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A bound on the index of exponent-4 algebras in terms of the u-invariant

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ABSTRACT. For a prime number p, an integer $e \geqslant 2$ and a field F containing a primitive p^e-th root of unity, the index of central simple F-algebras of exponent p^e is bounded in terms of the p-symbol length of F. For a nonreal field F of characteristic different from 2, the index of central simple algebras of exponent 4 is bounded in terms of the u-invariant of F. Finally, a new construction for nonreal fields of u-invariant 6 is presented.

CONTENTS

1. Introduction

Let F be a field and n a positive integer. A central simple F-algebra of degree n containing a subfield which is a cyclic extension of degree n of F is called *cyclic* or a *cyclic F-algebra*. Given a cyclic field extension K/F of degree n, a generator σ of its Galois group and an element $b \in F^\times$, the rules

$$j^n = b \quad \text{and} \quad xj = j\sigma(x) \quad \text{for all} \quad x \in K$$

determine a multiplication on the K-vector space $K \oplus jK \oplus \ldots \oplus j^{n-1}K$ turning it into a cyclic F-algebra of degree n, which is denoted by

$$[K/F, \sigma, b).$$

Any cyclic F-algebra is isomorphic to an algebra of this form; see [3, Theorem 5.9]. Furthermore, any central F-division algebra of degree 2 or 3 is cyclic; see [3, Theorem 11.5] for the degree-3 case.

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Central simple F-algebras of degree 2 are called *quaternion algebras*. We refer to [10, p. 25] for a discussion of quaternion algebras, including their standard presentation by symbols depending on two parameters from the base field. If $\mathsf{char}\, F \neq 2$, $a \in F^\times \smallsetminus F^{\times 2}$ and $b \in F^\times$, then the F-quaternion algebra $(a,b)_F$ is equal to $[K/F, \sigma, b)$ for $K = F(\sqrt{a})$ and the nontrivial automorphism σ of K/F.

We refer to [3] and [6] for the theory of central simple algebras, and to [4, Section 3] for a survey on the role of cyclic algebras in this context.

Before we approach the problem in the focus of our interest, we fix some notation. We set $\mathbb{N}^+ = \mathbb{N} \smallsetminus \{0\}$. We denote by $\mathsf{Br}(F)$ the Brauer group of F, and for $n \in \mathbb{N}^+$, we denote by $\mathsf{Br}_n(F)$ the n-torsion part of $\mathsf{Br}(F)$. Let p always denote a prime number.

The following question was asked by Albert in [1, p.126] and is still open in general.

Question 1.1. For $n \in \mathbb{N}^+$, is $\mathsf{Br}_n(F)$ generated by classes of cyclic algebras of degree dividing n?

In view of the Primary Decomposition Theorem for central simple algebras (see e.g. [6, Corollary 9.11]), any such question can be reduced to the case where n is a prime power. Each of the following two famous results gives a positive answer to Question 1.1 under additional hypotheses on F in relation to n.

Theorem 1.2 (Albert). *Let p be a prime number and assume that $\mathsf{char}\, F = p$. Let $e \in \mathbb{N}^+$. Then $\mathsf{Br}_{p^e}(F)$ is generated by classes of cyclic F-algebras of degree dividing p^e.*

Proof. See [3, Chapter VII, Section 9]. □

Theorem 1.3 (Merkurjev-Suslin). *Let $n \in \mathbb{N}^+$ and assume that F contains a primitive n-th root of unity. Then $\mathsf{Br}_n(F)$ is generated by the classes of cyclic F-algebras of degree dividing n.*

Proof. See [13]. □

If F contains a primitive n-th root of unity then $\mathsf{char}\, F$ does not divide n. Hence, the hypotheses of Theorem 1.2 and Theorem 1.3 are mutually exclusive.

For $n = 2$, Theorem 1.3 was obtained by Merkurjev in [12]. Note that the hypothesis of Theorem 1.3 for $n = 2$ just means that $\mathsf{char}\, F \neq 2$. Together with Theorem 1.2 this gives an unconditional positive answer to Question 1.1 for $n = 2$.

It was observed in [13, Proposition 16.6] that from the positive answer to Question 1.1 in the (highly nontrivial) case $n = 2$ one obtains (rather easily) an unconditional positive answer for $n = 4$. In Corollary 3.10, we obtain a different argument for this step.

Whenever we have a positive answer to Question 1.1, it is motivated to look at quantitative aspects of the problem. In the first place, this concerns the number of cyclic algebras needed for a tensor product representing a class in $\mathsf{Br}(F)$ of given exponent. This leads to the notion and the study of *symbol lengths*.

For a central simple F-algebra A, the *n-symbol length of A*, denoted by $\lambda_n(A)$, is the smallest $m \in \mathbb{N}^+$ such that A is Brauer equivalent to a tensor product of m cyclic algebras of degree dividing n, if such an integer m exists, otherwise we set $\lambda_n(A) = \infty$. The *n-symbol length of F* is defined as

$$\lambda_n(F) \quad = \quad \sup\{\lambda_n(A) \mid [A] \in \mathsf{Br}_n(F)\} \;\; \in \;\; \mathbb{N}^+ \cup \{\infty\}.$$

Note that the index of any central simple F-algebra of exponent n is at most $n^{\lambda_n(F)}$.

Let p be a prime number. It seems plausible to take the p-symbol length of F for a measure for the complexity of the whole p-primary part of the theory of central simple algebras over F. So in particular one might expect that $\lambda_{p^e}(F)$ can be bounded in terms of $\lambda_p(F)$ for all $e \in \mathbb{N}^+$. When F contains a primitive p^e-th root of unity, it follows from [17, Proposition 2.5] that $\lambda_{p^e}(F) \leqslant e\lambda_p(F)$, but in general, this problem is still open.

In this article, we consider the following question.

Question 1.4. Let $e \in \mathbb{N}^+$. Can one bound the index of a central simple F-algebra of exponent p^e in terms of e and $\lambda_p(F)$?

This is obviously true when $e = 1$. In the case where F contains a primitive p^e-th root of unity, one can distill from the proof of [17, Proposition 2.5] an argument showing that the index of any central simple F-algebra of exponent p^e is bounded by $p^{\frac{e(e+1)}{2}\lambda_p(F)}$. We retrieve this bound in Theorem 2.6 by means of a lifting argument formulated in Proposition 2.4.

