

On reducibility of induced representations of odd unitary groups: the depth zero case

Subha Sandeep Repaka

ABSTRACT. We study a problem concerning parabolic induction in certain p -adic unitary groups. More precisely, for E/F a quadratic extension of p -adic fields the associated unitary group $G = \mathrm{U}(n, n + 1)$ contains a parabolic subgroup P with Levi component L isomorphic to $\mathrm{GL}_n(E) \times \mathrm{U}_1(E)$. Let π be an irreducible supercuspidal representation of L of depth zero. We use Hecke algebra methods to determine when the parabolically induced representation $t_p^G \pi$ is reducible.

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1. Introduction

Let $G = \mathrm{U}(n, n + 1)$ be the odd unitary group over non-Archimedean local field E and π is an irreducible supercuspidal depth zero representation of the Siegel Levi component $L \cong \mathrm{GL}_n(E) \times \mathrm{U}_1(E)$ of the Siegel parabolic subgroup P of G . The terms $P, L, \pi, \mathrm{U}(n, n + 1)$ are described in much detail later in the paper. We use Hecke algebra methods to determine when the parabolically induced representation $t_p^G \pi$ is reducible. Harish-Chandra tells us to look not at an individual $t_p^G \pi$ but at the family $t_p^G(\pi\nu)$ as ν varies through the unramified characters of $L \cong \mathrm{GL}_n(E) \times \mathrm{U}_1(E)$. The unramified characters of L and the functor t_p^G are also described in greater detail later in the paper.

Before going any further, let us describe how the group $\mathrm{U}(n, n + 1)$ over non-Archimedean local fields looks like. Let E/F be a quadratic Galois extension of non-Archimedean local fields where $\mathrm{char} F \neq 2$. Write $-$ for the non-trivial element of $\mathrm{Gal}(E/F)$. The group $G = \mathrm{U}(n, n + 1)$ is given by

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$$U(n, n + 1) = \{g \in GL_{2n+1}(E) \mid {}^t\bar{g}Jg = J\}$$

for

$$J = \begin{bmatrix} 0 & 0 & Id_n \\ 0 & 1 & 0 \\ Id_n & 0 & 0 \end{bmatrix},$$

where each block is of size n and for $g = (g_{ij})$ we write $\bar{g} = (\bar{g}_{ij})$. We also write \mathfrak{O}_E and \mathfrak{O}_F for the ring of integers in E and F respectively. Similarly, \mathfrak{p}_E and \mathfrak{p}_F denote the maximal ideals in \mathfrak{O}_E and \mathfrak{O}_F and $k_E = \mathfrak{O}_E/\mathfrak{p}_E$ and $k_F = \mathfrak{O}_F/\mathfrak{p}_F$ denote the residue class fields of \mathfrak{O}_E and \mathfrak{O}_F . Let $|k_F| = q = p^r$ for some odd prime p and some integer $r \geq 1$.

There are two kinds of extensions of E over F . One is the unramified extension and the other one is the ramified extension. In the unramified case, we can choose uniformizers ϖ_E, ϖ_F in E, F such that $\varpi_E = \varpi_F$ so that we have $[k_E : k_F] = 2, \text{Gal}(k_E/k_F) \cong \text{Gal}(E/F)$. As $\varpi_E = \varpi_F$, so $\bar{\varpi}_E = \varpi_E$ since $\varpi_F \in F$. As $k_F = \mathbb{F}_q$, so $k_E = \mathbb{F}_{q^2}$ in this case. In the ramified case, we can choose uniformizers ϖ_E, ϖ_F in E, F such that $\varpi_E^2 = \varpi_F$ so that we have $[k_E : k_F] = 1, \text{Gal}(k_E/k_F) = 1$. As $\varpi_E^2 = \varpi_F$, we can further choose ϖ_E such that $\bar{\varpi}_E = -\varpi_E$. As $k_F = \mathbb{F}_q$, so $k_E = \mathbb{F}_q$ in this case.

