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Weight ergodic theorems for power bounded measures on locally compact groups

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Abstract. A complex sequence $\{a_n\}_{n\in\mathbb{N}}$ is called *good weight for the mean ergodic theorem* (briefly *good weight*) if for every Hilbert space $\mathcal{H}$ and every contraction T on $\mathcal{H}$ the limit

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}a_iT^ix \text{ exists in norm for every } x\in\mathcal{H}.$$

Let G be a locally compact group and let μ be a power bounded regular Borel measure on G. We study the behavior of the limit

$$\mathrm{w}^*-\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}a_i\mu^i$$

for the good weights $\{a_n\}$. Some related problems are also discussed.

Contents

1. Introduction

Let G be a locally compact group with the left Haar measure m_G (in the case when G is compact, m_G will denote normalized Haar measure on G) and let $M(G)$ be the convolution measure algebra of G. As usual, $C_0(G)$ will denote the space of all complex valued continuous functions on G vanishing at infinity. Since $C_0(G)^* = M(G)$, the space $M(G)$ carries the weak* topology $\sigma(M(G), C_0(G))$. In the following, the w*-topology on $M(G)$ always means

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this topology. Thus, a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ in $M(G)$ weak* converges to $\mu \in M(G)$ or w*-$\lim_{n\to\infty} \mu_n = \mu$ if:

$$\lim_{n\to\infty} \int_G f d\mu_n = \int_G f d\mu, \ \ \forall f \in C_0(G).$$

For a subset S of G, by $[S]$ we will denote the closed subgroup of G generated by S. A probability measure μ on G is said to be *adapted* if $[supp\mu] = G$. Also, a probability measure μ on G is said to be *strictly aperiodic* if the support of μ is not contained in a proper closed left cosets gH $(H \neq G,\ g \in G\setminus H)$ of G. For example, if $\mu \in M(G)$ is a probability measure with $e \in supp\mu$, then μ is strictly aperiodic, where e is the unit element of G.

Recall that the convolution product $\mu * \nu$ of two measures $\mu, \nu \in M(G)$ is defined by

$$(\mu * \nu)(B) = \int_G \mu\left(g^{-1}B\right) d\nu(g) \ \ \text{for every Borel subset } B \text{ of } G.$$

For $n \in \mathbb{N}$, by μ^n we will denote n-th convolution power of $\mu \in M(G)$, where $\mu^0 := \delta_e$ is the Dirac measure concentrated at the unit element of G. The classical Kawada-Itô theorem [14, Theorem 7] asserts that if μ is an adapted measure on a compact metrisable group G, then the sequence of probability measures $\left\{\frac{1}{n}\sum_{i=1}^{n} \mu^i\right\}_{n\in\mathbb{N}}$ weak* converges to the Haar measure on G (see also [11, Theorem 3.2.4]). If μ is an adapted and strictly aperiodic measure on a compact metrisable group G, then w*-$\lim_{n\to\infty} \mu^n = m_G$ [14, Theorem 8]. If μ is an adapted measure on a second countable non-compact locally compact group G, then w*-$\lim_{n\to\infty} \mu^n = 0$ [18, Theorem 2]. In [4, Théorème 8], it was proved that if μ is a strictly aperiodic measure on a non-compact locally compact group G, then w*-$\lim_{n\to\infty} \mu^n = 0$. For related results see, [1, 2, 9, 11, 19, 20, 21].

Let $\mu \in M(G)$ be a power bounded measure, that is, $\sup_{n\in\mathbb{N}_0} \|\mu^n\|_1 < \infty$. We study the behavior of the limit

$$\text{w}^* - \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} a_i \mu^i$$

for the good weights $\{a_n\}$.

2. Weighted ergodic theorems

Let X be a complex Banach space and let $B(X)$ be the algebra of all bounded linear operators on X. An operator $T \in B(X)$ is said to be *mean ergodic* if the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} T^i x \ \ \text{exists in norm for every } x \in X.$$

If T is mean ergodic, then

$$P_T x := \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} T^i x \ \ (x \in X)$$

is the projection onto $\ker(T-I)$. The projection P_T will be called *mean ergodic projection* associated with T.

If T is a mean ergodic operator, then T is Cesàro bounded, that is,

$$\sup_{n\in\mathbb{N}} \left\| \frac{1}{n} \sum_{i=1}^{n} T^i \right\| < \infty.$$

It follows from the spectral mapping theorem that if T is mean ergodic, then $r(T) \leq 1$, where $r(T)$ is the spectral radius of T.

The following result is a consequence of the Mean Ergodic Theorem [15, Ch.2, Theorem 1.1].

Proposition 2.1. *Let $T \in B(X)$ be Cesàro bounded and assume that $\frac{\|T^n x\|}{n} \to 0$ for all $x \in X$. If $u, v \in X$ and $\frac{1}{n}\sum_{i=1}^{n} T^i u \to v$ weakly, then*

$$\frac{1}{n} \sum_{i=1}^{n} T^i u \to v \ \ \textit{in norm, as } n \to \infty.$$

We will need also the following subsequential ergodic theorem [8, Theorem 21.14].

Theorem 2.2. *For a subsequence $(k_i)_{i\in\mathbb{N}}$ of $\mathbb{N}$, the following assertions are equivalent:*

(a) *For every contraction T on a Hilbert space H, the limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} T^{k_i} x \ \ \textit{exists in norm for every } x \in H.$$

(b) *The limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \xi^{k_i} \ \ \textit{exists for every } \xi \in \mathbb{T}.$$

An operator $T \in B(X)$ is said to be *power bounded* if

$$C_T := \sup_{n\in\mathbb{N}_0} \|T^n\| < \infty.$$

A power bounded operator T on a Banach space X is mean ergodic if and only if

$$X = \ker(T-I) \oplus \overline{\operatorname{ran}(T-I)}. \tag{2.1}$$

Recall [15, Chapter 2] that a power bounded operator on a reflexive Banach space is mean ergodic.

The following result is an immediate consequence of the identity (2.1).

Proposition 2.3. *Let T be a power bounded operator on a Banach space X and assume that $\lim_{n\to\infty}\left\|T^{n+1}x - T^n x\right\| = 0$ for all $x \in X$. If T is mean ergodic (so if X is reflexive), then $T^n \to P_T$ in the strong operator topology, where P_T is the mean ergodic projection associated with T.*

As usual, by $\sigma(T)$ and $\sigma_p(T)$ respectively, we denote the spectrum and the point spectrum of $T \in B(X)$. The open unit disc and the unit circle in the complex plane will be denoted by $\mathbb{D}$ and $\mathbb{T}$ respectively. If $T \in B(X)$ is power bounded then clearly, $\sigma(T) \subseteq \overline{\mathbb{D}}$. The classical Katznelson-Tzafriri theorem [13] states that if $T \in B(X)$ is power bounded, then $\lim_{n\to\infty}\left\|T^{n+1} - T^n\right\| = 0$ if and only if $\sigma(T) \cap \mathbb{T} \subseteq \{1\}$. For the normal operators on a Hilbert space, this fact is an immediate consequence of the Spectral Theorem.

Recall from [8, Section 21] that a sequence $\{a_n\}_{n\in\mathbb{N}}$ in $\mathbb{C}$ is called *good weight for the mean ergodic theorem* (briefly *good weight*) if for every (complex) Hilbert space $\mathcal{H}$ and every contraction T on $\mathcal{H}$ the limit

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} a_i T^i x \text{ exists in norm for every } x \in \mathcal{H}.$$

Let (Ω, Σ, m) be a probability space and let $\varphi : \Omega \to \Omega$ be a measure-preserving transformation. It follows from the Wiener-Wintner theorem [8, Corollary 21.6] that the sequence $(f(\varphi^n(\omega)))_{n\in\mathbb{N}}$ is a bounded good weight for all almost every $\omega \in \Omega$ and $f \in L^\infty(\Omega)$.

By [8, Theorem 21.2], a bounded sequence $\{a_n\}_{n\in\mathbb{N}}$ is a good weight if and only if the limit

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} a_i \xi^i =: a(\xi) \quad \text{exists for every } \xi \in \mathbb{T}.$$

If $\{a_n\}_{n\in\mathbb{N}}$ is a bounded good weight, then for every contraction T on a Hilbert space $\mathcal{H}$ and $x \in \mathcal{H}$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} a_i T^i x = \sum_{\xi\in\sigma_p(T)\cap\mathbb{T}} a(\xi) P_\xi x \text{ in norm,} \tag{2.2}$$

where P_ξ are orthogonal projections onto the mutually orthogonal eigenspaces $\ker(T - \xi I)$ for $\xi \in \sigma_p(T) \cap \mathbb{T}$ [8, Theorem 21.2] (it follows that $a(\xi) \neq 0$ for at most countably many $\xi \in \mathbb{T}$).

