

## On the invertibility of operators on a model space

Mubariz Garayev

ABSTRACT. For a scalar inner function  $\theta$ , the model space of Sz.-Nagy and Foias is the subspace  $K_\theta = H^2 \ominus \theta H^2$  of the classical Hardy space  $H^2 = H^2(\mathbb{D})$  over the unit disc  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ . For a bounded linear operator  $A$  on the model space  $K_\theta$ , its Berezin symbol is the function  $\tilde{A}^{K_\theta}$  defined on  $\mathbb{D}$  by  $\tilde{A}^{K_\theta}(\lambda) = \langle A\hat{k}_{\theta,\lambda}, \hat{k}_{\theta,\lambda} \rangle$ , where

$$\hat{k}_{\theta,\lambda}(z) = \left( \frac{1 - |\lambda|^2}{1 - |\theta(\lambda)|^2} \right)^{1/2} \frac{1 - \overline{\theta(\lambda)}\theta(z)}{1 - \bar{\lambda}z}$$

is the normalized reproducing kernel of the subspace  $K_\theta$ . We shall consider the following question: Let  $A : K_\theta \rightarrow K_\theta$  be an operator for which there exists a constant  $\delta > 0$  such that  $|\tilde{A}^{K_\theta}(\lambda)| \geq \delta > 0$ , for all  $\lambda \in \mathbb{D}$ . Under which additional conditions is  $A$  invertible? In this article we investigate this question in the case when  $\theta$  is an interpolation Blaschke product. In particular, the invertibility property of functions of model operators is investigated. Some other problems are also discussed.

### CONTENTS

1. Introduction	437
2. Notations and Preliminaries	438
3. Invertibility of operators on the model space $K_\theta$	440
4. Further Results	446
5. Acknowledgment	449
References	449

Received September 13, 2023.

2010 *Mathematics Subject Classification.* 47A35, 47B20.

*Key words and phrases.* Interpolation Blaschke product, reproducing kernel, Berezin symbol, invertibility of operators.

The author would like to extend his appreciation to the Distinguished Scientist Fellowship Program at King Saud University, Riyadh, Saudi Arabia, for funding this work through Researchers Supporting Project number RSPD2024R1056.

### 1. Introduction

In this paper we continue the investigation of a generalized Douglas problem started by the author in [18]. We consider the question of invertibility of operators on the model space  $K_\theta = H^2 \ominus \theta H^2$  of Sz.-Nagy and Foias, where  $H^2 = H^2(\mathbb{D})$  is the Hardy space of all analytic functions in the unit disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

for which

$$\|f\|_2^2 \stackrel{def}{=} \sup_{0 \leq r < 1} \int_{\mathbb{T}} |f(r\zeta)|^2 dm(\zeta) < \infty,$$

where  $\mathbb{T} = \partial\mathbb{D} = \{\zeta : |\zeta| = 1\}$  is the unit circle,  $m$  is the normalized Lebesgue measure on  $\mathbb{T}$ , and  $\theta$  is an inner function (i.e.,  $\theta \in H^2$  and  $|\theta(\zeta)| = 1$  a.a.  $\zeta \in \mathbb{T}$ ). Recall that  $H^2$  is a reproducing kernel Hilbert space, with the kernel

$$k_\lambda(z) = \frac{1}{1 - \bar{\lambda}z}, \lambda, z \in \mathbb{D},$$

known as the Szegő kernel. Thus  $\langle f, k_\lambda \rangle = f(\lambda)$  for all  $f \in H^2$  and  $\lambda \in \mathbb{D}$ . Therefore the function

$$k_{\theta,\lambda}(z) \stackrel{def}{=} P_\theta k_\lambda(z) = \frac{1 - \overline{\theta(\lambda)}\theta(z)}{1 - \bar{\lambda}z}, \lambda, z \in \mathbb{D},$$

where  $P_\theta$  is the orthogonal projection from  $H^2$  onto  $K_\theta$ , is the reproducing kernel for the space  $K_\theta$ . For any bounded linear operator  $A : K_\theta \rightarrow K_\theta$ , the Berezin symbol of  $A$  is the function  $\tilde{A}^{K_\theta}(\lambda)$  on  $\mathbb{D}$  defined by the formula

$$\tilde{A}^{K_\theta}(\lambda) \stackrel{def}{=} \langle A\hat{k}_{\theta,\lambda}, \hat{k}_{\theta,\lambda} \rangle, \lambda \in \mathbb{D},$$

where

$$\hat{k}_{\theta,\lambda}(z) \stackrel{def}{=} \frac{k_{\theta,\lambda}(z)}{\|k_{\theta,\lambda}(z)\|} = \left( \frac{1 - |\lambda|^2}{1 - |\theta(\lambda)|^2} \right)^{1/2} \frac{1 - \overline{\theta(\lambda)}\theta(z)}{1 - \bar{\lambda}z}$$

denotes the normalized reproducing kernel of  $K_\theta$ . Let  $\mathcal{B}(K_\theta)$  denote the algebra of all bounded linear operators on the space  $K_\theta$ . In this article we shall investigate the following question: Let  $A \in \mathcal{B}(K_\theta)$  satisfying

$$|\tilde{A}^{K_\theta}(\lambda)| \geq \delta > 0 \ (\forall \lambda \in \mathbb{D})$$

for some  $\delta > 0$ . Under which conditions is  $A$  invertible in  $K_\theta$ ?

This question is closely related to a problem of Douglas and works of Tolokonnikov, Nikolski and Wolff (see [21]), and author's paper [18]. Moreover, the Douglas type question for Bergman-Toeplitz operator with bounded harmonic symbols also has been widely investigated in recent years. For relevant references on this topic, one can consult Zhao-Zheng [28, 29], Guo-Zhao-Zheng [15] and Yoneda [27]. It is worth nothing that the Tolokonnikov-Nikolski type condition for Bergman-Toeplitz operators with bounded harmonic symbols was established in [28, 29]. In the present article, using the techniques of reproducing kernels and Berezin symbols, and an interpolation theorem of Shvedenko [22],

we obtain sufficient conditions ensuring the invertibility of a linear bounded operators on the model space  $K_B$  with a suitable interpolation Blaschke product  $B$  (see Theorem 3.1 below). In particular, we give a new proof for invertibility of some functions of model operators  $\varphi(M_B)$ , which does not use the Carleson's Corona Theorem (see Theorem 3.3 below). We also characterize in terms of Berezin symbols the normal operators on the Hardy space  $H^2$ , and study the compactness property of some products of Toeplitz operators on the Hardy and Bergman spaces.

