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A slope invariant and the A-polynomial of knots

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ABSTRACT. The A-polynomial is a knot invariant related to the space of $\mathrm{SL}_2(\mathbb{C})$ representations of the knot group. In this paper our interests lies in the *logarithmic Gauss map* of the A-polynomial. We develop a homological point of view on this function by extending the constructions of Degtyarev, the second author and Lecuona to the setting of non-abelian representations. It defines a rational function on the character variety, which unifies various known invariants such as the *change of curves* in the Reidemeister function, the *modulus* of boundary-parabolic representations, the *boundary slope* of some incompressible surfaces embedded in the exterior of the knot K or equivalently the slopes of the sides of the Newton polygon of the A-polynomial A_K. We also present a method to compute this invariant in terms of Alexander matrices and Fox calculus.

CONTENTS

1. Introduction

The set of all representations of a knot group in $\mathrm{SL}_2(\mathbb{C})$ carries naturally the structure of an algebraic set. This holds also for the characters of these representations, whose set is called the $\mathrm{SL}_2(\mathbb{C})$-character variety of the knot. Given a peripheral structure of the knot, the character variety is a plane curve in $\mathbb{C}^* \times \mathbb{C}^*$, whose coordinates M and L correspond to the eigenvalues of the meridian m and the preferred longitude ℓ. The polynomial $A_K(L, M)$ defining this curve is an invariant of the knot, called the A-polynomial. This invariant contains a lot of interesting information on the knot; in particular, Boyer and Zhang [4]

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and Dunfield and Garoufalidis [13] showed that $A_K = L - 1$ if and only if K is trivial.

In this paper, our motivations come, among others, from the following result of Boden:

Theorem 1.1 ([3]). *If the M-degree $\deg_M A_K(L,M)$ of the A-polynomial is zero, then K is the trivial knot.*

This result motivates the systematic study of the *logarithmic Gauss map* of the A-polynomial

$$\frac{M}{L} \cdot \frac{\partial_M A_K(L,M)}{\partial_L A_K(L,M)}, \tag{1}$$

where ∂_M and ∂_L denote the partial derivatives. By Theorem 1.1, this rational function vanishes identically on $\{A_K = 0\}$ if and only if K is trivial.

The logarithmic Gauss map was introduced in [14] by Guelfand, Kapranov and Zelevinsky in order to study some determinantial varieties. Then it has been used for instance by Mikhalkin in [17] for studying the topology of arrangements of real plane curves. In [15], Marché and Guilloux showed it is related with the volume function of the A-polynomial of knots, or more generally of exact polynomials.

Our proposal is to develop a homological point of view on this function, by extending the constructions of Degtyarev, the second author and Lecuona [10, 11] to the setting of non-abelian representations. Let K be an oriented knot in the 3-sphere S^3 with exterior M_K. Denote by $R(M_K)$ and $X(M_K)$ the $\mathrm{SL}_2(\mathbb{C})$-representation and character varieties of the knot K. We consider representations $\rho : \pi_1(M_K) \to \mathrm{SL}_2(\mathbb{C})$ composed with the adjoint action of $\mathrm{SL}_2(\mathbb{C})$ on the Lie algebra $\mathrm{Ad} : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Aut}(\mathfrak{sl}_2(\mathbb{C}))$, and show that there is a non-empty Zariski open subset of $X(M_K)$ such that for all ρ in this subset

- there is an element $v_\rho \in \mathfrak{sl}_2(\mathbb{C})$ such that $(v_\rho \otimes \ell, v_\rho \otimes m)$ is a basis of the homology group $H_1(\partial M_K, \mathrm{Ad} \circ \rho) \simeq \mathbb{C}^2$ with coefficients twisted by $\mathrm{Ad} \circ \rho$, and
- the kernel of the homomorphism induced by the inclusion:

  $$\mathcal{Z}(K, \mathrm{Ad} \circ \rho) = \ker\left(H_1(\partial M_K, \mathrm{Ad} \circ \rho) \xrightarrow{i_*} H_1(M_K, \mathrm{Ad} \circ \rho)\right)$$

  is generated by a single vector of the form $a\, v_\rho \otimes \ell + b\, v_\rho \otimes m$ for some $[b : a] \in \mathbb{CP}^1$.

The representations which verify these conditions are called *admissible*. We define the slope of K at the admissible representation ρ by

$$s_K(\rho) = -\frac{b}{a} \in \mathbb{CP}^1.$$

We prove that representations which restrict to *non-parabolic* representations of the boundary ∂M_K of M_K are admissible, see Lemma 3.4. If ρ is a boundary-parabolic representation, we define the slope $s_K(\rho)$ as the modulus of the euclidean structure induced by the restricted representation on $\pi_1(\partial M_K)$, see Section 3.3. It turns out that these two different definitions fit well and that the following holds.

Proposition 1.2. *The slope depends only on the conjugacy classes of the representations and induces a rational function*

$$s_K : X \subset X(M_K) \longrightarrow \mathbb{CP}^1$$

on each irreducible component X of the character variety.

Note that if the representation is real or unitary, then s_K takes values in $\mathbb{RP}^1$ (see Proposition 3.13). For any knot, the function s_K can be computed by Fox calculus, see Section 3.5. We illustrate the method in the case of the trefoil knot, and further compute the slope of the figure-eight knot.

The following theorem relates s_K to the original motivation; a precise statement is given in Theorem 4.1.

Theorem 1.3. *The slope function s_K equals minus the logarithmic Gauss map of the A-polynomial defined in Eq.* (1)*.*

We also relate s_K to the change of curve factor for the Reidemeister torsion. Let $\mathbb{T}_{M_K,\ell}(\rho)$ and $\mathbb{T}_{M_K,m}(\rho)$ be the Reidemeister torsions according to homology bases induced by the choices of the curves ℓ and m in ∂M_K, see Section 3.4.

Proposition 1.4. *The slope coincides with the quotient of Reidemeister torsion:*

$$s_K(\rho) = \frac{\mathbb{T}_{M_K,\ell}(\rho)}{\mathbb{T}_{M_K,m}(\rho)}$$

for all ρ such that this formula is well-defined.

Porti had already observed ([18, Corollary 4.9]) that the logarithmic Gauss map of the A-polynomial could be expressed as a ratio of torsions -up to a sign-, and that this ratio of torsions is equal to the modulus of ρ when it is a boundary-parabolic representation ([18, Proposition 4.7]). Our point of view permits to fix and compute the sign ambiguity. Moreover, our results Proposition 1.2 and Theorem 1.3 are more general, since they do not require the Reidemeister torsion to be well-defined, for instance they hold for high dimensional components of the character variety.

Finally, we consider *ideal* points of the A-polynomial, those are points added at infinity in a compactification of the curve $\{A(L,M)=0\}$ in $\mathbb{C}^2$. In [9], Culler and Shalen constructed incompressible surfaces in M_K associated to such points. Those surfaces have a non-empty boundary, whose slope is determined by rational number p/q. We prove the following theorem:

Theorem 1.5. *Let $y \in \mathbb{CP}^2$ be an ideal point of the curve $\{A_K(L,M) = 0\}$. The value of s_K at y equals* minus *the slope of the Culler–Shalen incompressible surface associated to y.*

This theorem sheds some light on the main theorem of [7], which states that the boundary slopes of the Culler–Shalen surfaces are boundary slopes of the Newton polygon of the A-polynomial. Indeed it is well-known that the logarithmic Gauss map converges at those ideal points to the value of the slope of the corresponding boundary of the Newton polygon.

To conclude this introduction, we mention that the slope invariant can be extended to orthogonal (real) representations of link groups. In this more general setting, the first twisted homology space $H_1(\partial M_K, \rho)$ can have an arbitrary dimension higher than 2 and the kernel $\mathcal{Z}(K, \rho)$ might not be a line anymore. However, the space $H_1(\partial M_K, \rho)$ carries a natural symplectic structure given by the (twisted) intersection form on ∂M_K, and $\mathcal{Z}(K, \rho)$ is still a Lagrangian subspace. A construction of Arnold [1] related to the Maslov index allows to construct a *generalized slope* for this context, lying in $S^1 \subset \mathbb{C}^*$. As it turns out, in the case of a representation $\rho : \pi_1(M_K) \to \mathrm{SU}(2)$, both theories coincide via the natural isomorphism $\mathbb{RP}^1 \simeq S^1$. We postpone rigorous definitions and further study of this invariant to an upcoming article.