In Section 3, we consider the case where $p^e = 4$ and make no assumption on roots of unity. For a nonreal field F, we obtain in Corollary 3.12 an upper bound on the index of exponent-4 algebras in terms of the u-invariant of F.

Section 4 is devoted to the construction of examples of nonreal fields with given u-invariant admitting a central simple algebra of given 2-primary exponent and of comparatively large index; see Proposition 4.3. If F is nonreal and $u(F) = 4$, then by Corollary 3.12 the index of a central simple F-algebra of exponent 4 is at most 8, and we see in Example 4.4 that this is optimal. This example provides at the same time quadratic field extensions K/F with $u(F) = 4$ and $u(K) = 6$; see Example 4.5. Hence, Section 4 provides also an alternative construction of fields of u-invariant 6.

2. Multiplication by a power of p in the Brauer group

For a finite field extension K/F, let $\mathsf{N}_{K/F} : K \to F$ denote the norm map.

Theorem 2.1. *Let $\zeta \in F$ be a primitive p-th root of unity. Let K/F be a cyclic field extension of degree p^{e-1}. Then K/F embeds into a cyclic field extension of degree p^e of F if and only if $\zeta = \mathsf{N}_{K/F}(x)$ for some $x \in K$.*

Proof. See [2, Theorem 9.11]. □

Let A and B be central simple F-algebras. We write $A \sim B$ to indicate that A and B are Brauer equivalent. For $n \in \mathbb{N}^+$ we denote by $A^{\otimes n}$ the n-fold tensor product $A \otimes_F \ldots \otimes_F A$.

Theorem 2.2 (Albert). *Let $n, m \in \mathbb{N}$ with $m \leqslant n$ and $b \in F^\times$. Let L/F be a cyclic field extension of degree p^n and let σ be a generator of its Galois group. Let K be the fixed field of $\sigma^{p^{n-m}}$ in L. Then*

$$[L/F, \sigma, b)^{\otimes p^m} \sim [K/F, \sigma|_K, b).$$

Proof. See [3, Theorem 7.14]. □

Corollary 2.3. *Let $\zeta \in F$ be a primitive p-th root of unity. Let $e \in \mathbb{N}^+$. For $\alpha \in \mathsf{Br}(F)$, the following are equivalent:*

- (i) *α is the class of a cyclic F-algebra of degree p^{e-1} containing a cyclic field extension K/F of degree p^{e-1} such that $\zeta = \mathsf{N}_{K/F}(x)$ for some $x \in K$.*
- (ii) *$\alpha = p\beta$ for the class $\beta \in \mathsf{Br}(F)$ of a cyclic F-algebra of degree p^e.*

Proof. $(i \Rightarrow ii)$ Assume that K/F is a cyclic field extension of degree p^{e-1}, σ a generator of its Galois group and $b \in F^\times$ is such that α is represented by $[K/F, \sigma, b)$. Assume further that $\zeta = \mathsf{N}_{K/F}(x)$ for some $x \in K$. By Theorem 2.1, there exists a field extension L/K of degree p such that L/F is cyclic. Then σ extends to an F-automorphism σ' of L, and it follows that σ' generates the Galois group of L/F. Let β be the class of the cyclic F-algebra $[L/F, \sigma', b)$. Since $[L : K] = p$ and $\sigma'|_K = \sigma$, we conclude by Theorem 2.2 that $p\beta = \alpha$.

$(ii \Rightarrow i)$ Assume that $\alpha = p\beta$ where $\beta \in \mathsf{Br}(F)$ is the class of a cyclic F-algebra of degree p^e. Then β is given by $[L/F, \sigma, b)$ for some cyclic field extension L/F of degree p^e, a generator σ of its Galois group and some $b \in F^\times$. Let K denote the fixed field of $\sigma^{p^{e-1}}$ in L. Then K/F is cyclic of degree p^{e-1}, and we obtain by Theorem 2.1 that $\zeta = \mathsf{N}_{K/F}(x)$ for some $x \in K$. By Theorem 2.2, we have $[L/F, \sigma, b)^{\otimes p} \sim [K/F, \sigma|_K, b)$. Hence, α is given by $[K/F, \sigma|_K, b)$. □

Given a central simple F-algebra A, we denote by $\deg A$, $\operatorname{ind} A$ and $\exp A$, the degree, index and exponent of A, respectively. For $\alpha \in \mathsf{Br}(F)$, we write $\operatorname{ind}\alpha$ and $\exp\alpha$ for the index and the exponent of any central simple F-algebra representing α.

Given a field extension F'/F and $\alpha \in \mathsf{Br}(F)$ we denote by $\alpha_{F'}$ the image of α under the natural map $\mathsf{Br}(F) \to \mathsf{Br}(F')$ induced by scalar extension.

Let $m \in \mathbb{N}^+$. We call $\alpha \in \mathsf{Br}(F)$ an *m-cycle* if $\exp\alpha = m = [K : F]$ for some cyclic field extension K/F for which $\alpha_K = 0$. Hence, given a central F-division algebra D, the class of D in $\mathsf{Br}(F)$ is an m-cycle if and only if D is cyclic and $\exp D = \deg D = m$.

Proposition 2.4. *Let $e, i \in \mathbb{N}^+$ with $i \leqslant e$ and such that every cyclic field extension of degree p^i of F embeds into a cyclic field extension of degree p^e of F. Then every p^i-cycle in $\mathsf{Br}(F)$ is of the form $p^{e-i}\beta$ for a p^e-cycle $\beta \in \mathsf{Br}(F)$.*

Proof. Let $\alpha \in \mathsf{Br}(F)$ be a p^i-cycle. Hence, α is given by $D = [K/F, \sigma, b)$ for a cyclic field extension K/F of degree p^i, a generator σ of its Galois group and some $b \in F^\times$. In particular $\mathsf{deg}\, D = p^i = \mathsf{exp}\,\alpha = \mathsf{exp}\, D$, whereby D is a division algebra. By the hypothesis, K/F embeds into a cyclic field extension L/F of degree p^e. Then σ extends to an F-automorphism σ' of L. It follows that σ' is a generator of the Galois group of L/F. We set $\Delta = [L/F, \sigma', b)$ and denote by β the class of Δ in $\mathsf{Br}(F)$. We obtain by Theorem 2.2 that $\Delta^{\otimes p^{e-i}} \sim D$, whereby $p^{e-i}\beta = \alpha$. Since $\mathsf{exp}\,\alpha = p^i$, it follows that $\mathsf{exp}\,\beta = p^e = \mathsf{deg}\,\Delta$. Since $\beta_L = 0$, we conclude that β is a p^e-cycle. □

An element $\alpha \in \mathsf{Br}(F)$ is called a *cycle* if it is an m-cycle for some $m \in \mathbb{N}^+$ (given by $m = \mathsf{exp}\,\alpha$).