Let N be a normal operator on a Hilbert space $\mathcal{H}$ with the spectral measure E. If N is mean ergodic, then $\|N\| = r(N) \leq 1$. If N is a normal contraction operator (a normal operator is power bounded if and only if it is a contraction), then for every $x \in \mathcal{H}$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} N^i x = E(\{1\})x \text{ in norm.}$$

If N is a normal contraction operator on a separable Hilbert space $\mathcal{H}$, then $\sigma_p(N) \cap \mathbb{T}$ is at most countable [3, Chapter IX] and

$$\sigma_p(N) \cap \mathbb{T} = \{\xi \in \mathbb{T} : E(\{\xi\}) \neq 0\}.$$

If $\sigma_p(N) \cap \mathbb{T} = \{\xi_1, \xi_2, ...\}$, then for every bounded good weight $\{a_n\}_{n\in\mathbb{N}}$ and $x \in \mathcal{H}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} a_i N^i x = \sum_{i=1}^{\infty} a(\xi_i) E(\{\xi_i\}) x \text{ in norm.}$$

In particular if $\sigma_p(N) \cap \mathbb{T} = \{1\}$, then for every $x \in \mathcal{H}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} a_i N^i x = \left(\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{\infty} a_i \right) E(\{1\}) x \text{ in norm.}$$

If $\sigma(N) \cap \mathbb{T} = \{1\}$, then as $\left\| N^{n+1} - N^n \right\| \to 0$, by Proposition 2.3, $N^n x \to E(\{1\}) x$ in norm for every $x \in \mathcal{H}$.

3. Generalized convolution operators

Let G be a locally compact group. A *representation* π of G on a Banach space X_π (the representation space of π) is a homomorphism from G into the group of invertible isometries on X_π. We will assume that π is strongly continuous. Then, for any $\mu \in M(G)$, we can define a bounded linear operator $\widehat{\mu}(\pi)$ on X_π, by

$$\widehat{\mu}(\pi) x = \int_G \pi(g) x d\mu(g), \ x \in X_\pi.$$

The map $\mu \to \widehat{\mu}(\pi)$ is linear, multiplicative, and contractive; $\|\widehat{\mu}(\pi)\| \leq \|\mu\|_1$, where $\|\mu\|_1$ is the total variation norm of $\mu \in M(G)$.

By $\widehat{G}$ we will denote unitary dual of G, the set of all equivalence classes of irreducible continuous unitary representations of G with the Fell topology. Recall that $\pi_0 \in \widehat{G}$ is a limit point of $M \subset \widehat{G}$ in the Fell topology, if the matrix function $g \to \langle \pi_0(g) x_0, x_0 \rangle$ $\left(x_0 \in \mathcal{H}_{\pi_0}\right)$ can be uniformly approximated on every compact subset of G by the matrix functions $g \to \langle \pi(g) x, x \rangle$ $(\pi \in M, \ x \in \mathcal{H}_\pi)$ (in the case when G is abelian, Fell topology coincides with the usual topology of $\widehat{G}$, the dual group of G).

The function $\pi \to \widehat{\mu}(\pi)$ $\left(\pi \in \widehat{G}\right)$ is called *Fourier-Stieltjes transform* of $\mu \in M(G)$. If $\widehat{\mu}(\pi) = 0$ for all $\pi \in \widehat{G}$, then $\mu = 0$ (for instance see, [6, §18]).

It is well known that if G is compact, then every $\pi \in \widehat{G}$ is finite dimensional. Also, we know that if G is compact (resp. compact and metrisable), then $\widehat{G}$ is discrete (resp. countable). These facts are consequences of the Peter-Weyl theory [17, Chapter 4].

By B_X and S_X respectively, we denote the closed unit ball and the unit sphere of a Banach space X. Notice that $\text{ext}B_X \subseteq S_X$, where $\text{ext}B_X$ is the set of all extreme points of B_X. X will be called *rotund Banach space* if $\text{ext}B_X = S_X$. For example, uniformly convex Banach spaces, in particular, Hilbert spaces and L^p $(1 < p < \infty)$ spaces are rotund Banach spaces.

The following result is a small variation of [5, Proposition 2.1].

Lemma 3.1. *Let μ be a probability measure on a locally compact group G and let π be a Banach representation of G. If the representation space X_π is a rotund Banach space, then for an arbitrary $\xi \in \mathbb{T}$, we have*

$$\ker[\widehat{\mu}(\pi) - \xi I_\pi] = \{x \in X_\pi : \pi(g)x = \xi x, \ \forall g \in supp\mu\}.$$

The following result was proved in [20, Lemma 2.3].

Lemma 3.2. *Let μ be a strictly aperiodic measure on a locally compact group G and let π be a Banach representation of G. If the representation space of π is a rotund Banach space, then the operator $\widehat{\mu}(\pi)$ cannot have unitary eigenvalues except $\xi = 1$.*

As a consequence of the above results, we have the following.

Corollary 3.3. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a bounded good weight and let π be a unitary representation of a locally compact group G on a Hilbert space $\mathcal{H}_\pi$. If $\mu \in M(G)$ is an adapted and strictly aperiodic measure, then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} a_i \widehat{\mu}(\pi)^i x = \left(\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} a_i \right) P_\mu^\pi x \ \text{ in norm for every } x \in \mathcal{H}_\pi,$$

where P_μ^π is the orthogonal projection onto the subspace

$$\{x \in \mathcal{H}_\pi : \pi(g)x = x : \forall g \in G\}.$$

If $\pi \in \widehat{G} \setminus id$, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} a_i \widehat{\mu}(\pi)^i x = 0 \ \text{ in norm for every } x \in \mathcal{H}_\pi,$$

where id is the trivial representation of G; $id(g) = I$ for all $g \in G$.

Proof. By Lemma 3.2, the operator $\widehat{\mu}(\pi)$ cannot have unitary eigenvalues except $\xi = 1$. From the identity (2.2), we get that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} a_i \widehat{\mu}(\pi)^i x = \left(\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} a_i \right) P_\mu^\pi x \ \text{ in norm for every } x \in \mathcal{H}_\pi,$$

where P_μ^π is the orthogonal projection onto $\ker[\widehat{\mu}(\pi) - I]$. On the other hand, by Lemma 3.1,

$$\ker[\widehat{\mu}(\pi) - I] = \{x \in \mathcal{H}_\pi : \pi(g)x = x : \forall g \in G\}.$$

Notice that

$$\{x \in \mathcal{H}_\pi : \pi(g)x = x : \forall g \in G\}$$

is a closed π-invariant subspace. As $\pi \in \widehat{G} \setminus id$, we have $P_\mu^\pi = 0$. $\square$

As is well known, equipped with the involution given by $d\widetilde{\mu}(g) = \overline{d\mu(g^{-1})}$, the algebra $M(G)$ becomes a Banach $*$-algebra. If μ is a probability measure on a locally compact group G, then as $\operatorname{supp}\widetilde{\mu} = (supp\mu)^{-1}$, we have

$$supp(\widetilde{\mu} * \mu) = \overline{\left\{(supp\mu)^{-1} \cdot (supp\mu)\right\}}.$$

Proposition 3.4. *If μ is a probability measure on a locally compact group G, then the following assertions hold:*
(a) If the measure $\widetilde{\mu} * \mu$ is adapted, then μ is strictly aperiodic.
(b) If μ is adapted and strictly aperiodic, then the measure $\widetilde{\mu} * \mu$ is adapted.

Proof. (a) Assume that μ is not strictly aperiodic. Then, $supp\mu \subseteq gH$ for some closed subgroup $H \neq G$ and $g \in G \diagdown H$. As $(supp\mu)^{-1} \subseteq Hg^{-1}$, we have

$$(supp\mu)^{-1} \cdot (supp\mu) \subseteq gH \cdot Hg^{-1} = H,$$

which implies $[supp(\widetilde{\mu} * \mu)] \subseteq H$. This shows that the measure $\widetilde{\mu} * \mu$ is not adapted.

(b) Let $H := [supp(\widetilde{\mu} * \mu)]$ and assume that $H \neq G$. If $supp\mu \subseteq H$, then as $G = [supp\mu] \subseteq H$, we have $G = H$. Hence, we may assume that $supp\mu \nsubseteq H$. Then there exists $s \in supp\mu$, but $s \notin H$. Since $s^{-1}g \in H$ for all $g \in supp\mu$, we get that $supp\mu \subseteq sH$. This show that μ is not strictly aperiodic. $\square$

Next, we have the following.