Organization of the paper. In Section 2 we collect basic definitions on character varieties and A-polynomials. In Section 3 we define the slope invariant and we prove Proposition 1.2 and Proposition 1.4. In Section 4 we prove Theorem 1.3. Finally, in Section 5 we prove Theorem 1.5.

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2. Representation varieties and A-polynomial

This section is devoted to definitions and properties of representations spaces and character varieties (Section 2.1). We compute the character variety of the group $\mathbb{Z}^2$ in Section 2.2 and define the A-polynomial of knots in Section 2.3. References for character varieties are [19, 20], the A-polynomial was first defined in [7], see also [6].

2.1. Representation and character varieties. Let Γ be a finitely generated group. The *representation variety* is the affine algebraic set

$$R(\Gamma) = \mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{C})).$$

If Γ is generated by n elements, the representation variety is an algebraic subset of $\mathrm{SL}_2(\mathbb{C})^n$ given by polynomial relations corresponding to the relations of the group Γ. Two different presentations yield naturally isomorphic algebraic sets.

A representation $\rho : \Gamma \to \mathrm{SL}_2(\mathbb{C})$ is *abelian* if $\rho(\Gamma)$ is an abelian subgroup of $\mathrm{SL}_2(\mathbb{C})$. A representation ρ is *reducible* if there exists a proper subspace of $\mathbb{C}^2$ invariant under the action of $\rho(\Gamma)$. Equivalently, $\rho(\Gamma)$ is conjugated to a subgroup of the group of upper-triangular matrices in $\mathrm{SL}_2(\mathbb{C})$. Abelian representations are reducible, but the converse does not hold. Non-reducible representations are *irreducible*.

Two representations ρ and ρ' in $R(\Gamma)$ are equivalent if they have the same trace:

$$\rho \sim \rho' \text{ if and only if } \operatorname{Tr}\rho(\gamma) = \operatorname{Tr}\rho'(\gamma), \text{ for any } \gamma \in \Gamma.$$

The set of equivalence classes of representations coincides with the algebro-geometric quotient of $R(\Gamma)$ by the action of $\mathrm{SL}_2(\mathbb{C})$ by conjugation. This quotient is usually constructed through invariant theory, and is denoted

$$X(\Gamma) = R(\Gamma) /\!/ \mathrm{SL}_2(\mathbb{C}).$$

Points of the character variety are called *characters*. The equivalence class of a representation ρ (the character of ρ) is denoted by $\chi_\rho : \Gamma \to \mathbb{C}$ with $\chi_\rho(\gamma) = \operatorname{Tr}(\rho(\gamma))$ for $\gamma \in \Gamma$. If Γ is the fundamental group of a manifold W, we simply write $R(W)$ and $X(W)$ for the representation and character varieties of the manifold W.

Despite the fact that being abelian is not a well-defined notion on the character variety, the notion of being reducible makes sense there, since a reducible representation $\rho : \Gamma \to \mathrm{SL}_2(\mathbb{C})$ can be characterized by the fact that for any $\gamma, \delta \in \Gamma$, the following equality holds (see for instance [9, Lemma 1.2.1]):

$$\operatorname{Tr}\rho\left(\gamma\delta\gamma^{-1}\delta^{-1}\right) = 2. \tag{2}$$

The character variety $X(\Gamma)$ can be decomposed as

$$X(\Gamma) = X^{\mathrm{irr}}(\Gamma) \cup X^{\mathrm{red}}(\Gamma),$$

where $X^{\mathrm{red}}(\Gamma)$ is the set of reducible characters, and its complement $X^{\mathrm{irr}}(\Gamma)$ is the set of irreducible characters. Eq. (2) implies that $X^{\mathrm{red}}(\Gamma)$ is a Zariski closed subset of $X(\Gamma)$.

An algebraic set is *reducible* if it can be written as a union of two proper algebraic subset, else it is *irreducible*. An irreducible component of an algebraic set is a maximal irreducible algebraic subset.

Remark 2.1. Despite $R(\Gamma)$ or $X(\Gamma)$ being called *varieties*, they are not quite algebraic varieties in general: they are actually reducible, and might not be reduced as schemes (some points or subspaces might have multiplicity). On the other hand, any irreducible component is irreducible, and in particular reduced, by definition.

Two representations ρ and ρ' are *conjugate* if there exists a matrix $M \in \mathrm{SL}_2(\mathbb{C})$ such that $\rho(\gamma) = M\rho'(\gamma)M^{-1}$ for every $\gamma \in \Gamma$. Two conjugate representations define the same character; the converse is false in general, but true for elements of $X^{\mathrm{irr}}(\Gamma)$. More precisely, the following holds.

Proposition 2.2. [9, Proposition 1.5.2] *If ρ and ρ' are two representations $\Gamma \to \mathrm{SL}_2(\mathbb{C})$ with ρ irreducible and $\chi_\rho = \chi_{\rho'}$, then ρ and ρ' are conjugate (and ρ' is irreducible as well).*

Two non-conjugate representations having the same character in $X(\Gamma)$ must be reducible. If Γ is a knot group, Burde and de Rham [5, 12] showed that the set of characters containing non-conjugate representations is finite.

2.2. The character variety of $\mathbb{Z}^2$. We describe explicitly the character variety of a 2-torus $S^1 \times S^1$. Pick a basis m, ℓ of $\pi_1(S^1 \times S^1) = \mathbb{Z}^2$. Any representation in $\mathrm{SL}_2(\mathbb{C})$ is conjugate to a representation ρ given by two commuting matrices of the form

$$\rho(m) = \begin{pmatrix} M & * \\ 0 & M^{-1} \end{pmatrix}, \qquad \rho(\ell) = \begin{pmatrix} L & * \\ 0 & L^{-1} \end{pmatrix}$$

for $M, L \in \mathbb{C}^*$. Each point of the character variety $X(S^1 \times S^1)$ has a pre-image in $R(S^1 \times S^1)$ of the form

$$\rho(m) = \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix}, \qquad \rho(\ell) = \begin{pmatrix} L & 0 \\ 0 & L^{-1} \end{pmatrix}, \qquad M, L \in \mathbb{C}^*. \tag{3}$$

This pre-image is unique up to the involution σ of $(\mathbb{C}^*)^2$ sending (L, M) to (L^{-1}, M^{-1}), and $X(S^1 \times S^1)$ can be identified with the singular affine complex surface $(\mathbb{C}^*)^2/\sigma$. It embeds in $\mathbb{C}^3$ as the zeros of the polynomial

$$\Delta = x^2 + y^2 + z^2 - xyz - 4.$$

Indeed, the function algebra of $X(S^1 \times S^1)$ naturally identifies with the σ-invariant sub-algebra $\mathbb{C}[M + M^{-1}, L + L^{-1}]$ of $\mathbb{C}[L^{\pm 1}, M^{\pm 1}]$. This algebra of invariant functions is isomorphic with $\mathbb{C}[x, y, z]/(\Delta)$ through

$$M + M^{-1} \mapsto x, \qquad L + L^{-1} \mapsto y, \qquad ML + (ML)^{-1} \mapsto z.$$

From this description, one sees that the singular locus of $X(S^1 \times S^1)$ consists on the four points $\{(L, M) = (\pm 1, \pm 1)\}$.

2.3. The A-polynomial. Let K be an oriented knot in S^3 with exterior M_K. The inclusion $\partial M_K \subset M_K$ induces an injective group homomorphism $\pi_1(\partial M_K) \hookrightarrow \pi_1(M_K)$. Let r be the restriction map:

$$r : X(M_K) \longrightarrow X(\partial M_K) \simeq X(S^1 \times S^1).$$

For short we denote by $\rho_\partial = r(\rho)$ the restriction of ρ to $\pi_1(\partial M_K)$. By Section 2.2, the choice of the longitude ℓ and the meridian m induces an identification of $X(S^1 \times S^1)$ with a quotient of $(\mathbb{C}^*)^2$. The image of r is a union of points and curves, possibly with multiplicities, see for instance [13, Lemma 2.1]. Discarding the 0-dimensional components, the *A-polynomial* of K is the unique polynomial $A_K(L, M)$ in $\mathbb{C}[L, M]$ whose zero locus in $\mathbb{C}^2$ is exactly mapped onto the image of r. Note that $A_K(L, M)$ is always divisible by $L - 1$. This factor corresponds to the curve of reducible characters. Boyer, Zhang, Dunfield and Garoufalidis have shown the following result.

