap Pq)\times(S\cap S)\\ &= Pz\times S\\ &= Pz\times Se_S = (P\times S)(z,e_S).\end{aligned}$$

Therefore, since U is bicovariant, for (5.4), we have

$$\begin{aligned}V_p^*V_pV_q^*V_q &= U_{(p,e_S)}^*U_{(p,e_S)}U_{(q,e_S)}^*U_{(q,e_S)}\\ &= U_{(z,e_S)}^*U_{(z,e_S)} = V_z^*V_z.\end{aligned}$$

If $Pp\cap Pq = \emptyset$, then it follows again by (5.1) that

$$(P\times S)(p,e_S)\cap(P\times S)(q,e_S) = (Pp\cap Pq)\times(S\cap S) = \emptyset\times S = \emptyset.$$

Therefore, for (5.4), we get

$$V_p^*V_pV_q^*V_q = U_{(p,e_S)}^*U_{(p,e_S)}U_{(q,e_S)}^*U_{(q,e_S)} = 0,$$

as U is bicovariant. A similar discussion shows that the representation V satisfies the Nica covariance condition (5.2), too. We skip it here. Finally, as we obviously have

$$V_pW_s = W_sV_p = U_{(p,s)}, \tag{5.5}$$

it is only left to show that $V_p^*W_s = W_sV_p^*$ for all $p\in P$ and $s\in S$. To do so, we first need to recall that the product vw of two partial isometries v and w is a partial isometry if and only if v^*v commutes with ww^* (see [14, Lemma 2]).

This fact can be applied to the partial isometries V_p and W_s due to (5.5). Now, we have

$$\begin{aligned} V_p^* W_s &= V_p^*[V_p V_p^* W_s W_s^*] W_s \\ &= U^*_{(p,e_S)}[U_{(p,e_S)} U^*_{(p,e_S)} U_{(e_P,s)} U^*_{(e_P,s)}] U_{(e_P,s)}. \end{aligned} \tag{5.6}$$

Since,

$$\begin{aligned} (p,e_S)(P\times S)\cap(e_P,s)(P\times S) &= (pP\times S)\cap(P\times sS) \\ &= (pP\cap P)\times(S\cap sS) \\ &= pP\times sS = (p,s)(P\times S), \end{aligned}$$

it follows that

$$\begin{aligned} V_p V_p^* W_s W_s^* &= U_{(p,e_S)} U^*_{(p,e_S)} U_{(e_P,s)} U^*_{(e_P,s)} \\ &= U_{(p,s)} U^*_{(p,s)} \\ &= U_{(p,e_S)} U_{(e_P,s)} [U_{(e_P,s)} U_{(p,e_S)}]^* \\ &= U_{(p,e_S)} U_{(e_P,s)} U^*_{(p,e_S)} U^*_{(e_P,s)} = V_p W_s V_p^* W_s^*, \end{aligned} \tag{5.7}$$

as U is bicovariant. Therefore, by (5.6) and (5.7), we get

$$\begin{aligned} V_p^* W_s &= V_p^*[V_p V_p^* W_s W_s^*] W_s \\ &= V_p^*[V_p W_s V_p^* W_s^*] W_s \\ &= [V_p^* V_p W_s W_s^*] W_s V_p^* [V_p V_p^* W_s^* W_s] \\ &= [W_s W_s^* V_p^* V_p] W_s V_p^* [W_s^* W_s V_p V_p^*] \\ &= W_s W_s^* V_p^* [V_p W_s V_p^* W_s^*] W_s V_p V_p^*] \\ &= W_s W_s^* V_p^* [V_p V_p^* W_s W_s^*] W_s V_p V_p^*] \quad \text{(by (5.7))} \\ &= W_s W_s^* V_p^* W_s V_p V_p^* \\ &= W_s[(V_p W_s)^* (V_p W_s)] V_p^* \\ &= W_s[U^*_{(p,s)} U_{(p,s)}] V_p^*, \end{aligned}$$

and hence,

$$V_p^* W_s = W_s[U^*_{(p,s)} U_{(p,s)}] V_p^*. \tag{5.8}$$

Moreover, since similarly

$$(P\times S)(e_P,s)\cap(P\times S)(p,e_S) = (P\times S)(p,s),$$

we have

$$U^*_{(e_P,s)} U_{(e_P,s)} U^*_{(p,e_S)} U_{(p,e_S)} = U^*_{(p,s)} U_{(p,s)},$$

as U is bicovariant. By applying this to (5.8), we finally get

$$\begin{aligned} V_p^* W_s &= W_s[U^*_{(e_P,s)} U_{(e_P,s)} U^*_{(p,e_S)} U_{(p,e_S)}] V_p^* \\ &= [W_s W_s^* W_s][V_p^* V_p V_p^*] \\ &= W_s V_p^*. \end{aligned}$$

This completes the proof. □

Remark 5.3. If P is the positive cone of a totally ordered group G, then every partial-isometric representation V of P automatically satisfies the left Nica covariance condition (5.2), such that

$$V_x V_x^* V_y V_y^* = V_{\max\{x,y\}} V_{\max\{x,y\}}^* \quad \text{for all } x, y \in P.$$

Therefore, every partial-isometric representation of P is automatically bicovariant (see Remark 3.2).

Remark 5.4. Recall that a partial isometry V is called a *power partial isometry* if V^n is a partial isometry for every $n \in \mathbb{N}$. One can see that every power partial isometry V generates a partial-isometric representation of $\mathbb{N}$ such that $V_n := V^n$ for every $n \in \mathbb{N}$, and every partial-isometric representation V of $\mathbb{N}$ arises this way, which means that it is actually generated by the power partial isometry V_1. Now, it follows by Lemma 5.2 that U is a bicovariant partial-isometric representation of $\mathbb{N}^2$ if and only if there are $*$-commuting power partial isometries V and W such that $U_{(m,n)} = V^m W^n$ for every $(m, n) \in \mathbb{N}^2$.

Lemma 5.5. *Consider the left quasi-lattice ordered group $(\mathbb{F}_n, \mathbb{F}_n^+)$ (see Example 2.4). A partial-isometric representation V of $\mathbb{F}_n^+$ satisfies the left Nica covariance condition (5.2) if and only if the initial projections $V_{a_i} V_{a_i}^*$ and $V_{a_j} V_{a_j}^*$ have orthogonal ranges, where $1 \leq i, j \leq n$ such that $i \neq j$.*

Proof. We skip the proof as it follows by a similar discussion to the proof of Lemma 3.3. □

Lemma 5.6. *Consider the abelian lattice-ordered group $(\mathbb{Q}_+^\times, \mathbb{N}^\times)$ (see Example 2.3). A partial-isometric representation V of $\mathbb{N}^\times$ is bicovariant if and only if $V_m^* V_n = V_n V_m^*$ for every relatively prime pair (m, n) of elements in $\mathbb{N}^\times$.*

Proof. Suppose that V is a bicovariant partial-isometric representation of $\mathbb{N}^\times$. So, we have

$$V_x^* V_x V_y^* V_y = V_{x\vee y}^* V_{x\vee y}, \tag{5.9}$$

and

$$V_x V_x^* V_y V_y^* = V_{x\vee y} V_{x\vee y}^* \tag{5.10}$$

for all $x, y \in \mathbb{N}^\times$. If (m, n) is a relatively prime pair of elements in $\mathbb{N}^\times$, then $n \vee m = nm$, and hence, we have

$$\begin{aligned} V_n V_m^* &= V_n (V_n^* V_n V_m^* V_m) V_m^* \\ &= V_n V_{nm}^* V_{nm} V_m^* \quad \text{(by (5.9))} \\ &= V_n (V_n V_m)^* V_n V_m V_m^* \\ &= V_n V_m^* (V_n^* V_n V_m V_m^*), \end{aligned}$$

where in the bottom line, by [14, Lemma 2], $V_n^*V_n$ commutes with $V_mV_m^*$ as $V_nV_m = V_{nm}$. We therefore get

$$\begin{aligned} V_nV_m^* &= V_n(V_m^*V_mV_m^*)V_n^*V_n \\ &= V_n(V_m^*V_n^*)V_n \\ &= V_n(V_nV_m)^*V_n \\ &= V_n(V_{nm})^*V_n \\ &= V_n(V_{mn})^*V_n \\ &= V_n(V_mV_n)^*V_n \\ &= V_n(V_n^*V_m^*)V_n \\ &= (V_nV_n^*V_m^*V_m)V_m^*V_n \\ &= (V_m^*V_mV_nV_n^*)V_m^*V_n \\ &= V_m^*V_{mn}(V_mV_n)^*V_n = V_m^*V_{mn}V_{mn}^*V_n. \end{aligned}$$

Finally, in the bottom line, since

$$V_{mn}V_{mn}^* = V_mV_m^*V_nV_n^*$$

by (5.10), it follows that

$$V_nV_m^* = V_m^*V_mV_m^*V_nV_n^*V_n = V_m^*V_n.$$

Conversely, assume that V is a partial-isometric representation of $\mathbb{N}^\times$ such that $V_m^*V_n = V_nV_m^*$ for every relatively prime pair (m,n) of elements in $\mathbb{N}^\times$. We want to prove that V is bicovariant. To do so, we only show that V satisfies the equation (5.9) as the other equation, namely (5.10), follows by a similar discussion. Let $x, y \in \mathbb{N}^\times$. If x and y are relatively prime, then one can easily see that V satisfies the equation (5.9) as V_x commutes with V_y^*. Now, assume that x and y are not relatively prime, and therefore, we must have $x, y > 1$. By the prime factorization theorem, x and y can be uniquely written as

$$x = (p_1^{n_1}p_2^{n_2}\cdots p_k^{n_k})r \quad \text{and} \quad y = (p_1^{m_1}p_2^{m_2}\cdots p_k^{m_k})s,$$

where $p_1 < p_2 < ... < p_k$ are primes, each n_i and m_i is a positive integer, and $r, s \in \mathbb{N}^\times$ such that all pairs $(r, \prod_{i=1}^k p_i^{n_i})$, (r,s), $(r, \prod_{i=1}^k p_i^{m_i})$, $(s, \prod_{i=1}^k p_i^{m_i})$, and $(s, \prod_{i=1}^k p_i^{n_i})$ are relatively prime. We let $a = \prod_{i=1}^k p_i^{n_i}$ and $b = \prod_{i=1}^k p_i^{m_i}$ for convenience. Now, we have

$$\begin{aligned} V_x^*V_xV_y^*V_y &= V_{ar}^*V_{ar}V_{bs}^*V_{bs} \\ &= (V_aV_r)^*V_aV_r(V_bV_s)^*V_bV_s \\ &= V_r^*V_a^*V_a(V_rV_s^*)V_b^*V_bV_s \\ &= V_r^*V_a^*(V_aV_s^*)(V_rV_b^*)V_bV_s \quad \text{(since g.c.d}(r,s)=1) \\ &= V_r^*(V_a^*V_s^*)V_aV_b^*(V_rV_b)V_s \quad \text{(since g.c.d}(a,s)=\text{g.c.d}(r,b)=1) \\ &= V_r^*(V_sV_a)^*V_aV_b^*V_{rb}V_s \\ &= V_r^*(V_{sa})^*V_aV_b^*V_{br}V_s \\ &= V_r^*(V_{as})^*V_aV_b^*V_bV_rV_s \\ &= V_r^*(V_aV_s)^*V_aV_b^*V_bV_{rs} \end{aligned} \tag{5.11}$$

$$
\begin{aligned}
&= (V_r^*V_s^*)(V_a^*V_aV_b^*V_b)V_{rs}\\
&= (V_sV_r)^*(V_a^*V_aV_b^*V_b)V_{rs}\\
&= V_{sr}^*(V_a^*V_aV_b^*V_b)V_{rs} = V_{rs}^*(V_a^*V_aV_b^*V_b)V_{rs}\,.
\end{aligned}
$$

Then, in the bottom line, for $V_a^*V_aV_b^*V_b$, since the pairs (V_t, V_u) and (V_t^*, V_u^*) obviously commute for all $t, u \in \mathbb{N}^{\times}$, and g.c.d$(p^m, q^n) = 1$ for distinct primes p and q and positive integers m and n, we have

$$
\begin{aligned}
&V_a^*V_aV_b^*V_b\\
&= (V^*_{p_k^{n_k}}\cdots V^*_{p_1^{n_1}})(V_{p_1^{n_1}}\cdots V_{p_k^{n_k}})(V^*_{p_k^{m_k}}\cdots V^*_{p_1^{m_1}})(V_{p_1^{m_1}}\cdots V_{p_k^{m_k}})\\
&= (V^*_{p_1^{n_1}}\cdots V^*_{p_k^{n_k}})(V_{p_1^{n_1}}\cdots V_{p_k^{n_k}})(V^*_{p_1^{m_1}}\cdots V^*_{p_k^{m_k}})(V_{p_1^{m_1}}\cdots V_{p_k^{m_k}})\\
&= (V^*_{p_1^{n_1}}V_{p_1^{n_1}}V^*_{p_1^{m_1}}V_{p_1^{m_1}})(V^*_{p_2^{n_2}}\cdots V^*_{p_k^{n_k}})(V_{p_2^{n_2}}\cdots V_{p_k^{n_k}})(V^*_{p_2^{m_2}}\cdots V^*_{p_k^{m_k}})\\
&\quad (V_{p_2^{m_2}}\cdots V_{p_k^{m_k}})\\
&\vdots\\
&= (V^*_{p_1^{n_1}}V_{p_1^{n_1}}V^*_{p_1^{m_1}}V_{p_1^{m_1}})(V^*_{p_2^{n_2}}V_{p_2^{n_2}}V^*_{p_2^{m_2}}V_{p_2^{m_2}})\cdots(V^*_{p_k^{n_k}}V_{p_k^{n_k}}V^*_{p_k^{m_k}}V_{p_k^{m_k}}).
\end{aligned}
$$

Let $t_i = \max\{n_i, m_i\}$ for every $1 \le i \le k$. If $\max\{n_i, m_i\} = n_i$, then

$$
\begin{aligned}
V^*_{p_i^{n_i}}V_{p_i^{n_i}}V^*_{p_i^{m_i}}V_{p_i^{m_i}} &= V^*_{p_i^{n_i}}(V_{p_i^{(n_i-m_i)}p_i^{m_i}})V^*_{p_i^{m_i}}V_{p_i^{m_i}}\\
&= V^*_{p_i^{n_i}}V_{p_i^{(n_i-m_i)}}(V_{p_i^{m_i}}V^*_{p_i^{m_i}}V_{p_i^{m_i}})\\
&= V^*_{p_i^{n_i}}V_{p_i^{(n_i-m_i)}}V_{p_i^{m_i}}\\
&= V^*_{p_i^{n_i}}V_{p_i^{(n_i-m_i)}p_i^{m_i}}\\
&= V^*_{p_i^{n_i}}V_{p_i^{n_i}} = V^*_{p_i^{t_i}}V_{p_i^{t_i}}.
\end{aligned}
$$