Proposition 3.5. *Let π be a unitary representation of a locally compact group G and let μ be a probability meeasure on G. If one of the measures $\widetilde{\mu} * \mu$ and $\mu * \widetilde{\mu}$ is adapted (in particular, if μ is adapted and strictly aperiodic), then for every $\pi \in \widehat{G} \diagdown id$,*

$$\widehat{\mu}(\pi)^n \to 0 \ \textit{in the weak operator topology.}$$

Proof. Recall that a contraction T on a Hilbert space is said to be *completely non-unitary* if it has no proper reducing subspace on which it acts as a unitary operator. By the Nagy-Foiaş theorem [7, Ch.II, Theorem 3.9], if T is a completely non-unitary contraction, then $T^n \to 0$ in the weak operator topology. Now, it suffices to show that $\widehat{\mu}(\pi)$ is a completely non-unitary contraction. Let $\mathcal{H}_\pi$ be the representation space of π. As $\widehat{\mu}(\pi)^* = \widehat{\widetilde{\mu}}(\pi)$, we must show that the identity $\widehat{\widetilde{\mu}}(\pi)\widehat{\mu}(\pi)x = x$, where $x \in \mathcal{H}_\pi$, implies $x = 0$. Since $\widehat{(\widetilde{\mu} * \mu)}(\pi)x = x$, by Lemma 3.1, $\pi(g)x = x$ for all $g \in [supp(\widetilde{\mu} * \mu)]$. As $[supp(\widetilde{\mu} * \mu)] = G$, we have $\pi(g)x = x$ for all $g \in G$. Since

$$E_\pi := \{x \in \mathcal{H}_\pi : \pi(g)x = x, \ \forall g \in G\}$$

is a closed π-invariant subspace and $\pi \in \widehat{G} \diagdown id$, we get that $E_\pi = \{0\}$. Hence $x = 0$. $\square$

4. Convolution operators

Let G be a locally compact group. The left convolution of $\mu \in M(G)$ and $f \in L^p(G)\,(1 \leq p < \infty)$, is given by

$$(\mu * f)(g) = \int_G f\left(s^{-1}g\right) d\mu(s).$$

For $f \in L^p(G)$, we put

$$f^\vee(g) := f\left(g^{-1}\right) \text{ and } \widetilde{f}(g) := \overline{f\left(g^{-1}\right)}.$$

Notice that for every $u, v \in L^2(G)$, the function $u * \widetilde{v}$ is in $C_0(G)$ and

$$\langle \mu,\ u * \widetilde{v}\rangle = \langle \mu * \overline{v},\ \overline{u}\rangle,\ \forall \mu \in M(G).$$

It follows that the set $\left\{u * \widetilde{v} : u, v \in L^2(G)\right\}$ is linearly dense in $C_0(G)$. Notice also that if $f \in L^p(G)\,(1 < p < \infty,\ p \neq 2)$ and $h \in L^q(G)\,(1/p + 1/q = 1)$, then $h * f^\vee \in C_0(G)$ and

$$\langle \mu,\ h * f^\vee\rangle = \langle \mu * f, h\rangle,\ \forall \mu \in M(G).$$

It follows that the set

$$\{h * f^\vee : h \in L^q(G),\ f \in L^p(G)\}$$

is linearly dense in $C_0(G)$.

Let π be the left regular representation of G on $L^p(G)\,(1 \leq p < \infty)$, where

$$\pi(g) f(s) = f\left(g^{-1}s\right) := f_g(s).$$

Then, π is continuous and for an arbitrary $\mu \in M(G)$, $\widehat{\mu}(\pi)$ is the left convolution operator on $L^p(G)$; $\widehat{\mu}(\pi) f = \mu * f$. We will denote this operator by $\lambda_p(\mu)$. It is well known that $\lambda_p(\mu)$ is a bounded linear operator on $L^p(G)$, that is,

$$\left\|\lambda_p(\mu) f\right\| \leq \|\mu\|_1 \|f\|_p \text{ and } \|\lambda_1(\mu)\| = \|\mu\|_1. \tag{4.1}$$

A measure $\mu \in M(G)$ is said to be *power bounded if*

$$C_\mu := \sup_{n \in \mathbb{N}_0} \|\mu^n\|_1 < \infty.$$

It follows from (4.1) that if $\mu \in M(G)$ is power bounded, then so is the operator $\lambda_p(\mu)$, that is,

$$\sup_{n \in \mathbb{N}_0} \left\|\lambda_p(\mu)^n\right\| \leq C_\mu.$$

The most comprehensive work on power bounded measures is Schreiber [22].

A measure $\mu \in M(G)$ is said to be *vague-ergodic* if there is a measure $\theta_\mu \in M(G)$ such that

$$\mathrm{w}^* - \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n \mu^i = \theta_\mu.$$

Probability measures are always vague-ergodic. Although, it is usually proved assuming the group is second countable [11, Theorem 3.0].

The following result was proved in [9, Theorem 3.4]. The same result for locally compact abelian groups was obtained earlier in [19, Proposition 2.5].

Proposition 4.1. *If μ is a power bounded measure on a locally compact group G, then there exists an idempotent measure $\theta_\mu \in M(G)$ such that*

$$w^* - \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mu^i = \theta_\mu.$$

The measure θ_μ will be called *limit measure associated with μ.*

In [21, Theorem 7.1], it was proved that if μ is a probability measure on a locally compact group G, then $\mathrm{w}^* - \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mu^i = 0$ if and only if the support of μ is not contained in a compact subgroups of G (see also, [20, Theorem 2.4]).

We have the following more general result.

Proposition 4.2. *For a subsequence $(k_i)_{i\in\mathbb{N}}$ of $\mathbb{N}$, the following assertions are equivalent:*

(a) *The limit*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \xi^{k_i} \text{ exists for every } \xi \in \mathbb{T}.$$

(b) *For an arbitrary power bounded measure μ on a locally compact group G, the limit*

$$w^* - \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mu^{k_i} \text{ exists.}$$

Proof. (a)⇒(b) Notice that $\lambda_2(\mu)$ is a power bounded operator. By changing to an equivalent norm, $\lambda_2(\mu)$ can be made a contraction. If $u, v \in L^2(G)$, then by Theorem 2.2, the limit

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \langle \lambda_2(\mu)^{k_i} u, v\rangle \text{ exists.}$$

As $u * \widetilde{v} \in C_0(G)$, we can write

$$\begin{aligned}
\lim_{n\to\infty} \langle \frac{1}{n}\sum_{i=1}^{n} \mu^{k_i}, u * \widetilde{v}\rangle &= \lim_{n\to\infty} \langle \frac{1}{n}\sum_{i=1}^{n} \mu^{k_i} * \overline{u}, \overline{v}\rangle \\
&= \lim_{n\to\infty} \langle \frac{1}{n}\sum_{i=1}^{n} \lambda_2(\mu)^{k_i} \overline{u}, \overline{v}\rangle.
\end{aligned}$$

Therefore, the limit

$$\lim_{n\to\infty} \langle \frac{1}{n}\sum_{i=1}^{n} \mu^{k_i}, u * \widetilde{v}\rangle \text{ exists for all } u, v \in L^2(G).$$

Since the sequence $\left\{\frac{1}{n}\sum_{i=1}^{n}\mu^{k_i}\right\}_{n\in\mathbb{N}}$ is bounded and the set $\{u * \widetilde{v} : u, v \in L^2(G)\}$ is linearly dense in $C_0(G)$, the limit

$$\mathrm{w}^* - \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mu^{k_i} \text{ exists.}$$

(b)⇒(a) Let $G = \mathbb{T}$ and let $\mu = \delta_\lambda$, where δ_λ is the Dirac measure concentrated at $\lambda \in \mathbb{T}$. Then as $\mu^n = \delta_{\lambda^n}$ $(\forall n \in \mathbb{N})$, the limit

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} f\left(\lambda^{k_i}\right) \text{ exists for every } f \in C(\mathbb{T}).$$

If we take $f \in C(\mathbb{T})$, defined by $f(\xi) = \xi$, then we get that the limit

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\xi^{k_i} \text{ exists.}$$

□

Next, we have the following.

Proposition 4.3. *Let μ be a power bounded measure on a locally compact group G and let θ_μ be the limit measure associated with μ. Then the following assertions hold:*

(a) For every $f \in L^p(G)$ $(1 < p < \infty)$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mu^i * f = \theta_\mu * f \text{ in } L^p\text{-norm,}$$

where $P_\mu f := \theta_\mu * f$ is the mean ergodic projection associated with $\lambda_p(\mu)$.

(b) If μ is a probability measure on G and if $[supp\mu]$ is compact, then for every $f \in L^1(G)$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mu^i * f = \theta_\mu * f \text{ in } L^1\text{-norm,}$$

where $P_\mu f := \theta_\mu * f$ is the mean ergodic projection associated with $\lambda_1(\mu)$.