Theorem 2.3 ([4, 13])**.** *Let K be a knot in S^3. The A-polynomial $A_K(L, M)$ is equal to $(L-1)^k$ for some k, if and only if K is the trivial knot (and in this case $k = 1$).*

3. The slope invariant

In this section we will define the slope of an admissible representation (Section 3.1), and observe that generic $\mathrm{SL}_2(\mathbb{C})$-representations are admissible. In Section 3.2 we show that the slope is invariant by conjugation of the representation. We prove in Section 3.3 that it yields a rational function on irreducible components of the character variety and that the slope of a real representation is a real number. Then we prove in Section 3.4 that the slope can be written as a quotient of Reidemeister torsions. Finally, in Section 3.5 we describe a procedure to compute the slope with an Alexander matrix.

3.1. Admissible representations. Let V be a finite dimensional $\mathbb{C}$-vector space, and $\rho\colon \pi_1(M_K) \to \mathrm{GL}(V)$ be a representation. The representation extends to a ring homomorphism and V can be viewed as a right $\mathbb{Z}[\pi_1(M_K)]$-module V_ρ. The twisted homology $H_*(M_K, \rho)$ is the homology of the complex of $\mathbb{C}$-vector spaces:

$$C_*(M_K;\rho) = V_\rho \otimes_{\mathbb{Z}[\pi_1(M_K)]} C_*(M_K;\mathbb{Z}[\pi_1(M_K)]).$$

Definition 3.1. A representation $\rho\colon \pi_1(M_K) \to \mathrm{GL}(V)$ is *admissible* if it satisfies:

- there exists $v_\rho \in V$ such that $\{v_\rho \otimes \ell, v_\rho \otimes m\}$ is a basis of the space $H_1(\partial M_K, V_\rho) \simeq \mathbb{C}^2$,
- the kernel of the homomorphism induced by the inclusion:

$$\mathcal{Z}(K,\rho) = \ker\Big(H_1(\partial M_K, V_\rho) \xrightarrow{i_*} H_1(M_K, V_\rho)\Big)$$

  has dimension one.

We restrict to representations $\rho\colon \pi_1(M_K) \to \mathrm{SL}_2(\mathbb{C})$. The composition of ρ with the adjoint action Ad of $\mathrm{SL}_2(\mathbb{C})$ on $\mathfrak{sl}_2(\mathbb{C})$ induces the following representation:

$$\begin{array}{rccc} \mathrm{Ad}\circ\rho\colon & \pi_1(M_K) & \longrightarrow & \mathrm{Aut}(\mathfrak{sl}_2(\mathbb{C})) \\ & \gamma & \longmapsto & \big(v \mapsto \rho(\gamma) v \rho(\gamma)^{-1}\big). \end{array}$$

Definition 3.2. Let $\rho\colon \pi_1(M_K) \to \mathrm{SL}_2(\mathbb{C})$ be such that $\mathrm{Ad}\circ\rho$ is admissible. Let $a\,(v_\rho \otimes \ell) + b\,(v_\rho \otimes m)$ be a generator of $\mathcal{Z}(K, \mathrm{Ad}\circ\rho)$ for some $[a : b] \in \mathbb{CP}^1$. The *slope* of the knot K at the representation ρ is

$$s_K(\rho) = -\frac{b}{a} \in \mathbb{C} \cup \infty.$$

Definition 3.3. A representation $\rho\colon \pi_1(M_K) \to \mathrm{SL}_2(\mathbb{C})$ is *boundary-parabolic* if the restriction $\rho_\partial : \pi_1(\partial M_K) \to \mathrm{SL}_2(\mathbb{C})$ is parabolic, that is $\mathrm{Tr}\,\rho(\gamma) = \pm 2$ for any $\gamma \in \pi_1(\partial M_K)$.

A boundary-parabolic character is the character of a boundary-parabolic representation.

Lemma 3.4. *Let $\rho\colon \pi_1(M_K) \to \mathrm{SL}_2(\mathbb{C})$ be a non-parabolic representation. The vector space $H_1(\partial M_K, \mathrm{Ad}\circ\rho)$ is isomorphic to $\mathbb{C}^2$, and the kernel subspace $\mathcal{Z}(M, \mathrm{Ad}\circ\rho)$ is one-dimensional. Moreover, if ρ is not boundary-parabolic, then $\mathrm{Ad}\circ\rho$ is admissible.*

Proof. The group $\pi_1(M_K)$ is generated by pairwise conjugate meridians. If ρ is non-parabolic, then the image of a meridian must differ to $\pm I_2$, otherwise we would have $\rho(\pi_1(M_K)) \subset \{\pm I_2\}$.

Consider the complex $C_*(\partial M_K, \mathrm{Ad}\circ\rho)$ with one 0-cell, two 1-cells corresponding to ℓ and m and one 2-cell. An explicit computation of the homology of shows that the dimension of $H_1(\partial M_K, \mathrm{Ad}\circ\rho)$ is two. Moreover, when ρ is not boundary-parabolic, for $v_\rho \in \mathfrak{sl}_2(\mathbb{C})$ invariant by $\mathrm{Ad}\circ\rho_\partial$, the pair of vectors $(v_\rho \otimes \ell, v_\rho \otimes m)$ forms a basis of $H_1(\partial M_K, \mathrm{Ad}\circ\rho)$. Since the Killing form on the local system $\mathfrak{sl}_2(\mathbb{C})$ yields a Poincaré duality isomorphism $H_*(M_K, \mathrm{Ad}\circ\rho) \simeq H_{3-*}(M_K, \partial M_K, \mathrm{Ad}\circ\rho)$, the following diagram is commutative

$$\begin{tikzcd}
H_2(M, \partial M_K, \mathrm{Ad}\circ\rho) \arrow[r, "\delta"] \arrow[d] & H_1(\partial M_K, \mathrm{Ad}\circ\rho) \arrow[r, "i_*"] \arrow[d] & H_1(M_K, \mathrm{Ad}\circ\rho) \arrow[d] \\
H_1(M_K, \mathrm{Ad}\circ\rho)^* \arrow[r, "i^*"] & H_1(\partial M_K, \mathrm{Ad}\circ\rho)^* \arrow[r, "\delta^*"] & H_2(M_K, \partial M_K, \mathrm{Ad}\circ\rho)^*
\end{tikzcd}$$

with exact rows and where the vertical arrows are isomorphisms. Exactness implies

$$\dim\ker i_* = \operatorname{rank}\delta = \operatorname{rank} i^* = \operatorname{rank} i_*$$

since the diagram commutes and transposition preserves the rank. Hence

$$\dim \mathcal{Z}(K, \mathrm{Ad}\circ\rho) = \dim\ker i_* = (1/2)\dim H_1(\partial M, \mathrm{Ad}\circ\rho). \qquad \square$$

As an example, we compute the slope for abelian non boundary-parabolic representations. Let $\varphi\colon \pi_1(M_K) \to H_1(M_K) = \mathbb{Z}$ be the abelianization. For any $\lambda \in \mathbb{C}^*$, there is an abelian representation

$$\begin{aligned} \rho_\lambda\colon \pi_1(M_K) &\longrightarrow \mathrm{SL}_2(\mathbb{C}) \\ \gamma &\longmapsto \begin{pmatrix} \lambda^{\varphi(\gamma)} & 0 \\ 0 & \lambda^{-\varphi(\gamma)} \end{pmatrix} \end{aligned} \tag{4}$$

and any abelian, non boundary-parabolic representation is conjugate to a representation of this form.

Lemma 3.5. *For any $\lambda \neq \pm 1$, the slope at the abelian representation ρ_λ vanishes:*

$$s_K(\rho_\lambda) = 0.$$

Proof. Up to conjugation, the representation $\mathrm{Ad}\circ\rho$ has the form

$$\mathrm{Ad}\circ\rho(\gamma) = \begin{pmatrix} \lambda^{2\varphi(\gamma)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-2\varphi(\gamma)} \end{pmatrix}$$

and $\mathfrak{sl}_2(\mathbb{C})$ splits as $\mathbb{Z}[\pi_1(M_K)]$-module as

$$\mathfrak{sl}_2(\mathbb{C}) = \mathbb{C}_{\lambda^2} \oplus \mathbb{C} \oplus \mathbb{C}_{\lambda^{-2}}.$$

This yields a splitting in twisted homology (with abelian coefficients), for $U = \partial M_K$ or $U = M_K$:

$$H_1(U, \mathrm{Ad} \circ \rho) = H_1(U, \mathbb{C}_{\lambda^2}) \oplus H_1(U, \mathbb{C}) \oplus H_1(U, \mathbb{C}_{\lambda^{-2}})$$

Since $\lambda \neq \pm 1$, for $U = \partial M_K$ the only non-trivial summand is $H_1(\partial M_K, \mathbb{C})$, and the map $H_1(\partial M_K, \mathrm{Ad} \circ \rho) \xrightarrow{i_*} H_1(M_K, \mathrm{Ad} \circ \rho)$ coincides with the corresponding map induced by the inclusion in homology with trivial coefficients

$$H_1(\partial M_K, \mathbb{C}) \to H_1(M_K, \mathbb{C}),$$

whose kernel is generated by ℓ. □

3.2. The slope of characters. By the following lemma, the slope does not depend on the conjugacy class of an irreducible representation. Combined with Proposition 2.2, it follows that the slope of an irreducible representation depends only on its character.