If $\max\{n_i, m_i\} = m_i$, then

$$
\begin{aligned}
V^*_{p_i^{n_i}}V_{p_i^{n_i}}V^*_{p_i^{m_i}}V_{p_i^{m_i}} &= V^*_{p_i^{n_i}}V_{p_i^{n_i}}(V_{p_i^{(m_i-n_i)}p_i^{n_i}})^*V_{p_i^{m_i}}\\
&= V^*_{p_i^{n_i}}V_{p_i^{n_i}}(V_{p_i^{(m_i-n_i)}}V_{p_i^{n_i}})^*V_{p_i^{m_i}}\\
&= (V^*_{p_i^{n_i}}V_{p_i^{n_i}}V^*_{p_i^{n_i}})V^*_{p_i^{(m_i-n_i)}}V_{p_i^{m_i}}\\
&= V^*_{p_i^{n_i}}V^*_{p_i^{(m_i-n_i)}}V_{p_i^{m_i}}\\
&= (V_{p_i^{(m_i-n_i)}}V_{p_i^{n_i}})^*V_{p_i^{m_i}}\\
&= (V_{p_i^{(m_i-n_i)}p_i^{n_i}})^*V_{p_i^{m_i}}\\
&= V^*_{p_i^{m_i}}V_{p_i^{m_i}} = V^*_{p_i^{t_i}}V_{p_i^{t_i}}.
\end{aligned}
$$

So, for every $1 \leq i \leq k$,

$$V^*_{p_i^{n_i}} V_{p_i^{n_i}} V^*_{p_i^{m_i}} V_{p_i^{m_i}} = V^*_{p_i^{t_i}} V_{p_i^{t_i}},$$

from which, it follows that

$$\begin{aligned}
V_a^* V_a V_b^* V_b &= (V^*_{p_1^{t_1}} V_{p_1^{t_1}})(V^*_{p_2^{t_2}} V_{p_2^{t_2}})(V^*_{p_3^{t_3}} V_{p_3^{t_3}}) \cdots (V^*_{p_k^{t_k}} V_{p_k^{t_k}}) \\
&= (V^*_{p_1^{t_1}} V^*_{p_2^{t_2}})(V_{p_1^{t_1}} V_{p_2^{t_2}})(V^*_{p_3^{t_3}} V_{p_3^{t_3}}) \cdots (V^*_{p_k^{t_k}} V_{p_k^{t_k}}) \\
&= (V^*_{p_1^{t_1}} V^*_{p_2^{t_2}} V^*_{p_3^{t_3}})(V_{p_1^{t_1}} V_{p_2^{t_2}} V_{p_3^{t_3}})(V^*_{p_4^{t_4}} V_{p_4^{t_4}}) \cdots (V^*_{p_k^{t_k}} V_{p_k^{t_k}}) \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
&= (V^*_{p_1^{t_1}} V^*_{p_2^{t_2}} V^*_{p_3^{t_3}} \cdots V^*_{p_k^{t_k}})(V_{p_1^{t_1}} V_{p_2^{t_2}} V_{p_3^{t_3}} \cdots V_{p_k^{t_k}}) \\
&= (V_{p_k^{t_k} \cdots p_3^{t_3} p_2^{t_2} p_1^{t_1}})^* (V_{p_1^{t_1} p_2^{t_2} p_3^{t_3} \cdots p_k^{t_k}}) \\
&= (V_{p_1^{t_1} p_2^{t_2} p_3^{t_3} \cdots p_k^{t_k}})^* (V_{p_1^{t_1} p_2^{t_2} p_3^{t_3} \cdots p_k^{t_k}}) \\
&= (V_{\prod_{i=1}^k p_i^{t_i}})^* (V_{\prod_{i=1}^k p_i^{t_i}}) = V^*_{a \vee b} V_{a \vee b}.
\end{aligned} \tag{5.12}$$

Finally, by applying (5.12) to (5.11), since $x \vee y = (\prod_{i=1}^k p_i^{t_i}) rs = (a \vee b) rs$, we get

$$\begin{aligned}
V_x^* V_x V_y^* V_y &= V_{rs}^* (V^*_{a \vee b} V_{a \vee b}) V_{rs} \\
&= (V_{a \vee b} V_{rs})^* (V_{a \vee b} V_{rs}) \\
&= (V_{(a \vee b) rs})^* (V_{(a \vee b) rs}) = V^*_{x \vee y} V_{x \vee y}.
\end{aligned}$$

So, we are done. □

Definition 5.7. Let (A, P, α) be a dynamical system, in which P and P^o are both left LCM semigroups. The action α is called *left-Nica covariant* if it satisfies

$$\overline{\alpha}_x(1) \overline{\alpha}_y(1) = \begin{cases} \overline{\alpha}_z(1) & \text{if } xP \cap yP = zP, \\ 0 & \text{if } xP \cap yP = \emptyset. \end{cases} \tag{5.13}$$

We should mention that the above definition is well-defined. This is due to the fact that if $tP = xP \cap yP = zP$, then there is an invertible element u of P such that $t = zu$. Since u is invertible, α_u becomes an automorphism of A, and hence $\overline{\alpha}_u(1) = 1$. So, it follows that

$$\overline{\alpha}_t(1) = \overline{\alpha}_{zu}(1) = \overline{\alpha}_z(\overline{\alpha}_u(1)) = \overline{\alpha}_z(1).$$

Remark 5.8. Let (A, P, α) be a dynamical system, in which P and P^o are both left LCM semigroups, and the action α is left Nica-covariant. If (π, V) is covariant partial-isometric of the system, then the representation V satisfies the left Nica-covariance condition (5.2). One can easily see this by applying the equation $V_x V_x^* = \overline{\pi}(\overline{\alpha}_x(1))$ (see Lemma 4.2). Thus, the representation V is actually bicovariant.

Let us also recall that for C^*-algebras A and B, there are nondegenerate homomorphisms

$$k_A : A \to \mathcal{M}(A \otimes_{\max} B) \text{ and } k_B : B \to \mathcal{M}(A \otimes_{\max} B)$$

such that

$$k_A(a)k_B(b) = k_B(b)k_A(a) = a \otimes b$$

for all $a \in A$ and $b \in B$ (see [27, Theorem B. 27]). Moreover, One can see that the extensions $\overline{k_A}$ and $\overline{k_B}$ of the nondegenerate homomorphisms k_A and k_B, respectively, have also commuting ranges. Therefore, there is a homomorphism

$$\overline{k_A} \otimes_{\max} \overline{k_B} : \mathcal{M}(A) \otimes_{\max} \mathcal{M}(B) \to \mathcal{M}(A \otimes_{\max} B),$$

which is the identity map on $A \otimes_{\max} B$ (see [18, Remark 2.2]).

Theorem 5.9. *Suppose that the unital semigroups P, P^o, S, and S^o are all left LCM. Let (A, P, α) and (B, S, β) be dynamical systems in which the actions α and β are both left Nica-covariant. Then, we have the following isomorphism:*

$$(A \otimes_{max} B) \times^{\text{piso}}_{\alpha\otimes\beta} (P \times S) \simeq (A \times^{\text{piso}}_{\alpha} P) \otimes_{max} (B \times^{\text{piso}}_{\beta} S). \tag{5.14}$$

Proof. Let the triples $(A \times^{\text{piso}}_{\alpha} P, i_A, i_P)$ and $(B \times^{\text{piso}}_{\beta} S, i_B, i_S)$ be the partial-isometric crossed products of the dynamical systems (A, P, α) and (B, S, β), respectively. Suppose that $(k_{A\times_\alpha P}, k_{B\times_\beta S})$ is the canonical pair of the algebras $A \times^{\text{piso}}_{\alpha} P$ and $B \times^{\text{piso}}_{\beta} S$ into the multiplier algebra $\mathcal{M}((A \times^{\text{piso}}_{\alpha} P) \otimes_{\max} (B \times^{\text{piso}}_{\beta} S))$. Define the map

$$j_{A\otimes_{\max}B} : A \otimes_{\max} B \to (A \times^{\text{piso}}_{\alpha} P) \otimes_{\max} (B \times^{\text{piso}}_{\beta} S)$$

by $j_{A\otimes_{\max}B} := i_A \otimes_{\max} i_B$ (see [27, Lemma B. 31]), and therefore, we have

$$j_{A\otimes_{\max}B}(a \otimes b) = i_A(a) \otimes i_B(b) = k_{A\times_\alpha P}(i_A(a))k_{B\times_\beta S}(i_B(b))$$

for all $a, b \in A$. Also, define a map

$$j_{P\times S} : P \times S \to \mathcal{M}((A \times^{\text{piso}}_{\alpha} P) \otimes_{\max} (B \times^{\text{piso}}_{\beta} S))$$

by

$$j_{P\times S}(x, t) = \overline{k_{A\times_\alpha P}} \otimes_{\max} \overline{k_{B\times_\beta S}}(i_P(x) \otimes i_S(t)) = \overline{k_{A\times_\alpha P}}(i_P(x))\overline{k_{B\times_\beta S}}(i_S(t))$$

for all $(x, t) \in P \times S$. We claim that the triple

$$\big((A \times^{\text{piso}}_{\alpha} P) \otimes_{\max} (B \times^{\text{piso}}_{\beta} S), j_{A\otimes_{\max}B}, j_{P\times S}\big)$$

is a partial-isometric crossed product of the system $(A \otimes_{\max} B, P \times S, \alpha \otimes \beta)$. To prove our claim, first note that, since the homomorphisms i_A and i_B are nondegenerate, so is the homomorphism $j_{A\otimes_{\max}B}$. Next, we show that the map $j_{P\times S}$ is a bicovariant partial-isometric representation of $P \times S$. To do so, first note that, since the actions α and β are left Nica-covariant, the representations i_P and i_S are bicovariant (see Remark 5.8). It follows that the maps

$$j_P : P \to \mathcal{M}((A \times_\alpha^{\text{piso}} P) \otimes_{\max} (B \times_\beta^{\text{piso}} S))$$

and

$$j_S : S \to \mathcal{M}((A \times_\alpha^{\text{piso}} P) \otimes_{\max} (B \times_\beta^{\text{piso}} S))$$

given by compositions

$$P \xrightarrow{i_P} \mathcal{M}(A \times_\alpha^{\text{piso}} P) \xrightarrow{\overline{k_{A\times_\alpha P}}} \mathcal{M}((A \times_\alpha^{\text{piso}} P) \otimes_{\max} (B \times_\beta^{\text{piso}} S))$$

and

$$S \xrightarrow{i_S} \mathcal{M}(B \times_\beta^{\text{piso}} S) \xrightarrow{\overline{k_{B\times_\beta S}}} \mathcal{M}((A \times_\alpha^{\text{piso}} P) \otimes_{\max} (B \times_\beta^{\text{piso}} S)),$$

respectively, are bicovariant partial-isometric representations of P and S in the multiplier algebra $\mathcal{M}((A \times_\alpha^{\text{piso}} P) \otimes_{\max} (B \times_\beta^{\text{piso}} S))$. Moreover, since the homomorphisms $\overline{k_{A\times_\alpha P}}$ and $\overline{k_{B\times_\beta S}}$ have commuting ranges, each $j_P(x)$ $*$-commutes with each $j_S(t)$. Therefore, as

$$j_{P\times S}(x,t) = \overline{k_{A\times_\alpha P}}(i_P(x))\overline{k_{B\times_\beta S}}(i_S(t)) = j_P(x)j_S(t),$$

it follows by Lemma (5.2) that the map $j_{P\times S}$ must be a bicovariant partial-isometric representation of $P\times S$. Moreover, by using the covariance equations of the pairs (i_A, i_P) and (i_B, i_S), and the commutativity of the ranges of the maps $\overline{k_{A\times_\alpha P}}$ and $\overline{k_{B\times_\beta S}}$, one can see that the pair $(j_{A\otimes_{\max}B}, j_{P\times S})$ satisfies the covariance equations

$$j_{A\otimes_{\max}B}((\alpha\otimes\beta)_{(x,t)}(a\otimes b)) = j_{P\times S}(x,t)j_{A\otimes_{\max}B}(a\otimes b)j_{P\times S}(x,t)^*$$

and

$$j_{P\times S}(x,t)^* j_{P\times S}(x,t) j_{A\otimes_{\max}B}(a\otimes b) = j_{A\otimes_{\max}B}(a\otimes b) j_{P\times S}(x,t)^* j_{P\times S}(x,t).$$

Next, suppose that the pair (π, U) is covariant partial-isometric representation of $(A\otimes_{\max} B, P\times S, \alpha\otimes\beta)$ on a Hilbert space H. We want to get a nondegenerate representation $\pi\times U$ of $(A \times_\alpha^{\text{piso}} P) \otimes_{\max} (B \times_\beta^{\text{piso}} S)$ such that

$$(\pi\times U)\circ j_{A\otimes_{\max}B} = \pi \quad\text{and}\quad (\overline{\pi\times U})\circ j_{P\times S} = U.$$

Let (k_A, k_B) be the canonical pair of the C^*-algebras A and B into the multiplier algebra $\mathcal{M}(A\otimes_{\max} B)$. The compositions

$$A \xrightarrow{k_A} \mathcal{M}(A\otimes_{\max} B) \xrightarrow{\overline{\pi}} B(H)$$

and

$$B \xrightarrow{k_B} \mathcal{M}(A\otimes_{\max} B) \xrightarrow{\overline{\pi}} B(H)$$

give us the nondegenerate representations π_A and π_B of A and B on H with commuting ranges, respectively. This is due to the fact that the ranges of i_A and i_B commute (see also [27, Corollary B. 22]). Also, define the maps

$$V : P \to B(H) \text{ and } W : S \to B(H)$$

by

$$V_x := U_{(x,e_S)} \text{ and } W_t := U_{(e_P,t)}$$

for all $x \in P$ and $t \in S$, respectively. Since the representation U already satisfies the right Nica covariance condition (3.1), if we show that it satisfies the left Nica covariance condition (5.2), too, then it is bicovariant. Therefore, it follows by Lemma 5.2 that the maps V and W are bicovariant partial-isometric representations such that each V_x $*$-commutes with each W_t for all $x \in P$ and $t \in S$. By Remark 5.8, we only need to verify that the action $\alpha \otimes \beta$ in the system $(A \otimes_{\max} B, P \times S, \alpha \otimes \beta)$ is left Nica-covariant. Firstly, we have

$$\begin{aligned} \overline{(\alpha \otimes \beta)}_{(x,t)} \circ (\overline{k_A} \otimes_{\max} \overline{k_B}) &= \overline{\alpha_x \otimes \beta_t} \circ (\overline{k_A} \otimes_{\max} \overline{k_B}) \\ &= (\overline{k_A} \circ \overline{\alpha}_x) \otimes_{\max} (\overline{k_B} \circ \overline{\beta}_t) \end{aligned} \tag{5.15}$$