Corollary 2.5. *Let $e \in \mathbb{N}^+$ be such that F contains a primitive p^e-th root of unity. Then every cycle in $\mathsf{Br}_{p^e}(F)$ is a multiple of a p^e-cycle.*

Proof. Let $\omega \in F$ be a primitive p^e-th root of unity and set $\zeta = \omega^{p^{e-1}}$. Then ζ is a primitive p-th root of unity. For any field extension K/F of degree p^i with $1 \leqslant i \leqslant e-1$, we have that $\zeta = (\omega^{p^{e-i-1}})^{p^i} = \mathsf{N}_{K/F}(\omega^{p^{e-i-1}})$. Hence, it follows by induction on i from Theorem 2.1 that every cyclic field extension of degree p^i of F embeds into a cyclic field extension of degree p^e. Now the conclusion follows by Proposition 2.4. □

The following bound can be easily derived from the proof of [17, Proposition 2.5]. To illustrate the general strategy, we include an argument.

Theorem 2.6. *Let $e \in \mathbb{N}^+$ be such that F contains a primitive p^e-th root of unity. Then $\mathsf{Br}_{p^e}(F)$ is generated by the p^e-cycles. Furthermore, for every $\alpha \in \mathsf{Br}_{p^e}(F)$, we have $\mathsf{ind}\,\alpha = p^n$ for some $n \in \mathbb{N}^+$ with*

$$n \leqslant \tfrac{e(e+1)}{2}\lambda_p(F)\,.$$

Proof. Consider $\alpha \in \mathsf{Br}_{p^e}(F)$. By induction on e we will show at the same time that α is a sum of p^e-cycles and that $\mathsf{ind}\,\alpha$ is of the claimed form.

We have $p^{e-1}\alpha \in \mathsf{Br}_p(F)$. It follows by Theorem 1.3 for $n = p$ and by the definition of $\lambda_p(F)$ that $p^{e-1}\alpha = \sum_{i=1}^m \gamma_i$ for some natural number $m \leqslant \lambda_p(F)$ and classes $\gamma_1, \dots, \gamma_m \in \mathsf{Br}(F)$ of cyclic F-division algebras of degree p. Then $\gamma_1, \dots, \gamma_m$ are p-cycles. By Corollary 2.5, for $1 \leqslant i \leqslant m$, we have $\gamma_i = p^{e-1}\beta_i$ for a p^e-cycle $\beta_i \in \mathsf{Br}(F)$.

We set $\alpha' = \alpha - \sum_{i=1}^m \beta_i$. Then $\alpha' \in \mathsf{Br}_{p^{e-1}}(F)$. If $e = 1$, then $\alpha' = 0$ and $\alpha = \sum_{i=1}^m \beta_i$, and we obtain that $\mathsf{ind}\,\alpha = p^n$ for some positive integer $n \leqslant m \leqslant \lambda_p(F)$, confirming the claims about α. Assume now that $e > 1$. By the induction hypothesis, α' is equal to a sum of p^{e-1}-cycles and $\mathsf{ind}\,\alpha' = p^{n'}$ for a natural number $n' \leqslant \frac{(e-1)e}{2}\lambda_p(F)$. By Corollary 2.5, every cycle in $\mathsf{Br}_{p^e}(F)$ is a multiple of a p^e-cycle, hence in particular, a sum of p^e-cycles. We conclude that α' is a sum of p^e-cycles, whereby α is a sum of p^e-cycles. Furthermore $\mathsf{ind}\,\alpha$ divides $\mathsf{ind}\,\alpha' \cdot \mathsf{ind}\,\beta_1 \cdots \mathsf{ind}\,\beta_m = p^{n'+em}$. Hence, $\mathsf{ind}\,\alpha = p^n$ for some positive integer

$$n \leqslant n' + em \leqslant \tfrac{(e-1)e}{2}\lambda_p(F) + e\lambda_p(F) = \tfrac{e(e+1)}{2}\lambda_p(F)\,.$$

This proves the claims about α. □

To obtain that $\mathsf{Br}_{p^e}(F)$ is generated by cycles, one can also conclude inductively on the basis of a weaker hypothesis on roots of unity than in Theorem 2.6.

Proposition 2.7. *Let $e \in \mathbb{N}^+$ be such that $p\,\mathsf{Br}(F) \cap \mathsf{Br}_{p^{e-1}}(F)$ is generated by elements $p\beta$ with cycles $\beta \in \mathsf{Br}_{p^e}(F)$. Then $\mathsf{Br}_{p^e}(F)$ is generated by cycles.*

Proof. Consider $\alpha \in \mathsf{Br}_{p^e}(F)$. Then $p\alpha \in p\,\mathsf{Br}(F) \cap \mathsf{Br}_{p^{e-1}}(F)$, so the hypothesis implies that $p\alpha = \sum_{i=1}^{n} p\beta_i$ for some $n \in \mathbb{N}$ and cycles $\beta_1, \dots, \beta_n \in \mathsf{Br}_{p^e}(F)$. Hence, $\alpha - \sum_{i=1}^{n} \beta_i \in \mathsf{Br}_p(F)$. By Theorem 1.3, $\alpha - \sum_{i=1}^{n} \beta_i = \sum_{i=1}^{m} \gamma_i$ for some $m \in \mathbb{N}$ and p-cycles $\gamma_1, \dots, \gamma_m \in \mathsf{Br}(F)$. Hence, α is a sum of cycles in $\mathsf{Br}_{p^e}(F)$. □

3. Multiplying by 2 in the Brauer group

From now on we assume that $\mathsf{char}\, F \neq 2$. We show that the hypotheses of Proposition 2.7 for $p = e = 2$ are satisfied to retrieve the positive answer to Question 1.1 in the case where $p^e = 4$. The argument also yields bounds on the index of exponent-4 algebras in terms of the 2-symbol length, and hence an affirmative answer to Question 1.4 for these algebras.

