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Zinbiel superalgebras

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ABSTRACT. Throughout the present work, we extend the study of Zinbiel algebras to Zinbiel superalgebras. In particular, we show that all Zinbiel superalgebras over an arbitrary field are nilpotent in the same way as occurs for Zinbiel algebras. In addition, and since the most important cases of nilpotent algebras or superalgebras are those with maximal nilpotency index, we study the complex null-filiform Zinbiel superalgebra proving that it is unique up to isomorphism. After that, we characterize the naturally graded filiform ones and obtain low-dimensional classifications.

CONTENTS

Introduction

Loday introduced a class of symmetric operads generated by one bilinear operation subject to one relation making each left-normed product of three elements equal to a linear combination of right-normed products:

$$(a_1a_2)a_3 = \sum_{\sigma\in\mathbb{S}_3} x_\sigma a_{\sigma(1)}(a_{\sigma(2)}a_{\sigma(3)});$$

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such an operad is called a parametrized one-relation operad. For a particular choice of parameters $\{x_\sigma\}$, this operad is said to be regular if each of its components is the regular representation of the symmetric group; equivalently, the corresponding free algebra on a vector space V is, as a graded vector space, isomorphic to the tensor algebra of V. Bremner and Dotsenko classified, over an algebraically closed field of characteristic zero, all regular parametrized one-relation operads. In fact, they proved that each such operad is isomorphic to one of the following five operads: the left-nilpotent operad, the associative operad, the Leibniz operad, the Zinbiel operad, and the Poisson operad [6]. Then, an algebra $\mathbf{Z}$ is called a (left) *Zinbiel algebra* if it satisfies the identity

$$(xy)z = x(yz + zy).$$

Zinbiel algebras were introduced by Loday in [26]. Under the Koszul duality, the operad of Zinbiel algebras is dual to the operad of Leibniz algebras. Zinbiel algebras are also known as pre-commutative algebras [25] and chronological algebras [24]. Remark that a Zinbiel algebra is equivalent to a commutative dendriform algebra [3]. Also, the variety of Zinbiel algebras is a proper subvariety in the variety of right commutative algebras and each Zinbiel algebra with the commutator multiplication gives a Tortkara algebra [16]. Zinbiel algebras also give an example of algebras of slowly growing length [19]. Recently, the notion of matching Zinbiel algebras was introduced in [18] and the defined identities for mono and binary Zinbliel algebras are studied in [21]. Moreover, Zinbiel algebras also appeared in a study of rack cohomology [14], number theory [12] and in the construction of a Cartesian differential category [20]. Thus, we can assert that in recent years, there has been a strong interest in the study of Zinbiel algebras in the algebraic and the operad context, see for instance [1, 2, 4, 7, 5, 15, 28, 18, 22, 16, 23, 30, 17, 9, 11, 21, 29, 27].

Free Zinbiel algebras were shown to be precisely the shuffle product algebra [27], which is under a certain interest until now [13]. Naurazbekova proved that, over a field of characteristic zero, free Zinbiel algebras are the free associative-commutative algebras (without unity) with respect to the symmetrization multiplication and their free generators are found; also she constructed examples of subalgebras of the two-generated free Zinbiel algebra that are free Zinbiel algebras of countable rank [28]. Nilpotent algebras play an important role in the class of Zinbiel algebras. So, Dzhumadildaev and Tulenbaev proved that each complex finite dimensional Zinbiel algebra is nilpotent [17]; this result was generalized by Towers for an arbitrary field [30]. Naurazbekova and Umirbaev proved that in characteristic zero any proper subvariety of the variety of Zinbiel algebras is nilpotent [29]. Finite-dimensional Zinbiel algebras with a "big" nilpotency index are classified in [1, 7]. Central extensions of three-dimensional Zinbiel algebras were calculated in [22] and of filiform Zinbiel algebras in [9]. The description of all degenerations in the variety of complex four-dimensional Zinbiel algebras is given in [23] and the geometric classification of complex five-dimensional Zinbiel algebras is given in [4]. After that, Ceballos and Towers studied abelian subalgebras and ideals of maximal dimension in Zinbiel algebras [11].

Our main goal then, for the present paper, is to extend the study of Zinbiel algebras to Zinbiel superalgebras. Thus, we prove that all the Zinbiel superalgebras over an arbitrary field are nilpotent, as occurs for Zinbiel algebras. The most important cases of nilpotent superalgebras are those with maximal nilpotency index, therefore we study the null-filiform and filiform cases. For the former, we show that they are unique up to isomorphism. For the latter, we characterise the naturally graded ones for arbitrary dimensions. Note that among all the gradations, the most important for nilpotent algebras or superalgebras is the natural gradation which comes from the filtration defined by the descending central sequence. Finally, we complete the study of Zinbiel superalgebras providing low-dimensional classifications.

1. Preliminaries and basic definitions

Zinbiel algebras. First, we recall some definitions and basic results regarding Zinbiel algebras.

Definition 1.1. *An algebra* $\mathbf{Z}$ *is called a Zinbiel algebra if it satisfies the identity*

$$(xy)z = x(yz + zy).$$

For a given Zinbiel (super)algebra $\mathbf{Z}$, the following sequence is defined:

$$\mathbf{Z}^1 = \mathbf{Z}, \quad \mathbf{Z}^{k+1} = \mathbf{Z}\mathbf{Z}^k.$$

Definition 1.2. *A Zinbiel (super)algebra* $\mathbf{Z}$ *is called nilpotent if there exists* $s \in \mathbb{N}$ *such that* $\mathbf{Z}^s = 0$*. The minimal number* s *satisfying this property is called the nilpotency index of the (super)algebra* $\mathbf{Z}$*.*

It is not difficult to see that the index of nilpotency of an arbitrary n-dimensional nilpotent Zinbiel (super)algebra does not exceed the number $n+1$. Since every finite-dimensional Zinbiel algebra over a field is nilpotent [30], it made perfect sense to start studying those with the maximal index of nilpotency, i.e. the null-filiform ones.

Definition 1.3. *An* n*-dimensional Zinbiel (super)algebra* $\mathbf{Z}$ *is called null-filiform if* $\dim \mathbf{Z}^i = n + 1 - i$*.*

Let us note that a Zinbiel algebra is null-filiform if and only if it is one-generated. The classification of complex null-filiform Zinbiel algebras was given in [2].

Theorem 1.4. *An arbitrary* n*-dimensional null-filiform Zinbiel algebra is isomorphic to the following algebra:*

$$e_i e_j = C_{i+j-1}^{j} e_{i+j}.$$

After having obtained the aforementioned Zinbiel algebras, the next case to consider is the filiform case. Let us denote by L_x the operator of left multiplication by an element x. Then, for the operator L_x, we can consider a descending sequence $C(x) = (n_1, n_2, \dots, n_k)$ with $n_1 + \dots + n_k = n$, which consists of the dimensions of the Jordan blocks of the operator L_x. In the set of such sequences, we consider the lexicographic order, that is, $C(x) = (n_1, n_2, \dots, n_k) < C(y) = (m_1, m_2, \dots, m_s)$ if there exists i such that $n_i < m_i$ and $n_j = m_j$ for $j < i$.

Definition 1.5. *The sequence $C(\mathbf{Z}) = max\{C(x) : x \in \mathbf{Z}^1 \backslash \mathbf{Z}^2\}$ is called the characteristic sequence of the Zinbiel algebra $\mathbf{Z}$.*

Definition 1.6. *The Zinbiel algebra $\mathbf{Z}$ is called p-filiform if $C(\mathbf{Z}) = (n - p, \underbrace{1, \dots, 1}_{p})$. If $p = 0$ (respectively, $p = 1$), then $\mathbf{Z}$ is called a null-filiform (respectively, filiform) Zinbiel algebra.*

Let $\mathbf{Z}$ be a finite-dimensional complex Zinbiel algebra with the nilpotency index equal to s. Let us consider $\mathbf{Z}_i = \mathbf{Z}^i / \mathbf{Z}^{i+1}$ and denote by $\mathrm{gr}(\mathbf{Z}) = \mathbf{Z}_1 \oplus \mathbf{Z}_2 \oplus \dots \oplus \mathbf{Z}_{s-1}$. It can be easily checked that $\mathrm{gr}(\mathbf{Z})$ is a graded Zinbiel algebra. If $\mathbf{Z}$ and $\mathrm{gr}(\mathbf{Z})$ are isomorphic, then $\mathbf{Z}$ is said to be naturally graded. The classification of complex naturally graded filiform Zinbiel algebras was given in [2]:

Theorem 1.7. *An arbitrary n-dimensional ($n \geq 5$) naturally graded complex filiform Zinbiel algebra is isomorphic to the following algebra:*

$$e_i e_j = C^j_{i+j-1} e_{i+j}, \quad \text{for} \quad 2 \leq i + j \leq n - 1.$$

Recently in [5], symmetric (left and right) Zinbiel superalgebras have been studied. We have extracted the following results concerning (left) Zinbiel superalgebras, which we will refer to just as Zinbiel superalgebras.

Zinbiel superalgebras. The notion of Zinbiel superalgebra can be obtained in the usual way.

Definition 1.8. *Let $\mathbf{Z} = \mathbf{Z}_{\bar{0}} \oplus \mathbf{Z}_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space with a bilinear map on $\mathbf{Z}$ such that $\mathbf{Z}_i \mathbf{Z}_j \subset \mathbf{Z}_{i+j}$. $\mathbf{Z}$ is called a Zinbiel superalgebra if, for all homogeneous elements $x, y, z \in \mathbf{Z}_{\bar{0}} \cup \mathbf{Z}_{\bar{1}}$, it satisfies*

$$(xy)z = x\big(yz + (-1)^{|y||z|} zy\big).$$

As usual, for $x \in \mathbf{Z}_{\bar{0}} \cup \mathbf{Z}_{\bar{1}}$, it is defined the corresponding endomorphism of $\mathbf{Z}$ by $L_x(y) = xy$ for all $y \in \mathbf{Z}$ which is called the left multiplication by x.

Remark 1.9. *In the same way that any Zinbiel algebra is a right-commutative algebra, any Zinbiel superalgebra is a right-commutative superalgebra. Namely, for any homogeneous elements x, y and z, the following superidentity is satisfied:*

$$(xy)z = (-1)^{|y||z|}(xz)y.$$

Next, we extend the definitions and first results of Zinbiel algebras to Zinbiel superalgebras. Thus, for a given Zinbiel superalgebra $\mathbf{Z} = \mathbf{Z}_{\bar{0}} \oplus \mathbf{Z}_{\bar{1}}$, we define the following sequence:

$$\mathbf{Z}^1 = \mathbf{Z}, \quad \mathbf{Z}^{k+1} = \mathbf{Z}\mathbf{Z}^k.$$

Let us note that also two sequences can be defined as follows:

$$\mathbf{Z}_{\bar{0}}^1 = \mathbf{Z}_{\bar{0}}, \quad \mathbf{Z}_{\bar{0}}^{k+1} = \mathbf{Z}_{\bar{0}}\mathbf{Z}_{\bar{0}}^k \quad \text{and} \quad \mathbf{Z}_{\bar{1}}^1 = \mathbf{Z}_{\bar{1}}, \quad \mathbf{Z}_{\bar{1}}^{k+1} = \mathbf{Z}_{\bar{0}}\mathbf{Z}_{\bar{1}}^k.$$

Along the last section of the present paper, we will show that all Zinbiel superalgebras over an arbitrary field are nilpotent, therefore the study of null-filiform and filiform is crucial for the understanding of finite-dimensional Zinbiel superalgebras. Now, let us introduce, following the spirit of the theory of nilpotent superalgebras for Lie and Leibniz cases (see for example [8] and reference therein), the concepts of characteristic sequence and filiform for superalgebras.