Proof. (a) By Proposition 4.1,

$$\mathrm{w}^* - \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mu^i = \theta_\mu.$$

On the other hand, by [9, Proposition 3.1], the mapping $\lambda_p : M(G) \to B(L^p(G))$ is w*-WOT continuous on norm bounded subsets of $M(G)$ for every $1 < p < \infty$. It follows that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mu^i * f = \theta_\mu * f \text{ weakly for every } f \in L^p(G).$$

Since the operator $\lambda_p(\mu)$ is mean ergodic, we get that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mu^i * f = \theta_\mu * f \text{ in } L^p\text{-norm.}$$

(b) By [9, Theorem 5.4], the operator $\lambda_1(\mu)$ is mean ergodic and therefore,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mu^i * f = P_\mu f \text{ in } L^1\text{-norm for every } f\in L^1(G),$$

where P_μ is the mean ergodic projection associated with the operator $\lambda_1(\mu)$. If $h\in C_0(G)$, then as $h * f^\vee \in C_0(G)$, we can write

$$\begin{aligned}
\langle P_\mu f, h\rangle &= \lim_{n\to\infty}\langle\frac{1}{n}\sum_{i=1}^{n}\mu^i * f, h\rangle \\
&= \lim_{n\to\infty}\langle\frac{1}{n}\sum_{i=1}^{n}\mu^i, h * f^\vee\rangle = \langle\theta_\mu,\ h * f^\vee\rangle \\
&= \langle\theta_\mu * f,\ h\rangle.
\end{aligned}$$

So we have $P_\mu f = \theta_\mu * f$. □

Let μ be a power bounded measure on a locally compact group G. For $\xi\in\mathbb{T}$, by θ_μ^ξ we will denote the limit measure associated with $\xi\mu$. By Proposition 4.1, θ_μ^ξ is an idempotent measure and

$$\mathrm{w}^* - \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\xi^i\mu^i = \theta_\mu^\xi.$$

Theorem 4.4. *Let G be a second countable locally compact group and let μ be a power bounded measure on G. Then the following assertions hold:*
(a) $\sigma_p(\lambda_2(\mu))\cap\mathbb{T}$ is at most countable.
(b) If $\{a_n\}_{n\in\mathbb{N}}$ is a bounded good weight and $\sigma_p(\lambda_2(\mu))\cap\mathbb{T} = \{\xi_1,\xi_2,...\}$, then

$$w^* - \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}a_i\mu^i = \sum_{i=1}^{\infty}a(\xi_i)\,\theta_\mu^{\xi_i},$$

where $\theta_\mu^{\xi_i}$ is the limit measure associated with $\xi_i\mu$ and

$$a(\xi_i) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}a_k\xi_i^k.$$

Proof. (a) Notice that $\lambda_2(\mu)$ is a power bounded operator on $L^2(G)$. It is no restriction to assume that $\lambda_2(\mu)$ is a contraction. Since $L^2(G)$ is separable, by the Jamison theorem [12], $\sigma_p(\lambda_2(\mu))\cap\mathbb{T}$ is at most countable set.

(b) Let $f \in L^2(G)$ and $\xi \in \mathbb{T}$ be given. By Proposition 4.3,

$$\frac{1}{n}\sum_{i=1}^{n} \xi^i \lambda_2(\mu)^i f \to \theta_\mu^\xi * f \text{ in } L^2\text{-norm.}$$

On the other hand, by the identity (2.2),

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} a_i \lambda_2(\mu)^i f = \sum_{i=1}^{\infty} a(\xi_i) P_{\xi_i} f \text{ in } L^2\text{-norm,}$$

where P_{ξ_i} is the orthogonal projection onto $\ker[\lambda_2(\mu) - \xi_i I]$. Since $P_{\xi_i} f = \theta_\mu^{\xi_i} * f$ (see, Proposition 4.3), for every $u, v \in L^2(G)$, we can write

$$\begin{aligned}
\lim_{n\to\infty} \langle \frac{1}{n}\sum_{i=1}^{n} a_i \mu^i, u * \widetilde{v}\rangle &= \lim_{n\to\infty} \langle \frac{1}{n}\sum_{i=1}^{n} a_i \lambda_2(\mu)^i \overline{u}, \overline{v}\rangle \\
&= \langle \sum_{i=1}^{\infty} a(\xi_i) P_{\xi_i} \overline{u}, \overline{v}\rangle \\
&= \langle \sum_{i=1}^{\infty} a(\xi_i) \theta_\mu^{\xi_i} * \overline{u}, \overline{v}\rangle \\
&= \langle \sum_{i=1}^{\infty} a(\xi_i) \theta_\mu^{\xi_i}, u * \widetilde{v}\rangle.
\end{aligned}$$

Since the sequence $\{a_n\}_{n\in\mathbb{N}}$ is bounded and the set $\{u * \widetilde{v} : u, v \in L^2(G)\}$ is linearly dense in $C_0(G)$, we get that

$$\mathrm{w}^* - \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} a_i \mu^i = \sum_{i=1}^{\infty} a(\xi_i) \theta_\mu^{\xi_i}.$$

□

If μ is a strictly aperiodic measure on a locally compact group G, then by Lemma 3.2, the operator $\lambda_2(\mu)$ cannot have unitary eigenvalues except $\xi = 1$.

The following result remains true without "second countability" condition.

Corollary 4.5. *If μ is a strictly aperiodic measure on a locally compact group G, then for a bounded good weight $\{a_n\}_{n\in\mathbb{N}}$, we have*

$$w^* - \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} a_i \mu^i = \left(\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{\infty} a_i\right) \theta_\mu,$$

where θ_μ is the limit measure associated with μ.

Remark 4.6. Let G be a locally compact abelian group and let $\mu \in M(G)$. The Fourier-Plancherel transform estabilishes unitary equivalence between convolution operator $\lambda_2(\mu)$ and the multiplication operator $M_{\widehat{\mu}}$ on $L^2\left(\widehat{G}\right)$, where $\widehat{\mu}$ is the Fourier-Stieltjes transform of μ. It follows that $\sigma(\lambda_2(\mu)) = \overline{\left\{\widehat{\mu}(\gamma) : \gamma \in \widehat{G}\right\}}$.

5. The sequence $\{\mu^n\}_{n\in\mathbb{N}}$

Recall that a linear operator T on a Banach space X is said to be *weakly almost periodic* if for every $x \in X$, the orbit $O_T(x) := \{T^n x : n \in \mathbb{N}_0\}$ is relatively weakly compact. Clearly, weakly almost periodic operator is power bounded. If T is a weakly almost periodic operator on a Banach space X, then by the Jacobs-Glicksberg-de Leeuw (JGdL) Decomposition Theorem [7, Ch.I, Theorem 1.15], there exist two T-invariant subspaces X_r and X_s such that $X = X_r \oplus X_s$, where

$$X_r = \overline{\operatorname{span}}\{x \in X : \exists \xi \in \mathbb{T},\ Tx = \xi x\} \tag{5.1}$$

and

$$X_s = \left\{x \in X : 0 \in \overline{\{T^n x : n \in \mathbb{N}_0\}}^{\text{weak}}\right\}. \tag{5.2}$$

The following result is a consequence of the JGdL Decomposition Theorem [7, Ch.II, Theorem 4.1].

Proposition 5.1. *Let T be a weakly almost periodic operator on a Banach space X and assume that T has no unitary eigenvalues. If X^* is separable, then there exists a subsequence $\{n_j\}_{j=1}^{\infty}$ of $\mathbb{N}$ such that $\lim_{j\to\infty} T^{n_j} = 0$ in the weak operator topology.*

As an application of Proposition 5.1, we have the following.

Proposition 5.2. *Let T be a weakly almost periodic operator on a Banach space X and assume that T has no unitary eigenvalues except $\xi = 1$. If X^* is separable, then there exists a subsequence $\{n_j\}_{j=1}^{\infty}$ of $\mathbb{N}$ such that $\lim_{j\to\infty} T^{n_j} = P$ in the weak operator topology, where P is the projection onto* $\ker(T - I)$.

Proof. By the JGdL Decomposition Theorem, $X = X_r \oplus X_s$, where the subspaces X_r and X_s are defined as in (5.1) and (5.2), respectively. Therefore, every $x \in X$ can be written as $x = x_r + x_s$, where $Tx_r = x_r$ for all $x_r \in X_r$ and

$$0 \in \overline{\{T^n x_s : n \in \mathbb{N}_0\}}^{\text{weak}} \quad \text{for all } x_s \in X_s.$$

Let $S := T|_{X_s}$ be the restriction of T to X_s. Notice that S has no unitary eigenvalues. Since X_s^* is separable, by Proposition 5.1, there exists a subsequence $\{n_j\}_{j=1}^{\infty}$ of $\mathbb{N}$ such that $\lim_{j\to\infty} S^{n_j} = 0$ in the weak operator topology. Now, for an arbitrary $\varphi \in X^*$, from the identity $T^{n_j}x = x_r + S^{n_j}x_s$, we can write

$$\langle\varphi, T^{n_j}x\rangle = \langle\varphi, x_r\rangle + \langle\varphi, S^{n_j}x_s\rangle \to \langle\varphi, x_r\rangle = \langle\varphi, Px\rangle\ (j \to \infty).$$

This shows that $T^{n_j} \to P\ (j \to \infty)$ in the weak operator topology. □

Next, we have the following.

Proposition 5.3. *Let G be a second countable locally compact group and let μ be a strictly aperiodic measure on G. Then there exists a subsequence $\{n_j\}_{j=1}^{\infty}$ of $\mathbb{N}$ such that*

$$w^*\text{-}\lim_{j\to\infty} \mu^{n_j} = \theta_\mu,$$

where θ_μ is the limit measure associated with μ.