## 2. Notations and Preliminaries

**2.1. Berezin symbol.** The Berezin symbol of a linear bounded operator  $T$  acting on a functional Hilbert space  $\mathcal{H} = \mathcal{H}(\mathbb{D})$  over the unit disk  $\mathbb{D}$ , with a reproducing kernel  $k_\lambda(z)$  is the complex-valued function

$$\widetilde{T}(\lambda) \stackrel{\text{def}}{=} \langle T\widehat{k}_\lambda, \widehat{k}_\lambda \rangle, \quad \lambda \in \mathbb{D},$$

where  $\widehat{k}_\lambda := k_\lambda / \|k_\lambda\|$  denotes the normalized reproducing kernel of  $\mathcal{H}$ . This notion has been introduced for the first time by Berezin [4, 5]. It is well known (see [9], [30]) that for a Toeplitz operator  $T_\varphi$ , with symbol  $\varphi \in L^\infty(\mathbb{T})$ , defined on  $H^2$  by  $T_\varphi f = P_+ \varphi f$ , where  $P_+$  is the orthogonal projection from  $L^2(\mathbb{T})$  onto  $H^2$ , known as the Riesz projection, its Berezin symbol  $\widetilde{T}_\varphi$  is the harmonic extension  $\widetilde{\varphi}$  of  $\varphi \in L^\infty(\mathbb{T})$  into  $\mathbb{D}$ .

It is natural to define the following two numerical characteristics for the operator  $T \in \mathcal{B}(\mathcal{H})$

$$\text{Ber}(T) \stackrel{\text{def}}{=} \text{Range}(\widetilde{T}) \text{ is the so-called Berezin set}$$

and

$$\text{ber}(T) \stackrel{\text{def}}{=} \sup\{|\lambda| : \lambda \in \text{Ber}(T)\} \text{ is the so-called Berezin number.}$$

Obviously,  $\text{Ber}(T) \subset W(T)$ , where  $W(T)$  is the numerical range of  $T$ . Also it is easy to see that  $\text{ber}(T) \leq w(T)$ , where  $w(T)$  denotes the numerical radius of  $T$ .

When  $T = T_\varphi$ , where  $\varphi$  is bounded, it is clear that  $\text{Ber}(T_\varphi) = \{\widetilde{\varphi}(z) : z \in \mathbb{D}\}$  and  $\text{ber}(T_\varphi) = \|\varphi\|_\infty$ . In particular,  $\text{Ber}(T_z) = \mathbb{D}$  and  $\text{ber}(T_z) = 1$ .

Note that on the most familiar functional Hilbert spaces, including the Hardy space and the Bergman space, the Berezin symbol uniquely determines the operator. In fact, if  $\widetilde{T}_1(\lambda) = \widetilde{T}_2(\lambda)$  for all  $\lambda$ , then  $T_1 = T_2$ . See for instance, Yang [26]; and for more general cases, see Fricain [10, Theorem 1.1.1]. In other words the Berezin symbol of a bounded operator contains a lot of information about the operator. It is one of the most useful tools in the study of Toeplitz operators. The Berezin notion is motivated by its connections with quantum physics and noncommutative geometry. For more details and references see [4, 5]. Other properties and applications of Berezin symbols and reproducing kernels can be found in [2, 3, 6, 7, 11, 12, 13].

**2.2. Carleson condition.** We now recall some well-known facts (see, for instance [21]) concerning reproducing kernels in  $H^2(\mathbb{D})$ . Let  $\Lambda = \{\lambda_n\}_{n \geq 1}$  be a sequence of distinct points in  $\mathbb{D}$ . We denote by

$$B = B_\Lambda = \prod_{n \geq 1} b_{\lambda_n},$$

where

$$b_{\lambda_n}(z) = \frac{|\lambda_n|}{\lambda_n} \frac{\lambda_n - z}{1 - \overline{\lambda_n}z},$$

the corresponding Blaschke product. Then we have

(i) If  $\{\lambda_n\}_{n \geq 1}$  satisfies the Blaschke condition, i.e.,  $\sum_{n=1}^\infty (1 - |\lambda_n|^2) < \infty$ , then  $\{k_{\lambda_n}\}_{n \geq 1}$  is a complete system in the model space  $K_B$ .

(ii) The family  $\mathcal{K} \stackrel{\text{def}}{=} \{\widehat{k}_{\lambda_n} : n \geq 1\}$  is a Riesz basis of  $K_B$  if and only if  $\{\lambda_n\}_{n \geq 1}$  satisfies the Carleson condition, namely

$$\inf_{n \geq 1} |B_n(\lambda_n)| > 0,$$

where  $B_n \stackrel{\text{def}}{=} \frac{B}{b_{\lambda_n}}$ . In this case, we will write  $\Lambda \in (C)$ .

**2.3. Riesz constant.** We recall (see [21]) that if  $H$  is a complex Hilbert space, and  $\{x_n\}_{n \geq 1} \subset H$ , then the set  $X \stackrel{\text{def}}{=} \{x_n : n \geq 1\}$  is called a Riesz basis of  $H$  if there exists an isomorphism  $U$  mapping  $X$  onto an orthonormal basis of  $H$ . In this case the operator  $U$  will be called the orthogonalizer of  $X$ . It is well known (see [21]) that  $X$  is a Riesz basis in its closed linear span if there are two positive constants  $C_1, C_2$  such that

$$C_1 \left( \sum_{n \geq 1} |a_n|^2 \right)^{1/2} \leq \left\| \sum_{n \geq 1} a_n x_n \right\| \leq C_2 \left( \sum_{n \geq 1} |a_n|^2 \right)^{1/2} \tag{2.1}$$

for all finite complex sequences  $\{a_n\}_{n \geq 1}$ . Note that if  $U$  is an orthogonalizer of the set  $X$ , then  $C_1 = \|U\|^{-1}$  and  $C_2 = \|U^{-1}\|$  are the best constants possible in the inequality (2.1). The product  $r(X) \stackrel{\text{def}}{=} \|U\| \|U^{-1}\|$  characterizes the deviation of the basis  $X$  from an orthonormal one.  $r(X)$  will be referred to as the Riesz constant of the family  $X$ . Clearly,  $r(X) \geq 1$ . For more detail, see [14, 21].

**2.4. Shvedenko constant.** Let  $\{L_k\}_{k=1}^\infty$  be a sequence of linear continuous functionals on the Hardy space  $H^p$ ,  $1 < p < \infty$ , (see Hoffman [16]). It is natural to try to describe the space of sequences

$$H^p\{L_k\} \stackrel{\text{def}}{=} \{ \{L_k(f)\}_{k=1}^\infty : f \in H^p \}.$$

In particular, it is not without interest to try to find conditions under which the inclusion  $S \subset H^p\{L_k\}$  is satisfied for a given space  $S$  of sequences of complex numbers. For some class of Banach spaces  $S$  of sequences, Shvedenko [22] gave a general criterion for such inclusion.

For  $1 < p < \infty$ , it is well known that the functionals  $L_k$  have the following representation

$$L_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) l_k(e^{it}) dt, f \in H^p,$$

where the functions  $l_k(z) \in H^q \left( \frac{1}{p} + \frac{1}{q} = 1 \right)$  depend only on  $L_k$ . Notice that for the evaluation functionals

$$L_k(f) = f(\lambda_k), k = 1, 2, \dots,$$

where  $\{\lambda_k\}_{k=1}^{\infty}$  is a sequence of different points of  $\mathbb{D}$ , it is easy to see that  $l_k(z) = \frac{1}{1-\lambda_k z}$ , which is the Szegő kernel.