Lemma 3.6. *Let ρ and $\rho' : \pi_1(M_K) \to \mathrm{SL}_2(\mathbb{C})$ be two irreducible, non boundary-parabolic representations. If ρ and ρ' are conjugate, then $s_K(\rho) = s_K(\rho')$.*

Proof. Let A be a matrix in $\mathrm{GL}_2(\mathbb{C})$ such that $\rho' = A\,\rho\,A^{-1}$. Any $\mathrm{Ad} \circ \rho$-invariant vector $v_\rho \in \mathfrak{sl}_2(\mathbb{C})$ yields an $\mathrm{Ad} \circ \rho'$-invariant vector $v'_\rho = A\,v_\rho\,A^{-1}$, and the conjugation by A induces an isomorphism

$$H_1(\partial M_K, \mathrm{Ad} \circ \rho) \longrightarrow H_1(\partial M_K, \mathrm{Ad} \circ \rho')$$

sending the basis $\{v_\rho \otimes \ell, v_\rho \otimes m\}$ to $\{v'_\rho \otimes \ell, v'_\rho \otimes m\}$ and the subspace $\mathcal{Z}(K, \mathrm{Ad} \circ \rho)$ to $\mathcal{Z}(K, \mathrm{Ad} \circ \rho')$. Hence $s_K(\rho) = s_K(\rho')$. □

Remark 3.7. There exist pairs of reducible, non-conjugate representations with the same character. Indeed, let χ be an arbitrary reducible character in $X(M_K)$. Consider a representation ρ of the form $\begin{pmatrix} \lambda(\gamma) & * \\ 0 & \lambda^{-1}(\gamma) \end{pmatrix}$, where $\lambda : \pi_1(M_K) \to \mathbb{C}^*$ is a group homomorphism, chosen such that $\chi(\rho) = \chi$. Note that λ can further be written $\lambda(\gamma) = \lambda^{\varphi(\gamma)}$ for some $\lambda \in \mathbb{C}^*$ and $\varphi : \pi_1(M_K) \to \mathbb{Z}$. Hence the abelian representation ρ_λ defined in Eq. (4) has also character χ, but is not conjugated in general to ρ. It turns out that they can have different slope values.

For example, consider the right-handed trefoil knot T in S^3. The character variety $X(M_T)$ is the union of a line X^{red} and a conic X^{irr} in the plane. The line contains only reducible characters, and any character in the conic is irreducible except the two intersection points $X^{\mathrm{red}} \cap X^{\mathrm{irr}}$. Let χ be a point in $X^{\mathrm{red}} \cap X^{\mathrm{irr}}$. Since χ is reducible, there exist $\lambda \in \mathbb{C}^*$ such that the abelian representation ρ_λ has character χ. By Lemma 3.5, one has $s_T(\rho_\lambda) = 0$. However, we show in Example 3.19 that the slope defines a constant function on X^{irr}, everywhere equal to -6.

3.3. Regularity and properties of the slope. We extend the slope to a rational function –locally a quotient of polynomials– on the character variety $X(M_K)$.

There is a component $X^{\mathrm{red}} \subset X(M_K)$ of reducible characters only. By Remark 3.7 any character in X^{red} is the character of an abelian representation. Hence the slope is identically zero on X^{red}, see Lemma 3.5. Suppose now that $X \subset X(M)$ is an irreducible component containing an irreducible character.

Proposition 3.8. *Let $X \subset X(M)$ be an irreducible component which contains an irreducible character. The slope extends to a rational function on X, still denoted s_K. Moreover, if $\chi \in X$ is a boundary-parabolic character then*

$$s_K(\chi) = \tau(\chi), \tag{5}$$

where the modulus $\tau(\chi) \in \mathbb{C}$ is defined by taking the representative ρ of χ satisfying

$$\rho(m) = \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}, \qquad \rho(\ell) = \begin{pmatrix} \pm 1 & \tau(\chi) \\ 0 & \pm 1 \end{pmatrix}.$$

Remark 3.9. If χ is the character of an irreducible representation and lies at the intersection of several irreducible components, then the value of the slope at χ is well-defined.

The rest of the section is devoted to the proof of Proposition 3.8. Lemma 3.10 asserts that the slope is a rational function in the neighborhood of any irreducible, non boundary-parabolic character. For boundary parabolic characters χ, we *define* the slope by the relation in Eq. (5) and we show that the result is still a rational function on X in Lemma 3.12.

Lemma 3.10. *Let χ_0 an irreducible, non-boundary-parabolic character in X. The slope is a rational function in a neighborhood of χ_0 in X.*

Proof. Let ρ_0 in $R(M_K)$ be a representation with character χ_0. By Lemma 3.4 one has $H_1(\partial M_K, \mathrm{Ad} \circ \rho) \simeq \mathbb{C}^2$. The set of complex lines

$$\mathbb{P}(\rho) = \mathbb{P}(H_1(\partial M_K, \mathrm{Ad} \circ \rho))$$

is a complex algebraic variety isomorphic to $\mathbb{C}P^1$. If ρ and ρ' are conjugate, then there is a natural algebraic isomorphism $\mathbb{P}(\rho) \simeq \mathbb{P}(\rho')$. It defines an algebraic $\mathbb{CP}^1$-fibration on a neighborhood of χ_{ρ_0}, and for any χ, the complex line $\mathcal{Z}(M_K, \mathrm{Ad} \circ \rho)$ is an algebraic section of this fibration, independent of the choice of representation ρ with character χ.

It remains to show that the identification $\mathbb{P}(\rho) \simeq \mathbb{CP}^1$ is algebraic, in other words, that the choice of the basis $(v_\rho \otimes \ell, v_\rho \otimes m)$ depends algebraically on ρ. Since ρ_0 is not boundary-parabolic, we can shrink the chosen neighborhood so that no representation ρ near ρ_0 is boundary-parabolic. Then, since ρ_∂ is conjugated to a diagonal representation, there is a unique $\mathrm{Ad} \circ \rho_\partial$-invariant vector v_ρ with norm 1 in $\mathfrak{sl}_2(\mathbb{C})$. This choice depends polynomially on the entries of the matrix $\mathrm{Ad} \circ \rho(m)$, and then the basis $(v_\rho \otimes \ell, v_\rho \otimes m)$ depends algebraically on ρ. $\square$

We now consider the case of boundary-parabolic characters.

Lemma 3.11. *Let ρ_0 be a boundary-parabolic representation whose character χ_{ρ_0} lies in X. Then ρ_0 is irreducible, in particular $\rho_0(m) \neq \pm I_2$.*

Proof. For ρ reducible in X, [5, 12] implies that $\rho(m)$ has eigenvalues λ, λ^{-1} in $\mathbb{C}$, whose square is a root of the Alexander polynomial $\Delta_{M_K}(t)$, in particular $\lambda \neq \pm 1$, and ρ is not boundary-parabolic. Now for irreducible ρ, the image of any meridian must be different of I_2, since meridians generate the group $\pi_1(M_K)$. $\square$

Lemma 3.12. *Let $X \subset X(M)$ be an irreducible component containing an irreducible character, and $\chi_0 \in X$ a boundary-parabolic character. Then the slope function s_K is rational in a neighborhood of χ_0.*

Proof. Suppose first that X contains only boundary-parabolic characters. Any $\chi \in X$ is the character of a representation ρ such that $\rho(m) = \left(\begin{smallmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{smallmatrix}\right)$. Hence $\chi \mapsto \tau(\chi)$ is rational.