Proof. Notice that $\lambda_2(\mu)$ is a weakly almost periodic operator on a separable Hilbert space $L^2(G)$. By Lemma 3.2, the operator $\lambda_2(\mu)$ has no unitary eigenvalues except $\xi = 1$. By Proposition 5.2, there exists a subsequence $\{n_j\}_{j=1}^{\infty}$ of $\mathbb{N}$ such that $\lambda_2(\mu)^{n_j} \to P_\mu$ $(j \to \infty)$ in the weak operator topology, where P_μ is the projection onto $\ker[\lambda_2(\mu) - I]$. On the other hand, by Proposition 4.3, $P_\mu f = \theta_\mu * f$, $f \in L^2(G)$, where θ_μ is the limit measure associated with μ. Now if $u, v \in L^2(G)$, then as $u * \widetilde{v} \in C_0(G)$, we can write

$$\begin{aligned}\lim_{j\to\infty} \langle \mu^{n_j}, u * \widetilde{v}\rangle &= \lim_{j\to\infty} \langle \mu^{n_j} * \overline{u}, \overline{v}\rangle \\ &= \lim_{j\to\infty} \langle \lambda_2(\mu)^{n_j}\, \overline{u}, \overline{v}\rangle = \langle P_\mu \overline{u}, \overline{v}\rangle \\ &= \langle \theta_\mu * \overline{u}, \overline{v}\rangle = \langle \theta_\mu, u * \widetilde{v}\rangle.\end{aligned}$$

Since the set $\{u * \widetilde{v} : u, v \in L^2(G)\}$ is linearly dense in $C_0(G)$, we have

$$\text{w}^*\text{-}\lim_{j\to\infty} \mu^{n_j} = \theta_\mu.$$

□

As we have noted above, $\|\lambda_1(\mu)\| = \|\mu\|_1$ for all $\mu \in M(G)$. Moreover, we have $\sigma(\lambda_1(\mu)) = \sigma(\mu)$ for all $\mu \in M(G)$, where $\sigma(\mu)$ is the spectrum of μ with respect to the algebra $M(G)$.

If G is a compact group, then the (normalized) Haar measure m_G is an idempotent measure on G with $supp m_G = G$. If H is a closed subgroup of G, then the measure m_H may be regarded as a measure on G by putting $\overline{m}_H(E) = m_H(E \cap H)$ for every Borel subset E of G. Notice that $supp \overline{m}_H = H$.

Theorem 5.4. (*a*) *Let μ be a power bounded measure on a locally compact group G. If $\sigma(\lambda_1(\mu)) \cap \mathbb{T} = \{1\}$, then*

$$w^* - \lim_{n\to\infty} \mu^n = \theta_\mu,$$

where θ_μ is the limit measure associated with μ.

(*b*) *Let μ be a probability measure on a compact group G. If $\sigma(\lambda_1(\mu)) \cap \mathbb{T} = \{1\}$, then*

$$w^* - \lim_{n\to\infty} \mu^n = \overline{m}_{[supp \mu]}.$$

Proof. (a) Let us first show that the sequence $\{\mu^n\}_{n\in\mathbb{N}}$ has only one weak* cluster point. Since $\sigma(\lambda_1(\mu)) \cap \mathbb{T} = \{1\}$, by the Katznelson-Tzafriri theorem,

$$\lim_{n\to\infty} \left\|\mu^{n+1} - \mu^n\right\|_1 = \lim_{n\to\infty} \left\|\lambda_1(\mu)^{n+1} - \lambda_1(\mu)^n\right\| = 0.$$

Assume that

$$\theta_1 = \text{w}^*\text{-}\lim_{\alpha} \mu^{n_\alpha} \text{ and } \theta_2 = \text{w}^*\text{-}\lim_{\beta} \mu^{m_\beta},$$

for two subnets $\{\mu^{n_\alpha}\}_\alpha$ and $\{\mu^{m_\beta}\}_\beta$ of $\{\mu^n\}_{n\in\mathbb{N}}$. Since the multiplication on $M(G)$ is separately w*-continuous, we have

$$\mu * \theta_1 = \theta_1 * \mu = \text{w}^*\text{-}\lim_\alpha \mu^{n_\alpha+1}.$$

Consequently,

$$\|\mu * \theta_1 - \theta_1\|_1 \leq \underline{\lim}_\alpha \left\|\mu^{n_\alpha+1} - \mu^{n_\alpha}\right\|_1 = 0.$$

Hence, $\mu * \theta_1 = \theta_1 * \mu = \theta_1$. Now, passing to the limit (in the w*-topology) in the identities

$$\mu^{m_\beta} * \theta_1 = \theta_1 * \mu^{m_\beta} = \theta_1,$$

we have $\theta_2 * \theta_1 = \theta_1 * \theta_2 = \theta_1$. Similarly, we can see that $\theta_2 * \theta_1 = \theta_1 * \theta_2 = \theta_2$. If $\theta := \theta_1 = \theta_2$, then $\theta^2 = \theta$. Thus we have

$$\text{w}^*\text{-}\lim_{n\to\infty} \mu^n = \theta.$$

By Proposition 4.1,

$$\text{w}^*\text{-}\lim_{n\to\infty} \mu^n = \theta_\mu,$$

where θ_μ is the limit measure associated with μ.

(b) Let $\pi \in \widehat{G}$ and let $\mathcal{H}_\pi$ be the representation space of π. Since G is a compact group, $\mathcal{H}_\pi$ is finite dimensional. Let $\dim \mathcal{H}_\pi := n_\pi$ and let $\left\{e_\pi^{(1)}, ..., e_\pi^{(n_\pi)}\right\}$ be the basic vectors in $\mathcal{H}_\pi$. Denote by $f_{i,j}^\pi$ the matrix functions of π, where

$$f_{i,j}^\pi(g) = \langle \pi(g) e_\pi^{(i)}, e_\pi^{(j)} \rangle \quad (i, j = 1, ..., n_\pi).$$

Notice that

$$\begin{aligned} \langle \mu^n, f_{i,j}^\pi \rangle &= \int_G \langle \pi(g) e_\pi^{(i)}, e_\pi^{(j)} \rangle d\mu^n \qquad (5.3) \\ &= \langle \widehat{\mu}(\pi)^n e_\pi^{(i)}, e_\pi^{(j)} \rangle, \forall n \in \mathbb{N}. \end{aligned}$$

As in the proof of (a),

$$\lim_{n\to\infty} \left\|\mu^{n+1} - \mu^n\right\|_1 = 0,$$

which implies

$$\left\|\widehat{\mu}(\pi)^{n+1} - \widehat{\mu}(\pi)^n\right\| \leq \left\|\mu^{n+1} - \mu^n\right\|_1 \to 0 \ (n \to \infty).$$

By Proposition 2.2,

$$\langle \widehat{\mu}(\pi)^n e_\pi^{(i)}, e_\pi^{(j)} \rangle \to \langle P_\mu^\pi e_\pi^{(i)}, e_\pi^{(j)} \rangle \ (n \to \infty),$$

where P_μ^π is an orthogonal projection onto $\ker[\widehat{\mu}(\pi) - I_\pi]$. In view of the identity (5.3), we have

$$\langle \mu^n, f_{i,j}^\pi \rangle \to \langle P_\mu^\pi e_\pi^{(i)}, e_\pi^{(j)} \rangle.$$

By the Peter-Weyl C-Theorem [17, Chapter 4], the system of matrix functions

$$\left\{f_{i,j}^\pi : \pi \in \widehat{G},\ i, j = 1, ..., n_\pi\right\}$$

is complete in $C(G)$. Consequently, the limit $\lim_{n\to\infty}\langle \mu^n, f\rangle$ exists for all $f \in C(G)$. Since

$$f \to \lim_{n\to\infty}\langle \mu^n, f\rangle$$

is a bounded linear functional on $C(G)$, there exists a measure $\vartheta_\mu \in M(G)$ such that

$$\lim_{n\to\infty}\langle \mu^n, f\rangle = \langle \vartheta_\mu, f\rangle,\ \forall f \in C(G).$$

So we have

$$\mathrm{w}^*\text{-}\lim_{n\to\infty} \mu^n = \vartheta_\mu.$$

By Proposition 4.1, ϑ_μ is the limit measure associated with μ. Therefore, ϑ_μ is an idempotent measure. Now let $H := [supp\mu]$. We must show that $\vartheta_\mu = \overline{m}_H$. Notice that

$$\widehat{\vartheta_\mu}(\pi) = P^\pi_\mu, \forall \pi \in \widehat{G}.$$

Further, since $\widehat{m_H}(\pi)$ is an orthogonal projection, by Lemma 3.2,

$$\begin{aligned}\widehat{\overline{m}_H}(\pi)\mathcal{H}_\pi &= \ker\left[\widehat{m_H}(\pi) - I_\pi\right]\\ &= \{x \in \mathcal{H}_\pi : \pi(g)x = x,\ \forall g \in H\}.\end{aligned}$$

For the same reasons,

$$\begin{aligned}\widehat{\vartheta_\mu}(\pi)\mathcal{H}_\pi &= P^\pi_\mu\mathcal{H}_\pi = \ker\left[\widehat{\mu}(\pi) - I_\pi\right]\\ &= \{x \in \mathcal{H}_\pi : \pi(g)x = x,\ \forall g \in H\}.\end{aligned}$$

Thus we have $\widehat{\vartheta_\mu}(\pi) = \widehat{\overline{m}_H}(\pi)$ for all $\pi \in \widehat{G}$. It follows that $\vartheta_\mu = \overline{m}_H$. □

Remark 5.5. If G is a locally compact amenable group, then for an arbitrary probability measure on G, $1 \in \sigma\left(\lambda_p(\mu)\right)$ $(1 \le p < \infty)$ [10, Theorem 3.2.2]. Recall also that compact groups are amenable.