Let  $S$  be a Banach space of sequences satisfying the following conditions

**(S1)**  $S$  be a *BK*-space [23], i.e., the map  $w \rightarrow w_k, w = \{w_k\}_{k=1}^{\infty} \in S$ , is continuous. This is equivalent to the inequality  $|w_k| \leq c_k \|w\|_S$ , where  $\|w\|_S$  is the norm in  $S$  and  $c_k > 0, k = 1, 2, \dots$ . In particular, this condition implies the inclusion  $E_{\infty} \subset S^*$ , where  $E_{\infty}$  is the space of sequences containing only a finite number of nonzero terms.

**(S2)**  $S$  is complexly conjugated, i.e., both of  $w = \{w_k\}_{k=1}^{\infty}$  and  $\bar{w} = \{\bar{w}_k\}_{k=1}^{\infty}$  belong to  $S$  and  $\|\bar{w}\|_S = \|w\|_S$ .

It is not difficult to verify that the classical weighted spaces  $l^p(w_n), p \geq 1$ , satisfy conditions (S1) and (S2). The following key lemma is due to Shvedenko [22].

**Lemma 2.1.** *For the Banach spaces  $S$  of sequences satisfying conditions (S1) and (S2), the inclusion  $S \subset H^p \{L_k\}, 1 < p < \infty$ , is fulfilled if and only if*

$$\inf_{\langle a_k \rangle \in E_{\infty}} \frac{\left\| \sum_k a_k k_{\lambda_k}(z) \right\|_q}{\|\langle a_k \rangle\|_{S^*}} > 0, \text{ where } \frac{1}{p} + \frac{1}{q} = 1.$$

In what follows we will call the number

$$\eta_{\Lambda} \stackrel{\text{def}}{=} \inf_{\langle a_k \rangle \in E_{\infty}} \frac{\left\| \sum_k a_k k_{\lambda_k}(z) \right\|_q}{\|\langle a_k \rangle\|_{S^*}}$$

the Shvedenko constant corresponding to the sequence  $\Lambda \stackrel{\text{def}}{=} \{\lambda_k\}_{k=1}^{\infty}$ .

### 3. Invertibility of operators on the model space $K_B$

**3.1. Main result.** The main result of this article is the following theorem, which essentially improves [18, Theorem 3.1 p. 185].

**Theorem 3.1.** *Suppose  $\Lambda = \{\lambda_n\}_{n \geq 1}$  is a Carleson sequence of distinct points of  $\mathbb{D}$ ,  $B$  is the interpolation Blaschke product associated to  $\{\lambda_n\}_{n \geq 1}$ ,  $\mathcal{K} = \{\widehat{k}_{\lambda_n} : n \geq 1\}$  is a corresponding Riesz basis of the space  $K_B$ , and  $r(\mathcal{K})$  is the corresponding Riesz constant associated the family  $\mathcal{K}$ . For any  $A \in \mathcal{B}(K_B)$  and any bounded sequence*

$b = \{b_n\}_{n \geq 1}$  of complex numbers  $b_n$ , for which there exists a constant  $\delta > 0$  such that  $|b_n| \geq \delta$  for all  $n \geq 1$ , we denote

$$\tau_{A,b} := \left( \sum_{n=1}^{\infty} (1 - |\lambda_n|^2) \left\| (A - b_n I) \widehat{k}_{\lambda_n} \right\|^2 \right)^{1/2}$$

and

$$\tau_{A,b}^* := \left( \sum_{n=1}^{\infty} (1 - |\lambda_n|^2) \left\| (A - b_n I)^* \widehat{k}_{\lambda_n} \right\|^2 \right)^{1/2}.$$

If

$$\frac{r(\mathcal{K})}{\eta_{\Lambda} \delta} \max \{ \tau_{A,b}, \tau_{A,b}^* \} < 1, \quad (\mathbf{H})$$

then  $A$  is an invertible operator in  $K_B$ . Moreover

$$\|A^{-1}\| \leq \frac{r(\mathcal{K}) \eta_{\Lambda}}{\delta \eta_{\Lambda} - r(\mathcal{K}) \tau_{A,b}},$$

where  $\eta_{\Lambda}$  is the Shvedenko constant corresponding to the sequence  $\Lambda = \{\lambda_n\}_{n \geq 1}$ .

**Proof.** Since  $\left\{ \left\| A \widehat{k}_{\lambda_n} - b_n \widehat{k}_{\lambda_n} \right\| \right\}_{n \geq 1}$  and  $\left\{ \left\| A^* \widehat{k}_{\lambda_n} - \bar{b}_n \widehat{k}_{\lambda_n} \right\| \right\}_{n \geq 1}$  are bounded sequences and  $\Lambda$  is a Blaschke sequence, the numbers  $\tau_{A,b}$  and  $\tau_{A,b}^*$  are finite. Also, the family  $\mathcal{K} = \{ \widehat{k}_{\lambda_n} : n \geq 1 \}$  is a Riesz basis in  $K_B$ , because  $\Lambda \in (C)$ . If  $U$  is an orthogonalizer of  $\mathcal{K}$ , then

$$\|U\|^{-1} \left( \sum_{n \geq 1} |a_n|^2 \right)^{1/2} \leq \left\| \sum_{n \geq 1} a_n \widehat{k}_{\lambda_n} \right\| \leq \|U^{-1}\| \left( \sum_{n \geq 1} |a_n|^2 \right)^{1/2} \quad (3.1)$$

for any finite complex sequence  $\{a_n\}_{n \geq 1}$ . Hence by considering the condition  $|b_n| \geq \delta > 0$ ,  $n \geq 1$ , we have that for any  $N > 0$

$$\begin{aligned} \left\| \sum_{n=1}^N a_n b_n \widehat{k}_{\lambda_n} \right\| &\geq \|U\|^{-1} \left( \sum_{n=1}^N |a_n b_n|^2 \right)^{1/2} \\ &\geq \delta \|U\|^{-1} \left( \sum_{n=1}^N |a_n|^2 \right)^{1/2} \\ &\geq \frac{\delta}{\|U\| \|U^{-1}\|} \left\| \sum_{n=1}^N a_n \widehat{k}_{\lambda_n} \right\|. \end{aligned}$$

Since  $r(\mathcal{K}) = \|U\| \|U^{-1}\|$ , we obtain

$$\left\| \sum_{n=1}^N a_n b_n \widehat{k}_{\lambda_n} \right\| \geq \frac{\delta}{r(\mathcal{K})} \left\| \sum_{n=1}^N a_n \widehat{k}_{\lambda_n} \right\|. \quad (3.2)$$