Firstly, for any even element $x \in \mathbf{Z}_{\bar{0}}$, we can consider the restricted map $L_x : \mathbf{Z}_{\bar{0}} \to \mathbf{Z}_{\bar{0}}$ and we can denote by $C_0(x)$ the corresponding descending sequence of the dimensions of Jordan blocks of the operator L_x acting on $\mathbf{Z}_{\bar{0}}$. Analogously, we can consider the restricted map $L_x : \mathbf{Z}_{\bar{1}} \to \mathbf{Z}_{\bar{1}}$ and we can denote by $C_1(x)$ the corresponding descending sequence of the dimensions of Jordan blocks of the operator L_x acting on $\mathbf{Z}_{\bar{1}}$. Then, with regard to the lexicographic order we have the following definition.

Definition 1.10. *The sequence*

$$C(\mathbf{Z}) = \left(\max_{x \in \mathbf{Z}_{\bar{0}} \setminus \mathbf{Z}_{\bar{0}}^2} C_0(x) \,\middle|\, \max_{y \in \mathbf{Z}_{\bar{0}} \setminus \mathbf{Z}_{\bar{0}}^2} C_1(y) \right),$$

is called the characteristic sequence of the Zinbiel superalgebra $\mathbf{Z}$.

Along the present work we assume that both characteristic sequences of the definition are obtained by the same generator element $x \in \mathbf{Z}_{\bar{0}} \setminus \mathbf{Z}_{\bar{0}}^2$ which is usually called the characteristic element.

Definition 1.11. *A Zinbiel superalgebra* $\mathbf{Z} = \mathbf{Z}_{\bar{0}} \oplus \mathbf{Z}_{\bar{1}}$*, with* $\dim(\mathbf{Z}_{\bar{0}}) = n$ *and* $\dim(\mathbf{Z}_{\bar{1}}) = m$*, is said to be filiform if its characteristic sequence is exactly* $C(\mathbf{Z}) = (n-1|\, m)$*.*

Remark 1.12. *Note that if* $\mathbf{Z}$ *is a filiform Zinbiel superalgebra, then* $\mathbf{Z}_{\bar{0}}$ *is a filiform Zinbiel algebra.*

Recall that among all the gradations, the most important for nilpotent structures is the natural gradation which comes from the filtration defined by the descending central sequence. Recently, it has been defined the concept of naturally graded for both nilpotent superalgebras, Lie and Leibniz [10]. We can extend this concept to Zinbiel superalgebras, which are all of them nilpotent.

Consider a Zinbiel superalgebra $\mathbf{Z} = \mathbf{Z}_{\bar{0}} \oplus \mathbf{Z}_{\bar{1}}$. It is clear that the sequences $\{\mathbf{Z}_{\bar{0}}^k\}$ and $\{\mathbf{Z}_{\bar{1}}^k\}$ define a filtration over $\mathbf{Z}_{\bar{0}}$ and $\mathbf{Z}_{\bar{1}}$, respectively. If we denote $\mathfrak{z}_{\bar{0}}^i := \mathbf{Z}_{\bar{0}}^{i-1}/\mathbf{Z}_{\bar{0}}^i$ and $\mathfrak{z}_{\bar{1}}^i := \mathbf{Z}_{\bar{1}}^{i-1}/\mathbf{Z}_{\bar{1}}^i$, then it is verified that $\mathfrak{z}_{\bar{0}}^i \mathfrak{z}_{\bar{0}}^j \subset \mathfrak{z}_{\bar{0}}^{i+j}$ and $\mathfrak{z}_{\bar{0}}^i \mathfrak{z}_{\bar{1}}^j \subset \mathfrak{z}_{\bar{1}}^{i+j}$. The definition of a natural gradation follows.

Definition 1.13. *Given a Zinbiel superalgebra* $\mathbf{Z} = \mathbf{Z}_{\bar{0}} \oplus \mathbf{Z}_{\bar{1}}$*, consider* $\mathfrak{z}^i = \mathfrak{z}^i_{\bar{0}} \oplus \mathfrak{z}^i_{\bar{1}}$*, with* $\mathfrak{z}^i_{\bar{0}} = \mathbf{Z}^{i-1}_{\bar{0}}/\mathbf{Z}^i_{\bar{0}}$ *and* $\mathfrak{z}^i_{\bar{1}} = \mathbf{Z}^{i-1}_{\bar{1}}/\mathbf{Z}^i_{\bar{1}}$*. Thus,* $\mathbf{Z}$ *is said to be naturally graded if the following conditions hold:*

1. $\mathrm{gr}(\mathbf{Z}) = \sum_{i\in\mathbb{N}} \mathfrak{z}^i$ *is a graded superalgebra* ($\mathfrak{z}^i\mathfrak{z}^j \subset \mathfrak{z}^{i+j}$)*,*
2. $\mathbf{Z}$ *and* $\mathrm{gr}(\mathbf{Z})$ *are isomorphic.*

2. Null-filiform Zinbiel superalgebras

Theorem 2.1. *Let* $\mathbf{Z}$ *be an n-dimensional null-filiform Zinbiel superalgebra with* $\dim(\mathbf{Z}_{\bar{1}}) \neq 0$*. Then* $\mathbf{Z}$ *is isomorphic to the following superalgebra which occurs only for the cases* $\dim(\mathbf{Z}_{\bar{0}}) = \dim(\mathbf{Z}_{\bar{1}})$ *and* $\dim(\mathbf{Z}_{\bar{1}}) = \dim(\mathbf{Z}_{\bar{0}}) + 1$:

$$\begin{array}{lcl} e_{2k+1}e_{2l} & = & C^k_{k+l}e_{2k+2l+1}, \\ e_{2k}e_{2l} & = & C^l_{k+l-1}e_{2k+2l}, \\ e_{2k+1}e_{2l+1} & = & C^l_{k+l}e_{2k+2l+2}, \end{array}$$

where $e_{2k}, e_{2l} \in \mathbf{Z}_{\bar{0}}$ *and* $e_{2k+1}, e_{2l+1} \in \mathbf{Z}_{\bar{1}}$.

Proof. It is clear, that each null-filiform Zinbiel superalgebra is one-generated. If it is generated by an even element, then it has zero odd part. Hence, we can suppose that our superalgebra is generated by an odd element e_1 and then, in the same way, as for null-filiform Zinbiel algebras, we can consider:

$$e_2 := e_1e_1, \quad e_3 := e_1(e_1e_1), \quad \dots, \quad e_n := e_1(e_1 \dots (e_1(e_1e_1))).$$

Let us remark that the elements above are linearly independent. This latter fact allows us to see $\{e_1, e_2, \dots, e_n\}$ as a basis of the superalgebra $\mathbf{Z}$ being, e_{2k+1} odd basis vectors and e_{2k} even ones. Moreover, we have only two possibilities: either $\dim(\mathbf{Z}_{\bar{0}}) = \dim(\mathbf{Z}_{\bar{1}})$ or $\dim(\mathbf{Z}_{\bar{1}}) = \dim(\mathbf{Z}_{\bar{0}}) + 1$.

Let us note that by construction we have

$$e_1e_i = e_{i+1}. \tag{1}$$

Now, we prove by induction that

$$e_{2k}e_1 = 0 \text{ and } e_{2k+1}e_1 = e_1e_{2k+1} = e_{2k+2}. \tag{2}$$

For $k = 1$ the equations hold by considering the following Zinbiel superidentity:

$$e_2e_1 = (e_1e_1)e_1 = e_1(e_1e_1) - e_1(e_1e_1) = 0$$

and then

$$e_3e_1 = (e_1e_2)e_1 = e_1(e_2e_1) + e_1(e_1e_2) = e_1e_3$$

Suppose that the equations hold for k, thus

$$e_{2k+2}e_1 = (e_1e_{2k+1})e_1 = e_1(e_{2k+1}e_1) - e_1(e_1e_{2k+1}) = 0$$

and then

$$e_{2k+3}e_1 = (e_1e_{2k+2})e_1 = e_1(e_{2k+2}e_1) + e_1(e_1e_{2k+2}) = e_1e_{2k+3}$$

and therefore we have equations (2) for $k+1$. Next, we prove, also by induction, the equations

$$e_{2k}e_2 = ke_{2k+2},\ e_{2k+1}e_2 = (k+1)e_{2k+3},\ e_2e_{2k} = e_{2k+2},\ e_2e_{2k+1} = 0. \quad (3)$$

The equations hold for $k = 1$ by considering the following Zinbiel superidentities:

$$\begin{array}{lllllllll}
e_2e_2 & = & (e_1e_1)e_2 & = & e_1(e_1e_2) + e_1(e_2e_1) & = & e_1e_3 & = & e_4, \\
e_3e_2 & = & (e_1e_2)e_2 & = & e_1(e_2e_2) + e_1(e_2e_2) & = & 2e_5, & & \\
0 & = & (e_2e_1)e_2 & = & e_2(e_1e_2) + e_2(e_2e_1) & = & e_2e_3, & &
\end{array}$$

supposing that the equations hold for k we get the equations for $k+1$:

$$\begin{array}{rclcl}
e_{2(k+1)}e_2 & = & (e_1e_{2k+1})e_2 & = & \\
& = & e_1(e_{2k+1}e_2) + e_1(e_2e_{2k+1}) & = & (k+1)e_{2(k+1)+2}, \\
e_2e_{2(k+1)} & = & (e_1e_1)e_{2k+2} & = & \\
& = & e_1(e_1e_{2k+2}) + e_1(e_{2k+2}e_1) & = & e_{2(k+1)+2}, \\
e_{2(k+1)+1}e_2 & = & (e_1e_{2k+2})e_2 & = & \\
& = & e_1(e_{2k+2}e_2) + e_1(e_2e_{2k+2}) & = & ((k+1)+1)e_{2(k+1)+3}, \\
0 & = & (e_2e_1)e_{2k+2} & = & \\
& = & e_2(e_1e_{2k+2}) + e_2(e_{2k+2}e_1) & = & e_2e_{2(k+1)+1}.
\end{array}$$