We denote by $\mathsf{S}_2(F)$ the set of nonzero sums of two squares in F. Note that $\mathsf{S}_2(F)$ is a subgroup of F.

The following statement is essentially contained in [11, Corollary 5.14]. We include the argument for convenience.

Proposition 3.1. *Let Q be an F-quaternion division algebra. The following are equivalent:*

(*i*) *-1 is a norm in a quadratic field extension of F contained in Q.*
(*ii*) *-1 is a reduced norm of Q.*
(*iii*) *$Q \sim C^{\otimes 2}$ for some cyclic F-algebra C of degree 4.*
(*iv*) *$Q \simeq (s,t)_F$ for certain $s \in \mathsf{S}_2(F)$ and $t \in F^\times$.*

Proof. Let $\mathsf{Nrd}_Q : Q \to F$ denote the reduced norm map. For any quadratic field extension K/F contained in Q and any $x \in K$ we have $\mathsf{Nrd}_Q(x) = \mathsf{N}_{K/F}(x)$. Therefore, the implication ($i \Rightarrow ii$) is obvious, and for ($ii \Rightarrow i$), it suffices to observe that, since Q is a division algebra, every maximal commutative subring of Q is a quadratic field extension of F.

The equivalence ($i \Leftrightarrow iii$) corresponds to the equivalence formulated in Corollary 2.3 in the case where $p = e = 2$, taking for $\alpha \in \mathsf{Br}(F)$ the class of Q.

To finish the proof, it suffices to show the equivalence ($i \Leftrightarrow iv$). As $\mathsf{char}\, F \neq 2$, any quadratic field extension of F is of the form $F(\sqrt{s})$ for some $s \in F^\times \smallsetminus F^{\times 2}$, and for such s, we have that -1 is a norm in $F(\sqrt{s})/F$ if and only if the quadratic form $X^2 + Y^2 - sZ^2$ over F is isotropic, if and only if $s \in \mathsf{S}_2(F)$. Finally, given a quadratic field extension K/F contained in Q and $s \in F^\times$ such that $K \simeq F(\sqrt{s})$, by [3, Theorem 5.9], we can find an element $t \in F^\times$ such that $Q \simeq (s,t)_F$. $\square$

We denote by $\mathsf{W}F$ the Witt ring of F and by $\mathrm{I}F$ its fundamental ideal. For $n \in \mathbb{N}^+$, we set $\mathrm{I}^n F = (\mathrm{I}F)^n$, and we call a regular quadratic form over F whose Witt equivalence class belongs to $\mathrm{I}^n F$ simply a *form in* $\mathrm{I}^n F$. Given a regular quadratic form q over F, we denote by $\dim q$ its dimension (rank). By a *torsion form* we shall mean a regular quadratic form over F whose class in $\mathsf{W}F$ has finite additive order. A quadratic form q such that $2 \times q$ is hyperbolic is called a 2-*torsion form*. The following statement describes 2-torsion forms in $\mathrm{I}^2 F$.

Lemma 3.2. *Let q be a form in I^2F. Let $m \in \mathbb{N}^+$ be such that $\dim q = 2m+2$. Then $2 \times q$ is hyperbolic if and only if q is Witt equivalent to $\perp_{i=1}^m a_i\langle\!\langle s_i, t_i\rangle\!\rangle$ for some $s_1,\dots,s_m \in \mathsf{S}_2(F)$ and $a_1,t_1,\dots,a_m,t_m \in F^\times$.*

Proof. For $s \in \mathsf{S}_2(F)$ and $t \in F^\times$, the form $2 \times \langle\!\langle s,t\rangle\!\rangle$ over F is hyperbolic. This proves the right-to-left implication.

We prove the opposite implication by induction on m. If $m = 0$, then q is a 2-dimensional quadratic form in I^2F and must therefore be hyperbolic. In particular, the statement holds in this case. Suppose now that $m \geqslant 1$. In view of the induction hypothesis, we may assume without loss of generality that q is anisotropic. As the quadratic form $2 \times q$ is hyperbolic and hence in particular isotropic, it follows by [7, Lemma 6.24] that $q \simeq q_1 \perp q_2$ for certain regular quadratic forms q_1 and q_2 over F such that $\dim q_1 = 2$ and $2 \times q_1$ is hyperbolic. We fix an element $a_1 \in F^\times$ represented by q_1. Then $q_1 \simeq \langle a_1, -a_1 s_1\rangle$ for some $s_1 \in F^\times$. As $2\times q_1$ is hyperbolic, so is $2\times\langle 1, -s_1\rangle$, whereby $s_1 \in \mathsf{S}_2(F)$. We write $q_2 \simeq \langle a\rangle \perp q'$ with $a \in F^\times$ and a $(2m-1)$-dimensional regular quadratic form q' over F. We set $q'' = q' \perp \langle s_1 a\rangle$ and $t_1 = -a_1 a$. We obtain that $q \perp -q''$ is Witt equivalent to $a_1\langle\!\langle s_1, t_1\rangle\!\rangle$. Since $s_1 \in \mathsf{S}_2(F)$, we have that $2 \times \langle\!\langle s_1,t_1\rangle\!\rangle$ is hyperbolic. Therefore, $2 \times q''$ is Witt equivalent to $2 \times q$, and hence equally hyperbolic. Furthermore, q'' is a form in I^2F. Since $\dim q'' = 2m$ and $2 \times q''$ is hyperbolic, the induction hypothesis yields that there exist $s_2,\dots,s_m \in \mathsf{S}_2(F)$ and $a_2,t_2,\dots,a_m,t_m \in F^\times$ such that q'' is Witt equivalent to $\perp_{i=2}^m a_i\langle\!\langle s_i,t_i\rangle\!\rangle$. Then q is Witt equivalent to $\perp_{i=1}^m a_i\langle\!\langle s_i,t_i\rangle\!\rangle$. This concludes the proof. $\square$

By [7, Theorem 14.3], associating to a quadratic form its Clifford algebra induces a homomorphism

$$e_2 : \mathrm{I}^2F \to \mathsf{Br}_2(F)\,.$$

By Merkurjev's Theorem [7, Theorem 44.1] together with [14, Theorem 4.1], the kernel of this homomorphism is precisely I^3F.