for all $(x, t) \in P \times S$ (see [18, Lemma 2.3]). It follows that

$$\begin{aligned} &\overline{(\alpha \otimes \beta)}_{(x,r)}(1_{\mathcal{M}(A \otimes_{\max} B)}) \overline{(\alpha \otimes \beta)}_{(y,s)}(1_{\mathcal{M}(A \otimes_{\max} B)}) \\ &= \big[\overline{\alpha_x \otimes \beta_r}\big(\overline{k_A}(1_{\mathcal{M}(A)})\overline{k_B}(1_{\mathcal{M}(B)})\big)\big]\big[\overline{\alpha_y \otimes \beta_s}\big(\overline{k_A}(1_{\mathcal{M}(A)})\overline{k_B}(1_{\mathcal{M}(B)})\big)\big] \\ &= \overline{k_A}\big(\overline{\alpha}_x(1_{\mathcal{M}(A)})\big)\overline{k_B}\big(\overline{\beta}_r(1_{\mathcal{M}(B)})\big)\overline{k_A}\big(\overline{\alpha}_y(1_{\mathcal{M}(A)})\big)\overline{k_B}\big(\overline{\beta}_s(1_{\mathcal{M}(B)})\big) \\ &= \overline{k_A}\big(\overline{\alpha}_x(1_{\mathcal{M}(A)})\big)\overline{k_A}\big(\overline{\alpha}_y(1_{\mathcal{M}(A)})\big)\overline{k_B}\big(\overline{\beta}_r(1_{\mathcal{M}(B)})\big)\overline{k_B}\big(\overline{\beta}_s(1_{\mathcal{M}(B)})\big) \\ &= \overline{k_A}\big(\overline{\alpha}_x(1_{\mathcal{M}(A)})\overline{\alpha}_y(1_{\mathcal{M}(A)})\big)\overline{k_B}\big(\overline{\beta}_r(1_{\mathcal{M}(B)})\overline{\beta}_s(1_{\mathcal{M}(B)})\big), \end{aligned}$$

and hence,

$$\begin{aligned} &\overline{(\alpha \otimes \beta)}_{(x,r)}(1_{\mathcal{M}(A \otimes_{\max} B)}) \overline{(\alpha \otimes \beta)}_{(y,s)}(1_{\mathcal{M}(A \otimes_{\max} B)}) \\ &= \overline{k_A}\big(\overline{\alpha}_x(1_{\mathcal{M}(A)})\overline{\alpha}_y(1_{\mathcal{M}(A)})\big)\overline{k_B}\big(\overline{\beta}_r(1_{\mathcal{M}(B)})\overline{\beta}_s(1_{\mathcal{M}(B)})\big) \end{aligned} \tag{5.16}$$

for all $(x, r), (y, s) \in P \times S$. Now, if

$$(x, r)(P \times S) \cap (y, s)(P \times S) = (z, t)(P \times S)$$

for some $(z, t) \in P \times S$, then, since

$$xP \cap yP = zP \quad \text{and} \quad rS \cap sS = tS,$$

and the actions α and β are left Nica-covariant, for (5.16), we get

$$\begin{aligned} &\overline{(\alpha \otimes \beta)}_{(x,r)}(1_{\mathcal{M}(A \otimes_{\max} B)}) \overline{(\alpha \otimes \beta)}_{(y,s)}(1_{\mathcal{M}(A \otimes_{\max} B)}) \\ &= \overline{k_A}\big(\overline{\alpha}_z(1_{\mathcal{M}(A)})\big)\overline{k_B}\big(\overline{\beta}_t(1_{\mathcal{M}(B)})\big) \\ &= (\overline{k_A} \circ \overline{\alpha}_z) \otimes_{\max} (\overline{k_B} \circ \overline{\beta}_t)(1_{\mathcal{M}(A)} \otimes 1_{\mathcal{M}(B)}) \\ &= \overline{\alpha_z \otimes \beta_t}\big(\overline{k_A} \otimes_{\max} \overline{k_B}(1_{\mathcal{M}(A)} \otimes 1_{\mathcal{M}(B)})\big) \quad \text{[by (5.15)]} \\ &= \overline{(\alpha \otimes \beta)}_{(z,t)}(1_{\mathcal{M}(A \otimes_{\max} B)}). \end{aligned}$$

If $(x,r)(P\times S)\cap(y,s)(P\times S)=\emptyset$, then

$$xP\cap yP=\emptyset\ \vee\ rS\cap sS=\emptyset,$$

which implies that

$$\overline{\alpha}_x(1_{\mathcal{M}(A)})\overline{\alpha}_y(1_{\mathcal{M}(A)})=0\ \vee\ \overline{\beta}_r(1_{\mathcal{M}(B)})\overline{\beta}_s(1_{\mathcal{M}(B)})=0$$

as α and β are left Nica-covariant. Thus, for (5.16), we get

$$\overline{(\alpha\otimes\beta)}_{(x,r)}(1_{\mathcal{M}(A\otimes_{\max}B)})\overline{(\alpha\otimes\beta)}_{(y,s)}(1_{\mathcal{M}(A\otimes_{\max}B)})=0.$$

So, the action $\alpha\otimes\beta$ is left Nica-covariant.

Now, consider the pairs (π_A,V) and (π_B,W). They are indeed the covariant partial-isometric representations of the systems (A,P,α) and (B,S,β) on H, respectively. We only show this for (π_A,V) as the proof for (π_B,W) follows similarly. We have to show that the pair (π_A,V) satisfies the covariance equations (4.1). We have

$$\begin{aligned}
V_x\pi_A(a)V_x^* &= U_{(x,e_S)}\overline{\pi}(k_A(a))U_{(x,e_S)}^*\\
&= U_{(x,e_S)}\overline{\pi}(k_A(a))\overline{\pi}(1_{\mathcal{M}(A\otimes_{\max}B)})U_{(x,e_S)}^*\\
&= U_{(x,e_S)}\overline{\pi}(\overline{k_A}(a))\overline{\pi}(\overline{k_B}(1_{\mathcal{M}(B)}))U_{(x,e_S)}^*\\
&= U_{(x,e_S)}\overline{\pi}(\overline{k_A}(a)\overline{k_B}(1_{\mathcal{M}(B)}))U_{(x,e_S)}^*\\
&= \overline{\pi}\big(\overline{(\alpha\otimes\beta)}_{(x,e_S)}(\overline{k_A}(a)\overline{k_B}(1_{\mathcal{M}(B)}))\big)\\
&= \overline{\pi}\big(\overline{\alpha_x\otimes\beta_{e_S}}(\overline{k_A}(a)\overline{k_B}(1_{\mathcal{M}(B)}))\big)\\
&= \overline{\pi}\big(\overline{k_A}(\overline{\alpha}_x(a))\overline{k_B}(\overline{\beta}_{e_S}(1_{\mathcal{M}(B)}))\big)\ \text{[by (5.15)]}\\
&= \overline{\pi}\big(k_A(\alpha_x(a))\overline{k_B}(\mathrm{id}_{\mathcal{M}(B)}(1_{\mathcal{M}(B)}))\big)\\
&= \overline{\pi}\big(k_A(\alpha_x(a))\overline{k_B}(1_{\mathcal{M}(B)})\big)\\
&= \overline{\pi}\big(k_A(\alpha_x(a))1_{\mathcal{M}(A\otimes_{\max}B)}\big)\\
&= \overline{\pi}\big(k_A(\alpha_x(a))\big)=\pi_A(\alpha_x(a))
\end{aligned}$$

for all $a\in A$ and $x\in P$. Also, by a similar calculation using the covariance of the pair $(\overline{\pi},U)$, it follows that

$$\pi_A(a)V_x^*V_x=V_x^*V_x\pi_A(a).$$

Consequently, there are nondegenerate representations $\pi_A\times V$ and $\pi_B\times W$ of the algebras $A\times_\alpha^{\mathrm{piso}}P$ and $B\times_\beta^{\mathrm{piso}}S$ on H, respectively, such that

$$(\pi_A\times V)\circ i_A=\pi_A,\ \overline{\pi_A\times V}\circ i_P=V\ \text{ and }\ (\pi_B\times W)\circ i_B=\pi_B,\ \overline{\pi_B\times W}\circ i_S=W.$$

Next, we aim to show that the representations $\pi_A\times V$ and $\pi_B\times W$ have commuting ranges, from which, it follows that there is a representation $(\pi_A\times V)\otimes_{\max}(\pi_B\times W)$ of $(A\times_\alpha^{\mathrm{piso}}P)\otimes_{\max}(B\times_\beta^{\mathrm{piso}}S)$, which is the desired (nondegenerate) representation $\pi\times U$. So, it suffices to see that the pairs (π_A,π_B), (V,W), (V^*,W), (π_A,W), and (π_B,V) all have commuting ranges. We already saw that this is indeed true for the first three pairs. So, we compute to show that this is also true for the pair (π_A, W) and skip the similar computation for the pair (π_B, V). We have

$$\begin{aligned}
W_t\pi_A(a) &= U_{(e_P,t)}\overline{\pi}(k_A(a)) \\
&= U_{(e_P,t)}\overline{\pi}(k_A(a)1_{\mathcal{M}(A\otimes_{\max}B)}) \\
&= U_{(e_P,t)}\overline{\pi}(\overline{k_A}(a)\overline{k_B}(1_{\mathcal{M}(B)})) \\
&= \overline{\pi}\big(\overline{(\alpha\otimes\beta)_{(e_P,t)}}(\overline{k_A}(a)\overline{k_B}(1_{\mathcal{M}(B)}))\big)U_{(e_P,t)} \\
&= \overline{\pi}\big(\overline{\alpha_{e_P}\otimes\beta_t}(\overline{k_A}(a)\overline{k_B}(1_{\mathcal{M}(B)}))\big)U_{(e_P,t)} \\
&= \overline{\pi}\big(\overline{k_A}(\overline{\alpha}_{e_P}(a))\overline{k_B}(\overline{\beta}_t(1_{\mathcal{M}(B)}))\big)U_{(e_P,t)} \quad \text{[by (5.15)]} \\
&= \overline{\pi}\big(k_A(\mathrm{id}_A(a))\overline{k_B}(\overline{\beta}_t(1_{\mathcal{M}(B)}))\big)U_{(e_P,t)} \\
&= (\overline{\pi}\circ k_A)(a)(\overline{\pi}\circ\overline{k_B})\big(\overline{\beta}_t(1_{\mathcal{M}(B)})\big)U_{(e_P,t)} \\
&= \pi_A(a)\overline{\pi_B}\big(\overline{\beta}_t(1_{\mathcal{M}(B)})\big)W_t \\
&= \pi_A(a)W_tW_t^*W_t \quad \text{[by the covariance of } (\overline{\pi_B}, W)] \\
&= \pi_A(a)W_t.
\end{aligned}$$

Thus, there is a representation $(\pi_A\times V)\otimes_{\max}(\pi_B\times W)$ of $(A\times_\alpha^{\mathrm{piso}} P)\otimes_{\max}(B\times_\beta^{\mathrm{piso}} S)$ on H such that

$$(\pi_A\times V)\otimes_{\max}(\pi_B\times W)(\xi\otimes\eta) = (\pi_A\times V)(\xi)(\pi_B\times W)(\eta)$$

for all $\xi\in(A\times_\alpha^{\mathrm{piso}} P)$ and $\eta\in(B\times_\beta^{\mathrm{piso}} S)$. Let

$$\pi\times U = (\pi_A\times V)\otimes_{\max}(\pi_B\times W),$$

which is nondegenerate as both representations $\pi_A\times V$ and $\pi_B\times W$ are. Then, we have

$$\begin{aligned}
\pi\times U(j_{A\otimes_{\max}B}(a\otimes b)) &= \pi\times U(i_A(a)\otimes i_B(b)) \\
&= (\pi_A\times V)(i_A(a))(\pi_B\times W)(i_B(b)) \\
&= \pi_A(a)\pi_B(b) \\
&= \overline{\pi}(k_A(a))\overline{\pi}(k_B(b)) \\
&= \overline{\pi}(k_A(a)k_B(b)) = \pi(a\otimes b).
\end{aligned}$$

To see $(\overline{\pi\times U})\circ j_{P\times S} = U$, we apply the equation

$$\begin{aligned}
&\overline{\pi\times U}\circ(\overline{k_{A\times_\alpha P}}\otimes_{\max}\overline{k_{B\times_\beta S}}) \\
&= \overline{(\pi_A\times V)\otimes_{\max}(\pi_B\times W)}\circ(\overline{k_{A\times_\alpha P}}\otimes_{\max}\overline{k_{B\times_\beta S}}) \\
&= \overline{(\pi_A\times V)}\otimes_{\max}\overline{(\pi_B\times W)},
\end{aligned}$$

which is valid by [18, Lemma 2.4]. Therefore, we have

$$\begin{aligned}
\overline{\pi\times U}(j_{P\times S}(x,t)) &= \overline{\pi\times U}\big(\overline{k_{A\times_\alpha P}}(i_P(x))\overline{k_{B\times_\beta S}}(i_S(t))\big) \\
&= \overline{\pi\times U}\big(\overline{k_{A\times_\alpha P}}\otimes_{\max}\overline{k_{B\times_\beta S}}(i_P(x)\otimes i_S(t))\big) \\
&= \overline{\pi\times U}\circ(\overline{k_{A\times_\alpha P}}\otimes_{\max}\overline{k_{B\times_\beta S}})(i_P(x)\otimes i_S(t)) \\
&= \overline{(\pi_A\times V)}\otimes_{\max}\overline{(\pi_B\times W)}(i_P(x)\otimes i_S(t)) \\
&= \overline{(\pi_A\times V)}(i_P(x))\overline{(\pi_B\times W)}(i_S(t)) \\
&= V_xW_t = U_{(x,e_S)}U_{(e_P,t)} = U_{(x,t)}.
\end{aligned}$$

Finally, as the algebras $A \times_{\alpha}^{\text{piso}} P$ and $B \times_{\beta}^{\text{piso}} S$ are spanned by the elements $i_P(x)^* i_A(a) i_P(y)$ and $i_S(r)^* i_B(b) i_S(t)$, respectively, the algebra

$$(A \times_{\alpha}^{\text{piso}} P) \otimes_{\max} (B \times_{\beta}^{\text{piso}} S)$$

is spanned by the elements

$$[i_P(x)^* i_A(a) i_P(y)] \otimes [i_S(r)^* i_B(b) i_S(t)],$$

where, by calculation, we have

$$[i_P(x)^* i_A(a) i_P(y)] \otimes [i_S(r)^* i_B(b) i_S(t)] = j_{P\times S}(x,r)^* j_{A\otimes_{\max} B}(a \otimes b) j_{P\times S}(y,t).$$

So, the triple

$$\big((A \times_{\alpha}^{\text{piso}} P) \otimes_{\max} (B \times_{\beta}^{\text{piso}} S), j_{A\otimes_{\max} B}, j_{P\times S}\big)$$

is a partial-isometric crossed product of the system $(A \otimes_{\max} B, P \times S, \alpha \otimes \beta)$. It thus follows that there is an isomorphism

$$\Gamma : \big((A \otimes_{\max} B) \times_{\alpha\otimes\beta}^{\text{piso}} (P \times S), i_{A\otimes_{\max} B}, i_{P\times S}\big) \to (A \times_{\alpha}^{\text{piso}} P) \otimes_{\max} (B \times_{\beta}^{\text{piso}} S)$$

such that

$$\begin{aligned}
&\Gamma\big(i_{P\times S}(x,r)^* i_{A\otimes_{\max} B}(a \otimes b) i_{P\times S}(y,t)\big) \\
&= j_{P\times S}(x,r)^* j_{A\otimes_{\max} B}(a \otimes b) j_{P\times S}(y,t) \\
&= [i_P(x)^* i_A(a) i_P(y)] \otimes [i_S(r)^* i_B(b) i_S(t)].
\end{aligned}$$