Next, on account of equations (2) together with the fact that $e_{2k}e_{2l}$ is a multiple of e_{2k+2l} we have

$$0 \quad = \quad (e_{2k}e_{2l})e_1 \quad = \quad e_{2k}(e_{2l}e_1) + e_{2k}(e_1e_{2l}) \quad = \quad e_{2k}e_{2l+1},$$

which leads to the equation $e_{2k}e_{2l+1} = 0$. Next, we prove by induction the equation

$$e_{2k+1}e_{2l} = C^k_{k+l}e_{2k+2l+1}. \quad (4)$$

From the equation (1) we get equation (4) for $k = 0$ and every l. Supposing that equation (4) holds for k and every l, we obtain that it also holds for $k+1$ and every l using the equation (3):

$$\begin{array}{rcl}
e_{2(k+1)+1}e_{2l} & = & \frac{1}{k+1}(e_{2k+1}e_2)e_{2l} \quad = \\
\frac{1}{k+1}\big(e_{2k+1}(e_2e_{2l}) + e_{2k+1}(e_{2l}e_2)\big) & = & \frac{1+l}{k+1}e_{2k+1}e_{2l+2},
\end{array}$$

but since $e_{2k+1}e_{2l+2} = C^k_{k+l+1}e_{2k+2l+3}$, then

$$\frac{1+l}{k+1}C^k_{k+l+1} \quad = \quad \frac{(1+l)(k+l+1)!}{(k+1)k!(l+1)!} \quad = \quad \frac{(k+l+1)!}{(k+1)!l!} \quad = \quad C^{k+1}_{(k+1)+l},$$

which concludes the proof of equation (4). Similarly, we prove the equation

$$e_{2k+1}e_{2l+1} = C^l_{k+l}e_{2k+2l+2}. \quad (5)$$

Equation (1) leads to equation (5) for $k = 0$ and every l. Supposing that equation (5) holds for k and every l, we obtain that it also holds for $k+1$ and every l using the equation (3):

$$e_{2(k+1)+1}e_{2l+1} = \frac{1}{k+1}(e_{2k+1}e_2)e_{2l+1} = \frac{1}{k+1}\big(e_{2k+1}(e_2e_{2l+1}) + e_{2k+1}(e_{2l+1}e_2)\big) = \frac{1+l}{k+1}e_{2k+1}e_{2l+3},$$

but since $e_{2k+1}e_{2l+3} = C_{k+l+1}^{l+1}e_{2k+2l+4}$, then

$$\frac{1+l}{k+1}C_{k+l+1}^{l+1} = \frac{(1+l)(k+l+1)!}{(k+1)k!(l+1)!} = \frac{(k+l+1)!}{(k+1)!l!} = C_{(k+1)+l}^{l},$$

which concludes the proof of equation (5). Finally, we prove the last equation

$$e_{2k}e_{2l} = C_{k+l-1}^{l}e_{2k+2l}, \quad k \geq 1,\ l \geq 1. \tag{6}$$

Equation (3) leads to equation (6) for $k = 1$ and every l. Supposing that equation (6) holds for k and every l, we obtain that it also holds for $k+1$ and every l using the equation (3):

$$e_{2(k+1)}e_{2l} = \frac{1}{k}(e_{2k}e_2)e_{2l} = \frac{1}{k}\big(e_{2k}(e_2e_{2l}) + e_{2k}(e_{2l}e_2)\big) = \frac{1+l}{k}e_{2k}e_{2l+2}$$

but as $e_{2k}e_{2l+2} = C_{k+l}^{l+1}e_{2k+2l+2}$, then

$$\frac{1+l}{k}C_{k+l}^{l+1} = \frac{(1+l)(k+l)!}{(k-1)!k(l+1)!} = \frac{(k+l)!}{k!l!} = C_{(k+1)+l-1}^{l},$$

which concludes the proof of equation (6) and of the statement of the Theorem. □

3. Naturally graded filiform Zinbiel superalgebras

Lemma 3.1. *Let* $\mathbf{Z}$ *be a complex naturally graded filiform Zinbiel superalgebra with* $\dim(\mathbf{Z}_{\bar{0}}) = n$, $n \geq 5$ *and* $\dim(\mathbf{Z}_{\bar{1}}) = m$. *Then, there are a basis* $\{e_1, \dots, e_n\}$ *for* $\mathbf{Z}_{\bar{0}}$ *and a basis* $\{f_1, \dots, f_m\}$ *for* $\mathbf{Z}_{\bar{1}}$, *for which we have the following multiplication table*

$$\begin{array}{ll} e_ie_j = C_{i+j-1}^{j}e_{i+j}, & 2 \leq i+j \leq n-1, \\ e_1f_j = f_{j+1}, & 1 \leq j \leq m-1. \end{array}$$

Proof. Consider $\mathbf{Z} = \mathbf{Z}_{\bar{0}} \oplus \mathbf{Z}_{\bar{1}}$ to be a naturally graded filiform Zinbiel superalgebra with $\dim(\mathbf{Z}_{\bar{0}}) = n$ and $\dim(\mathbf{Z}_{\bar{1}}) = m$. We set $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_m\}$ as bases of $\mathbf{Z}_{\bar{0}}$ and $\mathbf{Z}_{\bar{1}}$ respectively. It derives from the definition that $\mathbf{Z}_{\bar{0}}$ is a naturally graded Zinbiel algebra and then, for $n \geq 5$ we have from [2] completely determined its products $e_ie_j = C_{i+j-1}^{j}e_{i+j}$, for $2 \leq i+j \leq n-1$. Since e_1 is the characteristic element above then there is no loss of generality in supposing that $e_1f_j = f_{j+1}$, for $1 \leq j \leq m-1$. □

Let us assume the following convention $\prod_{k=0}^{-1} f(k) := 1$.

Theorem 3.2. *Let* $\mathbf{Z}$ *be a complex naturally graded filiform Zinbiel superalgebra with* $\dim(\mathbf{Z}_{\bar{0}}) = n$, $(n \geq 5)$, *and* $\dim(\mathbf{Z}_{\bar{1}}) = m$, $(m > 3)$. *Then* $\mathbf{Z}$ *is isomorphic to one of the following superalgebras:*

$$\mathfrak{nf}_1 \quad : \quad \begin{cases} e_i e_j = C_{i+j-1}^{j} e_{i+j}, & 2 \leq i+j \leq n-1. \\ e_i f_j = \dfrac{\prod_{k=0}^{i-2}(j+k-1)}{(i-1)!} f_{j+i}, & 1 \leq i \leq n-1. \\ f_j e_i = \dfrac{\prod_{k=0}^{i-1}(j+k-2)}{i!} f_{j+i}, & 1 \leq i \leq n-1. \\ e_n f_1 = f_2, & f_1 e_n = -f_2. \end{cases}$$

$$\mathfrak{nf}_2^{\alpha} \quad : \quad \begin{cases} e_i e_j = C_{i+j-1}^{j} e_{i+j}, & 2 \leq i+j \leq n-1. \\ e_i f_j = \dfrac{\prod_{k=0}^{i-2}(\alpha+j+k)}{(i-1)!} f_{j+i}, & 1 \leq i \leq n-1. \\ f_j e_i = \dfrac{\prod_{k=0}^{i-1}(\alpha+j+k-1)}{i!} f_{j+i}, & 1 \leq i \leq n-1. \end{cases}$$

$$\mathfrak{nf}_3 \quad : \quad \begin{cases} e_i e_j = C_{i+j-1}^{j} e_{i+j}, & 2 \leq i+j \leq n-1. \\ e_i f_j = \dfrac{\prod_{k=0}^{i-2}(2-m+j+k)}{(i-1)!} f_{j+i}, & 1 \leq i \leq n-1. \\ f_j e_i = \dfrac{\prod_{k=0}^{i-1}(1-m+j+k)}{i!} f_{j+i}, & 1 \leq i \leq n-1, \\ e_n f_{m-1} = f_m. & \end{cases}$$

If $m \geq n-2$, *the superalgebra* $\mathbf{Z}$ *is either isomorphic to one of the previous superalgebras or isomorphic to one of the next two:*

$$\mathfrak{nf}_4 \quad : \quad \begin{cases} e_i e_j = C_{i+j-1}^{j} e_{i+j}, & 2 \leq i+j \leq n-1. \\ e_i f_j = \dfrac{\prod_{k=0}^{i-2}(n-3+j+k)}{(i-1)!} f_{j+i}, & 1 \leq i \leq n-1. \\ f_j e_i = \dfrac{\prod_{k=0}^{i-1}(n-4+j+k)}{i!} f_{j+i}, & 1 \leq i \leq n-1. \\ f_1 f_{n-2} = e_{n-1}. & \end{cases}$$

$$\mathfrak{nf}_5 \quad : \quad \left\{ \begin{array}{lcll} e_i e_j & = & C_{i+j-1}^{j} e_{i+j}, & 2 \leq i+j \leq n-1. \\ e_i f_j & = & \dfrac{\prod\limits_{k=0}^{i-2}(3-n+j+k)}{(i-1)!} f_{j+i}, & 1 \leq i \leq n-1. \\ f_j e_i & = & \dfrac{\prod\limits_{k=0}^{i-1}(2-n+j+k)}{i!} f_{j+i}, & 1 \leq i \leq n-1. \\ e_n f_{n-2} & = & f_{n-1}, & f_1 f_{n-2} = e_{n-1}. \end{array} \right.$$

Proof. First, we consider $\mathbf{Z} = \mathbf{Z}_{\bar{0}} \oplus \mathbf{Z}_{\bar{1}}$ a complex naturally graded filiform Zinbiel superalgebra with $\dim(\mathbf{Z}_{\bar{0}}) = n$ with $n \geq 5$ and $\dim(\mathbf{Z}_{\bar{1}}) = m$ with $m \geq 3$. Then, there exists a basis $\{e_1, \dots, e_n, f_1, \dots, f_m\}$ and we have one of the following possibilities depending if $m < n-1$, $m = n-1$ or $m > n-1$.

$$\begin{array}{l} \underbrace{\langle e_1, e_n, f_1 \rangle}_{\mathfrak{z}^1} \oplus \underbrace{\langle e_2, f_2 \rangle}_{\mathfrak{z}^2} \oplus \dots \oplus \underbrace{\langle e_m, f_m \rangle}_{\mathfrak{z}^m} \oplus \underbrace{\langle e_{m+1} \rangle}_{\mathfrak{z}^{m+1}} \oplus \dots \oplus \underbrace{\langle e_{n-1} \rangle}_{\mathfrak{z}^{n-1}} \\ \underbrace{\langle e_1, e_n, f_1 \rangle}_{\mathfrak{z}^1} \oplus \underbrace{\langle e_2, f_2 \rangle}_{\mathfrak{z}^2} \oplus \dots \oplus \underbrace{\langle e_{n-1}, f_{n-1} \rangle}_{\mathfrak{z}^{n-1}} \\ \underbrace{\langle e_1, e_n, f_1 \rangle}_{\mathfrak{z}^1} \oplus \underbrace{\langle e_2, f_2 \rangle}_{\mathfrak{z}^2} \oplus \dots \oplus \underbrace{\langle e_{n-1}, f_{n-1} \rangle}_{\mathfrak{z}^{n-1}} \oplus \underbrace{\langle f_n \rangle}_{\mathfrak{z}^n} \oplus \dots \oplus \underbrace{\langle f_m \rangle}_{\mathfrak{z}^m} \end{array}$$

We will study these three cases together. We consider $f_1 e_1 = \alpha f_2$. By induction using the Zinbiel superidentity we obtain:

$$\begin{array}{lcllcll} e_i f_j & = & \dfrac{\prod\limits_{k=0}^{i-2}(\alpha+j+k)}{(i-1)!} f_{j+i}, & f_j e_i & = & \dfrac{\prod\limits_{k=0}^{i-1}(\alpha+j+k-1)}{i!} f_{j+i}, & 1 \leq i \leq n-1, \\ f_1 e_n & = & c_1 f_2. & e_n f_j & = & b_j f_{j+1}, & 1 \leq j \leq m-1. \end{array}$$