Now

$$\begin{aligned} \left\| A \sum_{n=1}^N a_n \widehat{k}_{\lambda_n} \right\| &= \left\| \sum_{n=1}^N a_n A \widehat{k}_{\lambda_n} \right\| \\ &= \left\| \sum_{n=1}^N a_n (A \widehat{k}_{\lambda_n} - b_n \widehat{k}_{\lambda_n} + b_n \widehat{k}_{\lambda_n}) \right\| \\ &\geq \left\| \sum_{n=1}^N a_n b_n \widehat{k}_{\lambda_n} \right\| - \sum_{n=1}^N |a_n| \|A \widehat{k}_{\lambda_n} - b_n \widehat{k}_{\lambda_n}\|. \end{aligned}$$

From inequality (3.2), it follows that

$$\begin{aligned} \left\| A \sum_{n=1}^N a_n \widehat{k}_{\lambda_n} \right\| &\geq \frac{\delta}{r(\mathcal{K})} \left\| \sum_{n=1}^N a_n \widehat{k}_{\lambda_n} \right\| \\ &\quad - \sum_{n=1}^N \frac{|a_n|}{(1 - |\lambda_n|^2)^{1/2}} (1 - |\lambda_n|^2)^{1/2} \|A \widehat{k}_{\lambda_n} - b_n \widehat{k}_{\lambda_n}\|. \end{aligned}$$

Using the Hölder inequality, we obtain

$$\begin{aligned} \left\| A \sum_{n=1}^N a_n \widehat{k}_{\lambda_n} \right\| &\geq \frac{\delta}{r(\mathcal{K})} \left\| \sum_{n=1}^N a_n \widehat{k}_{\lambda_n} \right\| \\ &\quad - \left( \sum_{n=1}^N \frac{|a_n|^2}{(1 - |\lambda_n|^2)} \right)^{1/2} \left( \|A \widehat{k}_{\lambda_n} - b_n \widehat{k}_{\lambda_n}\|^2 \sum_{n=1}^N (1 - |\lambda_n|^2) \right)^{1/2} \\ &\geq \frac{\delta}{r(\mathcal{K})} \left\| \sum_{n=1}^N a_n \widehat{k}_{\lambda_n} \right\| - \left( \sum_{n=1}^N \frac{|a_n|^2 (1 - |\lambda_n|^2)}{(1 - |\lambda_n|^2)^2} \right)^{1/2} \tau_{A,b}. \end{aligned}$$

Writing that

$$\left( \sum_{n=1}^N \frac{|a_n|^2 (1 - |\lambda_n|^2)}{(1 - |\lambda_n|^2)^2} \right)^{1/2} = \left\| \{a_n (1 - |\lambda|^2)^{1/2}\}_{n=1}^N \right\|_{l^2((1 - |\lambda_n|^2)^{-1})},$$

we have

$$\begin{aligned} \left\| A \sum_{n=1}^N a_n \widehat{k}_{\lambda_n} \right\| &\geq \\ &\quad \frac{\delta}{r(\mathcal{K})} \left\| \sum_{n=1}^N a_n \widehat{k}_{\lambda_n} \right\| - \left\| \{a_n (1 - |\lambda|^2)^{1/2}\}_{n=1}^N \right\|_{l^2((1 - |\lambda_n|^2)^{-1})} \eta_\Lambda \eta_\Lambda^{-1} \tau_{A,b}. \end{aligned}$$

Because  $\{\lambda_n\}_{n \geq 1}$  is assumed to be a Carleson sequence, it is well-known in this case that

$$H^2\{L_n\} = l^2(1 - |\lambda_n|^2).$$

Thus, by considering the obvious inclusion  $l^2\left((1 - |\lambda_n|^2)^{-1}\right) \subset l^2(1 - |\lambda_n|^2)$ , we have that  $l^2\left((1 - |\lambda_n|^2)^{-1}\right) \subset H^2\{L_n\}$ . Therefore, by setting  $p = 2$  and  $S = l^2\left((1 - |\lambda_n|^2)^{-1}\right)$  in Lemma 2.1, we obtain

$$\begin{aligned} \left\| A \sum_{n=1}^N a_n \widehat{k}_{\lambda_n} \right\| &\geq \frac{\delta}{r(\mathcal{K})} \left\| \sum_{n=1}^N a_n \widehat{k}_{\lambda_n} \right\| - \left\| \sum_{n=1}^N a_n (1 - |\lambda_n|^2)^{1/2} k_{\lambda_n} \right\| \eta_{\Lambda}^{-1} \tau_{A,b} \\ &= \frac{\delta}{r(\mathcal{K})} \left\| \sum_{n=1}^N a_n \widehat{k}_{\lambda_n} \right\| - \eta_{\Lambda}^{-1} \tau_{A,b} \left\| \sum_{n=1}^N a_n \widehat{k}_{\lambda_n} \right\|. \end{aligned}$$

Finally, we arrive to the following inequality

$$\left\| A \sum_{n=1}^N a_n \widehat{k}_{\lambda_n} \right\| \geq \left( \frac{\delta}{r(\mathcal{K})} - \frac{\tau_{A,b}}{\eta_{\Lambda}} \right) \left\| \sum_{n=1}^N a_n \widehat{k}_{\lambda_n} \right\| \tag{3.3}$$

for any complex numbers  $a_n, n = 1, 2, \dots, N$ , and all  $N > 0$ . Since the Carleson condition implies the Blaschke condition,  $\text{Span}(\mathcal{K}) = K_B$ , i.e.,  $\mathcal{K}$  is a complete system in  $K_B$ . Therefore, we deduce from (3.3) that

$$\|A f\| \geq \left( \frac{\delta}{r(\mathcal{K})} - \frac{\tau_A}{\eta_{\Lambda}} \right) \|f\|, \tag{3.4}$$

for any  $f \in K_B$ .

By a similar arguments, we prove that

$$\left\| A^* \sum_{n=1}^N a_n \widehat{k}_{\lambda_n} \right\| \geq \left( \frac{\delta}{r(\mathcal{K})} - \frac{\tau_{A,b}^*}{\eta_{\Lambda}} \right) \left\| \sum_{n=1}^N a_n \widehat{k}_{\lambda_n} \right\|,$$

which yields that

$$\|A^* f\| \geq \left( \frac{\delta}{r(\mathcal{K})} - \frac{\tau_{A,b}^*}{\eta_{\Lambda}} \right) \|f\| \tag{3.5}$$

for any  $f \in K_B$ . Now, combining hypothesis **(H)** with both inequalities (3.4) and (3.5), implies that  $A$  is invertible in  $K_B$  and that

$$\|A^{-1}\| \leq \frac{r(\mathcal{K}) \eta_{\Lambda}}{\eta_{\Lambda} \delta - r(\mathcal{K}) \tau_{A,b}},$$

which completes the proof. □

Since  $\widetilde{A}^{*K_B}(\lambda) = \overline{\widetilde{A}^{K_B}(\lambda)}$  and  $|\widetilde{A}^{*K_B}(\lambda)| = |\widetilde{A}^{K_B}(\lambda)| \leq \|A\|$  for every  $A$  in  $\mathcal{B}(K_B)$ , the following is an immediate corollary of Theorem 3.1.