This completes the proof. □

Let P be a unital semigroup such that itself and the opposite semigroup P^{o} are both left LCM. For every $y \in P$, define a map $1_y : P \to \mathbb{C}$ by

$$1_y(x) = \begin{cases} 1 & \text{if } x \in yP, \\ 0 & \text{otherwise,} \end{cases}$$

which is the characteristic function of yP. Each 1_y is obviously a function in $\ell^\infty(P)$. Then, since P is right LCM, one can see that we have

$$1_x 1_y = \begin{cases} 1_z & \text{if } xP \cap yP = zP, \\ 0 & xP \cap yP = \emptyset. \end{cases}$$

Note that, if $\tilde{z}P = xP \cap yP = zP$, then there is an invertible element u of P such that $\tilde{z} = zu$. It therefore follows that $s \in zP$ if and only if $s \in \tilde{z}P$ for all $s \in P$, which implies that we must have $1_z = 1_{\tilde{z}}$. So, the above equation is well-defined. Also, we clearly have $1_y^* = 1_y$ for all $y \in P$. Therefore, if B_P is the C^*-subalgebra of $\ell^\infty(P)$ generated by the characteristic functions $\{1_y : y \in P\}$, then we have

$$B_P = \overline{\text{span}}\{1_y : y \in P\}.$$

Note that the algebra B_P is abelian and unital, whose unit element is 1_e which is a constant function on P with the constant value 1. One can see that, in fact, $1_u = 1_e$ for every $u \in P^*$. In addition, the shift on $\ell^\infty(P)$ induces an action on B_P by injective endomorphisms. More precisely, for every $x \in P$, the map $\alpha_x : \ell^\infty(P) \to \ell^\infty(P)$ defined by

$$\alpha_x(f)(t) = \begin{cases} f(r) & \text{if } t = xr \text{ for some } r \in P \ (\equiv t \in xP), \\ 0 & \text{otherwise,} \end{cases}$$

for every $f \in \ell^\infty(P)$ is an injective endomorphism of $\ell^\infty(P)$. Also, the map

$$\alpha : P \to \mathrm{End}(\ell^\infty(P)); \ \ x \mapsto \alpha_x$$

is a semigroup homomorphism such that $\alpha_e = \mathrm{id}$, which gives us an action of P on $\ell^\infty(P)$ by injective endomorphisms. Since $\alpha_x(1_y) = 1_{xy}$ for all $x, y \in P$, $\alpha_x(B_P) \subset B_P$, and therefore the restriction of the action α to B_P gives an action

$$\tau : P \to \mathrm{End}(B_P)$$

by injective endomorphisms such that $\tau_x(1_y) = 1_{xy}$ for all $x, y \in P$. Note that $\tau_x(1_e) = 1_x \neq 1_e$ for all $x \in P \backslash P^*$. Consequently, we obtain a dynamical system (B_P, P, τ), for which, we want to describe the corresponding partial-isometric crossed product $(B_P \times_\tau^{\mathrm{piso}} P, i_{B_P}, i_P)$. More precisely, we want to show that the algebra $B_P \times_\tau^{\mathrm{piso}} P$ is universal for bicovariant partial-isometric representations of P. Once, we have done this, it would be proper to denote $B_P \times_\tau^{\mathrm{piso}} P$ by $C^*_{\mathrm{bicov}}(P)$. So, this actually generalizes [12, Proposition 9.6] from the positive cones of quasi lattice-ordered groups (in the sense of Nica [24]) to LCM semigroups.

To start, for our purpose, we borrow some notations from quasi lattice-ordered groups. For every $x, y \in P$, if $xP \cap yP = zP$ for some $z \in P$, which means that z is a right least common multiple of x and y, then we denote such an element z by $x \vee_{\mathrm{lt}} y$, which may not be unique. If $xP \cap yP = \emptyset$, then we denote $x \vee_{\mathrm{lt}} y = \infty$. Note that we are using the notation $\vee_{\mathrm{lt}}$ to indicate that we are treating P as a right LCM semigroup. But if we are treating P as a left LCM semigroup, then we use the notation $\vee_{\mathrm{rt}}$ to distinguish it from $\vee_{\mathrm{lt}}$. Moreover, if $F = \{x_1, x_2, ..., x_n\}$ is any finite subset of P, then σF is written for $x_1 \vee_{\mathrm{lt}} x_2 \vee_{\mathrm{lt}} ... \vee_{\mathrm{lt}} x_n$. Therefore, if $\bigcap_{x \in F} xP = \bigcap_{i=1}^n x_i P \neq \emptyset$, σF denotes an element in

$$\{y : \bigcap_{i=1}^n x_i P = yP\},$$

and if $\bigcap_{i=1}^n x_i P = \emptyset$, then $\sigma F = \infty$.

Lemma 5.10. *Let P be a unital semigroup such that itself and the opposite semigroup P^o are both left LCM. Let V be any bicovariant partial-isometric representation of P on a Hilbert space H. Then:*

(i) *there is a (unital) representation π_V of B_P on H such that $\pi_V(1_x) = V_x V_x^*$ for all $x \in P$;*
(ii) *the pair (π_V, V) is a covariant partial-isometric representation of (B_P, P, τ) on H.*

Proof. We prove (i) by extending [17, Proposition 1.3 (2)] to LCM semigroups for the particular family

$$\{L_x := V_x V_x^* : x \in P\}$$

of projections, which satifies

$$L_e = 1 \quad \text{and} \quad L_x L_y = L_{x \vee_{\mathrm{lt}} y},$$

where $L_\infty = 0$. To do so, we make some adjustment to the proof of [17, Proposition 1.3 (2)]. Define a map

$$\pi : \operatorname{span}\{1_x : x \in P\} \to B(H)$$

by

$$\pi\Big(\sum_{i=1}^{n} \lambda_{x_i} 1_{x_i}\Big) = \sum_{i=1}^{n} \lambda_{x_i} L_{x_i} = \sum_{i=1}^{n} \lambda_{x_i} V_{x_i} V_{x_i}^*,$$

where $\lambda_{x_i} \in \mathbb{C}$ for each i. It is obvious that π is linear. Next, we show that

$$\Big\| \sum_{x \in F} \lambda_x L_x \Big\| \leq \Big\| \sum_{x \in F} \lambda_x 1_x \Big\| \tag{5.17}$$

for any finite subset F of P. So, it follows that the map π is a well-defined bounded linear map, and therefore, it extends to a bounded linear map of B_p in $B(H)$ such that $1_x \mapsto V_x V_x^*$ for all $x \in P$. To see (5.17), we exactly follow [17, Lemma 1.4] to obtain an expression for the norm of the forms $\sum_{x \in F} \lambda_x L_x$ by using an appropriate set of mutually orthogonal projections. So, if F is any finite subset of P, then for every nonempty proper subset A of F, take $Q_A^L = \Pi_{x \in F \setminus A}(L_{\sigma A} - L_{\sigma A \vee_{\mathrm{lt}} x})$. Moreover, let $Q_\emptyset^L = \Pi_{x \in F}(1 - L_x)$ and $Q_F^L = \Pi_{x \in F} L_x = L_{\sigma F}$. Then, exactly by following the proof of [17, Lemma 1.4], we can show that $\{Q_A^L : A \subset F\}$ is a decomposition of the identity into mutually orthogonal projections, such that

$$\sum_{x \in F} \lambda_x L_x = \sum_{A \subset F} \Big(\sum_{x \in A} \lambda_x \Big) Q_A^L \tag{5.18}$$

and

$$\Big\| \sum_{x \in F} \lambda_x L_x \Big\| = \max \Big\{ \Big| \sum_{x \in A} \lambda_x \Big| : A \subset F \text{ and } Q_A^L \neq 0 \Big\}. \tag{5.19}$$

Also, we have a fact similar to [17, Remark 1.5]. Suppose that, similarly, $\{Q_A : A \subset F\}$ is the decomposition of the identity corresponding to the family of projections $\{1_x : x \in F\}$. Consider

$$Q_A = \Pi_{x \in F \setminus A}(1_{\sigma A} - 1_{\sigma A \vee_{\mathrm{lt}} x})$$

for any nonempty proper subset $A \subset F$. If $\sigma A \in x_0 P$ for some $x_0 \in F \setminus A$, then $(\sigma A)P = \bigcap_{y \in A} yP \subset x_0 P$ which implies that $(\sigma A)P \cap x_0 P = (\sigma A)P$. So, we have $\sigma A \vee_{\mathrm{lt}} x_0 = \sigma A$, and therefore,

$$1_{\sigma A} - 1_{\sigma A \vee_{\mathrm{lt}} x_0} = 1_{\sigma A} - 1_{\sigma A} = 0.$$

Thus, we get $Q_A = 0$. Note that when we say $\sigma A \in x_0 P$ (for some $x_0 \in F \backslash A$), it means that at least one element in

$$\{z : \bigcap_{y \in A} yP = zP\} \tag{5.20}$$

belongs to $x_0 P$, from which, it follows that all elements in (5.20) must belong to $x_0 P$. This is due to the fact that if $z, \tilde{z}$ are in (5.20), then $\tilde{z} = zu$ for some invertible element u of P. Now, conversely, suppose that

$$0 = Q_A = \Pi_{x \in F \backslash A}(1_{\sigma A} - 1_{\sigma A \vee_{\mathrm{lt}} x}).$$

This implies that we must have $Q_A(r) = 0$ for all $r \in P$, in particular, when $r = \sigma A$, and hence

$$0 = Q_A(r) = \Pi_{x \in F \backslash A}\big(1_r(r) - 1_{r \vee_{\mathrm{lt}} x}(r)\big) = \Pi_{x \in F \backslash A}\big(1 - 1_{r \vee_{\mathrm{lt}} x}(r)\big).$$

Therefore, there is at least one element $x_0 \in F \backslash A$ such that $1_{r \vee_{\mathrm{lt}} x_0}(r) = 1$, which implies that we must have $r \in (r \vee_{\mathrm{lt}} x_0)P = rP \cap x_0 P$. It follows that $\sigma A = r \in x_0 P$ and therefore, $r \vee_{\mathrm{lt}} x_0 = \sigma A \vee_{\mathrm{lt}} x_0 = \sigma A$. Consequently, we have $Q_A \neq 0$ if and only if

$$A = \{x \in F : \sigma A \in xP\}.$$

Eventually, we conclude that if $Q_A^L \neq 0$, then $Q_A \neq 0$. This is due to the fact that, if $Q_A = 0$, then there is $x_0 \in F \backslash A$ such that $\sigma A \in x_0 P$. Therefore, we get $Q_A^L = 0$ as the factor $L_{\sigma A} - L_{\sigma A \vee_{\mathrm{lt}} x_0}$ in Q_A^L becomes zero. Thus, it follows that

$$\Big\{\Big|\sum_{x \in A} \lambda_x\Big| : A \subset F \text{ and } Q_A^L \neq 0\Big\} \subset \Big\{\Big|\sum_{x \in A} \lambda_x\Big| : A \subset F \text{ and } Q_A \neq 0\Big\},$$

which implies that the inequality (5.17) is valid for any finite subset F of P. So, we have a bounded linear map $\pi_V : B_p \to B(H)$ (the extension of π) such that $\pi_V(1_x) = V_x V_x^*$ for all $x \in P$. Furthermore, since

$$\pi_V(1_x)\pi_V(1_y) = V_x V_x^* V_y V_y^* = V_{x \vee_{\mathrm{lt}} y} V_{x \vee_{\mathrm{lt}} y}^* = \pi_V(1_{x \vee_{\mathrm{lt}} y}) = \pi_V(1_x 1_y),$$

and obviously, $\pi_V(1_x)^* = \pi_V(1_x) = \pi_V(1_x^*)$, it follows that the map π_V is actually a $*$-homomorphism, which is clearly unital. This completes the proof of (i).

To see (ii), it is enough to show that the pair (π_V, V) satisfies the covariance equations (4.1) on the spanning elements of B_P. For all $x, y \in P$, we have

$$\pi_V(\tau_x(1_y)) = 1_{xy} = V_{xy} V_{xy}^* = V_x V_y [V_x V_y]^* = V_x V_y V_y^* V_x^* = V_x \pi_V(1_y) V_x^*.$$

Also, since the product of partial isometries V_x and V_y is a partial isometry, namely, $V_x V_y = V_{xy}$, by [14, Lemma 2], each $V_x^* V_x$ commutes with each $V_y V_y^*$. Hence, we have

$$V_x^* V_x \pi_V(1_y) = V_x^* V_x V_y V_y^* = V_y V_y^* V_x^* V_x = \pi_V(1_y) V_x^* V_x.$$

So, we are done with (ii), too. □

Proposition 5.11. *Suppose that P is a unital semigroup such that itself and the opposite semigroup P^o are both left LCM. Then, the map*

$$i_P : P \to B_P \times_\tau^{\text{piso}} P$$

is a bicovariant partial-isometric representation of P whose range generates the C^*-algebra $B_P \times_\tau^{\text{piso}} P$. Moreover, for every bicovariant partial-isometric representation V of P, there is a (unital) representation V_* of $B_P \times_\tau^{\text{piso}} P$ such that $V_* \circ i_P = V$.

Proof. To see that i_P is a bicovariant partial-isometric representation of P, we only need to show that it satisfies the left Nica covariance condition (5.2). Since

$$i_{B_P}(1_y) = i_{B_P}(\tau_y(1_e)) = i_P(y) i_{B_P}(1_e) i_P(y)^* = i_P(y) i_P(y)^*$$

for all $y \in P$, it follows that i_P indeed satisfies (5.2). Then, as the elements $\{1_y : y \in P\}$ generate the algebra B_P, the C^*-algebra $B_P \times_\tau^{\text{piso}} P$ is generated by the elements

$$i_{B_P}(1_y) i_P(x) = i_P(y) i_P(y)^* i_P(x),$$

which implies that $i_P(P)$ generates $B_P \times_\tau^{\text{piso}} P$.

Suppose that now V is a bicovariant partial-isometric representation of P on a Hilbert space H. Then, by Lemma 5.10, there is a covariant partial-isometric representation (π_V, V) of (B_P, P, τ) on H, such that $\pi_V(1_x) = V_x V_x^*$ for all $x \in P$. The corresponding (unital) representation $\pi_V \times V$ of $(B_P \times_\tau^{\text{piso}} P, i_{B_P}, i_P)$ on H is the desired representation V_* which satisfies $V_* \circ i_P = V$. □

So, as we mentioned earlier, we denote the algebra $B_P \times_\tau^{\text{piso}} P$ by $C^*_{\text{bicov}}(P)$, which is universal for bicovariant partial-isometric representations of P.

The following remark contains some point which will be applied in the next corollary and also in the next section.