For a quadratic field extension K/F, we denote by $\mathsf{cor}_{K/F}$ the corestriction homomorphism $\mathsf{Br}(K) \to \mathsf{Br}(F)$ defined in [10, Section 3.B] (where it is denoted by $\mathsf{N}_{K/F}$).

Proposition 3.3. *Let $\beta \in \mathsf{Br}_2(F)$. The following are equivalent:*

(*i*) $\beta \in 2\,\mathsf{Br}(F)$.

(*ii*) $\beta = e_2(q)$ *for some* 2*-torsion form* q *in* I^2F.
(*iii*) β *is given by* $\bigotimes_{i=1}^m (s_i, t_i)_F$ *for some* $m \in \mathbb{N}$, $s_1, \dots, s_m \in \mathsf{S}_2(F)$ *and* $t_1, \dots, t_m \in F^\times$.

Moreover, if these conditions are satisfied and $\operatorname{ind}\beta \leqslant 4$*, then one can choose* m *in* (*iii*) *such that* $\operatorname{ind}\beta = 2^m$.

Proof. The implication ($iii \Rightarrow i$) follows by Proposition 3.1.

For $m \in \mathbb{N}$, $s_1, \dots, s_m \in \mathsf{S}_2(F)$ and $a_1, t_1, \dots, a_m, t_m \in F^\times$, one has that $e_2(\perp_{i=1}^m a_i \langle\langle s_i, t_i \rangle\rangle) \sim \bigotimes_{i=1}^m (s_i, t_i)_F$. Hence, the equivalence ($ii \Leftrightarrow iii$) follows by Lemma 3.2.

We show now the implication ($i \Rightarrow iii$). If $-1 \in F^{\times 2}$, then $F^\times = \mathsf{S}_2(F)$, so (*iii*) holds by Theorem 1.3. Assume now that $-1 \in F^\times \setminus F^{\times 2}$ and that (*i*) holds. We set $K = F(\sqrt{-1})$. As $\beta \in \mathsf{Br}_2(F)$, it follows by Theorem 1.3 together with [11, Corollary A4] that $\beta \cup (-1) = 0$ in $H^3(F, \mu_2)$. By [7, Theorem 99.13], we obtain that $\beta = \mathsf{cor}_{K/F}\beta'$ for some $\beta' \in \mathsf{Br}_2(K)$. By Theorem 1.3 and [7, Proposition 100.2], $\mathsf{Br}_2(K)$ is generated by the classes of K-quaternion algebras $(x, t)_K$ with $x \in K^\times$ and $t \in F^\times$, and the corestriction with respect to K/F of such a class is given by $(\mathsf{N}_{K/F}(x), t)_F$. Since $\mathsf{N}_{K/F}(K^\times) \subseteq \mathsf{S}_2(F)$ and $\beta = \mathsf{cor}_{K/F}\beta'$, we obtain that β is given by $\bigotimes_{i=1}^m (s_i, t_i)_F$ for some $m \in \mathbb{N}$, $s_1, \dots, s_m \in \mathsf{S}_2(F)$ and $t_1, \dots, t_m \in F^\times$.

Hence, the equivalence of (*i*)–(*iii*) is established and it remains to prove the supplementary statement under the assumption that $\operatorname{ind}\beta \leqslant 4$. In this case β is the class of an F-biquaternion algebra. It follows by [10, Section 16.A] that $\beta = e_2(q')$ for a 6-dimensional form q' in I^2F. By (*ii*), there also exists a 2-torsion form q in I^2F with $\beta = e_2(q)$. Then $q' \perp -q$ is a form in I^2F with $e_2(q' \perp -q) = 0$. As mentioned above, this implies that $q' \perp -q$ is a form in I^3F. Since $2 \times q$ is hyperbolic, the Witt class of $2 \times q'$ lies in I^4F. Note that $\dim 2 \times q' < 16$. Thus, $2 \times q'$ is hyperbolic, by [7, Theorem 23.7], and hence Lemma 3.2 yields the result. □

By Proposition 3.3, for $p = e = 2$, the hypotheses of Proposition 2.7 on $2\,\mathsf{Br}(F) \cap \mathsf{Br}_2(F)$ are satisfied unconditionally. Hence, one gets a positive answer to Question 1.1 for $p^e = 4$. We will formulate this result together with a bound on the index of exponent-4 algebras in terms of the 2-symbol length.

For $\alpha \in \mathsf{Br}_4(F)$, we denote by $\mu(\alpha)$ the smallest $m \in \mathbb{N}$ for which there exist $s_1, \dots, s_m \in \mathsf{S}_2(F)$ and $t_1, \dots, t_m \in F^\times$ with $2\alpha = \sum_{i=1}^m [(s_i, t_i)_F]$, noticing that such a representation does exist in view of Proposition 3.3. We set further

$$\mu(F) = \sup\{\mu(\alpha) \mid \alpha \in \mathsf{Br}_4(F)\} \in \mathbb{N} \cup \{\infty\}.$$

Remark 3.4. If $\mathsf{S}_2(F) = F^\times$, then $\mu(F) = \lambda_2(F)$.

The invariants $\lambda_2(F)$ and $\mu(F)$ are related to the existence of anisotropic torsion (respectively 2-torsion) forms over F in certain dimensions. Recall that the u-invariant of F is defined as

$$u(F) = \sup\{\dim q \mid q \text{ anisotropic torsion form over } F\} \in \mathbb{N} \cup \{\infty\}.$$

We refer to [15, Chapter 8] for a general discussion of this invariant.