**Corollary 3.2.** *Under the same hypothesis as in Theorem 3.1, if the sequence  $b$  is chosen to be  $b = \{\widetilde{A}^{K_B}(\lambda_n)\}$  with  $|\widetilde{A}^{K_B}(\lambda_n)| \geq \delta > 0$ , for all  $n \geq 1$ . Then  $A$  is invertible in  $K_B$  and*

$$\|A^{-1}\| \leq \frac{r(\mathcal{K})\eta_\Lambda}{\delta\eta_\Lambda - r(\mathcal{K})\tau_{A,b}}.$$

**3.2. Model operators.** Consider the model operator  $M_\theta$  of Sz.-Nagy and Foias, acting on the subspace  $K_\theta$  in the following way

$$M_\theta f = P_\theta S f,$$

where  $S$  is the unilateral shift operator in the Hardy space  $H^2$  defined by  $Sg(z) = zg(z)$ ,  $\theta$  is an inner function and  $P_\theta = I - T_\theta T_\theta^*$  is the orthogonal projection from  $H^2$  onto  $K_\theta$ . It is well-known [24] that the operator  $M_\theta$  admits a functional calculus in the class  $H^\infty$ , i.e., for any function  $\varphi \in H^\infty$ , the operator  $\varphi(M_\theta)$  is defined by

$$\varphi(M_\theta) f = P_\theta \varphi f, \quad f \in K_\theta.$$

It is also known [21] that the operator  $\varphi(M_\theta)$  is invertible if and only if there exists a constant  $\delta > 0$  such that

$$|\varphi(z)| + |\theta(z)| \geq \delta, \tag{3.6}$$

for all  $z \in \mathbb{D}$ . The proof of this statement is based on the classical Carleson’s Corona Theorem [21]. When  $\theta$  is an interpolation Blaschke product  $B$ , i.e., Blaschke product with zeros  $\Lambda = \{\lambda_n\}_{n \geq 1} \in (C)$ , it is also known (see, for instance, Hoffman [16, Chapter 10]) that condition (3.6) becomes

$$|\varphi(\lambda_n)| \geq \delta, \quad \text{for all } n \geq 1.$$

In the next proposition, we present a new proof of the invertibility of the operator  $\varphi(M_B)$ , which does not use the Carleson’s corona theorem.

**Theorem 3.3.** *Let  $\Lambda = \{\lambda_n\}_{n \geq 1}$ ,  $\eta_\Lambda$ ,  $B$  and  $r(\mathcal{K})$  be as in Theorem 3.1. Let  $\varphi \in H^\infty$  be a function satisfying  $\|\varphi\|_\infty \leq 1$ , for which there exists a constant  $\delta > 0$  such that*

$$|\varphi(\lambda_n)| \geq \delta > r(\mathcal{K})\omega_\Lambda \sqrt{\frac{1}{\eta_\Lambda^2 + r(\mathcal{K})^2 \omega_\Lambda^2}}, \quad \text{for all } n \geq 1, \tag{3.7}$$

for some  $\delta > 0$ , where  $\omega_\Lambda \stackrel{\text{def}}{=} \left(\sum_{n=1}^\infty (1 - |\lambda_n|^2)\right)^{1/2}$ . Then the operator  $\varphi(M_B)$  is invertible in  $K_B$  and

$$\|(\varphi(M_B))^{-1}\| \leq \frac{r(\mathcal{K})}{\delta}.$$

**Proof.** The proof uses the following formula

$$\widetilde{\varphi(M_B)^{K_B}}(z) = \frac{\varphi(z) - B(z)\overline{\varphi B(z)}}{1 - |B(z)|^2} \quad (z \in \mathbb{D}),$$

from the author's paper [17, formula (1)], which implies that

$$b_n := \widehat{\varphi(M_B)^{K_B}}(\lambda_n) = \varphi(\lambda_n), \text{ for all } n \geq 1.$$

This shows that the inequality  $|\varphi(\lambda_n)| \geq \delta$  is in fact equivalent to

$$\left| \widehat{\varphi(M_B)^{K_B}}(\lambda_n) \right| \geq \delta > 0, \text{ for all } n \geq 1.$$

Moreover, using the fact that  $P_B \widehat{k}_{\lambda_n} = \widehat{k}_{\lambda_n}$ , we have

$$\begin{aligned} \left\| \varphi(M_B) \widehat{k}_{\lambda_n} - \varphi(\lambda_n) \widehat{k}_{\lambda_n} \right\|^2 &= \langle P_B T_\varphi \widehat{k}_{\lambda_n}, P_B T_\varphi \widehat{k}_{\lambda_n} \rangle - \overline{\varphi(\lambda_n)} \langle P_B T_\varphi \widehat{k}_{\lambda_n}, \widehat{k}_{\lambda_n} \rangle \\ &\quad - \varphi(\lambda_n) \langle \widehat{k}_{\lambda_n}, P_B T_\varphi \widehat{k}_{\lambda_n} \rangle + |\varphi(\lambda_n)|^2 \\ &= T_{\overline{\varphi}} P_B T_\varphi(\lambda_n) - \overline{\varphi(\lambda_n)} \langle T_\varphi \widehat{k}_{\lambda_n}, \widehat{k}_{\lambda_n} \rangle \\ &\quad - \varphi(\lambda_n) \langle \widehat{k}_{\lambda_n}, T_\varphi \widehat{k}_{\lambda_n} \rangle + |\varphi(\lambda_n)|^2 \\ &= T_{\overline{\varphi}} P_B T_\varphi(\lambda_n) - |\varphi(\lambda_n)|^2 \\ &\leq \text{ber}(T_{\overline{\varphi}} P_B T_\varphi) - \delta^2 \\ &\leq \|T_{\overline{\varphi}} P_B T_\varphi\| - \delta^2 \\ &\leq \|\varphi\|_\infty^2 - \delta^2 \\ &\leq 1 - \delta^2. \end{aligned}$$

Thus

$$\left\| \varphi(M_B) \widehat{k}_{\lambda_n} - \widehat{\varphi(M_B)^{K_B}}(\lambda_n) \widehat{k}_{\lambda_n} \right\|^2 \leq 1 - \delta^2. \tag{3.8}$$

Inequality (3.8) implies that (after rewriting the definition of  $\tau_{A,b}$  in Theorem 3.1, with  $A = \varphi(M_B)$  and  $b_n = \widehat{\varphi(M_B)^{K_B}}(\lambda_n)$ )

$$\tau_{\varphi(M_B),b} \leq \omega_\Lambda (1 - \delta^2)^{1/2}.$$

Since  $(\varphi(M_B))^* = T_{\overline{\varphi}} | K_B$ , it is easy to see that

$$(\varphi(M_B))^* \widehat{k}_{\lambda_n} - \widehat{(\varphi(M_B)^{K_B})^*}(\lambda_n) \widehat{k}_{\lambda_n} = 0.$$