Remark 5.12. Suppose that P is a left LCM semigroup. Let (A, P, α) and (B, P, β) be dynamical systems, and $\psi : A \to B$ a nondegenerate homomorphism such that $\psi \circ \alpha_x = \beta_x \circ \psi$ for all $x \in P$. Suppose that $(A \times_\alpha^{\text{piso}} P, i)$ and $(B \times_\beta^{\text{piso}} P, j)$ are the partial-isometric crossed products of the systems (A, P, α) and (B, P, β), respectively. Now, one can see that the pair $(j_B \circ \psi, j_P)$ is covariant partial-isometric representation of (A, P, α) in the algebra $B \times_\beta^{\text{piso}} P$. Hence, there is a nondegenerate homomorphism

$$\psi \times P := [(j_B \circ \psi) \times j_P] : A \times_\alpha^{\text{piso}} P \to B \times_\beta^{\text{piso}} P$$

such that

$$(\psi \times P) \circ i_A = j_B \circ \psi \quad \text{and} \quad \overline{\psi \times P} \circ i_P = j_P.$$

One can see that if ψ is an isomorphism, so is $\psi \times P$.

Corollary 5.13. *Suppose that the unital semigroups P, P^o, S, and S^o are all left LCM. Then,*

$$C^*_{\text{bicov}}(P \times S) \simeq C^*_{\text{bicov}}(P) \otimes_{\max} C^*_{\text{bicov}}(S). \tag{5.21}$$

Proof. Corresponding to the pairs (P, P^o) and (S, S^o) we have the dynamical systems (B_P, P, τ) and (B_S, S, β) along with their associated C^*-algebras

$$\big(C^*_{\text{bicov}}(P) = B_P \times_\tau^{\text{piso}} P, i_{B_P}, V\big)$$

and

$$\big(C^*_{\text{bicov}}(S) = B_S \times_\beta^{\text{piso}} S, i_{B_S}, W\big),$$

respectively. By Theorem 5.9, there is an isomorphism

$$\Gamma : \big((B_P \otimes B_S) \times_{\tau\otimes\beta}^{\text{piso}} (P \times S), i_{(B_P\otimes B_S)}, T\big) \to (B_P \times_\tau^{\text{piso}} P) \otimes_{\max} (B_S \times_\beta^{\text{piso}} S)$$

such that

$$\Gamma\big(i_{(B_P\otimes B_S)}(1_x \otimes 1_t)T_{(p,s)}\big) = [i_{B_P}(1_x)V_p] \otimes [i_{B_S}(1_t)W_s]$$

for all $x, p \in P$ and $t, s \in S$. Note that $B_P \otimes B_S = B_P \otimes_{\max} B_S = B_P \otimes_{\min} B_S$ as the algebras B_P and B_S are abelian. Now, since the unital semigroups $P \times S$ and $(P \times S)^o$ are both left LCM, we have a dynamical system $(B_{(P\times S)}, P \times S, \alpha)$ along with its associated C^*-algebra

$$\big(C^*_{\text{bicov}}(P \times S) = B_{(P\times S)} \times_\alpha^{\text{piso}} (P \times S), i_{B_{(P\times S)}}, U\big),$$

where the action $\alpha : P \times S \to \text{End}(B_{(P\times S)})$ is induced by the shift on $\ell^\infty(P \times S)$ such that $\alpha_{(p,s)}(1_{(x,t)}) = 1_{(p,s)(x,t)} = 1_{(px,st)}$. Moreover, since there is an isomorphism

$$\psi : (1_x \otimes 1_t) \in (B_P \otimes B_S) \mapsto 1_{(x,t)} \in B_{(P\times S)}$$

which satisfies $\psi\circ(\tau \otimes \beta)_{(p,s)} = \alpha_{(p,s)}\circ\psi$ for all $(p, s) \in P \times S$, we have an isomorphism

$$\Lambda : (B_P \otimes B_S) \times_{\tau\otimes\beta}^{\text{piso}} (P \times S) \to B_{(P\times S)} \times_\alpha^{\text{piso}} (P \times S)$$

such that

$$\Lambda\circ i_{(B_P\otimes B_S)} = i_{B_{(P\times S)}}\circ\psi \quad \text{and} \quad \Lambda\circ T = U \ \ \text{(see Remark 5.12)}.$$

Eventually, the composition

$$C^*_{\text{bicov}}(P \times S) \xrightarrow{\Lambda^{-1}} (B_P \otimes B_S) \times_{\tau\otimes\beta}^{\text{piso}} (P \times S) \xrightarrow{\Gamma} C^*_{\text{bicov}}(P) \otimes_{\max} C^*_{\text{bicov}}(S)$$

of isomorphisms gives the desired isomorphism (5.21), such that

$$U_{(p,s)} \mapsto V_p \otimes W_s$$

for all $(p, s) \in P \times S$. □

6. Ideals in tensor products

Suppose that P is a left LCM semigroup. Let α and β be the actions of P on C^*-algebras A and B by extendible endomorphisms, respectively. Then, there is an action

$$\alpha \otimes \beta : P \to \mathrm{End}(A \otimes_{\max} B)$$

of P on the maximal tensor product $A \otimes_{\max} B$ by extendible endomorphisms such that $(\alpha \otimes \beta)_x = \alpha_x \otimes \beta_x$ for all $x \in P$. Note that the extendibility of $\alpha \otimes \beta$ follows by the extendibility of the actions α and β (see [18, Lemma 2.3]). Therefore, we obtain a dynamical system $(A \otimes_{\max} B, P, \alpha \otimes \beta)$. Let $(A \otimes_{\max} B) \times_{\alpha\otimes\beta}^{\mathrm{piso}} P$ be the partial-isometric crossed product of $(A \otimes_{\max} B, P, \alpha \otimes \beta)$. Our main goal in this section is to obtain a composition series

$$0 \leq \mathcal{I}_1 \leq \mathcal{I}_2 \leq (A \otimes_{\max} B) \times_{\alpha\otimes\beta}^{\mathrm{piso}} P$$

of ideals, and then identify the subquotients

$$\mathcal{I}_1, \quad \mathcal{I}_2 / \mathcal{I}_1, \quad \text{and} \quad ((A \otimes_{\max} B) \times_{\alpha\otimes\beta}^{\mathrm{piso}} P) / \mathcal{I}_2$$

with familiar terms. To do so, we first need to recall the following lemma from [18]:

Lemma 6.1. [18, Lemma 3.2] *Suppose that α and β are extendible endomorphisms of C^*-algebras A and B, respectively. If I is an extendible α-invariant ideal of A and J is an extendible β-invariant ideal of B, then the ideal $I \otimes_{\max} J$ of $A \otimes_{\max} B$ is extendible $\alpha \otimes \beta$-invariant.*

Remark 6.2. It follows by Lemma 6.1 that if (A, P, α) and (B, P, β) are dynamical systems, and I is an extendible α_x-invariant ideal of A and J is an extendible β_x-invariant ideal of B for every $x \in P$, then $I \otimes_{\max} J$ is an extendible $(\alpha \otimes \beta)_x$-invariant ideal of $A \otimes_{\max} B$ for all $x \in P$. Therefore, by Theorem 4.8, the crossed product $(I \otimes_{\max} J) \times_{\alpha\otimes\beta}^{\mathrm{piso}} P$ sits in the algebra $(A \otimes_{\max} B) \times_{\alpha\otimes\beta}^{\mathrm{piso}} P$ as an ideal (this will be the ideal $\mathcal{I}_1$ shortly later). As an application of this fact, we observe that, by [27, Proposition B. 30], the short exact sequence

$$0 \longrightarrow J \longrightarrow B \xrightarrow{q^J} B/J \longrightarrow 0$$

gives rise to the short exact sequence

$$0 \longrightarrow A \otimes_{\max} J \longrightarrow A \otimes_{\max} B \xrightarrow{\mathrm{id}_A \otimes_{\max} q^J} A \otimes_{\max} B/J \longrightarrow 0, \qquad (6.1)$$

where $A \otimes_{\max} J$ is an extendible $(\alpha \otimes \beta)_x$-invariant ideal of $A \otimes_{\max} B$ for all $x \in P$. Thus, (6.1) itself by Theorem 4.8 gives rise to the following short exact sequence

$$0 \to (A \otimes_{\max} J) \times_{\alpha\otimes\beta}^{\mathrm{piso}} P \xrightarrow{\mu} \big((A \otimes_{\max} B) \times_{\alpha\otimes\beta}^{\mathrm{piso}} P, i\big) \xrightarrow{\phi} \big((A \otimes_{\max} B/J) \times_{\alpha\otimes\tilde{\beta}}^{\mathrm{piso}} P, j\big),$$

where $\tilde{\beta} : P \to \mathrm{End}(B/J)$ is the (extendible) action induced by β, and the surjective homomorphism ϕ is indeed the homomorphism $(\mathrm{id}_A \otimes_{\max} q^J) \times P$

(see Remark 5.12) such that $\overline{[(\mathrm{id}_A \otimes_{\max} q^J) \times P]} \circ i_P = j_P$ and $[(\mathrm{id}_A \otimes_{\max} q^J) \times P] \circ i_{(A \otimes_{\max} B)} = j_{(A \otimes_{\max} B/J)} \circ (\mathrm{id}_A \otimes_{\max} q^J)$.

In the following proposition and theorem, for the maximal tensor product between the C^*-algebras involved, we simply write $\otimes$ for convenience.

Proposition 6.3. *Let (A, P, α) and (B, P, β) be dynamical systems, and I an extendible α_x-invariant ideal of A and J an extendible β_x-invariant ideal of B for every $x \in P$. Assume that $\tilde{\alpha} : P \to \mathrm{End}(A/I)$ and $\tilde{\beta} : P \to \mathrm{End}(B/J)$ are the actions induced by α and β, respectively. Then, the following diagram*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (I \otimes J) \times^{\mathrm{piso}}_{\alpha \otimes \beta} P & \longrightarrow & (I \otimes B) \times^{\mathrm{piso}}_{\alpha \otimes \beta} P & \xrightarrow{\phi_1} & (I \otimes B/J) \times^{\mathrm{piso}}_{\alpha \otimes \tilde{\beta}} P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (A \otimes J) \times^{\mathrm{piso}}_{\alpha \otimes \beta} P & \longrightarrow & (A \otimes B) \times^{\mathrm{piso}}_{\alpha \otimes \beta} P & \xrightarrow{\phi_2} & (A \otimes B/J) \times^{\mathrm{piso}}_{\alpha \otimes \tilde{\beta}} P & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & \searrow^{q \times P} & \downarrow{\scriptstyle \varphi_3} & & \\
0 & \to & (A/I \otimes J) \times^{\mathrm{piso}}_{\tilde{\alpha} \otimes \beta} P & \to & (A/I \otimes B) \times^{\mathrm{piso}}_{\tilde{\alpha} \otimes \beta} P & \xrightarrow{\phi_3} & (A/I \otimes B/J) \times^{\mathrm{piso}}_{\tilde{\alpha} \otimes \tilde{\beta}} P & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{6.2}$$

commutes, where

$$\phi_1 := (\mathrm{id}_I \otimes q^J) \times P, \quad \phi_2 := (\mathrm{id}_A \otimes q^J) \times P, \quad \phi_3 := (\mathrm{id}_{A/I} \otimes q^J) \times P,$$

and

$$\varphi_1 := (q^I \otimes \mathrm{id}_J) \times P, \quad \varphi_2 := (q^I \otimes \mathrm{id}_B) \times P, \text{ and } \varphi_3 := (q^I \otimes \mathrm{id}_{B/J}) \times P.$$

Also, there is a surjective homomorphism $q : A \otimes B \to (A/I) \otimes (B/J)$ which intertwines the actions $\alpha \otimes \beta$ and $\tilde{\alpha} \otimes \tilde{\beta}$, and therefore, we have a homomorphism $q \times P$ of $(A \otimes B) \times^{\mathrm{piso}}_{\alpha \otimes \beta} P$ onto $(A/I \otimes B/J) \times^{\mathrm{piso}}_{\tilde{\alpha} \otimes \tilde{\beta}} P$ induced by q. Moreover, we have

$$\ker(q \times P) = (A \otimes J) \times^{\mathrm{piso}}_{\alpha \otimes \beta} P + (I \otimes B) \times^{\mathrm{piso}}_{\alpha \otimes \beta} P. \tag{6.3}$$

Proof. First of all, in the diagram, each row as well as each column is obtained by a similar discussion to Remark 6.2, and hence, it is exact.

Next, for the quotient maps $q^I : A \to A/I$ and $q^J : B \to B/J$, by [27, Lemma B. 31], there is a homomorphism $q^I \otimes q^J : A \otimes B \to (A/I) \otimes (B/J)$, which we denote it by q, such that

$$q(a \otimes b) = (q^I \otimes q^J)(a \otimes b) = q^I(a) \otimes q^J(b) = (a + I) \otimes (b + J)$$

for all $a \in A$ and $b \in B$. It is obviously surjective. Moreover,

$$\begin{aligned} q((\alpha \otimes \beta)_x(a \otimes b)) &= q((\alpha_x \otimes \beta_x)(a \otimes b)) \\ &= q(\alpha_x(a) \otimes \beta_x(b)) \\ &= (\alpha_x(a) + I) \otimes (\beta_x(b) + J) \\ &= \tilde{\alpha}_x(a + I) \otimes \tilde{\beta}_x(b + J) \\ &= (\tilde{\alpha}_x \otimes \tilde{\beta}_x)((a + I) \otimes (b + J)) \\ &= (\tilde{\alpha} \otimes \tilde{\beta})_x(q(a \otimes b)) \end{aligned} \tag{6.4}$$

for all $x \in P$. Therefore, by Remark 5.12, there is a (nondegenerate) homomorphism

$$q \times P : \big((A \otimes B) \times^{\text{piso}}_{\alpha\otimes\beta} P, i\big) \to \big((A/I \otimes B/J) \times^{\text{piso}}_{\tilde{\alpha}\otimes\tilde{\beta}} P, k\big)$$

such that

$$(q \times P) \circ i_{(A\otimes B)} = k_{(A/I\otimes B/J)} \circ q \quad \text{and} \quad \overline{q \times P} \circ i_P = k_P.$$

One can easily see that as q is surjective, so is $q \times P$.

Now, an inspection on spanning elements shows that the diagram commutes.