For every three homogeneous elements $a, b, c \in \mathbf{Z}$, we define

$$\mathfrak{s}\mathfrak{Z}\{a, b, c\} := (ab)c - a\big(bc + (-1)^{|a||b|} cb\big).$$

Now, applying the Zinbiel superidentity for basis elements gives the following relations.

$$\begin{array}{lcl} \Big\{\mathfrak{s}\mathfrak{Z}\{e_n, e_1, f_j\} = 0\Big\}_{2 \leq j \leq m-1} & \Rightarrow & (\alpha + j - 1) b_j = 0 \\ \Big\{\mathfrak{s}\mathfrak{Z}\{e_n, f_j, e_1\} = 0\Big\}_{1 \leq j \leq m-2} & \Rightarrow & (\alpha + j) b_j = 0 \Rightarrow \boxed{b_j = 0, \ 2 \leq j \leq m-2} \\ \Big\{\mathfrak{s}\mathfrak{Z}\{f_1, e_n, e_1\} = 0\Big\} & \Rightarrow & (\alpha + 1) c_1 = 0, \\ \Big\{\mathfrak{s}\mathfrak{Z}\{f_2, e_n, e_1\} = 0\Big\} & \Rightarrow & (\alpha + 2)(c_1 + b_1) = 0 \Rightarrow \boxed{c_1 = -b_1} \end{array}$$

thus, $f_1 e_n = -b_1 f_2$ and $f_j e_n = 0$ with $2 \leq j \leq m-1$. Then, we have the following products:

$$\begin{array}{lcll} e_i e_j & = & C_{i+j-1}^{j} e_{i+j}, & 2 \leq i+j \leq n-1. \\ e_i f_j & = & \dfrac{\prod\limits_{k=0}^{i-2}(\alpha+j+k)}{(i-1)!} f_{j+i}, & 1 \leq i \leq n-1, \end{array}$$

$$
\begin{array}{lclllll}
f_j e_i & = & \dfrac{\prod\limits_{k=0}^{i-1}(\alpha+j+k-1)}{i!} f_{j+i}, & 1 \leq i \leq n-1, & & & \\
e_n f_1 & = & b_1 f_2, & e_n f_{m-1} & = & b_{m-1} f_m, & \\
f_1 e_n & = & -b_1 f_2, & f_j e_n & = & 0, & 2 \leq j \leq m-1,
\end{array}
$$

with $(\alpha+1)b_1 = (\alpha+m-2)b_{m-1} = 0$.

Finally, only rest to compute the products $f_i f_j = h_{ij} e_{i+j}$. We compute them by induction:

- **Step 1:** We have:

$$
\begin{array}{lcl}
f_1 f_1 & = & h_{11} e_2, \\
f_2 f_1 & = & (e_1 f_1) f_1 = e_1(f_1 f_1) - e_1(f_1 f_1) = 0, \\
f_1 f_2 & = & h_{12} e_3.
\end{array}
$$

Moreover, we have

$$
\begin{array}{lll}
\mathfrak{s}\mathfrak{Z}\{f_1, f_1, e_1\} = 0 & \Rightarrow & 2h_{11} = (a+1)h_{12}, \\
\mathfrak{s}\mathfrak{Z}\{f_1, e_1, f_1\} = 0 & \Rightarrow & (\alpha+1)h_{12} = 0 \qquad \Rightarrow \quad h_{11} = (a+1)h_{12} = 0.
\end{array}
$$

- **Step 2:** Now, we can write:

$$
\begin{array}{lclclclcl}
 & & f_1 f_1 & = & f_2, & f_2 f_1 & = & 0, & \\
 & & f_1 f_2 & = & h_{12} e_3, & f_1 f_3 & = & h_{13} e_4, & \\
f_2 f_2 & = & (e_1 f_1) f_2 & = & e_1(f_1 f_2) - e_1(f_2 f_1) & = & h_{12}(e_1 e_3) & = & h_{12} e_4, \\
f_3 f_1 & = & (e_1 f_2) f_1 & = & e_1(f_2 f_1) - e_1(f_1 f_2) & = & -h_{12}(e_1 e_3) & = & -h_{12} e_4,
\end{array}
$$

with $(\alpha+1)h_{12} = 0$. Applying the Zinbiel superidentity, we get:

$$
\begin{array}{lll}
\mathfrak{s}\mathfrak{Z}\{f_1, f_2, e_1\} = 0 & \Rightarrow & 3h_{12} = (\alpha+2)h_{13}, \\
\mathfrak{s}\mathfrak{Z}\{f_1, e_1, f_2\} = 0 & \Rightarrow & (\alpha+2)h_{13} = \alpha h_{12}.
\end{array}
$$

Therefore, joining the following equations:

$$
\left.
\begin{array}{rcl}
(\alpha+1)h_{12} & = & 0 \\
3h_{12} & = & (\alpha+2)h_{13} \\
(\alpha+2)h_{13} & = & \alpha h_{12}
\end{array}
\right\} \Rightarrow h_{12} = 0,\ (\alpha+2)h_{13} = 0.
$$

- **Step 3:** We suppose that

$$
\begin{array}{rclclcl}
f_i f_{k+1-i} & = & 0, & & 2 \leq i \leq k, & & \\
f_1 f_k & = & h_{1k} e_{k+1}, & & & & \\
f_1 f_{k+1} & = & h_{1\,k+1} e_{k+2}, & & & & \\
f_{k+1} f_1 & = & (e_1 f_k) f_1 & = & e_1(f_k f_1) - e_1(f_1 f_k) & = & \\
 & = & -h_{1k}(e_1 e_{k+1}) & = & -h_{1k} e_{k+2}, & & \\
f_2 f_k & = & (e_1 f_1) f_k & = & e_1(f_1 f_k) - e_1(f_k f_1) & = & \\
 & = & h_{1k}(e_1 e_{k+1}) & = & h_{1k} e_{k+2}, & & \\
f_k f_2 & = & (e_1 f_{k-1}) f_2 & = & e_1(f_{k-1} f_2) - e_1(f_2 f_{k-1}) & = & 0,
\end{array}
$$

$$f_i f_{k+2-i} \;=\; (e_1 f_{i-1}) f_{k+2-i} \;=\; e_1(f_{i-1} f_{k+2-i}) - e_1(f_{k+2-i} f_{i-1}) \;=\; 0,$$
(by induction hypothesis),

with $(\alpha+k-1)h_{1k} = 0$. Applying the Zinbiel superidentity we obtain:

$$\begin{array}{lcl} \mathfrak{sZ}\{f_1, f_k, e_1\} = 0 \;\Rightarrow\; (k+1)h_{1k} & = & (\alpha+k)h_{1\,k+1}, \\ \mathfrak{sZ}\{f_1, e_1, f_k\} = 0 \;\Rightarrow\; (\alpha+k)h_{1\,k+1} & = & a h_{1k}. \end{array}$$

It follows that

$$\left.\begin{array}{rcl} (\alpha+k-1)h_{1k} & = & 0 \\ (k+1)h_{1k} & = & (\alpha+k)h_{1\,k+1} \\ (\alpha+k)h_{1\,k+1} & = & \alpha h_{1k} \end{array}\right\} \Rightarrow h_{1k} = 0,\ (\alpha+k)h_{1\,k+1} = 0.$$

This process is finite, so we get $f_1 f_m = h_{1m} e_{m+1}$ (otherwise $f_i f_j = 0$) with $(\alpha+m-1)h_{1m} = 0$ if $m < n-2$ and $f_1 f_{n-2} = h_{1\,n-2} e_{n-1}$ (otherwise $f_i f_j = 0$) with $(\alpha+n-3)h_{1\,n-2} = 0$ if $m \geq n-2$.

In the case $m < n-2$: $\mathfrak{sZ}\{f_1, f_m, e_1\} = 0$ gives $h_{1m} = 0$.

Thus, the Zinbiel superalgebra has the following multiplication table:

$$\begin{array}{lcllcll} e_i e_j & = & C_{i+j-1}^{j} e_{i+j}, & 2 \leq i+j \leq n-1, & & & \\ e_i f_j & = & \frac{\prod\limits_{k=0}^{i-2}(\alpha+j+k)}{(i-1)!} f_{j+i}, & 1 \leq i \leq n-1, & & & \\ f_j e_i & = & \frac{\prod\limits_{k=0}^{i-1}(\alpha+j+k-1)}{i!} f_{j+i}, & 1 \leq i \leq n-1, & & & \\ e_n f_1 & = & b_1 f_2, & e_n f_{m-1} & = & b_{m-1} f_m, & m \geq n-2 \\ f_1 e_n & = & -b_1 f_2, & f_1 f_{n-2} & = & h e_{n-1}, & \end{array}$$

with $(\alpha+1)b_1 = (\alpha+m-2)b_{m-1} = 0$ and $(\alpha+n-3)h = 0$ if $m \geq n-2$. Now, we consider $m > 3$ and $m < n-2$. We can distinguish the following cases:

Case 1: $b_1 \neq 0$. In this case, we have $\alpha = -1$ and $b_{m-1} = 0$. If $m \geq n-2$, we also have that $h = 0$. Then we obtain the Zinbiel superalgebra $\mathfrak{nf}_1$.

Case 2: $b_1 = 0$ and $b_{m-1} = 0$. In this case, we have the family of Zinbiel superalgebras $\mathfrak{nf}_2^{\alpha}$.

Case 3: $b_1 = 0$ and $b_{m-1} \neq 0$. Then, $\alpha = 2-m$. We have the superalgebra $\mathfrak{nf}_3$.