Remark 5.6. Let G be a locally compact abelian group and let $M_{reg}(G)$ be the greatest regular subalgebra of $M(G)$ [16, Theorem 4.3.6]. The algebra $L^1(G)$ and the discrete measure algebra $M_d(G)$ are regular subalgebras of $M(G)$ and therefore, $L^1(G) + M_d(G) \subseteq M_{reg}(G)$ (in general, $L^1(G) + M_d(G) \neq M_{reg}(G)$ [16, Example 4.3.11]). This shows that the algebra $M_{reg}(G)$ is remarkably large. For every $\mu \in M_{reg}(G)$, we have

$$\sigma(\lambda_1(\mu)) = \overline{\left\{\widehat{\mu}(\chi) : \chi \in \widehat{G}\right\}}$$

[16, Chapter 4]. It follows that if μ is a probability measure on G, then $\sigma(\lambda_1(\mu)) \cap \mathbb{T} = \{1\}$ if and only if for an arbitrary neighborhood U of 1, $\sup_{\chi\in U}|\widehat{\mu}(\chi)| < 1$.

As a consequence of Theorem 5.4, we have the following.

Proposition 5.7. *(a) Let μ be a power bounded measure on a locally compact group G. If $1 \in \sigma(\lambda_1(\mu))$, then*

$$w^* - \lim_{n\to\infty}\left(\frac{\delta_e + \mu}{2}\right)^n = \theta_{\frac{\delta_e+\mu}{2}},$$

where $\theta_{\frac{\delta_e+\mu}{2}}$ *is the limit measure associated with* $\frac{\delta_e+\mu}{2}$. *If* $1 \notin \sigma(\lambda_1(\mu))$, *then*

$$\lim_{n\to\infty} \left\| \left(\frac{\delta_e+\mu}{2} \right)^n \right\|_1 = 0.$$

(b) *If* μ *is an adapted measure on a compact group* G, *then*

$$w^* - \lim_{n\to\infty} \left(\frac{\delta_e+\mu}{2} \right)^n = m_G.$$

Proof. (a) Notice that the measure $\nu := \frac{\delta_e+\mu}{2}$ is power bounded, that is,

$$\sup_{n\in\mathbb{N}_0} \|\nu^n\|_1 \le C_\mu.$$

Consequently, the operator $\lambda_1(\nu)$ is power bounded and therefore, $\sigma(\lambda_1(\nu)) \subseteq \overline{\mathbb{D}}$. Notice also that if

$$h(z) := \frac{1+z}{2} \quad (z \in \mathbb{C}),$$

then $h(1) = 1$ and $|h(z)| < 1$ for all $z \in \overline{\mathbb{D}} \diagdown \{1\}$. Since $\lambda_1(\nu) = h(\lambda_1(\mu))$, by the spectral mapping theorem, $\sigma(\lambda_1(\nu)) \cap \mathbb{T} \subseteq \{1\}$. If $1 \in \sigma(\lambda_1(\mu))$, then $\sigma(\lambda_1(\nu)) \cap \mathbb{T} = \{1\}$ and by Theorem 5.4 (a),

$$\text{w}^*\text{-} \lim_{n\to\infty} \nu^n = \theta_\nu.$$

If $1 \notin \sigma(\lambda_1(\mu))$, then $\sigma(\lambda_1(\nu)) \cap \mathbb{T} = \emptyset$ and therefore, $\sigma(\lambda_1(\nu)) \subset \mathbb{D}$. It follows that $\left\|\lambda_1(\nu)^n\right\| = \|\nu^n\|_1 \to 0$ as $n \to \infty$.

(b) If μ is adapted, then as $supp\mu \subseteq supp\nu$, we have $[supp\nu] = G$. By Theorem 5.4 (b),

$$\text{w}^*\text{-} \lim_{n\to\infty} \nu^n = m_G.$$

□

Remark 5.8. If μ is a probability measure on a compact group G, then $\frac{\delta_e+\mu}{2}$ is a strictly aperiodic measure. Therefore, Proposition 5.6 (b) can be obtained from the Kawada-Itô theorem [14, Theorem 8].

We will need the following result.

Proposition 5.9. *Let* G *be a compact group and let* $\{\mu_n\}_{n\in\mathbb{N}}$ *be a norm bounded sequence in* $M(G)$. *The following conditions are equivalent:*

(a) $w^*\text{-}\lim_{n\to\infty} \mu_n = \mu$ *for some* $\mu \in M(G)$.

(b) $\lim_{n\to\infty} \mu_n * f = \mu * f$ *uniformly on* G *for every* $f \in C(G)$.

Proof. (a)⇒(b) Let $\mathcal{H}_\pi$ be the representation space of $\pi \in \widehat{G}$ and let

$$f^\pi_{x,y}(g) := \langle \pi(g)x, y\rangle \quad (x, y \in \mathcal{H}_\pi)$$

be the matrix functions of π. Notice that

$$\langle \theta,\ f^\pi_{x,y}\rangle = \langle \widehat{\theta}(\pi)x,\ y\rangle$$

and

$$\left(\theta * f^{\pi}_{x,y}\right)(g) = \langle \pi(g)x, \widehat{\theta}(\pi)y\rangle, \ \forall \theta \in M(G).$$

Consequently, we have

$$\langle \widehat{\mu_n}(\pi)x, y\rangle = \langle \mu_n, f^{\pi}_{x,y}\rangle \to \langle \mu, f^{\pi}_{x,y}\rangle = \langle \widehat{\mu}(\pi)x, y\rangle.$$

Since $\mathcal{H}_\pi$ is finite dimensional, $\widehat{\mu_n}(\pi) \to \widehat{\mu}(\pi)$ in the strong operator topology. Now let $f \in C(G)$ be given. Since the system of matrix functions is linearly dense in $C(G)$, for any $\varepsilon > 0$ there exist complex numbers $\lambda_1, ..., \lambda_k$ and $\pi_1, ..., \pi_k \in \widehat{G}$ such that

$$\left|f(g) - \lambda_1\langle \pi_1(g)x_1, y_1\rangle - ... - \lambda_k\langle \pi_k(g)x_k, y_k\rangle\right| < \varepsilon \ \ (\forall g \in G),$$

where $x_i, y_i \in \mathcal{H}_{\pi_i}$ $(i = 1, ..., k)$. It follows that

$$\left|(\mu_n * f)(g) - \lambda_1\langle \pi_1(g)x_1, \widehat{\mu_n}(\pi_1)y_1\rangle - ... - \lambda_k\langle \pi_k(g)x_k, \widehat{\mu_n}(\pi_k)y_k\rangle\right| < \varepsilon C$$

and

$$\left|(\mu * f)(g) - \lambda_1\langle \pi_1(g)x_1, \widehat{\mu}(\pi_1)y_1\rangle - ... - \lambda_k\langle \pi_k(g)x_k, \widehat{\mu}(\pi_k)y_k\rangle\right| < \varepsilon C,$$

where $C := \sup_{n\in\mathbb{N}} \|\mu_n\|$. So we have

$$\begin{aligned}\sup_{g\in G}\left|(\mu_n * f)(g) - (\mu * f)(g)\right| \ \leq \ & |\lambda_1| \, \|\widehat{\mu_n}(\pi_1)y_1 - \widehat{\mu}(\pi_1)y_1\| \, \|x_1\| + ... \\ & + |\lambda_k| \, \|\widehat{\mu_n}(\pi_k)y_k - \widehat{\mu}(\pi_k)y_k\| \, \|x_k\| + 2\varepsilon C.\end{aligned}$$

Since $\widehat{\mu_n}(\pi)x \to \widehat{\mu}(\pi)x$ in norm for all $\pi \in \widehat{G}$ and $x \in \mathcal{H}_\pi$, we have that $\mu_n * f \to \mu * f$ uniformly on G.

(b)⇒(a) For any $f \in C(G)$,

$$\int_G f\,d\mu_n - \int_G f\,d\mu = (\mu_n * f)(e) - (\mu * f)(e) \to 0.$$

□

Next, we have the following.