Finally, to see (6.3), we only show that

$$\ker(q \times P) \subset (A \otimes J) \times^{\text{piso}}_{\alpha\otimes\beta} P + (I \otimes B) \times^{\text{piso}}_{\alpha\otimes\beta} P$$

as the other inclusion can be verified easily. To do so, take a nondegenerate representation

$$\pi : (A \otimes B) \times^{\text{piso}}_{\alpha\otimes\beta} P \to B(H)$$

with

$$\ker \pi = (A \otimes J) \times^{\text{piso}}_{\alpha\otimes\beta} P + (I \otimes B) \times^{\text{piso}}_{\alpha\otimes\beta} P.$$

Then, define a map $\rho : (A/I \otimes B/J) \to B(H)$ by $\rho(q(\xi)) = \pi(i_{(A\otimes B)}(\xi))$ for all $\xi \in (A \otimes B)$. Since

$$(A \otimes J) + (I \otimes B) = \ker q \subset \ker(\pi \circ i_{(A\otimes B)}),$$

it follows that the map ρ is well-defined, which is actually a nondegenerate representation. Also, the composition

$$P \xrightarrow{i_P} \mathcal{M}\big((A \otimes B) \times^{\text{piso}}_{\alpha\otimes\beta} P\big) \xrightarrow{\overline{\pi}} B(H)$$

gives a (right) Nica partial-isometric representation $W : P \to B(H)$. Now, by applying the covariance equations of the pair $(i_{(A\otimes B)}, i_P)$ and (6.4), one can see that the pair (ρ, W) is a covariant partial-isometric representation of $(A/I \otimes B/J, P, \tilde{\alpha} \otimes \tilde{\beta})$ on H. The corresponding representation $\rho \times W$ lifts to π, which means that $(\rho \times W) \circ (q \times P) = \pi$, and therefore, we have

$$\ker(q \times P) \subset \ker \pi = (A \otimes J) \times^{\text{piso}}_{\alpha\otimes\beta} P + (I \otimes B) \times^{\text{piso}}_{\alpha\otimes\beta} P.$$

Thus, the equation (6.3) holds. □

Theorem 6.4. *Let (A, P, α) and (B, P, β) be dynamical systems, and I an extendible α_x-invariant ideal of A and J an extendible β_x-invariant ideal of B for every $x \in P$. Assume that $\tilde{\alpha} : P \to \mathrm{End}(A/I)$ and $\tilde{\beta} : P \to \mathrm{End}(B/J)$ are the actions induced by α and β, respectively. Then, there is a composition series*

$$0 \leq \mathcal{I}_1 \leq \mathcal{I}_2 \leq (A \otimes B) \times^{\mathrm{piso}}_{\alpha\otimes\beta} P$$

of ideals, such that:

(i) *the ideal $\mathcal{I}_1$ is (isomorphic to) $(I \otimes J) \times^{\mathrm{piso}}_{\alpha\otimes\beta} P$;*

(ii) $\mathcal{I}_2 / \mathcal{I}_1 \simeq (A/I \otimes J) \times^{\mathrm{piso}}_{\tilde{\alpha}\otimes\beta} P \oplus (I \otimes B/J) \times^{\mathrm{piso}}_{\alpha\otimes\tilde{\beta}} P$ *;*

(iii) *the surjection $q \times P$ induces an isomorphism of $\big((A \otimes B) \times^{\mathrm{piso}}_{\alpha\otimes\beta} P\big) / \mathcal{I}_2$ onto $(A/I \otimes B/J) \times^{\mathrm{piso}}_{\tilde{\alpha}\otimes\tilde{\beta}} P$.*

Proof. For (i), as we mentioned in Remark 6.2, $I \otimes J$ is an extendible $(\alpha\otimes\beta)_x$-invariant ideal of $A \otimes B$ for all $x \in P$. Therefore, by Theorem 4.8, the crossed product $(I \otimes J) \times^{\mathrm{piso}}_{\alpha\otimes\beta} P$ sits in the algebra $(A \otimes B) \times^{\mathrm{piso}}_{\alpha\otimes\beta} P$ as an ideal, which we denote it by $\mathcal{I}_1$.

To get (ii), we first define

$$\mathcal{I}_2 := (A \otimes J) \times^{\mathrm{piso}}_{\alpha\otimes\beta} P + (I \otimes B) \times^{\mathrm{piso}}_{\alpha\otimes\beta} P,$$

which is an ideal of $(A \otimes B) \times^{\mathrm{piso}}_{\alpha\otimes\beta} P$ as each summand is. Note that we have

$$[(A \otimes J) \times^{\mathrm{piso}}_{\alpha\otimes\beta} P] \cap [(I \otimes B) \times^{\mathrm{piso}}_{\alpha\otimes\beta} P] = (I \otimes J) \times^{\mathrm{piso}}_{\alpha\otimes\beta} P.$$

So, it follows that (see the diagram (6.2))

$$\begin{aligned}
&\mathcal{I}_2 / \mathcal{I}_1 \\
&= \big[(A \otimes J) \times^{\mathrm{piso}}_{\alpha\otimes\beta} P + (I \otimes B) \times^{\mathrm{piso}}_{\alpha\otimes\beta} P\big] / (I \otimes J) \times^{\mathrm{piso}}_{\alpha\otimes\beta} P \\
&= \big[(A \otimes J) \times^{\mathrm{piso}}_{\alpha\otimes\beta} P\big] / [(I \otimes J) \times^{\mathrm{piso}}_{\alpha\otimes\beta} P] \oplus \big[(I \otimes B) \times^{\mathrm{piso}}_{\alpha\otimes\beta} P\big] / [(I \otimes J) \times^{\mathrm{piso}}_{\alpha\otimes\beta} P] \\
&\simeq [(A \otimes J)/(I \otimes J)] \times^{\mathrm{piso}}_{\widetilde{\alpha\otimes\beta}} P \oplus [(I \otimes B)/(I \otimes J)] \times^{\mathrm{piso}}_{\widetilde{\alpha\otimes\beta}} P \\
&\simeq (A/I \otimes J) \times^{\mathrm{piso}}_{\tilde{\alpha}\otimes\beta} P \oplus (I \otimes B/J) \times^{\mathrm{piso}}_{\alpha\otimes\tilde{\beta}} P.
\end{aligned}$$

Finally, for (iii), we recall from Proposition (6.3) that we have a surjective homomorphism

$$q \times P : (A \otimes B) \times^{\mathrm{piso}}_{\alpha\otimes\beta} P \to (A/I \otimes B/J) \times^{\mathrm{piso}}_{\tilde{\alpha}\otimes\tilde{\beta}} P$$

with

$$\ker(q \times P) = (A \otimes J) \times^{\mathrm{piso}}_{\alpha\otimes\beta} P + (I \otimes B) \times^{\mathrm{piso}}_{\alpha\otimes\beta} P = \mathcal{I}_2 .$$

Therefore, we have

$$\begin{aligned}
\big((A \otimes B) \times^{\mathrm{piso}}_{\alpha\otimes\beta} P\big) / \mathcal{I}_2 &= \big((A \otimes B) \times^{\mathrm{piso}}_{\alpha\otimes\beta} P\big) / \ker(q \times P) \\
&\simeq (A/I \otimes B/J) \times^{\mathrm{piso}}_{\tilde{\alpha}\otimes\tilde{\beta}} P.
\end{aligned}$$

□

7. An application

In this section, as an application, we will consider the dynamical system $(C^*(G_{p,q}), \mathbb{N}^2, \beta)$ studied in [19], where $\mathbb{N}^2$ is the positive cone of the abelian lattice-ordered group $\mathbb{Z}^2$. Let p and q be distinct odd primes, and consider the subgroup

$$G_{p,q} := \{np^{-k}q^{-l} : n, k, l \in \mathbb{Z}\}/\mathbb{Z}$$

of $\mathbb{Q}/\mathbb{Z}$. There is an averaging type action β of $\mathbb{N}^2$ on the group C^*-algebra $C^*(G_{p,q})$ by endomorphisms, such that on the canonical generating unitaries $\{u_r : r \in G_{p,q}\}$ of $C^*(G_{p,q})$ we have

$$\beta_{(m,n)}(u_r) = \frac{1}{p^m q^n} \sum_{\{s \in G_{p,q} : p^m q^n s = r\}} u_s \tag{7.1}$$

for all $(m,n) \in \mathbb{N}^2$. Let $\mathbb{Z}_p$ be the compact topological ring of p-adic integers (similarly for $\mathbb{Z}_q$). See in [19, Lemma 1.1] that by the Fourier transform $C^*(G_{p,q}) \simeq C(\mathbb{Z}_p \times \mathbb{Z}_q)$, the action β corresponds to the action α of $\mathbb{N}^2$ on $C(\mathbb{Z}_p \times \mathbb{Z}_q)$ by endomorphisms, such that

$$\alpha_{(m,n)}(f)(x,y) = \begin{cases} f(p^{-m}q^{-n}x, p^{-m}q^{-n}y) & \text{if } x \in p^m q^n \mathbb{Z}_p \text{ and } y \in p^m q^n \mathbb{Z}_q, \\ 0 & \text{otherwise} \end{cases} \tag{7.2}$$

for all $(m,n) \in \mathbb{N}^2$ and $f \in C(\mathbb{Z}_p \times \mathbb{Z}_q)$. Therefore, to study the partial-isometric crossed product $C^*(G_{p,q}) \times_\beta^{\text{piso}} \mathbb{N}^2$ of the system $(C^*(G_{p,q}), \mathbb{N}^2, \beta)$, it is enough to study the crossed product $C(\mathbb{Z}_p \times \mathbb{Z}_q) \times_\alpha^{\text{piso}} \mathbb{N}^2$ of the corresponding system $(C(\mathbb{Z}_p \times \mathbb{Z}_q), \mathbb{N}^2, \alpha)$. Firstly, we have $C(\mathbb{Z}_p \times \mathbb{Z}_q) \simeq C(\mathbb{Z}_p) \otimes C(\mathbb{Z}_q)$, and recall that the action α decomposes as the tensor product $\gamma \otimes \delta : \mathbb{N}^2 \to \operatorname{End}\big(C(\mathbb{Z}_p) \otimes C(\mathbb{Z}_q)\big)$ of two actions of $\mathbb{N}^2$, such that

$$\gamma_{(m,n)}(g)(x) = \begin{cases} g(p^{-m}q^{-n}x) & \text{if } x \in p^m q^n \mathbb{Z}_p, \\ 0 & \text{otherwise} \end{cases} \tag{7.3}$$

for all $(m,n) \in \mathbb{N}^2$ and $g \in C(\mathbb{Z}_p)$ (similarly for $\delta : \mathbb{N}^2 \to \operatorname{End}\big(C(\mathbb{Z}_q)\big)$). It thus follows that $C(\mathbb{Z}_p \times \mathbb{Z}_q) \times_\alpha^{\text{piso}} \mathbb{N}^2 \simeq \big(C(\mathbb{Z}_p) \otimes C(\mathbb{Z}_q)\big) \times_{\gamma\otimes\delta}^{\text{piso}} \mathbb{N}^2$, for which we want to apply Theorem 6.4. To do so, consider the extendible γ-invariant ideal $I := C_0(\mathbb{Z}_p \setminus \{0\})$ of $C(\mathbb{Z}_p)$ and the extendible δ-invariant ideal $J := C_0(\mathbb{Z}_q \setminus \{0\})$ of $C(\mathbb{Z}_q)$ as in [19]. Now, by applying Theorem 6.4 to the systems $(C(\mathbb{Z}_p), \mathbb{N}^2, \gamma)$ and $(C(\mathbb{Z}_q), \mathbb{N}^2, \delta)$ along with the ideals I and J, we get the following theorem which is the partial-isometric version of [19, Theorem 2.2]:

Theorem 7.1. *There are ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ in*

$$C(\mathbb{Z}_p \times \mathbb{Z}_q) \times_{\alpha}^{\text{piso}} \mathbb{N}^2 \simeq \big(C(\mathbb{Z}_p) \otimes C(\mathbb{Z}_q)\big) \times_{\gamma\otimes\delta}^{\text{piso}} \mathbb{N}^2$$

which form the composition series

$$0 \leq \mathcal{I}_1 \leq \mathcal{I}_2 \leq C(\mathbb{Z}_p \times \mathbb{Z}_q) \times_{\alpha}^{\text{piso}} \mathbb{N}^2 \tag{7.4}$$

of ideals, such that:

(a) $\mathcal{I}_1 \simeq \mathcal{A} \otimes_{\max} \mathcal{A} \otimes C(\mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q))$,

(b) $\mathcal{I}_2 / \mathcal{I}_1 \simeq \big(\mathcal{A} \otimes_{\max} [C(\mathcal{U}(\mathbb{Z}_p)) \times_{\sigma^{p,q}}^{\text{piso}} \mathbb{N}]\big) \oplus \big(\mathcal{A} \otimes_{\max} [C(\mathcal{U}(\mathbb{Z}_q)) \times_{\sigma^{q,p}}^{\text{piso}} \mathbb{N}]\big)$, *and*

(c) $\big(C(\mathbb{Z}_p \times \mathbb{Z}_q) \times_{\alpha}^{\text{piso}} \mathbb{N}^2\big) / \mathcal{I}_2 \simeq \mathcal{T}(\mathbb{Z}^2) \simeq \mathcal{T}(\mathbb{Z}) \otimes \mathcal{T}(\mathbb{Z})$,

where $\mathcal{U}(\mathbb{Z}_p)$ is the group of the multiplicatively invertible elements in $\mathbb{Z}_p$ (similarly for $\mathcal{U}(\mathbb{Z}_q)$), $\sigma^{p,q}$ is the action of $\mathbb{N}$ on the algebra $C(\mathcal{U}(\mathbb{Z}_p))$ by automorphisms such that $\sigma_n^{p,q}(f)(x) = f(q^{-n}x)$ (similarly for $\sigma^{q,p}$), and the algebra $\mathcal{A}$ is a full corner in the algebra $\mathcal{K}(\ell^2(\mathbb{N}) \otimes \mathbf{c})$ of compact operators, in which $\mathbf{c} = B_{\mathbb{N}} = \overline{\text{span}}\{1_n : n \in \mathbb{N}\}$.

Proof. For the proof, we apply Theorem 6.4 to the systems $(C(\mathbb{Z}_p), \mathbb{N}^2, \gamma)$ and $(C(\mathbb{Z}_q), \mathbb{N}^2, \delta)$ with the ideals $I = C_0(\mathbb{Z}_p \setminus \{0\})$ and $J = C_0(\mathbb{Z}_q \setminus \{0\})$. Therefore, we have

$$\mathcal{I}_1 := \big[C_0(\mathbb{Z}_p \setminus \{0\}) \otimes C_0(\mathbb{Z}_q \setminus \{0\})\big] \times_{\gamma\otimes\delta}^{\text{piso}} \mathbb{N}^2,$$

and

$$\mathcal{I}_2 := \big[C_0(\mathbb{Z}_p \setminus \{0\}) \otimes C(\mathbb{Z}_q)\big] \times_{\gamma\otimes\delta}^{\text{piso}} \mathbb{N}^2 + \big[C(\mathbb{Z}_p) \otimes C_0(\mathbb{Z}_q \setminus \{0\})\big] \times_{\gamma\otimes\delta}^{\text{piso}} \mathbb{N}^2,$$

from which, we obtain the composition series (7.4) of ideals.