Proposition 3.5. *If F is nonreal, then $\lambda_2(F) \leqslant \max\left\{0, \frac{1}{2}u(F) - 1\right\}$.*

Proof. See [9, Théorème 2]. □

In [15, Section 8.2], the following relative of the u-invariant is studied.

$$u'(F) = \sup\{\dim q \mid q \text{ anisotropic 2-torsion form over } F\} \in \mathbb{N} \cup \{\infty\}.$$

Note that clearly $u'(F) \leqslant u(F)$.

Proposition 3.6. *We have $\mu(F) \leqslant \max\left\{0, \frac{1}{2}u'(F) - 1\right\}$.*

Proof. We need to show that $\mu(\alpha) \leqslant m$ holds for any $\alpha \in \mathsf{Br}_4(F)$ and any $m \in \mathbb{N}^+$ with $u'(F) \leqslant 2m + 2$. Let $m \in \mathbb{N}^+$ be such that $u'(F) \leqslant 2m + 2$. Let $\alpha \in \mathsf{Br}_4(F)$. By Proposition 3.3, we have $2\alpha = e_2(q)$ for some 2-torsion form q in I^2F. Then $\dim q \leqslant u'(F) \leqslant 2m + 2$. Hence, q is even-dimensional and we obtain that q is Witt equivalent to a quadratic form of dimension $2m + 2$. It follows by Lemma 3.2 that q is Witt equivalent to $\perp_{i=1}^m a_i\langle\langle s_i, t_i\rangle\rangle$ for some $s_1, \dots, s_m \in \mathsf{S}_2(F)$ and $a_1, t_1, \dots, a_m, t_m \in F^\times$. Then

$$2\alpha = e_2(q) = e_2\left(\mathop{\perp}_{i=1}^{m} a_i\langle\langle s_i, t_i\rangle\rangle\right) = \sum_{i=1}^{m}[(s_i, t_i)]\,,$$

whereby $\mu(\alpha) \leqslant m$. □

The last statements motivate the following question.

Question 3.7. Is $\mu(F) \leqslant \lambda_2(F)$?

If $\lambda_2(F) \leqslant 2$, then a positive answer to Question 3.7 is obtained by Proposition 3.3. In the following example, the inequality in Proposition 3.6 is strict.

Example 3.8. Consider the iterated power series field $F = \mathbb{C}((x))((y))((z))$. The 8-dimensional quadratic form $\varphi = \langle 1, x, y, z, xy, xz, yz, xyz\rangle$ over F is anisotropic. Since -1 is square in F and $F^\times/F^{\times 2}$ is generated by the square-classes of x, y and z, it is easy to see that every anisotropic quadratic form over F is a subform of φ. This implies on the one hand that $u(F) = 8$, on the other hand that $\lambda_2(F) = 1$, because there is no anisotropic 6-dimensional form in I^2F. Furthermore $-1 \in F^{\times 2}$, so $u'(F) = u(F) = 8$ and $\mu(F) = \lambda_2(F) = 1$.

Proposition 3.9. *Let $\alpha \in \mathsf{Br}_4(F)$. There exist a natural number $m \leqslant \mu(F)$ and 4-cycles $\alpha_1, \dots, \alpha_m \in \mathsf{Br}(F)$ such that $\alpha \equiv \sum_{i=1}^m \alpha_i \mod \mathsf{Br}_2(F)$.*

Proof. By Proposition 3.3 and the definition of $\mu(F)$, there exist a natural number $m \leqslant \mu(F)$ and $s_1, \dots, s_m \in \mathsf{S}_2(F)$ and $t_1, \dots, t_m \in F^\times$ such that $2\alpha = \sum_{i=1}^m[(s_i, t_i)_F]$. By Proposition 3.1, for $1 \leqslant i \leqslant m$, we can find a 4-cycle $\alpha_i \in \mathsf{Br}_4(F)$ such that $2\alpha_i = [(s_i, t_i)_F]$. We obtain that $2\alpha - \sum_{i=1}^m 2\alpha_i = 0$, whereby $\alpha - \sum_{i=1}^m \alpha_i \in \mathsf{Br}_2(F)$. Therefore, $\alpha \equiv \sum_{i=1}^m \alpha_i \mod \mathsf{Br}_2(F)$. □

We retrieve [13, Proposition 6.16]:

Corollary 3.10. $\mathrm{Br}_4(F)$ *is generated by cycles.*

Proof. By Theorem 1.3, $\mathrm{Br}_2(F)$ is generated by classes of F-quaternion division algebras and thus by 2-cycles. The statement now follows by combining this fact with Proposition 3.9. □

Theorem 3.11. *We have* $\lambda_4(F) \leqslant \lambda_2(F) + \mu(F)$. *Furthermore, for* $\alpha \in \mathrm{Br}_4(F)$, *there exist* $\beta \in \mathrm{Br}_4(F)$ *with* $\lambda_4(\beta) \leqslant \mu(\alpha)$ *and* $\gamma \in \mathrm{Br}_2(F)$ *such that* $\alpha = \beta + \gamma$, *and in particular* $\operatorname{ind}\alpha = 2^n$ *for some natural number* $n \leqslant \lambda_2(F) + 2\mu(F)$.

Proof. Let $\alpha \in \mathrm{Br}_4(F)$ and set $m = \mu(\alpha)$. By Proposition 3.9, we obtain that $\alpha = \sum_{i=1}^m \alpha_i + \gamma$ for some 4-cycles $\alpha_1, \dots, \alpha_m \in \mathrm{Br}_4(F)$ and some $\gamma \in \mathrm{Br}_2(F)$. Set $\beta = \sum_{i=1}^m \alpha_i$. Then $\beta \in \mathrm{Br}_4(F)$ and

$$\lambda_4(\alpha) \leqslant \lambda_4(\gamma) + \lambda_4(\beta) \leqslant \lambda_2(\gamma) + m \leqslant \lambda_2(F) + \mu(F)\,.$$

Note that $\operatorname{ind}\beta$ divides $\prod_{i=1}^m \operatorname{ind}\alpha_i = 2^{2m}$. Since $\operatorname{ind}\gamma$ divides $2^{\lambda_2(\gamma)}$ and $\operatorname{ind}\alpha$ divides $\operatorname{ind}\beta \cdot \operatorname{ind}\gamma$, we obtain that $\operatorname{ind}\alpha = 2^n$ for some $n \in \mathbb{N}$ with $n \leqslant \lambda_2(F) + 2\mu(F)$. □

Note that when F contains a primitive 4-th root of unity, the bounds in Theorem 3.11 coincide with those in Theorem 2.6.