Now we assume that X contains a non boundary-parabolic character. By definition, boundary-parabolic characters form a Zariski closed subset of X. By Lemma 3.10, the slope function is rational on the open, non-empty subset of X consisting of non boundary-parabolic characters. By analytic continuation, it is enough to show that

$$\lim_{\chi \to \chi_0} s_K(\chi) = \tau(\chi_0).$$

By Lemma 3.11 any boundary-parabolic representation ρ_0 with character χ_0 is irreducible. Moreover, since $\rho_0(m)$ can not be trivial, we can chose such a ρ_0 satisfying

$$\rho_0(m) = \left(\begin{smallmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{smallmatrix}\right).$$

For any χ close to χ_0, we chose similarly a representation ρ with character χ such that

$$\rho(m) = \left(\begin{smallmatrix} M & 1 \\ 0 & M^{-1} \end{smallmatrix}\right),$$

with M close to ± 1 in $\mathbb{C}^*$. For such ρ, let $v_\rho = \left(\begin{smallmatrix} M - M^{-1} & 2 \\ 0 & M^{-1} - M \end{smallmatrix}\right)$ be an $\mathrm{Ad} \circ \rho_\partial$-invariant vector. The limit at ρ_0 of v_ρ is the $(\mathrm{Ad} \circ \rho_0)_\partial$-invariant vector $v_{\rho_0} = \left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right)$. However, a direct computation shows that $v_{\rho_0} \otimes \ell$ and $v_{\rho_0} \otimes m$ are linearly dependent in $H_1(\partial M_K, \mathrm{Ad} \circ \rho_0)$, and we cannot compute the slope of the boundary parabolic representation ρ_0 by means of Definition 3.2. Nevertheless, the subspace $\mathcal{Z}(K, \mathrm{Ad} \circ \rho_0)$ is one-dimensional by Lemma 3.4.

It implies that the map $i_* : H_1(\partial M_K, \mathrm{Ad} \circ \rho) \to H_1(M_K, \mathrm{Ad} \circ \rho)$ has rank one at any representation ρ near ρ_0, and at ρ_0 as well. In particular, for any ρ near ρ_0, the slope can be computed as the ratio of $i_*(v_\rho \otimes \ell)$ and $i_*(v_\rho \otimes m)$ in

$i_*(H_1(\partial M_K, \mathrm{Ad}\circ\rho))$. This actually makes sense for $\rho = \rho_0$ as well. An explicit computation of the boundary operator

$$\partial_1 : C_2(\partial M_K, \mathrm{Ad}\circ\rho_0) \to C_1(\partial M_K, \mathrm{Ad}\circ\rho_0)$$

shows that the vector $v_{\rho_0} \otimes \ell - \tau(\chi_0)\, v_{\rho_0} \otimes m$ belongs to $\mathrm{im}\,\partial_2$, and the equality

$$v_{\rho_0} \otimes \ell = \tau(\chi_0)\, v_{\rho_0} \otimes m$$

holds in $H_1(\partial M_K, \mathrm{Ad}\circ\rho_0)$. This implies that the ratio of $i_*(v_{\rho_0} \otimes \ell)$ and $i_*(v_{\rho_0} \otimes m)$ coincides with the modulus $\tau(\chi_0)$. This proves the lemma, and achieves the proof of Proposition 3.8. □

We end up this section with the following observation.

Proposition 3.13. *Let $X \subset X(M)$ be an irreducible component which contains a non boundary-parabolic representation. If $\rho \in X$ is a* real *representation*

$$\rho : \pi_1(M_K) \to \mathrm{SL}_2(\mathbb{R}) \quad \text{or} \quad \rho : \pi_1(M_K) \to \mathrm{SU}(2),$$

then the slope is a real number in $\mathbb{RP}^1$.

Proof. First assume that ρ is non-boundary-parabolic. If ρ is real, denoting by $\mathrm{Ad}\circ\rho_{\mathbb{R}}$ the action of ρ on the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ (resp. $\mathfrak{su}(2)$) of $\mathrm{SL}_2(\mathbb{R})$ (resp. $\mathrm{SU}(2)$), then obviously the Lagrangian sub-space

$$\mathcal{Z}(M_K, \mathrm{Ad}\circ\rho) \subset H_1(\partial M_K, \mathrm{Ad}\circ\rho)$$

is the complexification of the real Lagrangian sub-space $\mathcal{Z}(M_K, \mathrm{Ad}\circ\rho_{\mathbb{R}})$ in the real symplectic vector space $H_1(\partial M_K, \mathrm{Ad}\circ\rho_{\mathbb{R}})$ and the slope of this real Lagrangian is the slope of its complexification, a real number. If ρ is boundary-parabolic and real, then it takes value into $\mathrm{SL}_2(\mathbb{R})$ and the proposition follows from the definition of the modulus τ. □

3.4. Slope and Reidemeister torsion. In this section we show that the slope coincides with the "change of curve term" for the Reidemeister torsion as stated in Proposition 1.4.

If ρ is an irreducible representation in $X(M_K)$, we consider the torsion of the complex $C_*(M_K, \mathrm{Ad}\circ\rho)$ defined in Section 3.1. This complex is naturally based from a cell decomposition of M_K and a choice of a basis of $\mathfrak{sl}_2(\mathbb{C})$, but not acyclic. The Reidemeister torsion is usually defined for acyclic complexes. In the case we are considering, one needs to make some additional choices to define it, namely a basis of each homology group $H_*(M_K, \mathrm{Ad}\circ\rho)$.

According to [18], one can still define the Reidemeister torsion of the cellular complex $C_*(M_K, \mathrm{Ad}\circ\rho)$ for representations ρ in $R(M_K)$ such that $H_1(M_K, \mathrm{Ad}\circ\rho)$ has dimension 1. For a given curve $\gamma \in \pi_1(\partial M_K)$, the representation ρ is γ-*regular* if there exists a vector $v_\rho \in \mathfrak{sl}_2(\mathbb{C})$ such that $v_\rho \otimes \gamma$ spans $H_1(M_K, \mathrm{Ad}\circ\rho)$. In this case, since there is a natural choice of a basis of $H_2(M_K, \mathrm{Ad}\circ\rho)$, the curve γ determines a homology basis of the complex $C_*(M_K, \mathrm{Ad}\circ\rho)$ and the torsion $\mathbb{T}_{M_K,\gamma}(\mathrm{Ad}\circ\rho) \in \mathbb{C}^*$ is defined. Note that this torsion depends only on the conjugacy class of ρ, as well as the property of being γ-regular.

Let $X \subset X(M_K)$ the component containing χ, the torsion function is the rational function

$$\mathbb{T}_{M_K,\gamma} : X \longrightarrow \mathbb{C}$$

defined as the Reidemeister torsion of the complex $C_*(M_K, \mathrm{Ad} \circ \rho)$ if χ is γ-regular, and by $T_{M_K,\gamma}(\chi) = 0$ otherwise.

We start with the following lemma, which provides the genuine setting to define the Reidemeister torsion.

Lemma 3.14. *If X has dimension one and contains the character of a scheme-smooth representation ρ, then* $\dim H_1(M_K, \mathrm{Ad} \circ \rho) = 1$.

Proof. The proof of Lemma 3.14 follows from the isomorphism between $H^1(M_K, \mathrm{Ad} \circ \rho)$ and the Zariski tangent space of $X(M_K)$ at ρ, see [20, Theorem 1] . Scheme-smoothness implies that the Zariski tangent space is the actual tangent space, which is one-dimensional because X is. □

Note that scheme-smoothness is a Zariski open condition.

It turns out that the character variety $X(M_K)$ of a knot exterior is often one-dimensional. This is the case if the knot is *small* (if it does not contains a closed incompressible surface [7, Proposition 2.4]). This is also the case for any component $X \subset X(M_K)$ containing the character of a lift of the holonomy representation $\overline{\rho} : \pi_1(M_K) \to \mathrm{PSL}_2(\mathbb{C})$, provided that the interior of M_K admits a hyperbolic structure.

The following proposition is the main result of this section.

Proposition 3.15. *Let $X \subset X(M)$ be an irreducible one-dimensional component which contains a scheme-smooth, non-boundary parabolic character. For all $\chi \in X$ the following holds*

$$s_K(\chi) = \frac{\mathbb{T}_{M_K,\ell}(\chi)}{\mathbb{T}_{M_K,m}(\chi)}.$$

We provide two different proofs of this result: one uses the natural definition of the torsion while the other relies directly on some results on the *torsion form* proved by the first author in [2].

3.4.1. Torsion and chain complexes. This section is devoted to the proof of Proposition 3.15 by using the chain complex of M_K. The proof is very similar to [10, Theorem 3.21] or [10, Theorem 6.7]. We use the following technical lemma.