Now, we consider $m > 3$ and $m \geq n-2$. Analyzing the cases above, we have a new superalgebra in **Case 2** (when $\alpha = n-3$ and $h \neq 0$, we can consider $h = 1$), $\mathfrak{nf}_4$, and another one in **Case 3** (when $m = n-1$, $\alpha = 3-n$, and $h \neq 0$, we can consider $h = 1$), $\mathfrak{nf}_5$. □

Finally, we study the particular case $m = 3$. Let $\mathbf{Z}$ be a complex naturally graded filiform Zinbiel superalgebra with $\dim(\mathbf{Z}_{\bar{0}}) = n$, $n \geq 5$ and $\dim(\mathbf{Z}_{\bar{1}}) = 3$. Then, there exists a basis $\{e_1, \dots, e_n, f_1, f_2, f_3\}$ and we have the gradation:

$$\underbrace{\langle e_1, e_n, f_1\rangle}_{\mathfrak{z}^1} \oplus \underbrace{\langle e_2, f_2\rangle}_{\mathfrak{z}^2} \oplus \underbrace{\langle e_3, f_3\rangle}_{\mathfrak{z}^3} \oplus \underbrace{\langle e_4\rangle}_{\mathfrak{z}^4} \oplus \underbrace{\langle e_5\rangle}_{\mathfrak{z}^5} \oplus \dots$$

Theorem 3.3. *Let $\mathbf{Z}$ be a complex naturally graded filiform Zinbiel superalgebra with $\dim(\mathbf{Z}_{\bar{0}}) = n$, $(n \geq 5)$, and $\dim(\mathbf{Z}_{\bar{1}}) = 3$. Then $\mathbf{Z}$ is isomorphic to one of the following superalgebras:*

- *If $n > 5$, then $\mathbf{Z}$ is isomorphic either to $\mathfrak{nf}_1$, $\mathfrak{nf}_2^{\alpha}$, $\mathfrak{nf}_3$, or*

$$\mathfrak{a}_1 : \begin{cases} e_ie_j = C_{i+j-1}^{j} e_{i+j}, & 2 \leq i+j \leq n-1, \\ e_1f_1 = f_2, & e_1f_2 = f_3, \qquad f_1e_1 = -f_2, \\ e_nf_1 = f_2, & e_nf_2 = f_3, \qquad f_1e_n = -f_2. \end{cases}$$

- *If $n = 5$, then $\mathbf{Z}$ is isomorphic either to $\mathfrak{nf}_1$, $\mathfrak{nf}_2^{\alpha\neq -2}$, $\mathfrak{nf}_3$, $\mathfrak{a}_1$ or*

$$\mathfrak{a}_2 : \begin{cases} e_ie_j = C_{i+j-1}^{j} e_{i+j}, & 2 \leq i+j \leq n-1, \\ f_1e_1 = -2f_2, & f_2e_1 = -f_3, \qquad f_1e_2 = f_3, \\ e_1f_1 = f_2, & e_1f_2 = f_3, \\ e_2f_1 = -f_2, & f_1f_{n-2} = e_{n-1}. \end{cases}$$

Proof. Similar to the general case (see the proof of Theorem 3.2) we get that $\mathbf{Z}$ is isomorphic to

$$\begin{array}{lllllll} e_ie_j & = & C_{i+j-1}^{j} e_{i+j}, & 2 \leq i+j \leq n-1, & & \\ f_1e_1 & = & af_2, & f_1e_2 & = & \frac{\alpha(\alpha+1)}{2} f_3, \\ f_2e_1 & = & (\alpha+1)f_3, & e_nf_1 & = & b_1f_2, \\ e_1f_1 & = & f_2, & e_2f_1 & = & (\alpha+1)f_2, \\ e_1f_2 & = & f_3, & e_nf_2 & = & b_2f_3, \\ f_1e_n & = & -b_1f_2, & (f_1f_3 = he_4, \text{ if } n = 5.), & & \end{array}$$

with $(\alpha+1)b_1 = (\alpha+1)b_2 = 0$ and $(\alpha+2)h = 0$ if $n = 5$.

First, we consider $n > 5$. We can distinguish the following cases:

- $b_1 \neq 0$ and $b_2 \neq 0$. In this case, $\alpha = -1$ and by a change of basis we can consider $b_1 = b_2 = 1$ obtaining the superalgebra $\mathfrak{a}_1$.
- $b_1 \neq 0$ and $b_2 = 0$. We obtain $\mathfrak{nf}_1$.
- $b_1 = 0$. We obtain $\mathfrak{nf}_3$ (for $b_2 \neq 0$) and $\mathfrak{nf}_2^{\alpha}$ (for $b_2 = 0$)

Finally, the case when $n = 5$ leads to a similar study, obtaining the superalgebras in the statement. $\square$

4. Classification of low-dimensional complex Zinbiel superalgebras

In this section, we obtain the classification of the complex Zinbiel superalgebras up to dimension three. Let us begin by proving the following important lemma.

Lemma 4.1. *Given a n-dimensional Zinbiel superalgebra $\mathbf{Z}$ of type $(n-1,1)$, i.e.* $\dim(\mathbf{Z}_{\bar{0}}) = n-1$ *and* $\dim(\mathbf{Z}_{\bar{1}}) = 1$*, then* $\mathbf{Z}_{\bar{0}}\mathbf{Z}_{\bar{1}} = \mathbf{Z}_{\bar{1}}\mathbf{Z}_{\bar{0}} = \{0\}$*. Moreover,* $\mathbf{Z}_{\bar{1}}\mathbf{Z}_{\bar{1}}$ *is a subspace of* $\mathrm{Ann}_{\mathrm{L}}(\mathbf{Z}_{\bar{0}})$*.*

Proof. Let $e_1, \dots, e_{n-1}$ be a basis of $\mathbf{Z}_{\bar{0}}$ and let f_1 be a basis of $\mathbf{Z}_{\bar{1}}$. Denote by $e_i f_1 = a_i f_1$ and $f_1 e_i = b_i f_1$, for $a_i, b_i \in \mathbb{C}$ and $i = 1, \dots, n-1$. On the one hand, for $i = 1, \dots, n-1$ we have

$$(e_i f_1)e_i = e_i(f_1 e_i) + e_i(e_i f_1); \quad a_i b_i f_1 = a_i b_i f_1 + a_i^2 f_1.$$

Hence, we have $a_i = 0$ and $\mathbf{Z}_{\bar{0}}\mathbf{Z}_{\bar{1}} = \{0\}$.

On the other hand, since $\mathbf{Z}_{\bar{0}}$ is a Zinbiel algebra, it is nilpotent. Suppose $\mathbf{Z}_{\bar{0}}^{s} = 0$ and proceed by induction. For $x \in \mathbf{Z}_{\bar{0}}^{s-1}$, we have

$$(f_1 x)x = f_1(xx + xx) = 2f_1(xx),$$

and since $xx \in \mathbf{Z}_{\bar{0}}^{s} = \{0\}$, we obtain $(f_1 x)x = 0$, which implies $f_1 x = 0$, and we may write $f_1 \mathbf{Z}_{\bar{0}}^{s-1} = \{0\}$. Now, suppose $f_1 \mathbf{Z}_{\bar{0}}^{s-k+1} = \{0\}$, then for $x \in \mathbf{Z}_{\bar{0}}^{s-k}$, we have $(f_1 x)x = 2f_1(xx) = 0$, because $xx \in \mathbf{Z}_{\bar{0}}^{s-k+1}$. Therefore, $f_1 x = 0$, and we have $f_1 \mathbf{Z}_{\bar{0}}^{s-k} = \{0\}$. For $k = s-1$, we conclude that $\mathbf{Z}_{\bar{1}}\mathbf{Z}_{\bar{0}}^{1} = \mathbf{Z}_{\bar{1}}\mathbf{Z}_{\bar{0}} = \{0\}$.

Further, we have $(f_1 f_1)x = f_1(f_1 x) + f_1(x f_1) = 0$ for $x \in \mathbf{Z}_{\bar{0}}$. Therefore, $(f_1 f_1) \in \mathrm{Ann}_{\mathrm{L}}(\mathbf{Z}_{\bar{0}})$ and $\mathbf{Z}_{\bar{1}}\mathbf{Z}_{\bar{1}}$ is a subspace of $\mathrm{Ann}_{\mathrm{L}}(\mathbf{Z}_{\bar{0}})$, where $\mathrm{Ann}_{\mathrm{L}}(\mathbf{Z}_{\bar{0}}) = \{x \in A : x\mathbf{Z}_{\bar{0}} = 0\}$. □

The converse is a straightforward verification.

Remark 4.2. *Given a $(n-1)$-dimensional Zinbiel algebra $\mathbf{Z}_{\bar{0}}$. If we construct a superalgebra $\mathbf{Z}$ such that $\mathbf{Z}_{\bar{0}}\mathbf{Z}_{\bar{1}} = \mathbf{Z}_{\bar{1}}\mathbf{Z}_{\bar{0}} = \{0\}$ and such that $\mathbf{Z}_{\bar{1}}\mathbf{Z}_{\bar{1}}$ is a subspace of* $\mathrm{Ann}_{\mathrm{L}}(\mathbf{Z}_{\bar{0}})$*. Then, $\mathbf{Z}$ is a Zinbiel superalgebra of type $(n-1,1)$.*

Remark 4.3. *Given a non-zero n-dimensional Zinbiel superalgebra $\mathbf{Z}$ of type $(n-1,1)$ such that $\mathbf{Z}_{\bar{0}}$ is the zero algebra. Then it is isomorphic to $\mathbf{Z}_{n,0} : f_1 f_1 = e_1$, simply choosing $\phi : A \to A_{n,0}$ such that $\phi(f_1 f_1) = e_1$. The classification of the 2-dimensional Zinbiel superalgebras follows by this statement.*

Now, we recover the classification of the n-dimensional Zinbiel algebras [2, 4], as it will be required.

Theorem 4.4. *Given an n-dimensional, for $n \leq 3$, non-trivial Zinbiel algebra, then it is isomorphic to only one of the following*

- *If $n = 2$, then it is*
  (1) $\mathfrak{Z}_{2,1} : e_1 e_1 = e_2$.

- *If $n = 3$, then it is isomorphic to $\mathfrak{Z}_{3,1} = \mathfrak{Z}_{2,1} \oplus \mathbb{C}$ or to*
  (1) $\mathfrak{Z}_{3,2} : e_1e_1 = e_2, e_1e_2 = \frac{1}{2}e_3, e_2e_1 = e_3.$
  (2) $\mathfrak{Z}_{3,3} : e_1e_2 = e_3, e_2e_1 = -e_3.$
  (3) $\mathfrak{Z}_{3,4} : e_1e_1 = e_3, e_1e_2 = e_3, e_2e_2 = \beta e_3.$
  (4) $\mathfrak{Z}_{3,5} : e_1e_1 = e_3, e_1e_2 = e_3, e_2e_1 = e_3.$

4.1. 3-dimensional Zinbiel superalgebras. In our classification, we will not consider non-proper superalgebras. So type $(n, 0)$, which corresponds to Zinbiel algebras, and type $(0, n)$ (zero algebra) are omitted. Also, we omit the superalgebras with $\mathfrak{Z}_0\mathfrak{Z}_1 = \mathfrak{Z}_1\mathfrak{Z}_0 = \mathfrak{Z}_1^2 = 0$, as they are split algebras.