Corollary 3.12. *Assume that F is nonreal. Let* $\alpha \in \mathrm{Br}_4(F)$. *Then* $\operatorname{ind}\alpha = 2^n$ *for some natural number* $n \leqslant \max\left\{0, 3\left(\frac{1}{2}u(F) - 1\right)\right\}$.

Proof. Since $u'(F) \leqslant u(F)$, this follows by Theorem 3.11 together with Proposition 3.5 and Proposition 3.6. □

Proposition 3.13. *Let* $l = \lambda_2(F)$ *and* $m = \mu(F)$ *and assume that* $l + m < \infty$. *Let D be a central F-division algebra of degree* 2^{l+2m} *for which* $D^{\otimes 4}$ *is split. There exist F-quaternion algebras* $Q_1, \dots, Q_l$ *and cyclic F-algebras* $C_1, \dots, C_m$ *of degree* 4 *such that*

$$D \simeq \left(\bigotimes_{i=1}^{l} Q_i\right) \otimes \left(\bigotimes_{i=1}^{m} C_i\right).$$

Proof. By Theorem 3.11, the class of D in $\mathrm{Br}(F)$ is represented by such a tensor product, and since the degrees coincide, the statement follows. □

Corollary 3.14. *Assume that F is nonreal and let* $m \in \mathbb{N}$ *be such that* $u(F) = 2m + 2$. *Let D be a central F-division algebra such that* $D^{\otimes 4}$ *is split and* $\deg D = 2^{3m}$. *Then D is decomposable into a tensor product of m F-quaternion algebras and m cyclic F-algebras of degree* 4.

Proof. Since $u(F) = 2m+2$, we have $\lambda_2(F) \leqslant m$, by Proposition 3.5, and further $\mu(F) \leqslant m$, by Proposition 3.6. The statement follows by Proposition 3.13. □

Theorem 3.15. *Assume that F is nonreal with $u(F) = 4$. Let D be a central F-division algebra of degree 8 such that $D^{\otimes 4}$ is split. Then D decomposes into a tensor product of a cyclic F-algebra of degree 4 and an F-quaternion algebra. Furthermore,* $\operatorname{ind} D^{\otimes 2} = 2$*, and $u(K) = 6$ holds for every quadratic field extension K/F such that $(D^{\otimes 2})_K$ is split.*

Proof. The first part follows by Corollary 3.14 applied with $m = 1$.

Since $u(F) = 4$, we have $\lambda_2(F) \leqslant 1$, by Proposition 3.5. Hence, $\operatorname{ind} C \leqslant 2$ for every central simple F-algebra C such that $C^{\otimes 2}$ is split. Since $\operatorname{ind} D > 2$ and $D^{\otimes 4}$ is split, we conclude that $\operatorname{ind} D^{\otimes 2} = 2$.

Consider now a quadratic field extension K/F such that $(D^{\otimes 2})_K$ is split. Note that $(D^{\otimes 2})_K \simeq (D_K)^{\otimes 2}$ and $\operatorname{ind} D_K \geqslant \frac{1}{2} \operatorname{ind} D = 4$. Hence, D_K represents an element of $\operatorname{Br}_2(K)$ which is not given by any K-quaternion algebra. This shows that $\lambda_2(K) \geqslant 2$. It follows by Proposition 3.5 that $u(K) \geqslant 6$. On the other hand, since $u(F) = 4$ and $[K : F] = 2$, it follows by [8, Theorem 4.3] that $u(K) \leqslant \frac{3}{2} u(F) \leqslant 6$. Therefore, $u(K) = 6$. □

4. Examples of fields with u-invariant 6

In this section, we provide a construction leading to an example which shows that the bound in Corollary 3.12 is optimal for fields of u-invariant 4. In particular this construction provides examples of nonreal fields of u-invariant 6.

Let q be a regular quadratic form over F of dimension $n \geqslant 2$. If $n = 2$, then assume that q is not hyperbolic. Then as a polynomial in $F[X_1, \dots, X_n]$, the quadratic form $q(X_1, \dots, X_n)$ is irreducible. Thus, the ideal generated by $q(X_1, \dots, X_n)$ in the polynomial ring $F[X_1, \dots, X_n]$ is a prime ideal, and hence the quotient ring $F[X_1, \dots, X_n]/(q(X_1, \dots, X_n))$ is a domain. Its fraction field is denoted by $F(q)$ and called the *function field of q over F*.

Lemma 4.1. *Let $m, n \in \mathbb{N}^+$. Let $\alpha \in \operatorname{Br}(F)$ be such that $\operatorname{ind} \alpha = 2^n$. Let q be a regular $(2m+1)$-dimensional quadratic form over F such that $\operatorname{ind} \alpha_{F(q)} < \operatorname{ind} \alpha$. Then $n \geqslant m$. Moreover, if $n > m$, then $\operatorname{ind} 2\alpha \leqslant 2^{n-m-1}$.*

Proof. Let D be the central F-division algebra representing α in $\operatorname{Br}(F)$. Then $\deg D = \operatorname{ind} \alpha = 2^n$. Let $C_0(q)$ denote the even Clifford algebra of q. By [7, Proposition 11.6], the F-algebra $C_0(q)$ is central simple. As $\dim_F C_0(q) = 2^{2m}$, we have $\deg C_0(q) = 2^m$. By [7, Example 11.3 and Proposition 11.4 (*b*)], $C_0(q)$ carries an F-linear involution. Therefore, $(C_0(q))^{\otimes 2}$ is split.

Since $\operatorname{ind} D_{F(q)} = \operatorname{ind} \alpha_{F(q)} < \operatorname{ind} \alpha = \deg D$, it follows by [7, Proposition 30.5], that there exists an F-algebra homomorphism $C_0(q) \to D$. As $C_0(q)$ and D are central simple F-algebras, it follows that $D \simeq C_0(q) \otimes_F B$ for a central F-division algebra B. Hence, $2^n = \deg D = 2^m \cdot \deg B$, so in particular $n \geqslant m$.