Lemma 3.16. *Let γ be a curve in $\pi_1(\partial M_K)$, and χ be an irreducible γ-regular character in $X(M_K)$. There exists a Zariski open neighborhood of χ such that any character in this neighborhood is irreducible and γ-regular.*

Proof. Note that being irreducible is a Zariski open condition, see Eq. (2). The γ-regularity follows from lower semi-continuity of the rank of a linear map. Indeed the dimension of $H_1(M_K, \mathrm{Ad} \circ \rho)$ is upper semi-continuous. It is at least one (the dimension of X) again because it is isomorphic to the Zariski tangent space hence it is locally constant equal to one. On the other hand, the rank of the linear map $H_1(\gamma, \mathrm{Ad}\circ\rho) \to H_1(M_K, \mathrm{Ad}\circ\rho)$ sending $v_\rho \otimes \gamma$ to itself is lower semi-continuous. It is at most one (the dimension of $H_1(\gamma, \mathrm{Ad}\circ\rho)$ and it cannot decrease on a neighborhood of χ. We deduce that $H_1(\gamma, \mathrm{Ad}\circ\rho) \to H_1(M_K, \mathrm{Ad}\circ\rho)$ is an isomorphism on a Zariski open subset. □

Proof of Proposition 3.15. Let χ be an irreducible, scheme-smooth, and non boundary-parabolic character, and let ρ be a representation in $R(M_K)$ with character χ. We first assume that ρ is ℓ and m-regular, that is for $v \in \mathfrak{sl}_2(\mathbb{C})$ an $\mathrm{Ad}\circ\rho_\partial$-invariant vector, both $v \otimes \ell$ and $v \otimes m$ provide a basis of the space $H_1(M_K, \mathrm{Ad}\circ\rho)$.

The calculation of the torsions $\mathbb{T}_{M_K,\ell}(\chi)$ and $\mathbb{T}_{M_K,m}(\chi)$ involves different choices of homology basis of $C_*(M_K, \mathrm{Ad}\circ\rho)$. By [18, Proposition 3.18], the bases of $H_2(M_K, \mathrm{Ad}\circ\rho)$ are determined by the fundamental class of $H_2(\partial M_K; \mathbb{C})$ and can be chosen to be the same. Hence, if b_1 is a basis of $\mathrm{im}(\partial_1)$, the ratio of torsions corresponding to the choice of m or of ℓ is reduced to

$$\frac{\mathbb{T}_{M_K,\ell}(\chi)}{\mathbb{T}_{M_K,m}(\chi)} = \frac{\det(b_1 \oplus (v \otimes \ell), c_1)}{\det(b_1 \oplus (v \otimes m), c_1)}.$$

In parallel, consider the affine equation in $C_1(M_K, \mathrm{Ad}\circ\rho)$:

$$y\, b_1 + x\, v \otimes m = v \otimes \ell,$$

with at least a solution $y = 0$ and $x = s_K(\rho)$. The Cramer determinants expressed in the common basis c_1 show that $s_K(\rho)$ coincides with the ratio of torsions.

If there exists a character in X which is ℓ-regular and a character in X which is m-regular, then Proposition 3.15 holds on the whole component X by Lemma 3.16.

Assume that X contains only characters that are not (say) ℓ-regular. Since the map $H_1(\partial M_K, \mathrm{Ad}\circ\rho) \to H_1(M_K, \mathrm{Ad}\circ\rho)$ is not trivial (by Lemma 3.4), it is onto on a Zariski open subset $U \in X$, again because $H_1(M_K, \mathrm{Ad}\circ\rho)$ has dimension one generically. Thus all characters in U must be m-regular, and it follows from the definition that the slope and the quotient of torsions are identically zero on X. A similar argument works replacing ℓ by m and zero by infinity. □

3.4.2. The torsion form. In this paragraph, we present an alternative proof of Proposition 3.15. We follow a slightly different point of view on the torsion, as a volume form on the character variety. The following lemma asserts that the cotangent space of the character variety [20, Section 8] is isomorphic to the first $\mathrm{Ad}\circ\rho$-twisted homology group.

Lemma 3.17. *Let χ be an irreducible character in $X(M_K)$, and a representation ρ with character χ. Let $T^*_\chi X(M_K)$ be the Zariski tangent space of $X(M_K)$ at χ. There is a natural isomorphism*

$$H_1(M_K, \mathrm{Ad}\circ\rho) \simeq T^*_\chi X(M_K).$$

Moreover, if χ is not boundary-parabolic, then

$$H_1(\partial M_K, \mathrm{Ad}\circ\rho) \simeq T^*_{r(\chi)} X(\partial M_K).$$

The proof of Lemma 3.17 follows from [20, Theorem 1]. Note that through the isomorphism, the space $\mathcal{Z}(K, \mathrm{Ad}\circ\rho)$ is the Zariski conormal bundle of $r(X(M_K))$ in $X(\partial M_K)$.

If $X \subset X(M_K)$ is a one-dimensional component of the character variety which contains a scheme-smooth character, the first author proved in [2, Proposition 5.1] that the *torsion form* can be written as

$$\mathrm{tor}(M_K) = \frac{1}{\mathbb{T}_{M_K,\ell}}\, r^*\left(\frac{dL}{L}\right) = \frac{1}{\mathbb{T}_{M_K,m}}\, r^*\left(\frac{dM}{M}\right) \tag{6}$$

where r^* is the cotangent map

$$r^* :\ T^*X(\partial M_K) \longrightarrow T^*X(M_K).$$

Proof of Proposition 3.15. By Eq. (6), the ratio of torsions can be written as

$$\frac{\mathbb{T}_{M_K,\ell}}{\mathbb{T}_{M_K,m}} = \frac{r^*(dL/L)}{r^*(dM/M)}.$$

If χ is a non boundary-parabolic character, the character variety $X(\partial M_K)$ is diffeomorphic to $(\mathbb{C}^*)^2$ in a neighborhood of $r(\chi)$. A local chart of $X(\partial M_K)$ is given by taking $\mathfrak{l}, \mathfrak{m} \in \mathbb{C}$ satisfying $\exp \mathfrak{l} = L$ and $\exp \mathfrak{m} = M$. The latter ratio of torsions can be written

$$\frac{\mathbb{T}_{M_K,\ell}}{\mathbb{T}_{M_K,m}} = \frac{r^*(d\,\mathfrak{l})}{r^*(d\,\mathfrak{m})}.$$

Lemma 3.17 implies that the cotangent map

$$r^* :\ T^*_{r(\chi)}X(\partial M_K) \longrightarrow T^*_{\chi}X(M_K)$$

coincides with the homomorphism in homology:

$$H_1(\partial M_K, \mathrm{Ad}\circ\rho) \longrightarrow H_1(M_K, \mathrm{Ad}\circ\rho),$$

thus by Lemma 3.4 the range of the map r^* is one-dimensional, and the images of the elements $d\,\mathfrak{l}, d\,\mathfrak{m}$ are collinear. It turns out that the ratio $\frac{r^*(d\,\mathfrak{l})}{r^*(d\,\mathfrak{m})}$ coincides with the slope by its very definition.

Finally, the formula extends to the entirety of X since irreducible and non boundary-parabolic characters are Zariski dense in X. $\square$

3.5. Compute the slope. In this section we compute the slope $s_K(\rho)$ when ρ is an irreducible non-boundary parabolic representation, with Fox calculus, similarly to [10]. Note that for the boundary-parabolic case, the slope can be computed directly from the representation using Proposition 3.8.