4.1.1. (1, 2) superalgebras. Let $\{e_1, f_1, f_2\}$ be a basis of a superalgebra of $\mathbf{Z} = \mathbf{Z}_{\bar{0}} \oplus \mathbf{Z}_{\bar{1}}$. Since $\mathbf{Z}_{\bar{0}}$ is the trivial one dimensional algebra, we have the following multiplication table for $\mathbf{Z}$:

$$e_1f_i = a_i^1 f_1 + a_i^2 f_2, f_ie_1 = b_i^1 f_1 + b_i^2 f_2, f_if_j = c_{ij}e_1,$$

where $1 \leq i, j \leq 2$. We find the equations on variables the structural constants studying case by case:

$$\begin{array}{lccl}
\mathfrak{s}\mathfrak{Z}\{e_1,e_1,f_1\} & = 0 & \Rightarrow & (a_1^1)^2 + a_1^1b_1^1 + a_1^2a_2^1 + a_2^1b_1^2 = 0 \\
 & & & \text{and } a_1^1a_1^2 + a_1^2a_2^2 + a_1^2b_1^1 + a_2^2b_1^2 = 0 \\
\mathfrak{s}\mathfrak{Z}\{e_1,e_1,f_2\} & = 0 & \Rightarrow & a_1^1a_2^1 + a_1^1b_2^1 + a_2^1a_2^2 + a_2^1b_2^2 = 0 \\
 & & & \text{and } a_1^2a_2^1 + a_1^2b_2^1 + (a_2^2)^2 + a_2^2b_2^2 = 0 \\
\mathfrak{s}\mathfrak{Z}\{e_1,f_1,e_1\} & = 0 & \Rightarrow & (a_1^1)^2 + a_1^2a_2^1 - a_1^2b_2^1 + a_2^1b_1^2 = 0 \\
 & & & \text{and } a_1^1a_1^2 - a_1^1b_1^2 + a_1^2a_2^2 + a_1^2b_1^1 - a_1^2b_2^2 + a_2^2b_1^2 = 0 \\
\mathfrak{s}\mathfrak{Z}\{e_1,f_2,e_1\} & = 0 & \Rightarrow & a_1^1a_2^1 + a_1^1b_2^1 + a_2^1a_2^2 - a_2^1b_1^1 + a_2^1b_2^2 - a_2^2b_2^1 = 0 \\
 & & & \text{and } a_1^2a_2^1 + a_1^2b_2^1 - a_2^1b_1^2 + (a_2^2)^2 = 0 \\
\mathfrak{s}\mathfrak{Z}\{e_1,f_1,f_1\} & = 0 & \Rightarrow & a_1^1c_{11} + a_1^2c_{21} = 0 \\
\mathfrak{s}\mathfrak{Z}\{e_1,f_1,f_2\} & = 0 & \Rightarrow & a_1^1c_{12} + a_1^2c_{22} = 0 \\
\mathfrak{s}\mathfrak{Z}\{e_1,f_2,f_1\} & = 0 & \Rightarrow & a_2^1c_{11} + a_2^2c_{21} = 0 \\
\mathfrak{s}\mathfrak{Z}\{e_1,f_2,f_2\} & = 0 & \Rightarrow & a_2^1c_{12} + a_2^2c_{22} = 0 \\
\mathfrak{s}\mathfrak{Z}\{f_1,e_1,e_1\} & = 0 & \Rightarrow & (b_1^1)^2 + b_1^2b_2^1 = 0 \text{ and } b_1^1b_1^2 + b_1^2b_2^2 = 0 \\
\mathfrak{s}\mathfrak{Z}\{f_2,e_1,e_1\} & = 0 & \Rightarrow & b_1^1b_2^1 + b_2^1b_2^2 = 0 \text{ and } b_1^2b_2^1 + (b_2^2)^2 = 0 \\
\mathfrak{s}\mathfrak{Z}\{f_1,e_1,f_1\} & = 0 & \Rightarrow & a_1^1c_{11} + a_1^2c_{12} + b_1^2c_{12} - b_1^2c_{21} = 0 \\
\mathfrak{s}\mathfrak{Z}\{f_1,e_1,f_2\} & = 0 & \Rightarrow & a_2^1c_{11} + a_2^2c_{12} - b_1^1c_{12} - b_1^2c_{22} + b_2^1c_{11} + b_2^2c_{12} = 0 \\
\mathfrak{s}\mathfrak{Z}\{f_2,e_1,f_1\} & = 0 & \Rightarrow & a_1^1c_{21} + a_1^2c_{22} + b_1^1c_{21} + b_1^2c_{22} - b_2^1c_{11} - b_2^2c_{21} = 0 \\
\mathfrak{s}\mathfrak{Z}\{f_2,e_1,f_2\} & = 0 & \Rightarrow & a_2^1c_{21} + a_2^2c_{22} - b_2^1c_{12} + b_2^1c_{21} = 0 \\
\mathfrak{s}\mathfrak{Z}\{f_1,f_1,e_1\} & = 0 & \Rightarrow & a_1^1c_{11} + a_1^2c_{12} + b_1^1c_{11} + b_1^2c_{12} = 0 \\
\mathfrak{s}\mathfrak{Z}\{f_1,f_2,e_1\} & = 0 & \Rightarrow & a_2^1c_{11} + a_2^2c_{12} + b_2^1c_{11} + b_2^2c_{12} = 0 \\
\mathfrak{s}\mathfrak{Z}\{f_2,f_1,e_1\} & = 0 & \Rightarrow & a_1^1c_{21} + a_1^2c_{22} + b_1^1c_{21} + b_1^2c_{22} = 0 \\
\mathfrak{s}\mathfrak{Z}\{f_2,f_2,e_1\} & = 0 & \Rightarrow & a_2^1c_{21} + a_2^2c_{22} + b_2^1c_{21} + b_2^2c_{22} = 0 \\
\mathfrak{s}\mathfrak{Z}\{f_1,f_1,f_1\} & = 0 & \Rightarrow & a_1^1c_{11} = 0 \text{ and } a_1^2c_{11} = 0 \\
\mathfrak{s}\mathfrak{Z}\{f_1,f_1,f_2\} & = 0 & \Rightarrow & a_2^1c_{11} - b_1^1c_{12} + b_1^1c_{21} = 0 \\
 & & & \text{and } a_2^2c_{11} - b_1^2c_{12} + b_1^2c_{21} = 0 \\
\mathfrak{s}\mathfrak{Z}\{f_1,f_2,f_1\} & = 0 & \Rightarrow & a_1^1c_{12} + b_1^1c_{12} - b_1^1c_{21} = 0
\end{array}$$

$$\begin{array}{rcl} & & \text{and } a_1^2c_{12}+b_1^2c_{12}-b_1^2c_{21}=0 \\ \mathfrak{s}\mathfrak{Z}\{f_1,f_2,f_2\} = 0 & \Rightarrow & a_2^1c_{12}=0 \text{ and } a_2^2c_{12}=0 \\ \mathfrak{s}\mathfrak{Z}\{f_2,f_1,f_1\} = 0 & \Rightarrow & a_1^1c_{21}=0 \text{ and } a_1^2c_{21}=0 \\ \mathfrak{s}\mathfrak{Z}\{f_2,f_1,f_2\} = 0 & \Rightarrow & a_2^1c_{21}-b_2^1c_{12}+b_2^1c_{21}=0 \\ & & \text{and } a_2^2c_{21}-b_2^2c_{12}+b_2^2c_{21}=0 \\ \mathfrak{s}\mathfrak{Z}\{f_2,f_2,f_1\} = 0 & \Rightarrow & a_1^1c_{22}+b_2^1c_{12}-b_2^1c_{21}=0 \\ & & \text{and } a_1^2c_{22}+b_2^2c_{12}-b_2^2c_{21}=0 \\ \mathfrak{s}\mathfrak{Z}\{f_2,f_2,f_2\} = 0 & \Rightarrow & a_2^1c_{22}=0 \text{ and } a_2^2c_{22}=0. \end{array}$$

By solving the system of equations, we have one of the following possibilities for **Z**:

(1) $f_1f_1 = \lambda_{11}e_1,\quad f_1f_2 = \lambda_{12}e_1,\quad f_2f_1 = \lambda_{21}e_1,\quad f_2f_2 = \lambda_{22}e_1.$

(2) $f_1e_1 = \mu f_1 - \frac{\lambda_{11}}{\lambda_{12}}\mu f_2,\quad f_2e_1 = \frac{\lambda_{12}}{\lambda_{11}}\mu f_1 - \mu f_2,\quad f_1f_1 = \lambda_{11}e_1,\quad f_1f_2 = \lambda_{12}e_1,\quad f_2f_1 = \lambda_{12}e_1,\quad f_2f_2 = \frac{\lambda_{12}^2}{\lambda_{11}}e_1.$

(3) $e_1f_1 = \mu f_1 - \frac{\mu^2}{\mu'}f_2,\quad e_1f_2 = \mu' f_1 - \mu f_2.$

(4) $e_1f_1 = \mu f_1 - \frac{\mu\nu}{\nu'}f_2,\quad e_1f_2 = \frac{\mu\nu'}{\nu}f_1 - \mu f_2,\quad f_1e_1 = \nu f_1 - \frac{\nu^2}{\nu'}f_2,\quad f_2e_1 = \nu' f_1 - \nu f_2.$

(5) $f_1e_1 = \mu f_2,\quad f_1f_1 = \mu' e_1.$

(6) $f_2e_1 = \mu f_1,\quad f_2f_2 = \mu' e_1.$

(7) $e_1f_1 = \mu f_2,\quad f_1e_1 = \mu' f_2.$

(8) $e_1f_2 = \mu f_1,\quad f_2e_1 = \mu' f_1.$

Lemma 4.5. *The isomorphism clases of the eight cases above can be reduced to the study of the cases (a), (e) and (g).*

(1) *The superalgebras (e) and (f) are isomorphic.*
(2) *The superalgebras (g) and (h) are isomorphic.*
(3) *The superalgebras (b) are isomorphic to (e) for* $\mu' = \frac{\lambda_{12}^2}{\lambda_{11}}$.
(4) *The superalgebras (c) are isomorphic to (g) for* $\mu' = 0$.
(5) *The superalgebras (d) are isomorphic to (g) for* $\mu' = \nu$.

Proof. The statements (1) and (2) are trivial. The statements (3), (4) and (5) can be shown, respectively, using the following maps.

$$\begin{array}{llllll} \phi_1(e_1) = e_1, & \phi_1(e_2) = \frac{\lambda_{11}}{\lambda_{12}}e_2, & \phi_1(e_3) = e_2+e_3, \\ \phi_2(e_1) = e_1, & \phi_2(e_2) = \frac{\mu^2}{\mu'}e_2+\frac{\mu}{\mu'}e_3, & \phi_2(e_3) = e_2, \\ \phi_3(e_1) = e_1, & \phi_3(e_2) = e_2, & \phi_3(e_3) = \frac{\nu'}{\nu}e_2-\frac{\nu'}{\nu}e_3. \end{array}$$

□

Next, we study the remaining cases.
Case (a) is equivalent to one of the following, depending on the parameter.