Corollary 5.10. (*a*) *Let μ be a power bounded measure on a locally compact group G. If $1 \in \sigma\left(\lambda_p(\mu)\right)$, then for every $f \in L^p(G)$ $(1 < p < \infty)$,*

$$\left(\frac{\delta_e + \mu}{2}\right)^n * f \to \theta_{\frac{\delta_e+\mu}{2}} * f \ \ \textit{in } L^p\textit{-norm.}$$

(*b*) *Let μ be a probability measure on a locally compact group G and assume that $[supp\mu]$ is compact. If $1 \in \sigma(\lambda_1(\mu))$, then for every $f \in L^1(G)$,*

$$\left(\frac{\delta_e + \mu}{2}\right)^n * f \to \theta_{\frac{\delta_e+\mu}{2}} * f \ \ \textit{in } L^1\textit{-norm.}$$

(*c*) *If μ is an adapted measure on a compact group G, then for every $f \in C(G)$,*

$$\left(\frac{\delta_e + \mu}{2}\right)^n * f \to \left(\int_G f\,dm_G\right)\mathbf{1} \ \ \textit{uniformly on } G,$$

where $\mathbf{1}$ is the identity one function on G.

Proof. (a) As in the proof of Proposition 5.7, the measure $\nu := \frac{\delta_e+\mu}{2}$ is power bounded and $\sigma\left(\lambda_p(\nu)\right) \cap \mathbb{T} = \{1\}$. By the Katznelson-Tzafriri theorem,

$$\lim_{n\to\infty} \left\| \lambda_p(\nu)^{n+1} - \lambda_p(\nu)^n \right\| = 0.$$

Since the operator $\lambda_p(\nu)$ is mean ergodic, by Proposition 2.3,

$$\lambda_p(\nu)^n f \to P_\nu f \text{ in } L^p\text{-norm, for every } f \in L^p(G),$$

where P_ν is the projection associated with the operator $\lambda_p(\nu)$. By Proposition 4.3, $P_\nu f = \theta_\nu * f$. Hence, $\nu^n * f \to \theta_\nu * f$ in L^p-norm.

(b) Since $\sigma(\lambda_1(\nu)) \cap \mathbb{T} = \{1\}$, by the Katznelson-Tzafriri theorem,

$$\lim_{n\to\infty} \left\| \lambda_1(\nu)^{n+1} - \lambda_1(\nu)^n \right\| = 0.$$

Since the operator $\lambda_1(\nu)$ is mean ergodic [9, Theorem 5.4], by Proposition 2.3,

$$\lambda_1(\nu)^n f \to P_\nu f \text{ in } L^1\text{-norm, for every } f \in L^1(G).$$

By Propositions 3.2, $P_\nu f = \theta_\nu * f$. Hence, $\nu^n * f \to \theta_\nu * f$ in L^1-norm.

(c) follows from Propositions 5.7 (b) and 5.9 (b). □

Recall from [7, Ch.IV, Proposition 2.6] that a mean ergodic operator T on a Banach space X is said to be *weakly mixing* if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left| \langle \varphi, T^i x \rangle - \langle \varphi, P_T x \rangle \right| = 0 \text{ for all } x \in X \text{ and } \varphi \in X^*,$$

where P_T is the mean ergodic projection associated with T.

Proposition 5.11. *Let T be a power bounded operator on a reflexive Banach space X and assume that T has no unitary eigenvalues except $\xi = 1$. Then T is weakly mixing.*

Proof. Notice that T is a mean ergodic operator. Since T is weakly almost periodic, there exist two T-invariant subspaces X_r and X_s such that $X = X_r \oplus X_s$, where $Tx_r = x_r$ for all $x_r \in X_r$ and $S := T\,|_{X_s}$ has no unitary eigenvalues (see the proof of Proposition 5.2). On the other hand, it follows from the JGdL Decomposition Theorem that S has no unitary eigenvalues if and only if $0 \in \overline{\{T^n x_s : n \in \mathbb{N}_0\}}^{\text{weak}}$ for all $x_s \in X_s$. By [7, Ch.II, Theorem 4.1], this is equivalent to the fact that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left| \langle \varphi, S^i x_s \rangle \right| = 0 \text{ for all } \varphi \in X^* \text{ and } x_s \in X_s.$$

If $x \in X$, then as $x = x_r + x_s$, $T^i x = x_r + S^i x_s$, and $P_T x = x_r$, we get that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left| \langle \varphi, T^i x \rangle - \langle \varphi, P_T x \rangle \right| = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left| \langle \varphi, S^i x_s \rangle \right| = 0 \ \ (\forall \varphi \in X^*).$$

□

If μ is a strictly aperiodic measure on a locally compact group G, then by Lemma 3.2, the operator $\lambda_p(\mu)$ $(1 < p < \infty)$ has no unitary eigenvalues except $\xi = 1$.

Corollary 5.12. *If μ is a strictly aperiodic measure on a locally compact group G, then the operator $\lambda_p(\mu)$ $(1 < p < \infty)$ is weakly mixing.*

6. Weak convergence

Let G be a locally compact group and let $C_b(G)$ be the space of all complex valued bounded continuous functions on G. A sequence $\{\mu_n\}_{n\in\mathbb{N}}$ in $M(G)$ *weak converges* to $\mu \in M(G)$, denoted by w-$\lim_{n\to\infty} \mu_n = \mu$ if

$$\lim_{n\to\infty} \int_G f d\mu_n = \int_G f d\mu, \ \ \forall f \in C_b(G).$$

Clearly, w-$\lim_{n\to\infty} \mu_n = \mu$ implies w*-$\lim_{n\to\infty} \mu_n = \mu$.

Recall that a subset $\mathcal{M}$ of $M(G)$ is called *uniformly tight* if for each $\varepsilon > 0$, there is a compact subset K_ε of G such that $|\mu|(G\diagdown K_\varepsilon) < \varepsilon$ for all $\mu \in \mathcal{M}$.

The following result probably is known. Since we couldn't find a suitable reference, we include its proof.

Lemma 6.1. *Let $\mathcal{M}$ be a uniformly tight subset of $M(G)$ and let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}$. If w^*-$\lim_{n\to\infty} \mu_n = \mu$ for some $\mu \in M(G)$, then w-$\lim_{n\to\infty} \mu_n = \mu$.*

Proof. For an arbitrary $\varepsilon > 0$, there is a compact subset K_ε of G such that $|\mu|(G\diagdown K_\varepsilon) < \varepsilon$ and $|\mu_n|(G\diagdown K_\varepsilon) < \varepsilon$ for all $n \in \mathbb{N}$. If $\nu_n := \mu_n - \mu$, then

$$|\nu_n|(G\diagdown K_\varepsilon) < 2\varepsilon, \forall n \in \mathbb{N}.$$

Let U_ε be a neighborhood of K_ε such that $\overline{U_\varepsilon}$ is compact. By the Urysohn lemma, there exists a continuous function h_ε on G such that $h_\varepsilon = 1$ on K_ε, $h_\varepsilon = 0$ on $G\diagdown U_\varepsilon$, and $0 \le h_\varepsilon \le 1$. Now let $f \in C_b(G)$ be given. If $f_\varepsilon := h_\varepsilon f$, then $f_\varepsilon \in C_0(G)$, $\|f_\varepsilon\|_\infty \le \|f\|_\infty$, and $f = f_\varepsilon$ on K_ε. From the identity

$$\int_G f d\nu_n = \int_{G\diagdown K_\varepsilon} (f - f_\varepsilon) d\nu_n + \int_{K_\varepsilon} (f - f_\varepsilon) d\nu_n + \int_G f_\varepsilon d\nu_n,$$

we get

$$\begin{aligned} \left| \int_G f d\mu_n - \int_G f d\mu \right| &= \left| \int_G f d\nu_n \right| \le 2\|f\|_\infty |\nu_n|(G\diagdown K_\varepsilon) + \left| \int_G f_\varepsilon d\nu_n \right| \\ &\le 4\|f\|_\infty \varepsilon + \left| \int_G f_\varepsilon d\mu_n - \int_G f_\varepsilon d\mu \right|. \end{aligned}$$

Since

$$\int_G f_\varepsilon d\mu_n \to \int_G f_\varepsilon d\mu,$$

we have

$$\int_G f\,d\mu_n \to \int_G f\,d\mu.$$

□

For the sake of convenience, we will call $\mu \in M(G)$ *weakly compact measure* if the sequence $\{\mu^n\}_{n\in\mathbb{N}}$ is relatively compact in the $\sigma(M(G), C_b(G))$ topology. Clearly, weakly compact measure is power bounded.

Proposition 6.2. *Let π be a representation of a second countable locally compact group G on a Banach space X_π. If $\mu \in M(G)$ is a weakly compact measure, then the following assertions hold:*

(a) The operator $\widehat{\mu}(\pi)$ is mean ergodic, that is,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \widehat{\mu}(\pi)^i x = \widehat{\theta_\mu}(\pi)\, x \text{ in norm for all } x \in X_\pi.$$

(b) If $\sigma(\lambda_1(\mu)) \cap \mathbb{T} = \{1\}$, then

$$\lim_{n\to\infty} \widehat{\mu}(\pi)^n x = \widehat{\theta_\mu}(\pi)\, x \text{ in norm for all } x \in X_\pi,$$

where θ_μ is the limit measure associated with μ.