Consider a presentation of the knot group

$$\pi_1(M_K) = \langle x_1, \dots, x_p \mid r_1, \dots, r_q \rangle$$

obtained from a Wirtinger presentation. We also assume that $m = x_1$ is the preferred meridian and add the preferred longitude $\ell = x_2$, with the relation $[m, \ell] = 1$. Consider the complex of $\mathbb{Z}[\pi_1(M_K)]$-modules

$$S_* := S_2 \xrightarrow{\partial_1} S_1 \xrightarrow{\partial_0} S_0$$

where

$$S_2 = \bigoplus_{j=1}^{q} \mathbb{Z}[\pi_1(M_K)] \otimes r_j, \quad S_1 = \bigoplus_{j=1}^{p} \mathbb{Z}[\pi_1(M_K)] \otimes dx_i, \quad S_0 = \mathbb{Z}[\pi_1(M_K)]$$

and dx_i is a formal generator corresponding to x_i. Let $\partial/\partial x_i : \mathbb{Z}[\pi_1(M_K)] \to \mathbb{Z}[\pi_1(M_K)]$ denote the Fox derivatives. For every $i \in \{1, \dots, p\}$ and $j \in \{1, \dots, q\}$ the boundary operators are defined by

$$\partial_1 : r_j \longmapsto dr_j, \qquad \partial_0 : dx_i \longmapsto x_i,$$

where dw is the Fox differential of the word $w \in \pi_1(M_K)$:

$$dw := \sum_{i=1}^{p} \frac{\partial w}{\partial x_i} dx_i \in S_1.$$

Now consider the $\mathrm{Ad} \circ \rho$-twisted chain complex $S_*(\rho) := \mathfrak{sl}_2(\mathbb{C}) \otimes_{\mathbb{Z}[\pi_1(M_K)]} S_*$, where elements of $\mathfrak{sl}_2$ are identified with line vectors in $\mathbb{C}^3$. The ($\mathrm{Ad} \circ \rho$-twisted) *Alexander matrix* of M_K associated with the presentation is the matrix of $\partial_1(\rho)$, with coefficients in $\mathbb{C}$ and given by the blockwise definition:

$$\left((\mathrm{Ad} \circ \rho) \left(\frac{\partial r_i}{\partial x_j} \right) \right)_{1 \le i \le q,\, 1 \le j \le p}$$

The computation of the slope using $S_*(\rho)$ is achieved with the following result:

Proposition 3.18. *If ρ is irreducible and non-boundary parabolic, then there exist $a, b \in \mathbb{C}$ and an $\mathrm{Ad} \circ \rho_\partial$-invariant vector $v_\rho \in \mathfrak{sl}_2(\mathbb{C})$ such that*

$$\mathrm{im}\,(\partial_1(\rho)) \cap \langle v_\rho \otimes d\ell, v_\rho \otimes dm \rangle = \langle a\,(v_\rho \otimes d\ell) + b\,(v_\rho \otimes dm) \rangle$$

and the slope is $s_K(\rho) = -\frac{b}{a}$.

Proof. Set a base point p on ∂M_K. Following Crowell [8], the homology space $H_1(S_*(\rho))$ is isomorphic to $H_1(M_K, p, \mathrm{Ad} \circ \rho)$. The sub-complex $S_*(\rho_\partial)$ defined by considering only the generators $x_1 = m$, $x_2 = \ell$ and the relation $[m, \ell] = 1$ computes the space $H_1(\partial M_K, p; \mathrm{Ad} \circ \rho)$ as well. There are natural identifications

$$H_1(\partial M_K, p; \mathrm{Ad} \circ \rho) = H_1(\partial M_K, \mathrm{Ad} \circ \rho)$$
$$H_1(\partial M_K, p; \mathrm{Ad} \circ \rho) \hookrightarrow H_1(\partial M_K, \mathrm{Ad} \circ \rho)$$

and the following diagram commutes:

$$\begin{tikzcd}
S_1(\rho_\partial) \arrow[rr, hook] \arrow[d, two heads, "h_{\partial M_K}"'] & & S_1(\rho) \arrow[d, two heads, "h"] \\
H_1(\partial M_K, \mathrm{Ad}\circ\rho) \arrow[r, "i_*"] & H_1(M_K, \mathrm{Ad}\circ\rho) \arrow[r, hook] & H_1(M_K, p, \mathrm{Ad}\circ\rho)
\end{tikzcd}$$

where h and $h_{\partial M_K}$ are the quotient maps.

Let $u \in \mathfrak{sl}_2(\mathbb{C})$ be an $\mathrm{Ad}\circ\rho$-invariant vector and $\gamma \in \pi_1(M_K)$. Any element $u \otimes \gamma$ of $H_1(\partial M_K, \mathrm{Ad}\circ\rho)$ can be lifted to $u \otimes dw$ in $S_1(\rho_\partial)$. Since ρ is admissible, there exists $a, b \in \mathbb{C}$ such that $\mathcal{Z}(K, \mathrm{Ad}\circ\rho) = \ker i_* = \langle a\,(v_\rho \otimes \ell) + b\,(v_\rho \otimes m)\rangle$. Then $a\,(v_\rho \otimes d\ell) + b\,(v_\rho \otimes dm) \in \ker(h) = \mathrm{im}\,(\partial_1(\rho))$.

Reciprocally, suppose that there exists complex numbers $a, b \in \mathbb{C}$ such that $dz := a\,(v_\rho \otimes d\ell) + b\,(v_\rho \otimes dm)$ is a non-zero vector belonging to $\mathrm{im}\,(\partial_1(\rho))$. Then $h_{\partial M_K}(dz) = a\,(v_\rho \otimes \ell) + b\,(v_\rho \otimes m)$ must be non-zero since $(v_\rho \otimes \ell, v_\rho \otimes m)$ is a free basis of $H_1(\partial M_K, \mathrm{Ad}\circ\rho)$. However, $h(dz) = i_*\left(h_{\partial M_K}(dz)\right) = 0$; hence $h_{\partial M_K}(dz) \in \ker i_*$. Since $\ker i_*$ is one-dimensional, then $\ker i_* = \langle h_{\partial M_K}(dz)\rangle$, and the slope is $-\frac{b}{a}$. □

Example 3.19. *The trefoil knot.* Let T be the exterior of the right-handed trefoil knot, with group $\pi_1(M_K) = \langle u, v \mid uvu = vuv\rangle$. Any irreducible representation is conjugate to ρ with

$$\rho(u) = \begin{pmatrix} M & 1 \\ 0 & M^{-1} \end{pmatrix}, \qquad \rho(v) = \begin{pmatrix} M^{-1} & 0 \\ -1 & M \end{pmatrix}$$

where $M \in \mathbb{C}$. If $\ell = vuv^{-1}uvu^{-3}$ is the preferred longitude with corresponding meridian $m = u$, we obtain

$$\rho(\ell) = \begin{pmatrix} -M^{-6} & M^5+M^3+M+M^{-1}+M^{-3}+M^{-5} \\ 0 & -M^6 \end{pmatrix}.$$

Whenever $M \neq \pm 1$, the vector $v_\rho = \left(0, 1, \frac{1}{M-M^{-1}}\right)$ is right $\mathrm{Ad}\circ\rho_\partial$-invariant. By Section 3.5, the Alexander matrix (acting on the right on the coefficients) whose row-space is generating $\mathrm{im}\,(\partial_1)$ is given by

$$\begin{array}{cc}
 & \begin{array}{ccc} \quad\mathfrak{sl}_2(\mathbb{C})\otimes d\ell & \quad\mathfrak{sl}_2(\mathbb{C})\otimes dm & \qquad\mathfrak{sl}_2(\mathbb{C})\otimes dv \end{array} \\
\begin{array}{c} dr_1 \\ \\ \\ dr_2 \end{array} &
\left(\begin{array}{ccc:ccc:ccc}
0 & 0 & 0 & 1 & 0 & 0 & -M^{-2} & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -M^{-1} & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 2M & -M^2 \\
\hdashline
-1 & 0 & 0 & -2(2-M^{-2}) & 0 & 0 & 1+M^{-2} & 0 & 0 \\
0 & -1 & 0 & 2M^{-1} & -2 & 0 & M^{-1} & 2 & 0 \\
0 & 0 & -1 & -2 & -4M & 2M^2-4 & -1 & -2M & M^2+1
\end{array}\right)
\end{array}$$

where r_1 is $uvu = vuv$ and r_2 is the longitude definition. By Proposition 3.18, the space $\mathcal{Z}(M_K, \mathrm{Ad}\circ\rho)$ has generator

$$\left(0, 1, \frac{1}{M-M^{-1}}, 0, 6, \frac{6}{M-M^{-1}}, 0, 0, 0\right)$$

in the 2-dimensional subspace spanned by

$$\left\{\left(0, 1, \frac{1}{M-M^{-1}}, 0, 0, 0, 0, 0, 0\right), \left(0, 0, 0, 0, 1, \frac{1}{M-M^{-1}}, 0, 0, 0\right)\right\}$$

and the slope is $s_T(\mathrm{Ad}\circ\rho) = -6$. In particular it does not depend on ρ.