Next, to identify the subquotients with familiar terms, we start with (c). First, by (iii) in Theorem 6.4, we have

$$\begin{aligned}&\big(C(\mathbb{Z}_p \times \mathbb{Z}_q) \times_{\alpha}^{\text{piso}} \mathbb{N}^2\big) / \mathcal{I}_2 \\ &\simeq \big(\big[C(\mathbb{Z}_p)/C_0(\mathbb{Z}_p \setminus \{0\})\big] \otimes \big[C(\mathbb{Z}_q)/C_0(\mathbb{Z}_q \setminus \{0\})\big]\big) \times_{\tilde{\gamma}\otimes\tilde{\delta}}^{\text{piso}} \mathbb{N}^2.\end{aligned}$$

Then, as

$$C(\mathbb{Z}_p)/C_0(\mathbb{Z}_p \setminus \{0\}) \simeq \mathbb{C} \simeq C(\mathbb{Z}_q)/C_0(\mathbb{Z}_q \setminus \{0\}) \ \text{(see [19])},$$

we get

$$\big(C(\mathbb{Z}_p \times \mathbb{Z}_q) \times_{\alpha}^{\text{piso}} \mathbb{N}^2\big) / \mathcal{I}_2 \simeq (\mathbb{C} \otimes \mathbb{C}) \times_{\text{id}\otimes\text{id}}^{\text{piso}} \mathbb{N}^2 \simeq \mathbb{C} \times_{\text{id}}^{\text{piso}} \mathbb{N}^2 \simeq \mathcal{T}(\mathbb{Z}^2),$$

where the bottom line follows from Example 4.9 as $(\mathbb{Z}^2, \mathbb{N}^2)$ is abelian (see also the remark prior to [28, Lemma 5.4]). Also, by applying Theorem 5.9,

$$\begin{aligned}\mathcal{T}(\mathbb{Z}^2) \simeq \mathbb{C} \times_{\text{id}}^{\text{piso}} \mathbb{N}^2 \ &\simeq \ (\mathbb{C} \otimes \mathbb{C}) \times_{\text{id}\otimes\text{id}}^{\text{piso}} \mathbb{N}^2 \\ &\simeq \ (\mathbb{C} \times_{\text{id}}^{\text{piso}} \mathbb{N}) \otimes (\mathbb{C} \times_{\text{id}}^{\text{piso}} \mathbb{N}) \simeq \mathcal{T}(\mathbb{Z}) \otimes \mathcal{T}(\mathbb{Z}),\end{aligned}$$

where $\mathbb{C} \times_{\text{id}}^{\text{piso}} \mathbb{N} \simeq \mathcal{T}(\mathbb{Z})$ is known by [4, Example 4.3]. So, we are done with (c).

To get (a), first, by [19, Corollary 2.4], there is an isomorphism

$$C_0(\mathbb{Z}_p \setminus\{0\}) \simeq \mathbf{c}_0 \otimes C(\mathcal{U}(\mathbb{Z}_p)),$$

where $\mathbf{c}_0 = \overline{\text{span}}\{1_n - 1_m : n, m \in \mathbb{N} \text{ with } n < m\} = C_0(\mathbb{N})$. By this isomorphism, the action γ corresponds to the tensor product action $\tau \otimes \sigma^{p,q}$, where τ is the action of $\mathbb{N}$ on $\mathbf{c_0}$ by forward shifts (similarly for $C_0(\mathbb{Z}_q \setminus\{0\})$ and the action δ). Therefore, we have an isomorphism (see the ideal $\mathcal{I}_1$)

$$\begin{aligned} C_0(\mathbb{Z}_p \setminus\{0\}) \otimes C_0(\mathbb{Z}_q \setminus\{0\}) &\simeq \mathbf{c}_0 \otimes \mathbf{c}_0 \otimes C(\mathcal{U}(\mathbb{Z}_p)) \otimes C(\mathcal{U}(\mathbb{Z}_q)) \\ &\simeq C_0(\mathbb{N}\times\mathbb{N}) \otimes C(\mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q)) \\ &\simeq C_0(\mathbb{N}\times\mathbb{N}\times\mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q)) \end{aligned}$$

which takes each endomorphism $(\gamma \otimes \delta)_{(m,n)}$ to $\tau_m \otimes \tau_n \otimes \sigma_n^{p,q} \otimes \sigma_m^{q,p}$. So, it follows that

$$\mathcal{I}_1 \simeq \big[C_0(\mathbb{N}\times\mathbb{N}) \otimes C(\mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q))\big] \times^{\text{piso}}_{(\tau\otimes\tau)\otimes(\sigma^{p,q}\otimes\sigma^{q,p})} \mathbb{N}^2 .$$

Moreover, there is an automorphism Φ of $C_0(\mathbb{N}\times\mathbb{N}) \otimes C(\mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q))$ such that we have

$$\Phi\circ(\tau_m \otimes \tau_n \otimes \sigma_n^{p,q} \otimes \sigma_m^{q,p}) = \tau_m \otimes \tau_n \otimes \text{id} \otimes \text{id} \ \ \text{(see again [19])}.$$

The automorphism Φ then induces the isomorphism

$$\mathcal{I}_1 \simeq \big[C_0(\mathbb{N}\times\mathbb{N}) \otimes C(\mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q))\big] \times^{\text{piso}}_{(\tau\otimes\tau)\otimes\text{id}} \mathbb{N}^2 .$$

Next, we need to show that

$$\begin{aligned} &\big[C_0(\mathbb{N}\times\mathbb{N}) \otimes C(\mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q))\big] \times^{\text{piso}}_{(\tau\otimes\tau)\otimes\text{id}} \mathbb{N}^2 \\ &\simeq \big[C_0(\mathbb{N}\times\mathbb{N}) \times^{\text{piso}}_{\tau\otimes\tau} \mathbb{N}^2\big] \otimes C(\mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q)). \end{aligned}$$

We skip the proof as it is routine and refer readers to Remark 7.4 for an indication on the proof. Also, by applying Theorem 5.9,

$$C_0(\mathbb{N}\times\mathbb{N}) \times^{\text{piso}}_{\tau\otimes\tau} \mathbb{N}^2 \simeq (\mathbf{c}_0 \otimes \mathbf{c}_0) \times^{\text{piso}}_{\tau\otimes\tau} \mathbb{N}^2 \simeq (\mathbf{c}_0 \times^{\text{piso}}_{\tau} \mathbb{N}) \otimes_{\max} (\mathbf{c}_0 \times^{\text{piso}}_{\tau} \mathbb{N}),$$

and hence, we get

$$\mathcal{I}_1 \simeq (\mathbf{c}_0 \times^{\text{piso}}_{\tau} \mathbb{N}) \otimes_{\max} (\mathbf{c}_0 \times^{\text{piso}}_{\tau} \mathbb{N}) \otimes C(\mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q)).$$

Finally, see in [4, Example 4.3] that the algebra $\mathbf{c}_0 \times^{\text{piso}}_{\tau} \mathbb{N}$ is a full corner in the algebra $\mathcal{K}(\ell^2(\mathbb{N}) \otimes \mathbf{c})$ of compact operators. More precisely, let P be the projection in $\mathcal{M}(\mathcal{K}(\ell^2(\mathbb{N}) \otimes \mathbf{c})) \simeq \mathcal{L}(\ell^2(\mathbb{N}) \otimes \mathbf{c})$ defined by

$$P(\xi)(n) = \tau_n(1)\xi(n) = 1_n\xi(n) \ \text{ for all } \xi \in \ell^2(\mathbb{N}) \otimes \mathbf{c},$$

where τ is the action of $\mathbb{N}$ on the algebra $\mathbf{c}$ by forward shifts. Then, $\mathbf{c}_0 \times^{\text{piso}}_{\tau} \mathbb{N}$ is isomorphic to the full corner $P\,\mathcal{K}(\ell^2(\mathbb{N}) \otimes \mathbf{c})P$, which we denote it by $\mathcal{A}$. Thus, we have

$$\mathcal{I}_1 \simeq \mathcal{A} \otimes_{\max} \mathcal{A} \otimes C(\mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q)).$$

At last, to see (b), we first apply (ii) in Theorem 6.4 to get

$$\mathcal{I}_2 / \mathcal{I}_1 \simeq \big[C_0(\mathbb{Z}_p \setminus\{0\}) \otimes C(\mathbb{Z}_q)/J\big] \times^{\text{piso}}_{\gamma\otimes\tilde{\delta}} \mathbb{N}^2 \oplus \big[C(\mathbb{Z}_p)/I \otimes C_0(\mathbb{Z}_q \setminus\{0\})\big] \times^{\text{piso}}_{\tilde{\gamma}\otimes\delta} \mathbb{N}^2,$$

and since $C(\mathbb{Z}_q)/J \simeq \mathbb{C} \simeq C(\mathbb{Z}_p)/I$, we have

$$\begin{aligned}\mathcal{I}_2/\mathcal{I}_1 &\simeq \big[C_0(\mathbb{Z}_p\setminus\{0\})\otimes\mathbb{C}\big]\times^{\text{piso}}_{\gamma\otimes\text{id}}\mathbb{N}^2 \oplus \big[\mathbb{C}\otimes C_0(\mathbb{Z}_q\setminus\{0\})\big]\times^{\text{piso}}_{\text{id}\otimes\delta}\mathbb{N}^2\\ &\simeq \big(C_0(\mathbb{Z}_p\setminus\{0\})\times^{\text{piso}}_{\gamma}\mathbb{N}^2\big)\oplus\big(C_0(\mathbb{Z}_q\setminus\{0\})\times^{\text{piso}}_{\delta}\mathbb{N}^2\big).\end{aligned}$$

Then, by applying [19, Corollary 2.4] and Theorem 5.9 (see also the proof of (a)), for the crossed product $\big(C_0(\mathbb{Z}_p\setminus\{0\})\times^{\text{piso}}_{\gamma}\mathbb{N}^2\big)$, we have

$$\begin{aligned}C_0(\mathbb{Z}_p\setminus\{0\})\times^{\text{piso}}_{\gamma}\mathbb{N}^2 &\simeq \big(\mathbf{c}_0\otimes C(\mathcal{U}(\mathbb{Z}_p))\big)\times^{\text{piso}}_{\tau\otimes\sigma^{p,q}}(\mathbb{N}\times\mathbb{N})\\ &\simeq \big[\mathbf{c}_0\times^{\text{piso}}_{\tau}\mathbb{N}\big]\otimes_{\max}\big[C(\mathcal{U}(\mathbb{Z}_p))\times^{\text{piso}}_{\sigma^{p,q}}\mathbb{N}\big]\\ &\simeq \mathcal{A}\otimes_{\max}\big[C(\mathcal{U}(\mathbb{Z}_p))\times^{\text{piso}}_{\sigma^{p,q}}\mathbb{N}\big].\end{aligned}$$

Similarly,

$$C_0(\mathbb{Z}_q\setminus\{0\})\times^{\text{piso}}_{\delta}\mathbb{N}^2 \simeq \mathcal{A}\otimes_{\max}\big[C(\mathcal{U}(\mathbb{Z}_q))\times^{\text{piso}}_{\sigma^{q,p}}\mathbb{N}\big].$$

Therefore, it follows that

$$\mathcal{I}_2/\mathcal{I}_1 \simeq \big(\mathcal{A}\otimes_{\max}\big[C(\mathcal{U}(\mathbb{Z}_p))\times^{\text{piso}}_{\sigma^{p,q}}\mathbb{N}\big]\big)\oplus\big(\mathcal{A}\otimes_{\max}\big[C(\mathcal{U}(\mathbb{Z}_q))\times^{\text{piso}}_{\sigma^{q,p}}\mathbb{N}\big]\big).$$

This completes the proof. □

Remark 7.2. Recall that, if m and n are relatively prime integers, then $m \in \mathcal{U}(\mathbb{Z}/n\mathbb{Z})$ (similarly $n \in \mathcal{U}(\mathbb{Z}/m\mathbb{Z})$). Let $o_n(m)$ denote the order of m in $\mathcal{U}(\mathbb{Z}/n\mathbb{Z})$. Since p and q in Theorem 7.1 are distinct odd primes, it follows by [19, Theorem 3.1] that there is a positive integer $L = L_p(q)$ such that

$$o_{p^\ell}(q) = \begin{cases} o_p(q) & \text{if } 1 \le \ell \le L,\\ p^{\ell-L}o_p(q) & \text{if } \ell > L.\end{cases} \tag{7.5}$$

Now, since the action $\sigma^{p,q}$ of $\mathbb{N}$ on the algebra $C(\mathcal{U}(\mathbb{Z}_p))$ is given by automorphisms, it follows by [4, Theorem 4.1] that the algebra of compact operators $\mathcal{K}(\ell^2(\mathbb{N}))\otimes C(\mathcal{U}(\mathbb{Z}_p))$ sits in $C(\mathcal{U}(\mathbb{Z}_p))\times^{\text{piso}}_{\sigma^{p,q}}\mathbb{N}$ as an ideal, such that

$$\begin{aligned}&\big[C(\mathcal{U}(\mathbb{Z}_p))\times^{\text{piso}}_{\sigma^{p,q}}\mathbb{N}\big]/\big[\mathcal{K}(\ell^2(\mathbb{N}))\otimes C(\mathcal{U}(\mathbb{Z}_p))\big]\\ &\simeq C(\mathcal{U}(\mathbb{Z}_p))\times^{\text{iso}}_{\sigma^{p,q}}\mathbb{N}\simeq C(\mathcal{U}(\mathbb{Z}_p))\times_{\sigma^{p,q}}\mathbb{Z}.\end{aligned}$$

But, again by [19, Theorem 3.1], the classical crossed product $C(\mathcal{U}(\mathbb{Z}_p))\times_{\sigma^{p,q}}\mathbb{Z}$ is the direct sum of $p^{L_p(q)-1}(p-1)/o_p(q)$ Bunce-Deddens algebras with supernatural number $o_p(q)p^\infty$. Similarly,

$$\begin{aligned}&\big[C(\mathcal{U}(\mathbb{Z}_q))\times^{\text{piso}}_{\sigma^{q,p}}\mathbb{N}\big]/\big[\mathcal{K}(\ell^2(\mathbb{N}))\otimes C(\mathcal{U}(\mathbb{Z}_q))\big]\\ &\simeq C(\mathcal{U}(\mathbb{Z}_q))\times^{\text{iso}}_{\sigma^{q,p}}\mathbb{N}\simeq C(\mathcal{U}(\mathbb{Z}_q))\times_{\sigma^{q,p}}\mathbb{Z},\end{aligned}$$

where $C(\mathcal{U}(\mathbb{Z}_q))\times_{\sigma^{q,p}}\mathbb{Z}$ is the direct sum of $q^{L_q(p)-1}(q-1)/o_q(p)$ Bunce-Deddens algebras with supernatural number $o_q(p)q^\infty$. Readers are referred to [8, 9, 10] on Bunce-Deddens algebras. However, we will provide a quick recall on these algebras shortly later.