Thus,

$$\tau_{\varphi(M_B),b}^* = 0.$$

Now, combining the equation above with the inequality (3.8) and using the same argument as in the proof of Theorem 3.1, one can show that

$$\|\varphi(M_B) f\| \geq \left[ \frac{\delta}{r(\mathcal{K})} - \frac{\omega_\Lambda}{\eta_\Lambda} (1 - \delta^2)^{1/2} \right] \|f\|$$

and

$$\|(\varphi(M_B))^* f\| \geq \frac{\delta}{r(\mathcal{K})} \|f\|,$$

for any  $f \in K_B$ . Therefore Corollary 3.2 implies that  $\varphi(M_B)$  is invertible, and since  $\|(\varphi(M_B))^{-1}\| = \|(\varphi(M_B)^*)^{-1}\|$ , we have that

$$\|(\varphi(M_B))^{-1}\| \leq \frac{r(\mathcal{K})}{\delta},$$

which completes the proof.  $\square$

**Remark 3.4.** If  $\Lambda = \{\lambda_n\} \in (C)$  is a sequence such that

$$\eta_\Lambda^2 + r(\mathcal{K})^2 w_\Lambda^2 = (r(\mathcal{K}) w_\Lambda + 1)^2$$

(i.e., if  $\eta_\Lambda = \sqrt{1 + 2r(\mathcal{K}) w_\Lambda}$ ), then condition (3.7), satisfied by  $\delta$  in Theorem 3.3, becomes

$$\delta > \frac{r(\mathcal{K}) w_\Lambda}{r(\mathcal{K}) w_\Lambda + 1}. \quad (3.9)$$

In particular, if  $r(\mathcal{K}) w_\Lambda = 45$ , then  $\eta_\Lambda = \sqrt{91}$ . Therefore condition (3.9) becomes

$$\delta > \frac{45}{46}, \text{ which is a "Tolokonnikov type" condition [25].}$$

If  $r(\mathcal{K}) w_\Lambda = 23$ , we obtain that

$$\delta > \frac{23}{24}, \text{ which is a "Nikolski type" condition [21].}$$

More details about "Tolokonnikov-Nikolski type" invertibility conditions for the Toeplitz operators on the Hardy space  $H^2$  can be found in [21].

## 4. Further Results

**4.1. Normal and Toeplitz operators.** We shall characterize normal operators on the Hardy space  $H^2$  in terms of Berezin symbols. Also, we shall discuss compactness properties of products of some Toeplitz operators acting in Hardy and Bergman spaces.

**Theorem 4.1.** Let  $A$  be a bounded operator on  $H^2$  and  $\tilde{A}(\lambda) = \langle A\hat{k}_\lambda(z), \hat{k}_\lambda(z) \rangle$  be its Berezin symbol, where  $\hat{k}_\lambda(z) = \frac{\sqrt{1-|\lambda|^2}}{1-\bar{\lambda}z}$  is a normalized reproducing kernel of  $H^2$ . Then

(i)  $A$  is a normal operator on  $H^2$  if and only if

$$\|(A - \tilde{A}(\lambda)I)\hat{k}_\lambda\| = \|(A - \tilde{A}(\lambda)I)^*\hat{k}_\lambda\|, \text{ for all } \lambda \in \mathbb{D}.$$

(ii) In particular, if  $A = T_\varphi$ , where  $\varphi \in L^\infty$ , then the product  $T_\varphi^*T_\varphi$  (or  $T_\varphi T_\varphi^*$ ) is compact if and only if  $\varphi = 0$ .

**Proof.** An easy computation shows that

$$\|A\hat{k}_\lambda - \tilde{A}(\lambda)\hat{k}_\lambda\|^2 = \widetilde{A^*A}(\lambda) - |\tilde{A}(\lambda)|^2 \quad (4.1)$$

and

$$\|A^*\hat{k}_\lambda - \widetilde{A^*}(\lambda)\hat{k}_\lambda\|^2 = \widetilde{AA^*}(\lambda) - |\tilde{A}(\lambda)|^2. \quad (4.2)$$

Since the Berezin symbol uniquely determines the operator  $A$ , it follows from formulas (4.1), (4.2) that  $A$  is a normal operator on  $H^2$  if and only if

$$\|(A - \tilde{A}(\lambda)I) \hat{k}_\lambda\| = \|(A - \tilde{A}(\lambda)I)^* \hat{k}_\lambda\| \quad (\forall \lambda \in \mathbb{D}),$$

which proves (i).

On the other hand, it is known that (see Engliš [9] and Karaev [19])

$$\|T_\varphi \hat{k}_\lambda - \tilde{\varphi}(\lambda) \hat{k}_\lambda\| \rightarrow 0 \text{ as } \lambda \rightarrow \mathbb{T} \text{ radially.}$$

In fact, the functions  $\hat{k}_\lambda$  ( $\lambda \in \mathbb{D}$ ) are "loosely speaking" asymptotic eigenfunctions for the Toeplitz operator  $T_\varphi$ , with asymptotic eigenvalues  $\tilde{\varphi}(\lambda)$ . Since  $\tilde{T}_\varphi(\lambda) = \tilde{\varphi}(\lambda)$  is the harmonic extension of  $\varphi$  onto unit disk  $\mathbb{D}$ , it follows from equalities (4.1) and (4.2) that

$$\|T_\varphi \hat{k}_\lambda - \tilde{\varphi}(\lambda) \hat{k}_\lambda\|^2 = \overline{T_\varphi^* T_\varphi}(\lambda) - |\tilde{\varphi}(\lambda)|^2 \tag{4.3}$$

and

$$\|T_\varphi^* \hat{k}_\lambda - \tilde{\varphi}(\lambda) \hat{k}_\lambda\|^2 = \overline{T_\varphi T_\varphi^*}(\lambda) - |\tilde{\varphi}(\lambda)|^2. \tag{4.4}$$

Now, using the fact that the Berezin symbol of any compact operator on  $H^2$  vanishes at the boundary  $\mathbb{T}$ , and considering the above mentioned facts, we deduce from (4.3) and (4.4) that if  $T_\varphi^* T_\varphi$  (or  $T_\varphi T_\varphi^*$ ) is a compact operator on  $H^2$  then

$$\lim_{r \rightarrow 1^-} |\tilde{\varphi}(re^{it})|^2 = |\varphi(e^{it})|^2 = 0,$$

for almost all  $t \in [0, 2\pi)$ , and hence  $\varphi = 0$ . This proves (ii) and the proposition is then proved.  $\square$

In the Bergman space, as usual, things are much more complicated. The analogous of assertion (ii) of Theorem 4.1 is not true for the Bergman space Toeplitz operators. In Example 4.2, we were able to find a **nonzero** radial symbol  $f$  such that the product  $T_f^2$  is equal to a **compact** Toeplitz operator  $T_g$ .