- If $\lambda_{11} \neq 0$ and $\lambda_{12} \neq \lambda_{21}$. Now, choose
$$\phi(e_1) = \frac{\lambda_{11}}{(\lambda_{12}-\lambda_{21})^2}e_1, \phi(e_2) = \frac{\lambda_{11}}{\lambda_{21}-\lambda_{12}}e_2 \text{ and } \phi(e_3) = \frac{\lambda_{12}}{\lambda_{21}-\lambda_{12}}e_2 + e_3,$$
then
$$\mathbf{z}_{3,1}^{\alpha} : f_1f_1 = e_1, \quad f_2f_1 = e_1, \quad f_2f_2 = \alpha e_1.$$
- If $\lambda_{11} \neq 0$ and $\lambda_{12} = \lambda_{21}$. Then, by choosing the map
$$\phi(e_1) = \lambda_{11}^{-1}e_1, \phi(e_2) = e_3, \phi(e_3) = -\frac{\sqrt{\lambda_{11}\lambda_{22}-\lambda_{12}^2}}{\lambda_{11}}e_2 + \lambda_{12}\lambda_{11}^{-1}e_3,$$
we obtain
$$\mathbf{z}_{3,2} : f_1f_1 = e_1, \quad f_2f_2 = e_1.$$
Note that, if $\lambda_{22} = \lambda_{12}^2\lambda_{11}^{-1}$, then this is equivalent to some case in (e).
- If $\lambda_{11} = 0$, $\lambda_{12} \neq \lambda_{21}$ and ($\lambda_{12} \neq -\lambda_{21}$ or $\lambda_{22} \neq 0$). Choose the map
$$\phi(e_1) = \frac{\lambda_{22}}{(\lambda_{12}-\lambda_{21})^2}e_1, \phi(e_2) = \frac{\lambda_{21}}{\lambda_{12}-\lambda_{21}}e_2 + e_3 \text{ and } \phi(e_3) = \frac{\lambda_{22}}{\lambda_{12}-\lambda_{21}},$$
we obtain $\mathbf{z}_{3,1}^{\alpha}$.
- If $\lambda_{11} = 0$ and $\lambda_{12} = \lambda_{21} \neq 0$, then we have $\mathbf{z}_{3,2}$.
- If $\lambda_{11} = \lambda_{12} = \lambda_{21} = 0$, then we obtain
$$\mathbf{z}_{3,3} : f_1f_1 = e_1,$$
choosing $\phi(e_1) = \lambda_{22}^{-1}e_1$, $\phi(e_2) = e_3$ and $\phi(e_3) = e_2$.
- If $\lambda_{11} = \lambda_{22} = 0$ and $\lambda_{12} = -\lambda_{21}$. Choose the map
$$\phi(e_1) = \lambda_{12}^{-1}e_1, \phi(e_2) = e_2 \text{ and } \phi(e_3) = e_3$$
to obtain
$$\mathbf{z}_{3,4} : f_1f_2 = e_1, \quad f_2f_1 = -e_1.$$

Case (e) is equivalent to one of the following, depending on the parameters.

- If $\mu \neq 0$ and $\mu' \neq 0$, then the map given by
$$\phi(e_1) = \mu'^{-1}e_1, \phi(e_2) = e_2 \text{ and } \phi(e_3) = \mu^{-1}\mu'^{-1}e_3$$
shows that it is isomorphic to
$$\mathbf{z}_{3,5} : f_1e_1 = f_2, \quad f_1f_1 = e_1.$$
- If $\mu = 0$ and $\mu' \neq 0$, then, choosing $\phi(e_1) = \mu'^{-1}e_1$, $\phi(e_2) = e_2$ and $\phi(e_3) = e_3$, we have $\mathbf{z}_{3,3}$.
- If $\mu \neq 0$ and $\mu' = 0$, choose $\phi(e_1) = e_1$, $\phi(e_2) = e_2$ and $\phi(e_3) = \mu^{-1}e_3$ to obtain
$$\mathbf{z}_{3,6} : f_1e_1 = f_2.$$

Case (g) is equivalent to one of the following, depending on the parameters.

- If $\mu' \neq 0$, then, with ϕ such that $\phi(e_1) = e_1$, $\phi(e_2) = e_2$ and $\phi(e_3) = \mu'^{-1}e_3$, we have
$$\mathbf{z}_{3,7} : e_1f_1 = \alpha f_2, f_1e_1 = f_2.$$

- If $\mu' = 0$, choosing $\phi(e_1) = e_1$, $\phi(e_2) = e_2$ and $\phi(e_3) = \mu^{-1}e_3$ then we obtain

$$\mathbf{z}_{3,8} : e_1 f_1 = f_2.$$

4.1.2. (2, 1) superalgebras.

- Even part $\mathfrak{Z}_{2,0}$. Then, by Remark 4.3, we have $\mathbf{z}_{3,0}$.
- Even part $\mathfrak{Z}_{2,1}$. By Lemma 4.1 and Remark 4.2, we have that every superalgebra constructed on $\mathfrak{Z}_{2,1}$ is of the form

$$e_1e_1 = e_2, \quad f_1f_1 = \lambda_2 e_2.$$

Choose the linear map ϕ such that $\phi(e_1) = \lambda_2^{-\frac{1}{2}} e_1, \phi(e_2) = \lambda_2^{-1}e_2, \phi(e_3) = e_3$, to obtain the superalgebra

$$\mathbf{z}_{3,9} : e_1e_1 = e_2, f_1f_1 = e_2.$$

Summing up, we have the classification of the 3-dimensional Zinbiel superalgebras.

Theorem 4.6. *Given a* 3*-dimensional complex non-split Zinbiel superalgebra* **Z***, then it is isomorphic to a* 3*-dimensional Zinbiel algebra or to only one of the following algebras.*

$$\begin{array}{lllllllllll}
\mathbf{z}_{3,1}^{\alpha} & : & f_1f_1 & = & e_1 & f_2f_1 & = & e_1 & f_2f_2 & = & \alpha e_1 \\
\mathbf{z}_{3,2} & : & f_1f_1 & = & e_1 & f_2f_2 & = & e_1 & & & \\
\mathbf{z}_{3,3} & : & f_1f_1 & = & e_1 & & & & & & \\
\mathbf{z}_{3,4} & : & f_1f_2 & = & e_1 & f_2f_1 & = & -e_1 & & & \\
\mathbf{z}_{3,5} & : & f_1e_1 & = & f_2 & f_1f_1 & = & e_1 & & & \\
\mathbf{z}_{3,6} & : & f_1e_1 & = & f_2 & & & & & & \\
\mathbf{z}_{3,7} & : & e_1f_1 & = & \alpha f_2 & f_1e_1 & = & f_2 & & & \\
\mathbf{z}_{3,8} & : & e_1f_1 & = & f_2 & & & & & & \\
\mathbf{z}_{3,9} & : & e_1e_1 & = & e_2 & f_1f_1 & = & e_2. & & &
\end{array}$$

5. Finite-dimensional Zinbiel superalgebras are nilpotent

It is well-known that finite-dimensional Zinbiel algebras are nilpotent over an arbitrary field [30] (also see [17] for context). It is a natural question to wonder if this is also true in the case of Zinbiel superalgebras. Note, for instance, that all the 3-dimensional Zinbiel superalgebras are nilpotent (Theorem 4.6). It turns out that the answer is positive, as we will see in this section.

Definition 5.1. *Given an algebra* **Z** *we define the right annihilator of an element* $a \in \mathbf{Z}$ *as the set*

$$RC(a) = \{x \in \mathbf{Z} : ax = 0\}.$$

Lemma 5.2. *Given a right-commutative superalgebra* **Z***, then for homogeneous elements* $a_1, a_2 \in \mathbf{Z}$*, we have* $RC(a_1) \subseteq RC(a_1a_2)$*.*

Proof. Given $x \in RC(a_1)$ and suppose $x = x_0 + x_1$, for $x_i \in \mathbf{Z}_{\bar{i}}$. Then, since a_1 is homogeneous $a_1 x = a_1 x_0 + a_1 x_1 = 0$ implies $a_1 x_0 = 0$ and $a_1 x_1 = 0$. Hence, we have

$$(a_1 a_2)x = (a_1 a_2)x_0 + (a_1 a_2)x_1 = (a_1 x_0)a_2 + (-1)^{|a_2|}(a_1 x_1)a_2 = 0.$$

□

The first key lemma of this section is the following.

Lemma 5.3. *Given a finite-dimensional Zinbiel superalgebra* $\mathbf{Z}$*, there exists a homogeneous element* e *such that* $e\mathbf{Z} = 0$*.*

Proof. Since $\mathbf{Z}_0$ is a Zinbiel algebra, it is nilpotent. Assume it has nilpotency index N, then we have some non-zero element $e_0 \in \mathbf{Z}_{\bar{0}}^{N-1}$ such that $e_0 \mathbf{Z}_{\bar{0}} = \mathbf{Z}_{\bar{0}} e_0 = 0$. Construct e as follows.

(1) Fix $e = e_0$.
(2) If there is some $x \in \mathbf{Z}_{\bar{1}}$ such that $ex \neq 0$, set $e_0 = ex$. Then $x \in RC(e_0)$, by the Zinbiel superidentity. Otherwise, set $e = e_0$ and finish the iteration.
(3) Repeat from (1).

Note that the element obtained in each iteration is homogeneous, so by Lemma 5.2, in each iteration, the right annihilator becomes bigger. Also, since the algebra is finite-dimensional, this process is finite, as it is enough to run it for a basis of $\mathbf{Z}_{\bar{1}}$. So we conclude $RC(e) = \mathbf{Z}$, that is $e\mathbf{Z} = 0$. □

Lemma 5.4. *Given* I *a right ideal of a Zinbiel superalgebra* $\mathbf{Z}$*, then* $\mathbf{Z}I$ *is an ideal.*

Proof. We have $\mathbf{Z}(\mathbf{Z}I) \subseteq \mathbf{Z}^2 I + \mathbf{Z}(I\mathbf{Z}) \subseteq \mathbf{Z}I$ and $(\mathbf{Z}I)\mathbf{Z} \subseteq \mathbf{Z}^2 I \subseteq \mathbf{Z}I$. □

The next result follows by the previous lemmas.

Lemma 5.5. *Any Zinbiel superalgebra of dimension* $n > 1$ *has a proper graded ideal.*

Proof. Given a finite-dimensional Zinbiel superalgebra $\mathbf{Z}$, by Lemma 5.3, there exists an element $e \in \mathbf{Z}_{\bar{i}}$, for $i = 0$ or $i = 1$, such that $e\mathbf{Z} = 0$. Now, if $\mathbf{Z}e = 0$, then the vector space generated by E is a proper graded ideal. Conversely, if $\mathbf{Z}e \neq 0$, choose $I = \mathbf{Z}e$, then since the linear spam of e is a right ideal, I is an ideal, by Lemma 5.4.

To show that it is a proper ideal, we have to prove that its dimension is lower than n. Choose a basis $e_1, e_2, \dots, e_n$ of $\mathbf{Z}$ such that $e_1 = E$, then the ideal I is linearly generated by the elements $e_1 E = 0, e_2 E, \dots, e_n E$, therefore, at most it has dimension $n - 1$.

The ideal I is graded as a consequence of e being homogeneous. □

Now, we can prove the first main result of this section.

Lemma 5.6. *Any finite-dimensional Zinbiel superalgebra is solvable.*

Proof. Let $\mathbf{Z}$ be a finite-dimensional Zinbiel superalgebra. We proceed by induction on the dimension n. If $n = 1$, we have the trivial one-dimensional algebra, which is solvable. Now, if $n > 1$ and the statement is true for up to dimension $n - 1$, then A has a proper graded ideal I, so I and $\mathbf{Z}/I$ are Zinbiel superalgebras of dimension lower than n, therefore they are solvable. Hence, $\mathbf{Z}$ itself is solvable. □

Proposition 5.7. *Let I be a minimal ideal of a finite-dimensional Zinbiel superalgebra $\mathbf{Z}$. Let J be a minimal right ideal of $\mathbf{Z}$ such that $J \subseteq I$. Then $I = J$.*

Proof. The proof is identical to the proof of [30, Proposition 2.2]. □

Corollary 5.8. *Let I be a minimal ideal of a finite-dimensional Zinbiel superalgebra $\mathbf{Z}$, then we have $I\mathbf{Z} = \mathbf{Z}I = 0$. Hence, we have* $\dim I = 1$.