Next, we would like to analyze the crossed product $C(\mathcal{U}(\mathbb{Z}_p))\times^{\mathrm{piso}}_{\sigma^{p,q}}\mathbb{N}$ in Theorem 7.1 more. Recall from [19] that if Γ is the closed subgroup of $\mathcal{U}(\mathbb{Z}_p)$ generated by q, then it is invariant under multiplication by powers of q. It thus follows that the ideal $C(\Gamma)$ of $C(\mathcal{U}(\mathbb{Z}_p))$ is $\sigma^{p,q}$-invariant. Note that the same facts hold for each closed subset $x\Gamma$ of $\mathcal{U}(\mathbb{Z}_p)$ and each ideal $C(x\Gamma)$ of $C(\mathcal{U}(\mathbb{Z}_p))$, where $x\in\mathcal{U}(\mathbb{Z}_p)$. Moreover, since $\mathcal{U}(\mathbb{Z}_p)$ is the disjoint union of $N := p^{L_p(q)-1}(p-1)/o_p(q)$ closed invariant subsets of the form $x\Gamma$, $C(\mathcal{U}(\mathbb{Z}_p))$ is the direct sum of N $\sigma^{p,q}$-invariant ideals of the form $C(x\Gamma)$. Now, since each ideal $C(x\Gamma)$, as well as the algebra $C(\mathcal{U}(\mathbb{Z}_p))$, is unital and the action $\sigma^{p,q}$ of $\mathbb{N}$ is given by automorphisms, it is not difficult to see that each algebra $C(x\Gamma)$ is actually an extendible $\sigma^{p,q}$-invariant ideal of $C(\mathcal{U}(\mathbb{Z}_p))$. Therefore, each crossed product $C(x\Gamma)\times^{\mathrm{piso}}_{\sigma^{p,q}}\mathbb{N}$ sits in $C(\mathcal{U}(\mathbb{Z}_p))\times^{\mathrm{piso}}_{\sigma^{p,q}}\mathbb{N}$ as an ideal by Theorem 4.8 or [5, Theorem 3.1]. Also, calculation shows that

$$C(\mathcal{U}(\mathbb{Z}_p))\times^{\mathrm{piso}}_{\sigma^{p,q}}\mathbb{N}\simeq\big(C(x_1\Gamma)\times^{\mathrm{piso}}_{\sigma^{p,q}}\mathbb{N}\big)\oplus\cdots\oplus\big(C(x_N\Gamma)\times^{\mathrm{piso}}_{\sigma^{p,q}}\mathbb{N}\big), \tag{7.6}$$

where one may take x_1 to be 1, the unit element of the group $\mathcal{U}(\mathbb{Z}_p)$. Now, since for every $x\in\mathcal{U}(\mathbb{Z}_p)$, the closed subsets Γ and $x\Gamma$ of $\mathcal{U}(\mathbb{Z}_p)$ are homeomorphic, there is an isomorphism $\psi : C(\Gamma)\to C(x\Gamma)$ of C^*-algebras such that $\sigma^{p,q}_n\circ\psi=\psi\circ\sigma^{p,q}_n$ for all $n\in\mathbb{N}$. The isomorphism ψ then induces an isomorphism between crossed products $C(\Gamma)\times^{\mathrm{piso}}_{\sigma^{p,q}}\mathbb{N}$ and $C(x\Gamma)\times^{\mathrm{piso}}_{\sigma^{p,q}}\mathbb{N}$ (see Remark 5.12). So, this fact and (7.6) imply that the crossed product $C(\mathcal{U}(\mathbb{Z}_p))\times^{\mathrm{piso}}_{\sigma^{p,q}}\mathbb{N}$ is actually isomorphic to the direct sum of N ideals $C(\Gamma)\times^{\mathrm{piso}}_{\sigma^{p,q}}\mathbb{N}$. At last, we want to have a familiar description for the crossed product $C(\Gamma)\times^{\mathrm{piso}}_{\sigma^{p,q}}\mathbb{N}$. To do so, we need to recall on Bunce-Deddens algebras quickly. These algebras were first defined in [8] as the C^*-algebras related to certain weighted shift operators on the Hilbert space $\ell^2(\mathbb{N})$. They were then identified as the classical crossed products by (automorphic) actions induced by the odometer map (see [10, 19]). Suppose that $\{n_i\}_{i=0}^{\infty}$ is a strictly increasing sequence of positive integers, such that n_i divides n_{i+1} for all i. Note that one can assume that $n_0=1$ without loss of generality. For every $i\geq 0$, let $m_i=n_{i+1}/n_i$, and then consider the Cantor set $\mathbf{K}$ given by the model

$$\mathbf{K}=\prod_{i=0}^{\infty}\{0,1,...,m_i-1\}.$$

The odometer map on $\mathbf{K}$, namely $\mathcal{O} : \mathbf{K}\to\mathbf{K}$, is given by addition of $(1,0,0,...)$ with carry over to the right. For example,

$$\mathcal{O}(m_0-1,m_1-1,0,0,0,...)=(0,0,1,0,0,...).$$

So, it induces an action τ of $\mathbb{Z}$ on the algebra $C(\mathbf{K})$ by automorphisms, such that the classical crossed product $C(\mathbf{K})\times_\tau\mathbb{Z}$ is a Bunce-Deddens algebra with supernatural number $\Pi_{i\geq 0}\, m_i$. Now, in particular, for the sequence

$$\{n_i\}_{i=0}^{\infty}=\{1,d,dp,dp^2,dp^3,...\},$$

the Cantor set $\mathbf{K}$ is

$$\mathbf{K} = \{0, 1, ..., d-1\} \times \prod_{i=1}^{\infty} \{0, 1, ..., p-1\},$$

where $d = o_p(q)$. Then, there is a homeomorphism of $\mathbf{K}$ onto the closed subgroup Γ, which induces an isomorphism φ of $C(\mathbf{K})$ onto $C(\Gamma)$ such that $\varphi \circ \tau_n = \sigma_n^{p,q} \circ \varphi$ for all $n \in \mathbb{N}$ (see [19]). Therefore, each ideal $C(\Gamma) \times_{\sigma^{p,q}}^{\text{piso}} \mathbb{N}$ is actually isomorphic to the partial-isometric crossed product $C(\mathbf{K}) \times_{\tau}^{\text{piso}} \mathbb{N}$ (see Remark 5.12). Consequently, the algebra $C(\mathcal{U}(\mathbb{Z}_p)) \times_{\sigma^{p,q}}^{\text{piso}} \mathbb{N}$ is in fact isomorphic to the direct sum of N crossed products $C(\mathbf{K}) \times_{\tau}^{\text{piso}} \mathbb{N}$. Note that, since the action τ is automorphic, by [4, Theorem 4.1], $C(\mathbf{K}) \times_{\tau}^{\text{piso}} \mathbb{N}$ contains the algebra of compact operators $\mathcal{K}(\ell^2(\mathbb{N})) \otimes C(\mathbf{K})$ as an ideal, such that the quotient algebra

$$\big[C(\mathbf{K}) \times_{\tau}^{\text{piso}} \mathbb{N}\big] / \big[\mathcal{K}(\ell^2(\mathbb{N})) \otimes C(\mathbf{K})\big] \simeq C(\mathbf{K}) \times_{\tau}^{\text{iso}} \mathbb{N} \simeq C(\mathbf{K}) \times_{\tau} \mathbb{Z}$$

is a Bunce-Deddens algebra with supernatural number $o_p(q)p^{\infty}$. By swapping the roles of p and q, similarly, the algebra $C(\mathcal{U}(\mathbb{Z}_q)) \times_{\sigma^{q,p}}^{\text{piso}} \mathbb{N}$ is indeed isomorphic to the direct sum of $q^{L_q(p)-1}(q-1)/o_q(p)$ crossed products $C(\widetilde{\mathbf{K}}) \times_{\tilde{\tau}}^{\text{piso}} \mathbb{N}$, where

$$\widetilde{\mathbf{K}} = \{0, 1, ..., o_q(p)-1\} \times \prod_{i=1}^{\infty} \{0, 1, ..., q-1\},$$

and $\tilde{\tau}$ is the action of $\mathbb{Z}$ on $C(\widetilde{\mathbf{K}})$ by automorphisms induced by the odometer map on $\widetilde{\mathbf{K}}$. Also, the quotient algebra (again by [4, Theorem 4.1])

$$\big[C(\widetilde{\mathbf{K}}) \times_{\tilde{\tau}}^{\text{piso}} \mathbb{N}\big] / \big[\mathcal{K}(\ell^2(\mathbb{N})) \otimes C(\widetilde{\mathbf{K}})\big] \simeq C(\widetilde{\mathbf{K}}) \times_{\tilde{\tau}}^{\text{iso}} \mathbb{N} \simeq C(\widetilde{\mathbf{K}}) \times_{\tilde{\tau}} \mathbb{Z}$$

is a Bunce-Deddens algebra with supernatural number $o_q(p)q^{\infty}$. Therefore, we have actually proved the following corollary as a refinement of Theorem 7.1:

Corollary 7.3. *There are ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ in*

$$C(\mathbb{Z}_p \times \mathbb{Z}_q) \times_{\alpha}^{\text{piso}} \mathbb{N}^2 \simeq \big(C(\mathbb{Z}_p) \otimes C(\mathbb{Z}_q)\big) \times_{\gamma \otimes \delta}^{\text{piso}} \mathbb{N}^2$$

which form the composition series

$$0 \leq \mathcal{I}_1 \leq \mathcal{I}_2 \leq C(\mathbb{Z}_p \times \mathbb{Z}_q) \times_{\alpha}^{\text{piso}} \mathbb{N}^2 \tag{7.7}$$

of ideals, such that:

(a) $\mathcal{I}_1 \simeq \mathcal{A} \otimes_{\max} \mathcal{A} \otimes C(\mathcal{U}(\mathbb{Z}_p) \times \mathcal{U}(\mathbb{Z}_q))$,
(b) $\mathcal{I}_2 / \mathcal{I}_1 \simeq (\mathcal{A} \otimes_{\max} \mathcal{C}) \oplus (\mathcal{A} \otimes_{\max} \mathcal{D})$, *and*
(c) $\big(C(\mathbb{Z}_p \times \mathbb{Z}_q) \times_{\alpha}^{\text{piso}} \mathbb{N}^2\big) / \mathcal{I}_2 \simeq \mathcal{T}(\mathbb{Z}^2) \simeq \mathcal{T}(\mathbb{Z}) \otimes \mathcal{T}(\mathbb{Z})$,

where $\mathcal{U}(\mathbb{Z}_p)$ is the group of the multiplicatively invertible elements in $\mathbb{Z}_p$ (similarly for $\mathcal{U}(\mathbb{Z}_q)$), the algebra $\mathcal{A}$ is a full corner in the algebra $\mathcal{K}(\ell^2(\mathbb{N}) \otimes \mathbf{c})$ of compact operators, the algebra $\mathcal{C}$ is the direct sum of $p^{L_p(q)-1}(p-1)/o_p(q)$ crossed products $C(\mathbf{K}) \times_{\tau}^{\text{piso}} \mathbb{N}$, and the algebra $\mathcal{D}$ is the direct sum of $q^{L_q(p)-1}(q-1)/o_q(p)$ crossed products $C(\widetilde{\mathbf{K}}) \times_{\tilde{\tau}}^{\text{piso}} \mathbb{N}$.

Remark 7.4. To see that for any C^*-algebra A,

$$(C_0(\mathbb{N}\times\mathbb{N}) \otimes A) \times^{\text{piso}}_{(\tau\otimes\tau)\otimes \text{id}} \mathbb{N}^2 \simeq \big(C_0(\mathbb{N}\times\mathbb{N}) \times^{\text{piso}}_{\tau\otimes\tau} \mathbb{N}^2\big) \otimes_{\max} A,$$

let $(j_{C_0(\mathbb{N}\times\mathbb{N})}, j_{\mathbb{N}^2})$ be the canonical covariant partial-isometric pair of the system $(C_0(\mathbb{N}\times\mathbb{N}), \mathbb{N}^2, \tau\otimes\tau)$ in the algebra $B := C_0(\mathbb{N}\times\mathbb{N}) \times^{\text{piso}}_{\tau\otimes\tau} \mathbb{N}^2$. Suppose that i_B and i_A are the canonical nondegenerate homomorphisms of the algebras B and A in the multiplier algebra $\mathcal{M}(B\otimes_{\max} A)$, respectively (see [27, Theorem B.27]). Consider the homomorphism given by the composition

$$C_0(\mathbb{N}\times\mathbb{N}) \xrightarrow{j_{C_0(\mathbb{N}\times\mathbb{N})}} B \xrightarrow{i_B} \mathcal{M}(B\otimes_{\max} A).$$

Then, the ranges of $(i_B\circ j_{C_0(\mathbb{N}\times\mathbb{N})})$ and i_A commute, and therefore, there is a homomorphism $k_{C_0(\mathbb{N}\times\mathbb{N})\otimes A} := (i_B\circ j_{C_0(\mathbb{N}\times\mathbb{N})}) \otimes_{\max} i_A$ of $C_0(\mathbb{N}\times\mathbb{N})\otimes A$ in $\mathcal{M}(B\otimes_{\max} A)$ such that

$$k_{C_0(\mathbb{N}\times\mathbb{N})\otimes A}(f\otimes a) = i_B(j_{C_0(\mathbb{N}\times\mathbb{N})}(f))i_A(a) = j_{C_0(\mathbb{N}\times\mathbb{N})}(f)\otimes a$$

for all $f\in C_0(\mathbb{N}\times\mathbb{N})$ and $a\in A$. One can see that $k_{C_0(\mathbb{N}\times\mathbb{N})\otimes A}$ is indeed nondegenerate. Next, let $k_{\mathbb{N}^2}$ be the map defined by the composition

$$\mathbb{N}^2 \xrightarrow{j_{\mathbb{N}^2}} \mathcal{M}(B) \xrightarrow{\overline{i_B}} \mathcal{M}(B\otimes_{\max} A).$$

It is not difficult to see that $k_{\mathbb{N}^2}$ is a (right) Nica partial-isometric representation. Now, it is routine for one to check that the triple $(B\otimes_{\max} A, k_{C_0(\mathbb{N}\times\mathbb{N})\otimes A}, k_{\mathbb{N}^2})$ is a partial-isometric crossed product for the system $\big(C_0(\mathbb{N}\times\mathbb{N})\otimes A, \mathbb{N}^2, (\tau\otimes\tau)\otimes \text{id}\big)$. Therefore, there is an isomorphism

$$\big((C_0(\mathbb{N}\times\mathbb{N}) \otimes A) \times^{\text{piso}}_{(\tau\otimes\tau)\otimes \text{id}} \mathbb{N}^2, i_{C_0(\mathbb{N}\times\mathbb{N})\otimes A}, i_{\mathbb{N}^2}\big) \overset{\Upsilon}{\simeq} \big(C_0(\mathbb{N}\times\mathbb{N}) \times^{\text{piso}}_{\tau\otimes\tau} \mathbb{N}^2\big) \otimes_{\max} A$$

such that

$$\begin{aligned}
\Upsilon\big(i_{C_0(\mathbb{N}\times\mathbb{N})\otimes A}(f\otimes a)i_{\mathbb{N}^2}(m,n)\big) &= k_{C_0(\mathbb{N}\times\mathbb{N})\otimes A}(f\otimes a)k_{\mathbb{N}^2}(m,n) \\
&= i_B(j_{C_0(\mathbb{N}\times\mathbb{N})}(f))i_A(a)\overline{i_B}(j_{\mathbb{N}^2}(m,n)) \\
&= i_B(j_{C_0(\mathbb{N}\times\mathbb{N})}(f))\overline{i_B}(j_{\mathbb{N}^2}(m,n))i_A(a) \\
&= i_B(j_{C_0(\mathbb{N}\times\mathbb{N})}(f)j_{\mathbb{N}^2}(m,n))i_A(a) \\
&= [j_{C_0(\mathbb{N}\times\mathbb{N})}(f)j_{\mathbb{N}^2}(m,n)]\otimes a.
\end{aligned}$$

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