Let  $dA = r dr \frac{d\theta}{\pi}$ , where  $(r, \theta)$  are the polar coordinates in the complex plane  $\mathbb{C}$ , denote the normalized Lebesgue area measure on the unit disk  $\mathbb{D}$ , so that the measure of  $\mathbb{D}$  equals 1. The Bergman space  $L_a^2(\mathbb{D})$  is the Hilbert space consisting of the analytic functions on  $\mathbb{D}$  that are also square integrable with respect to the measure  $dA$ . We denote the inner product in  $L^2(\mathbb{D}, dA)$  by  $\langle, \rangle$ . It is well known that  $L_a^2(\mathbb{D})$  is a closed subspace of the Hilbert space  $L^2(\mathbb{D}, dA)$ , and has the set  $\{\sqrt{n+1}z^n \mid n \geq 0\}$  as an orthonormal basis. We let  $P$  be the orthogonal projection from  $L^2(\mathbb{D}, dA)$  onto  $L_a^2(\mathbb{D})$ . For a bounded function  $f$  on  $\mathbb{D}$ , the Toeplitz operator  $T_f$  with symbol  $f$  is defined by

$$T_f(h) = P(\phi h) \text{ for } h \in L_a^2(\mathbb{D}).$$

It is well known that if the symbol  $f$  is a radial function, i.e.  $f(z) = f(|z|)$ , then the matrix of the Toeplitz operator  $T_f$ , with respect to the orthonormal

basis  $\{\sqrt{n+1}z^n \mid n \geq 0\}$  of  $L_a^2(\mathbb{D})$ , is a diagonal matrix with the sequence  $\left\{2(n+1) \int_0^1 f(r)r^{2n+1} dr\right\}_{n \geq 0}$  as elements of the main diagonal. In fact

$$\begin{aligned} \langle T_f(\sqrt{n+1}z^n), \sqrt{m+1}z^m \rangle &= \langle P(f\sqrt{n+1}z^n), \sqrt{m+1}z^m \rangle \\ &= \sqrt{n+1}\sqrt{m+1} \langle fz^n, z^m \rangle \\ &= \sqrt{n+1}\sqrt{m+1} \int_0^1 \int_0^{2\pi} f(r)r^{n-m+1} e^{i(n-m)\theta} \frac{d\theta}{2\pi} dr \\ &= \begin{cases} 2(n+1) \int_0^1 f(r)r^{2n+1} dr & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \end{aligned}$$

Moreover, it has been shown in [20, Proposition 4.3. p 530], that the product  $T_f^2$  is equal to a Toeplitz operator  $T_g$  if and only if there exist a **radial** symbol  $g$  solution to the following Mellin convolution equation

$$\int_r^1 g(t) \frac{dt}{t} = \int_r^1 f\left(\frac{r}{t}\right) f(t) \frac{dt}{t}. \quad (4.5)$$

Now we are ready to present our counter example to condition (ii) of Theorem 4.1 in the case of Bergman space Toeplitz operators.

**Example 4.2.** Let  $f(r) = r \ln r$ . By solving equation (4.5) for  $g$ , we obtain

$$g(r) = \frac{r}{2} \left( \frac{1}{3} \ln r - 1 \right) (\ln r)^2.$$

Hence  $T_f^2 = T_g$ . Obviously  $f$  and  $g$  are not bounded but they are the so-called "nearly bounded functions" [1, p.204]. Thus the Toeplitz operators associated to these two symbols are bounded. Since  $g$  is a radial symbol,  $T_g$  is a diagonal operator with the sequence  $\left\{2(n+1) \int_0^1 g(r)r^{2n+1} dr\right\}_{n \geq 0}$  as elements of the main diagonal. In this case, it is well known that  $T_g$  will be compact if and only if

$$\lim_{n \rightarrow +\infty} 2(n+1) \left| \int_0^1 g(r)r^{2n+1} dr \right| = 0.$$

Now since  $T_g = T_f^2$ , a direct calculation shows that

$$\begin{aligned} 2(n+1) \int_0^1 g(r)r^{2n+1} dr &= \left( 2(n+1) \int_0^1 f(r)r^{2n+1} dr \right)^2 \\ &= \frac{4(n+1)^2}{(2n+3)^4}, \text{ for all } n \geq 0. \end{aligned}$$

It is clear that the fraction above will tend to zero as  $n$  goes to infinity. Hence  $T_g$  and therefore  $T_f^2$  is compact. But  $f$  is not the zero function.

## 5. Acknowledgment

The author would like to thank the Editor and the anonymous reviewer for their insightful suggestions and careful reading of the manuscript.

## References

- [1] AHERN, PATRICK; ČUČKOVIĆ, ŽELJKO A theorem of Brown-Halmos type for Bergman space Toeplitz operators. *J. Funct. Anal.* **187** (2001), no. 1, 200–210. [MR2072710](#), [Zbl 0996.47037](#), doi: [10.1006/jfan.2001.3811](#). 448
- [2] ARAZY, JONATHAN; ENGLIŠ, MIROSLAV. Iterates and the boundary behavior of the Berezin transform. *Ann. Inst. Fourier (Grenoble)* **51** (1982), no. 4, 1101–1133. [MR1849217](#), [Zbl 0989.47027](#), doi: [10.5802/aif.1847](#). 438
- [3] AXLER, SHELDON; ZHENG, DECHAO. The Berezin transform on the Toeplitz algebra. *Stud. Math.* **127** (1998), no. 2, 113–136. [MR1488147](#), [Zbl 0915.47022](#). 438
- [4] BEREZIN, FELIKS ALEKSANDROVICH. Covariant and contravariant symbols of operators. *Izv. Akad. Nauk SSSR Ser. Mat.* **6** (1972), 1117–1151. [MR0350504](#), [Zbl 0259.47004](#), doi: [10.1070/IM1972v006n05ABEH001913](#). 438
- [5] BEREZIN, FELIKS ALEKSANDROVICH. Quantization. *Izv. Akad. Nauk SSSR Ser. Mat.* **8** (1974), 1109–1165. [MR0395610](#), [Zbl 1113.46021](#), doi: [10.1070/IM1974v008n05ABEH002140](#). 438
- [6] CHALENDAR, ISABELLE; FRICAIN, EMMANUEL; PARTINGTON, JONATHAN R.. Over-completeness of sequences of reproducing kernels in model spaces. *Integr. Equ. Oper. Theory* **56** (2006), no. 1, 45–56. [MR2256996](#), [Zbl 0915.47022](#), doi: [10.1007/s00020-005-1413-1](#). 438
- [7] CHALENDAR, ISABELLE; FRICAIN, EMMANUEL; TIMOTIN, DAN. Functional models and asymptotically orthonormal sequences. *Ann. Inst. Fourier (Grenoble)* **535** (2003), no. 5, 1527–1549. [MR2032942](#), [Zbl 1113.46021](#), doi: [10.1007/s00020-005-1413-1](#). 438
- [8] DOUGLAS, RONALD GEORGE. Banach algebra techniques in the theory of Toeplitz operators. *Amer. Math. Soc., Providence, R.I.* **15** (1973), pp. 53. [MR0361894](#), [Zbl 0252.47025](#).
- [9] ENGLIŠ, MIROSLAV. Toeplitz operators and the Berezin transform on  $H^2$ . *Linear Algebra Appl.* **223/224** (1995), 171–204. [MR1340692](#), [Zbl 0827.47017](#), doi: [10.1016/0024-3795\(94\)00056-J](#). 438, 447
- [10] FRICAIN, EMMANUEL. Uniqueness theorems for analytic vector-valued functions. *J. Math. Sci.(New York)* **101** (2000), no. 3, 3193–3210. [MR1692671](#), [Zbl 0961.46029](#), doi: [10.1007/BF02673744](#). 438
- [11] FRICAIN, EMMANUEL. Bases of reproducing kernels in model spaces. *J. Operator Theory* **46** (2001), no. 3, 517–543. [MR1897152](#), [Zbl 0995.46021](#). 438
- [12] FRICAIN, EMMANUEL. Complétude des noyaux reproduisants dans les espaces modèles. *Ann. Inst. Fourier (Grenoble)* **52** (2002), no. 2, 661–686. [MR1906486](#), [Zbl 1032.46040](#), doi: [10.5802/aif.1897](#). 438
- [13] FRICAIN, EMMANUEL. Bases of reproducing kernels in de-Branges spaces. *J. Funct. Anal.* **226** (2005), no. 2, 373–405. [MR2159461](#), [Zbl 1086.46018](#), doi: [10.1016/j.jfa.2004.11.014](#). 438
- [14] GOHBERG, ISRAEL C.; KREIN, MARK GRIGOREVICH. Introduction to the Theory of Linear Nonselfadjoint Operators. *American Mathematical Society, Providence, RI* **18** (1969), pp. 378.. [MR0246142](#), [Zbl 0181.13504](#). 439
- [15] GUO, KUNYU; ZHAO, XIANFENG; ZHENG, DECHAO. The spectral picture of Bergman Toeplitz operators with harmonic polynomial symbols. *Ark. Mat.* **61** (2023), no. 2, 343–374. [MR4666991](#), [Zbl 7768173](#), doi: [10.4310/ARKIV.2023.v61.n2.a5](#). 437
- [16] HOFFMAN, KENNETH. Banach spaces of analytic functions. *Prentice-Hall, Inc., Englewood Cliffs, NJ* (1962), pp. 217. [MR0133008](#), [Zbl 0117.34001](#). 439, 444