Proof. The proof is identical to the proof of [30, Corollary 2.3]. □

Observe that the previous result implies that any finite-dimensional Zinbiel superalgebra has a non-trivial annihilator. However, this is not enough to prove that any finite-dimensional Zinbiel superalgebra is nilpotent, we need the next straightforward remark.

Remark 5.9. *Let I be a minimal ideal of a Zinbiel superalgebra $\mathbf{Z}$. Suppose it is generated by some element $e = e_0 + e_1 \in \mathbf{Z}$, where $e_i \in \mathbf{Z}_i$. Note that $e \in$ $\mathrm{Ann}(\mathbf{Z})$. Then $e\mathbf{Z}_{\bar{i}} = 0$ implies $e_0\mathbf{Z}_{\bar{i}} = 0$ and $e_1\mathbf{Z}_{\bar{i}} = 0$ (resp. $\mathbf{Z}_{\bar{i}}e = 0$ implies $\mathbf{Z}_{\bar{i}}e_0 = 0$ and $\mathbf{Z}_{\bar{i}}e_1 = 0$), and we have $e_i \in \mathrm{Ann}(A)$. Moreover, $I_i = \langle e_i \rangle$ is an ideal. Furthermore, if $e_i \neq 0$, then I_i is a proper graded ideal. Hence, we have the following corollary.*

Corollary 5.10. *Any finite-dimensional Zinbiel superalgebra $\mathbf{Z}$ has a minimal ideal which is graded. Moreover, there exist a homonegeous element $e \in \mathbf{Z}$ such that $e \in \mathrm{Ann}(\mathbf{Z})$.*

Finally, Corollary 5.10 enables us to prove the main result of this section.

Theorem 5.11. *Any finite-dimensional Zinbiel superalgebra is nilpotent.*

Proof. Let $\mathbf{Z}$ be a finite-dimensional Zinbiel superalgebra. We proceed by induction on the dimension n. If $n = 1$, we have the trivial one-dimensional algebra, which is nilpotent. Now, suppose we have that any finite-dimensional Zinbiel superalgebra is nilpotent up to dimension $n - 1$ for $n > 1$. Since $\mathbf{Z}$ has a graded ideal I of dimension one (generated by a homogeneous element) such that $\mathbf{Z}I = I\mathbf{Z} = 0$, then $\mathbf{Z}/I$ is a Zinbiel superalgebra of dimension $n - 1$, and so it is nilpotent. Therefore, $\mathbf{Z}$ is nilpotent. □

References

1. Adashev J., Camacho L. M., Gómez-Vidal S., Karimjanov I., Naturally graded Zinbiel algebras with nilindex $n - 3$, *Linear Algebra and Its Applications.* **443** (2014), 86–104. MR3496758, Zbl 1317.17002. doi: 10.1016/j.laa.2013.11.021. 1342

2. ADASHEV J., KHUDOYBERDIYEV A., OMIROV B., Classifications of some classes of Zinbiel algebras, *Journal of Generalized Lie Theory and Applications.* **4** (2010), S090601. MR2645324, Zbl 1241.17003. doi: 10.4303/jglta/S090601. 1342, 1343, 1344, 1348, 1354
3. AGUIAR M., Pre-Poisson algebras, *Letters in Mathematical Physics.* **54** (2000), 263–277. MR1846958, Zbl 1032.17038. doi: 10.1023/A:1010818119040. 1342
4. ÁLVAREZ A., FEHLBERG JÚNIOR R., KAYGORODOV I., The algebraic and geometric classification of Zinbiel algebras, *Journal of Pure and Applied Algebra.* **226** (2022), 11, 107106. MR4412231, Zbl 1506.17001. doi: 10.1016/j.jpaa.2022.107106. 1342, 1343, 1354
5. BENAYADI S., KAYGORODOV I., MHAMDI F., Symmetric Zinbiel superalgebras, *Communications in Algebra.* **51** (2023), 1, 224–238. MR4525294, Zbl 07637666. doi: 10.1080/00927872.2022.2096224. 1342, 1344
6. BREMNER M., DOTSENKO V., Classification of regular parametrized one-relation operads, *Canadian Journal of Mathematics.* **69** (2017), 5, 992–1035. MR3693146, Zbl 1388.18010. doi: 10.4153/CJM-2017-018-3. 1342
7. CAMACHO L. M., CAÑETE E. M., GÓMEZ-VIDAL S., OMIROV B., p-Filiform Zinbiel algebras, *Linear Algebra and its Applications.* **438** (2013), 7, 2958–2972. MR3018050, Zbl 1300.17002. doi: 10.1016/j.laa.2012.11.030. 1342
8. CAMACHO L.M., GÓMEZ J.R., NAVARRO R.M., OMIROV B.A., Classification of some nilpotent class of Leibniz superalgebras, *Acta Mathematica Sinica, English Series.* **26** (2010), 5, 799–816. MR2644010, Zbl 1260.17003. doi: 10.1007/s10114-010-8358-2. 1345
9. CAMACHO L., KARIMJANOV I., KAYGORODOV I., KHUDOYBERDIYEV A., Central extensions of filiform Zinbiel algebras, *Linear and Multilinear Algebra.* **70** (2022), 2, 1479–1495. MR4420911, Zbl 1493.17004. doi: 10.1080/03081087.2020.1764903. 1342
10. CAMACHO L.M., NAVARRO R.M., SÁNCHEZ J.M., On Naturally Graded Lie and Leibniz Superalgebras, *Bulletin of the Malaysian Mathematical Sciences Society.* **43** (2020), 5, 3411–3435. MR4152838, Zbl 1477.17011. doi: 10.1007/s40840-019-00876-9. 1345
11. CEBALLOS M., TOWERS D., Abelian subalgebras and ideals of maximal dimension in Zinbiel algebras, *Communications in Algebra.* **51** (2023), 4, 1323–1333. MR4552895, Zbl 07664512. doi: 10.1080/00927872.2022.2134409. 1342, 1343
12. CHAPOTON F., Zinbiel algebras and multiple zeta values, *Documenta Mathematica.* **27** (2022), 519–533. MR4424028, Zbl 1498.17006. doi: 10.25537/dm.2022v27.519-533. 1342
13. COLMENAREJO L., DIEHL J., SOREA M., A quadratic identity in the shuffle algebra and an alternative proof for de Bruijn's formula, *European Journal of Combinatorics.* **99** (2022), 103406. MR4304214, Zbl 1483.17023. doi: 10.1016/j.ejc.2021.103406. 1342
14. COVEZ S., FARINATI M., LEBED V., MANCHON D., Bialgebraic approach to rack cohomology, *Algebraic & Geometric Topology.* **23** (2023), 4, 1551–1582. MR4602407, Zbl 07706504. doi: 10.2140/agt.2023.23.1551. 1342
15. DOKAS I., Zinbiel algebras and commutative algebras with divided powers, *Glasgow Mathematical Journal.* **52** (2010), 2, 303–313. MR2610974, Zbl 1250.17002. doi: 10.1017/S0017089509990358. 1342
16. DZHUMADILDAEV A., Zinbiel algebras under q-commutators, *Journal of Mathematical Sciences (New York).* **144** (2007), 2, 3909–3925. MR2176680, Zbl 1119.17001. doi: 10.1007/s10958-007-0244-9. 1342
17. DZHUMADILDAEV A., TULENBAEV K., Nilpotency of Zinbiel algebras, *Journal of Dynamical and Control Systems.* **11** (2005), 2, 195–213. MR2131808, Zbl 1063.17002. doi: 10.1007/s10883-005-4170-1. 1342, 1358
18. GAO X., GUO L., ZHANG YI., Commutative matching Rota-Baxter operators, shuffle products with decorations and matching Zinbiel algebras, *Journal of Algebra.* **586** (2021), 402–432. MR4287778, Zbl 1496.17015. doi: 10.1016/j.jalgebra.2021.06.032. 1342
19. GUTERMAN A., KUDRYAVTSEV D., Algebras of slowly growing length, *International Journal of Algebra and Computation.* **32** (2022), 7, 1307–1325. MR4514265, Zbl 1519.17001. doi: 10.1142/S0218196722500564. 1342

20. IKONICOFF S., PACAUD LEMAY J.-P., Cartesian Differential Comonads and New Models of Cartesian Differential Categories, *Cahiers de Topologie et Géométrie Différentielle Catégoriques.* **64** (2023), 2, 198–239. MR4605864, Zbl 1511.18002. 1342
21. ISMAILOV N., MASHUROV F., SMADYAROV N., Defining Identities for mono and binary Zinbiel algebras, *Journal of Algebra and Its Applications.* **22** (2023), 8, 2350165. MR4598670, Zbl 07709974. doi: 10.1142/S0219498823501657. 1342
22. KAYGORODOV I., ALVAREZ M.A., CASTILHO DE MELLO T., Central extensions of 3-dimensional Zinbiel algebras, *Ricerche di Matematica.* **72** (2023), 2, 921–947. MR4649473, Zbl 07754375. doi: 10.1007/s11587-021-00604-1. 1342
23. KAYGORODOV I., POPOV YU., POZHIDAEV A., VOLKOV YU., Degenerations of Zinbiel and nilpotent Leibniz algebras, *Linear and Multilinear Algebra.* **66** (2018), 4, 704–716. [Corrigendum to "Degenerations of Zinbiel and nilpotent Leibniz algebras", Linear and Multilinear Algebra, 70 (2022), 5, 993–995.] MR3779144, Zbl 1472.17100. doi: 10.1080/03081087.2017.1319457. 1342
24. KAWSKI M., Chronological algebras: combinatorics and control, *Journal of Mathematical Sciences (New York).* **103** (2001), 6, 725–744. MR1871128, Zbl 1157.93353. doi: 10.1023/A:1009502501461. 1342
25. KOLESNIKOV P., Commutator algebras of pre-commutative algebras, *Matematicheskii Zhurnal.* **16** (2016), 2, 56–70. Zbl 1479.17006. 1342
26. LODAY J.-L., Cup product for Leibniz cohomology and dual Leibniz algebras, *Mathematica Scandinavica.* **77** (1995), 2, 189–196. MR1379265, Zbl 0859.17015. doi: 10.7146/math.scand.a-12560. 1342
27. LODAY J.-L., On the algebra of quasi-shuffles, *Manuscripta mathematica.* **123** (2007), 1, 79–93. MR2300061, Zbl 1126.16029. doi: 10.1007/s00229-007-0086-2. 1342
28. NAURAZBEKOVA A., On the structure of free dual Leibniz algebras, *Eurasian Mathematical Journal.* **10** (2019), 3, 40–47. MR4034423, Zbl 1463.17009. doi: 10.32523/2077-9879-2019-10-3-40-47. 1342
29. NAURAZBEKOVA A., UMIRBAEV U., Identities of dual Leibniz algebras, *TWMS Journal of Pure and Applied Mathematics.* **1** (2010), 1, 86–91. MR2745589, Zbl 1223.17003. 1342
30. TOWERS D., Zinbiel algebras are nilpotent, *Journal of Algebra and Its Applications.* **22** (2023), 8, 2350166. MR4598671, Zbl 07709975. doi: 10.1142/S0219498823501669. 1342, 1343, 1358, 1360

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