Example 3.20. *The figure-eight knot.* Let K be the figure-eight knot. There is a unique component $X \subset X(M_K)$ containing irreducible characters (see for instance [2, Examples 1.6.2 and 5.5]). This component is a plane curve given by the equation

$$\{2x^2 + y^2 - x^2y - y - 1 = 0\} \subset \mathbb{C}^2,$$

where x it the coordinate function given by $\chi \mapsto \chi(m)$. Note that the coordinate function of the longitude is $\chi \mapsto \chi(\ell) = x^4 - 5x^2 + 2$. Using [18, Théorème 4.1 (ii)] and Proposition 3.15 we compute

$$s_K(x,y)^2 = \frac{x^2-4}{(x^4-5x^2+2)^2-4}(4x^3-10x)^2 = \frac{4(2x^2-5)^2}{(x^2-5)(x^2-1)}$$

Expanding the denominator with the relation $x^2 = \frac{y^2-y-1}{y-2}$, we obtain, up to sign

$$s_K(x,y) = \pm\frac{2(2x^2-5)(y-2)}{(y-1)(y-3)}.$$

4. Slope and A-polynomial

In this section, we express the slope function in terms of the A-polynomial of the knot. As mentioned in Section 2.3, $r(X)$ might have 0-dimensional components but they are omitted in the definition of the A-polynomial.

Theorem 4.1. *Let $X \subset X(M_K)$ be an irreducible component such that $r(X)$ has dimension* 1*. For all $\chi \in X$ with $r(\chi) = (L, M)$, the following holds*

$$s_K(\chi) = -\frac{M}{L}\cdot\frac{\partial_M A(L,M)}{\partial_L A(L,M)},$$

where $A(L,M) = A_K(L,M)$ and ∂_L and ∂_M are the partial derivatives.

Remark 4.2. Combining Proposition 3.15 with [18, Corollaire 4.9], the result of Theorem 4.1 follows directly, up to sign, in the case where X has itself dimension 1. We resolve those two issues. Moreover Theorem 4.1 does not require the characters in X to be scheme-reduced, and the factors of the A-polynomial might have multiplicities greater than 1.

Proof. From Lemma 3.17 it follows that the Lagrangian $\mathcal{Z}(M_K, \mathrm{Ad}\circ\rho)$ generically identifies with the Zariski conormal bundle of $r(X(M_K))$ inside $X(\partial M_K)$. Picking local coordinates $\mathfrak{l} = \log L$, $\mathfrak{m} = \log M$ around $r(\chi)$, the kernel of the cotangent map is generated by

$$dA(e^{\mathfrak{l}}, e^{\mathfrak{m}}) = \partial_{\mathfrak{l}}A(e^{\mathfrak{l}}, e^{\mathfrak{m}})d\mathfrak{l} + \partial_{\mathfrak{m}}A(e^{\mathfrak{l}}, e^{\mathfrak{m}})d\mathfrak{m}$$

in $\mathbb{C}^2 = \langle d\mathfrak{l}, d\mathfrak{m}\rangle$. Using the chain rule, we obtain that it is generated by the vector

$$\left(L\frac{\partial A(M,L)}{\partial L}, M\frac{\partial A(M,L)}{\partial M}\right)$$

and the proposition follows. $\square$

Remark 4.3. Let T be the right-handed trefoil knot, with $A_T(L,M) = 1+LM^6$. Theorem 4.1 gives

$$s_T = -\frac{M}{L}\cdot\frac{6M^5L}{M^6} = -6.$$

Compare to Example 3.19.

5. The slope at an ideal point

In this section we prove Theorem 1.5. The context of this result is the work of Culler–Shalen (see for instance [19]) which associates incompressible surfaces in M_K to ideal points of curves of $X(M_K)$.

Let $X \subset X(M_K)$ be an irreducible component whose image $r(X) = Y$ is a curve in $X(\partial M_K)$, defined as the zero locus of an irreducible factor P of $A_K(L,M)$. Its function ring is usually denoted by $\mathbb{C}[Y] = \mathbb{C}[L,M]/(P)$, and its function field is $\mathbb{C}(Y) = \mathrm{Frac}(\mathbb{C}[Y])$.

To any point y in Y one can associate a *discrete valuation* v on the multiplicative group $\mathbb{C}(Y)^*$ in the field $\mathbb{C}(Y)$ of rational functions on Y. A discrete valuation $v : \mathbb{C}(Y)^* \to \mathbb{Z}$ is a group epimorphism satisfying $v(f+g) \geq \min(v(f), v(g))$. The valuation associated to a smooth point y is simply the map

$$f \longmapsto v_y(f) = \mathrm{ord}_y\, f$$

given the vanishing order of f at the point y. More generally, the smooth projective model $\overline{Y}$ of Y is smooth compact curve bi-rational to Y, canonically defined up to isomorphism, and the points of $\overline{Y}$ are bijectively associated to discrete valuations on the function field $\mathbb{C}(Y) \simeq \mathbb{C}(\overline{Y})$.

An *ideal* point y of Y is a point added "at infinity" in the smooth projective model $\overline{Y}$, it corresponds to a valuation v_y on $\mathbb{C}(Y)$ such that not every regular function $f \in \mathbb{C}[Y]$ has non-negative valuation $v_y(f)$. In other words, some regular functions (at least one) should have poles at y.

In [9], Marc Culler and Peter Shalen gave a procedure to construct an *incompressible* surface Σ in M_K from the data of an ideal point x in a sub-curve C of $X(M_K)$ together with the valuation $v_x : \mathbb{C}(C)^* \to \mathbb{Z}$. Not any ideal point $x \in X(M_K)$ yields an ideal point $y = r(x) \in X(\partial M_K)$.

In this special case, the ideal point y in Y gives an incompressible surface in M_K of a particular kind: as observed in [7, Proposition 3.1], the incompressible surface Σ must have non-empty boundary $\partial\Sigma \subset \partial M_K$. The curve $\partial\Sigma$ is a finite union of parallel circles in ∂M_K and uniquely determines a *boundary slope* in $\mathbb{Q}\cup\{\infty\}$: the slope of $a\ell + bm$ in $H_1(\partial M_K;\mathbb{Z})$ is the rational number $\frac{b}{a}$.

On the other hand, the Newton polygon of $A_K(L,M) = \sum_{i,j} a_{i,j}L^iM^j$ is the convex hull in $\mathbb{C}^2$ of the points $\{(i,j) \in \mathbb{Z}^2 \mid a_{i,j} \neq 0\}$. It is a convex polygon of $\mathbb{C}^2$ with integral vertices, whose sides have a slope in $\mathbb{Q}\cup\{\infty\}$. In [7], Culler, Cooper, Gillet, Long and Shalen proved the following result:

Theorem 5.1. [7, Theorem 3.4] *The slopes of the sides of the Newton polygon of the A-polynomial $A_K(L,M)$ are boundary slopes of incompressible surfaces in M_K*

which correspond to ideal points of one-dimensional components of $r^*(X(M_K))$ *in* $X(\partial M_K)$.

Our next statement (Theorem 1.5 in the introduction) states that the slope invariant studied in this paper coincides with the slopes of [7] at ideal points.

Theorem 5.2. *Let* y *be an ideal point in a one-dimensional component* Y *of the* A*-polynomial. Then the value of the slope function at the ideal point* y *equals* minus *the boundary slope of an incompressible surface corresponding to* y *or* minus *the slope of the corresponding side of the Newton polygon of the* A*-polynomial.*

Proof. The coordinate functions L, M define rational functions on Y, in particular their valuations $v_y(L)$ and $v_y(M)$ are well-defined. Since y is an ideal point and L, M generate the coordinate ring $\mathbb{C}[Y]$ of the curve Y, at least one of this valuation must be negative, and at least one of these coordinate functions must have a pole at y.

Claim. *The value of* s_K *at the ideal point* y *is* $\frac{v_y(L)}{v_y(M)}$.

Proof of the claim. From the proof of Proposition 3.15, we deduce that the value of the slope at y is given by

$$s_K(y) = \lim_{(L,M)\to y} \frac{r^*(dL/L)}{r^*(dM/M)}.$$

The following argument is an algebraic analogue of taking Taylor expansion of the functions L and M around the ideal point y. We pick t a local coordinate around y. It is characterized by $v_y(t) = 1$, and we can write

$$L = u_1 t^{v_y(L)}$$

for $u_1 \in \mathbb{C}(Y), v_y(u_1) = 0$, and similarly

$$M = u_2 t^{v_y(M)}$$

for $u_2 \in \mathbb{C}(Y), v_y(u_2) = 0$. Moreover, near y it follows that

$$\frac{r^*(dL/L)}{r^*(dM/M)} = \frac{v_y(L)/t}{v_y(M)/t} = \frac{v_y(L)}{v_y(M)}$$

and the claim follows. □

Now Theorem 1.5 follows directly from the claim, because it is proven in [7, Proposition 3.1] that the quantity $-\frac{v_y(L)}{v_y(M)}$ is the boundary slope of an incompressible surface corresponding to y. □

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