- [17] KARAEV, MUBARIZ T.. Berezin symbols and Schatten-von Neumann classes. *Mat. Zametki* **72** (2002), no. 2, 207–215. [MR1942546](#), [Zbl 1026.47013](#), doi: [10.1023/A:1019893710815.445](#)
- [18] KARAEV, MUBARIZ T.. Berezin symbol and invertibility of operators on the functional Hilbert spaces. *J. Funct. Anal.* **238** (2006), no. 1, 181–192. [MR2253012](#), [Zbl 1102.47018](#), doi: [10.1016/j.jfa.2006.04.030.437,440](#)
- [19] KARAEV, MUBARIZ T.. On the Riccati equations. *Monatsh. Math.* **155** (2008), no. 3-4, 213–215. [MR2461577](#), [Zbl 1148.47013](#), doi: [10.1007/s00605-008-0542-4.447](#)
- [20] LOUHICHI, ISSAM; STROUSE, ELIZABETH; ZAKARIASY, LOVA. Products of Toeplitz operators on the Bergman space. *Integral Equations Operator Theory* **54** (2006), no. 4, 525–539. [MR2222982](#), [Zbl 1109.47023](#), doi: [10.1007/s00020-005-1369-1.448](#)
- [21] NIKOL'SKIJ, NIKOLAI K.. Treatise on the shift operator. *Springer-Verlag, Berlin* **273** (1986), pp. 491. [MR0827223](#), [Zbl 0587.47036.437,439,444,446](#)
- [22] SHVEDENKO, S. V.. Functional interpolation in the Hardy classes  $H^p$ . (Russian) *Izvestiya VUZ, Ser. Matematika* **155** (1975), no. 4, 108–111. [MR0390233](#), [Zbl 0214.37401](#), doi: [10.4064/sm-39-2-137-153.437,439,440](#)
- [23] SNYDER, A. K.. Sequence spaces and interpolation problems for analytic functions. *Stud. Math.* **39** (1971), 137–153. [MR0306924](#), [Zbl 0214.37401](#), doi: [10.4064/sm-39-2-137-153.440](#)
- [24] SZ.-NAGY, BÉLA, FOIAŞ, CIPRIAN. Harmonic analysis of operators on Hilbert space. *North-Holland, Amsterdam* (1970) xiii, 389 p. [MR0275190](#), [Zbl 0201.45003.444](#)
- [25] TOLOKONNIKOV, VADIM A.. Estimates in Carleson corona theorem, ideals of the algebra  $H^\infty$ , a problem of Sz.Nagy. *J. Soviet Math.* **22** (1983), 1814–1828. [MR0629839](#), [Zbl 0515.46032](#), doi: [10.1007/BF01882580.446](#)
- [26] YANG, RONGWEI. Beurling's phenom in two variables. *Integral Equations Operator Theory* **48** (2004), no. 3, 411–423. [MR2038510](#), [Zbl 1061.46023](#), doi: [10.1007/s00020-002-1195-7.438](#)
- [27] YONEDA, RIKIO. Invertibility of Toeplitz operators on the Bergman spaces with harmonic symbols. *J. Math. Anal. Appl.* **516** (2022), no. 1, Paper No. 126515, 15 pp. [MR4456512](#), [Zbl 7574903](#), doi: [10.1016/j.jmaa.2022.126515.437](#)
- [28] ZHAO, XIANFENG; ZHENG, DECHAO. Invertibility of Toeplitz operators via Berezin transforms. *J. Operator Theory* **75** (2016), no. 2, 475–496. [MR3509138](#), [Zbl 1389.47087](#), doi: [10.7900/jot.2015jul07.2082.437](#)
- [29] ZHAO, XIANFENG; ZHENG, DECHAO. Toeplitz operators on the Bergman space and the Berezin transform. *Chapman and Hall/CRC, New York* (2019) 287–318. [MR3966622](#), [Zbl 1429.30052.437](#)
- [30] ZHU, KEHE. Operator Theory in Function Spaces. Monographs and Textbooks in Pure and Applied Mathematics. *Marcel Dekker, Inc., New York* **139** (1990), xii+258 pp.. [MR1074007](#), [Zbl 0706.47019](#), doi: [10.1007/s00020-005-1369-1.438](#)

(Mubariz Garayev) DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, KING SAUD UNIVERSITY, RIYADH, SAUDI ARABIA  
[mgarayev@ksu.edu.sa](mailto:mgarayev@ksu.edu.sa)

This paper is available via <http://nyjm.albany.edu/j/2024/30-17.html>.