Assume now that $n > m$. Then $\operatorname{ind} B = \deg B = 2^{n-m} \geqslant 2$. Since $(C_0(q))^{\otimes 2}$ is split, the class $2\alpha \in \operatorname{Br}_2(F)$ is given by $B^{\otimes 2}$. Hence, $\operatorname{ind} 2\alpha = \operatorname{ind} B^{\otimes 2}$. By [3, Lemma 5.7], we have $\operatorname{ind} B^{\otimes 2} \leqslant \frac{1}{2} \operatorname{ind} B$. Therefore, $\operatorname{ind} 2\alpha \leqslant 2^{n-m-1}$. □

Theorem 4.2. *Let $\mathcal{C}$ be a class of field extensions of F with the following properties:*

(i) *$\mathcal{C}$ is closed under direct limits,*
(ii) *if $L/F \in \mathcal{C}$ and K/F is a subextension of L/F then $K/F \in \mathcal{C}$,*
(iii) *$F/F \in \mathcal{C}$.*

Then there exists a field extension $K/F \in \mathcal{C}$ such that $K(\varphi)/F \notin \mathcal{C}$ for any anisotropic quadratic form φ over K of dimension at least 2.

Proof. See [5, Theorem 6.1]. □

The following statement and its hypotheses are motivated by an application which we obtain in Example 4.4.

Proposition 4.3. *Let $m, e \in \mathbb{N}^+$ with $m \geqslant 2$. Let $\alpha \in \mathsf{Br}(F)$ be such that $\exp \alpha = 2^e$, $\operatorname{ind} \alpha = 2^{me-1}$ and $\operatorname{ind} 2^i\alpha = 2^{me-1-i}$ for $0 \leqslant i \leqslant e-1$. There exists a field extension K/F such that $u(K) \leqslant 2m$, $\exp \alpha_K = 2^e$ and $\operatorname{ind} \alpha_K = 2^{me-1}$.*

Proof. Let $\mathcal{C}$ be the class of field extensions K/F such that $\operatorname{ind} 2^i\alpha_K \geqslant 2^{me-mi-1}$ for $0 \leqslant i \leqslant e-1$. Then $\mathcal{C}$ satisfies the conditions of Theorem 4.2. Hence, there exists a field extension $K/F \in \mathcal{C}$ such that $K(\varphi)/F \notin \mathcal{C}$ for any anisotropic quadratic form φ over K of dimension at least 2. As $\operatorname{ind} 2^{e-1}\alpha_K \geqslant 2^{m-1}$, $m \geqslant 2$ and $\exp \alpha = 2^e$, we get that $\exp \alpha_K = 2^e$. Since $\operatorname{ind} \alpha_K \geqslant 2^{me-1}$ and $\operatorname{ind} \alpha = 2^{me-1}$, we have that $\operatorname{ind} \alpha_K = 2^{me-1}$.

Let φ be an arbitrary $(2m+1)$-dimensional quadratic form over K. We claim that φ is isotropic. Let $\alpha_i = 2^i\alpha_K$ for $0 \leqslant i \leqslant e-1$. We will check for $0 \leqslant i \leqslant e-1$ that the inequality $\operatorname{ind} \alpha_i \geqslant 2^{me-mi-1}$ is preserved under scalar extension from K to $K(\varphi)$. Consider first the case where $i = e-1$. If $\operatorname{ind} \alpha_{e-1} = 2^{m-1}$, then $\operatorname{ind}(\alpha_{e-1})_{K(\varphi)} = 2^{m-1}$, by Lemma 4.1. Otherwise, $\operatorname{ind} \alpha_{e-1} \geqslant 2^m$, and therefore $\operatorname{ind}(\alpha_{e-1})_{K(\varphi)} \geqslant 2^{m-1}$. Consider now the case where $0 \leqslant i \leqslant e-2$. Note that $me - mi - 1 \geqslant m+1$, because $m \geqslant 2$. If $\operatorname{ind} \alpha_i = 2^{me-mi-1}$, then since $\operatorname{ind} 2\alpha_i = \operatorname{ind} \alpha_{i+1} \geqslant 2^{me-mi-1-m}$, we conclude by Lemma 4.1 that $\operatorname{ind}(\alpha_i)_{K(\varphi)} = \operatorname{ind} \alpha_i$. Otherwise, $\operatorname{ind} \alpha_i \geqslant 2^{me-mi}$, and hence $\operatorname{ind}(\alpha_i)_{K(\varphi)} \geqslant 2^{me-mi-1}$. Therefore, we have $\operatorname{ind}(\alpha_i)_{K(\varphi)} \geqslant 2^{me-mi-1}$ for $0 \leqslant i \leqslant e-1$. This shows that $K(\varphi)/F \in \mathcal{C}$. In view of the choice of K, this implies that φ is isotropic. This argument shows that $u(K) \leqslant 2m$. □

We can now show that the bound in Corollary 3.12 is optimal when $u(F) \leqslant 4$.

Example 4.4. Let $m, e \in \mathbb{N}^+$ with $m \geqslant 2$. By [16, Construction 2.8], there exist a nonreal field F of characteristic different from 2 and a central F-division algebra D such that $\exp D = 2^e$, $\deg D = 2^{me-1}$ and $\operatorname{ind} D^{\otimes 2^i} = 2^{me-1-i}$ for $1 \leqslant i \leqslant e-1$. Then Proposition 4.3 (applied to the the Brauer equivalence class of D) yields a field extension F'/F such that $u(F') \leqslant 2m$, $\exp D_{F'} = 2^e$ and $\operatorname{ind} D_{F'} = 2^{me-1}$. In the case where $m = 2$, it follows that $u(F') = 4$.

Example 4.5. By Example 4.4, there exist a nonreal field F with $\operatorname{char} F \neq 2$ together with an F-division algebra D of degree 8 such that $u(F) = 4$ and $D^{\otimes 4}$

is split. By Theorem 3.15, it follows that $\operatorname{ind} D^{\otimes 2} = 2$ and that $u(K) = 6$ for every quadratic field extension K/F such that $(D^{\otimes 2})_K$ is split.

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