Proof. (a) By the Prokhorov theorem [2, Theorem 8.6.2], the set $\{\mu^n : n \in \mathbb{N}\}$ is uniformly tight. It follows that the set

$$\left\{\frac{1}{n}\sum_{i=1}^{n} \mu^i : n \in \mathbb{N}\right\}$$

is also uniformly tight. Since μ is power bounded, by Proposition 4.1,

$$\mathrm{w}^* - \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mu^i = \theta_\mu.$$

In view of Lemma 6.1,

$$\mathrm{w} - \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mu^i = \theta_\mu.$$

Let an arbitrary $x \in X_\pi$ and $\varphi \in X_\pi^*$ be given. Since $g \to \varphi(\pi(g)x)$ is a bounded continuous function on G, we can write

$$\begin{aligned}\lim_{n\to\infty} \langle \varphi, \frac{1}{n}\sum_{i=1}^{n} \widehat{\mu}(\pi)^i x\rangle &= \lim_{n\to\infty} \langle \frac{1}{n}\sum_{i=1}^{n} \mu^i,\ \varphi(\pi(g)x)\rangle \\ &= \langle \theta_\mu, \varphi(\pi(g)x)\rangle = \langle \varphi, \widehat{\theta_\mu}(\pi)\, x\rangle.\end{aligned}$$

This shows that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \widehat{\mu}(\pi)^i x = \widehat{\theta_\mu}(\pi)\, x \text{ weakly.}$$

By Proposition 2.1,

$$\frac{1}{n}\sum_{i=1}^{n}\widehat{\mu}(\pi)^{i}x \to \widehat{\theta_{\mu}}(\pi)x \text{ in norm for all } x \in X.$$

(b) By (a), $\widehat{\mu}(\pi)$ is a mean ergodic operator and $\widehat{\theta_{\mu}}(\pi)$ is the mean ergodic projection associated with $\widehat{\mu}(\pi)$. Since

$$\left\|\widehat{\mu}(\pi)^{n+1} - \widehat{\mu}(\pi)^{n}\right\| \leq \left\|\mu^{n+1} - \mu^{n}\right\|_{1} \to 0 \ (n \to \infty),$$

by Proposition 2.3,

$$\lim_{n\to\infty}\widehat{\mu}(\pi)^{n}x = \widehat{\theta_{\mu}}(\pi)x \text{ in norm for all } x \in X_{\pi}.$$

□

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References

[1] Billingsley, Patrick. Convergence of probability measures. Second edition. Wiley Series in Probability and Statistics. *John Wiley & Sons, Inc., New York*, 1999. x+277 pp. ISBN: 0-471-19745-9. MR1700749, Zbl 0944.60003, doi: 10.1002/9780470316962. 1057

[2] Bogachev, Vladimir I. Measure theory. I, II. *Springer-Verlag, Berlin*, 2007. Vol. I: xviii+500 pp., Vol. II: xiv+575 pp. ISBN:978-3-540-34513-8, ISBN:3-540-34513-2. MR2267655, Zbl 1120.28001, doi: 10.1007/978-3-540-34514-5. 1057, 1076

[3] Conway, John B. A course in functional analysis. Graduate Texts in Mathematics, 96. *Springer-Verlag, New York*, 1985. xiv+404 pp. ISBN: 0-387-96042-2. MR0768926, Zbl 0558.46001, doi: 10.1007/978-1-4757-4383-8. 1060

[4] Derriennic, Yves. Lois "zéro ou deux" pour les processus de Markov. Applications aux marches aléatoires. *Ann. Inst. H. Poincaré, Sect. B (N.S.)* **12** (1976), no. 2, 111–129. MR0423532, Zbl 0353.60075. 1057

[5] Derriennic, Yves; Lin, Michael. Convergence of iterates of averages of certain operator representations and of convolution powers. *J. Funct. Anal.* **85** (1989), no. 1, 86–102. MR1005857, Zbl 00712.22008, doi: 10.1016/0022-1236(89)90047-5. 1061

[6] Dixmier, Jacques. Les C^*-algèbres et leurs représentations. *Gauthier-Villars and Cie, Éditeur-Imprimeur, Paris*, 1964. xi+382 pp. MR0171173, Zbl 0152.32902. 1060

[7] Eisner, Tanja. Stability of operators and operator semigroups. Operator Theory Advances and Applications, 209. *Birkhäuser Verlag, Basel*, 2010. viii+204 pp. ISBN: 978-3-0346-0194-8. MR2681062, Zbl 1205.47002, doi: 10.1007/978-3-0346-0195-5. 1062, 1068, 1074

[8] Eisner, Tanja; Farkas, Bálint; Haase, Markus; Nagel, Rainer. Operator theoretic aspects of ergodic theory. Graduate Texts in Mathematics, 272. *Springer, Cham*, 2015. xviii+628 pp. ISBN: 978-3-0346-0194-8. MR3410920, Zbl 1353.37002, doi: 10.1007/978-3-319-16898-2. 1058, 1059

[9] Galindo, Jorge; Jordá, Enrique. Ergodic properties of convolution operators. *J. Operator Theory* **86** (2021), no. 2, 469–501. MR4373146, Zbl 1524.43001, arXiv:2004.07622, doi: 10.7900/jot.2020jun25.2303. 1057, 1064, 1065, 1066, 1074

[10] Greenleaf, Frederick P. Invariant means on topological groups and their applications. Van Nostrand Mathematical Studies, No. 16. *Van Nostrand Reinhold Co., New York-Toronto-London*, 1969. ix+113 pp. MR0251549, Zbl 0252.43005. 1071

[11] GRENANDER, ULF. Probabilities on algebraic structures. Second edition. *Almqvist and Wiksell, Stockholm; John Wiley and Sons, Inc., New York-London*, 1968. 218 pp. MR0259969, Zbl 0139.33401. 1057, 1063

[12] JAMISON, BENTON. Eigenvalues of modulus 1. *Proc. Amer. Math. Soc.* **16** (1965), no. 3, 375–377. MR0176332, Zbl 0133.07202, doi: 10.2307/2034656. 1066

[13] KATZNELSON, YITZHAK; TZAFRIRI, LIOR. On power bounded operators. *J. Funct. Anal.* **68** (1986), no.3, 313–328. MR0859138, Zbl 0611.47005, doi: 10.1016/0022-1236(86)90101-1. 1059

[14] KAWADA, YUKIYOSI; ITÔ, KIYOSI. On the probability distribution on a compact group. I. *Proc. Phys.-Math. Soc. Japan* **22** (1940), 977–998. MR0003462, Zbl 0026.13801. 1057, 1072

[15] KRENGEL, ULRICH. Ergodic theorems. Gruyter Studies in Mathematics, 6. *Walter de Gruyter and Co., Berlin*, 1985. viii+357 pp. ISBN:3-11-008478-3. MR0797411, Zbl 0575.28009, doi: 10.1515/9783110844641. 1058

[16] LAURSEN, KJELD B.; NEUMANN, MICHAEL M. An introduction to local spectral theory. London Mathematical Society Monographs. New Series, 20. *The Clarendon Press, Oxford University Press, New York*, 2000. xii+591 pp. ISBN:0-19-852381-5. MR1747914, Zbl 0957.47004. 1071

[17] LYUBICH, YURII I. Introduction to the theory of Banach representations of groups. Operator Theory: Advances and Applications, 30. *Birkhäuser Verlag, Basel*, 1988. x+223 pp. ISBN:3-7643-2207-1. MR1015717, Zbl 0635.22001, doi: 10.1007/978-3-0348-9169-1. 1060, 1070

[18] MUKHERJEA, ARUNAVA. Limit theorems for probability measures on non-compact groups and semi-groups. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **33** (1976), no. 4, 273–284. MR0400334, Zbl 0304.60004, doi: 10.1007/BF00534779. 1057

[19] MUSTAFAYEV, HEYBETKULU. Mean ergodic theorems for multipliers on Banach algebras. *J. Fourier Anal. Appl.* **25** (2019), no. 2, 393–426. MR3917951, Zbl 1429.46034, doi: 10.1007/s00041-017-9587-x. 1057, 1064

[20] MUSTAFAYEV, HEYBETKULU. A note on the Kawada-Itô theorem. *Statist. Probab. Lett.* **181** (2022), Paper No. 109261, 6 pp. MR4330970, Zbl 1481.60010, doi: 10.1016/j.spl.2021.109261. 1057, 1061, 1064

[21] NEUFANG, MATTHIAS; SALMI, PEKKA; SKALSKI, ADAM; SPRONK, NICO. Fixed points and limits of convolution powers of contractive quantum measures. *Indiana Univ. Math. J.* **70** (2021), no. 5, 1971–2009. MR4340487, Zbl 1492.46066, arXiv:1907.07337, doi: 10.1512/iumj.2021.70.8692. 1057, 1064

[22] SCHREIBER, BERTRAM M. Measures with bounded convolution powers. *Trans. Amer. Math. Soc.* **151** (1970), 405–431. MR0264335, Zbl 0202.14202, doi: 10.1090/S0002-9947-1970-0264335-3. 1063

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