We write P for the Siegel parabolic subgroup of G . Write L for the Siegel Levi component of P and U for the unipotent radical of P . Thus $P = L \ltimes U$ with

$$L = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & {}^t\bar{a}^{-1} \end{bmatrix} \mid a \in GL_n(E), \lambda \in E^\times, \lambda\bar{\lambda} = 1 \right\}$$

and

$$U = \left\{ \begin{bmatrix} Id_n & u & X \\ 0 & 1 & -{}^t\bar{u} \\ 0 & 0 & Id_n \end{bmatrix} \mid X \in M_n(E), u \in M_{n \times 1}(E), X + {}^t\bar{X} + u{}^t\bar{u} = 0 \right\}.$$

Note that $L \cong GL_n(E) \times U_1(E)$ and $U_1(E) \cong U_1(\mathfrak{O}_E)$. Let $\bar{P} = L \ltimes \bar{U}$ be the L -opposite of P where

$$\bar{U} = \left\{ \begin{bmatrix} Id_n & 0 & 0 \\ -{}^t\bar{u} & 1 & 0 \\ X & u & Id_n \end{bmatrix} \mid X \in M_n(E), u \in M_{n \times 1}(E), X + {}^t\bar{X} + u{}^t\bar{u} = 0 \right\}.$$

Let $K_0 = GL_n(\mathfrak{O}_E)$ and $K_1 = Id_n + \varpi_E M_n(\mathfrak{O}_E)$. Note $K_1 = Id_n + \varpi_E M_n(\mathfrak{O}_E)$ is the kernel of the surjective group homomorphism

$$(g_{ij}) \longrightarrow (g_{ij} + \mathfrak{p}_E) : GL_n(\mathfrak{O}_E) \longrightarrow GL_n(k_E)$$

As π is a depth zero representation of $L \cong GL_n(E) \times U_1(E)$, so $\pi = \lambda\chi$ where λ is a depth zero representation of $GL_n(E)$ and χ is a depth zero character of $U_1(E)$. We say π is a depth zero representation of the Siegel Levi component L of P if $\lambda^{K_1} \neq 0$ and $\chi|_{U_1(1+\mathfrak{p}_E)} = 1$.

Let (ρ, V) be a smooth representation of the group H which is a subgroup of K . The smoothly induced representation from H to K is denoted by $Ind_H^K(\rho, V)$ or $Ind_H^K(\rho)$. Let us denote $c\text{-}Ind_H^K(\rho, V)$ or $c\text{-}Ind_P^G(\rho)$ for smoothly induced compact induced representation from H to K .

The normalized induced representation from P to G is denoted by $i_P^G(\rho, V)$ or $i_P^G(\rho)$ where $i_P^G(\rho) = Ind_P^G(\rho \otimes \delta_P^{1/2})$, δ_P is a character of P defined as $\delta_P(p) = \|\det(\text{Ad } p)|_{\text{Lie } U}\|_F$ for $p \in P$ and $\text{Lie } U$ is the Lie-algebra of U . In this paper, we work with normalized induced representations rather than induced representations as results look more appealing (for example, such representations commute with taking duals).

Write L° for the smallest subgroup of L containing the compact open subgroups of L . We say a character $\nu : L \rightarrow \mathbb{C}^\times$ is unramified if $\nu|_{L^\circ} = 1$. Observe that if ν is an unramified character of L then $\nu = \nu' \beta$ where ν' is an unramified character of $\text{GL}_n(E)$ and β is an unramified character of $\text{U}_1(E)$. But as $\text{U}_1(E) = \text{U}_1(\mathfrak{O}_E)$, so β is trivial. Hence, ν can be viewed as an unramified character of $\text{GL}_n(E)$. Let the group of unramified characters of L be denoted by $X_{nr}(L)$.

1.1. Question. The question we answer in this paper is, given π an irreducible supercuspidal representation of L of depth zero, we look at the family of representations $i_P^G(\pi\nu)$ for $\nu \in X_{nr}(L)$ and we want to determine the set of such ν for which this induced representation is reducible for both ramified and unramified extensions. By general theory, this is a finite set.

Recall that $\pi = \lambda\chi$ where λ is an irreducible supercuspidal depth zero representation of $\text{GL}_n(E)$ and χ is a supercuspidal depthzero character of $\text{U}_1(E)$. Now $\lambda|_{K_0}$ contains an irreducible representation τ of K_0 such that $\tau|_{K_1}$ is trivial. So τ can be viewed as an irreducible representation of $K_0/K_1 \cong \text{GL}_n(k_E)$ inflated to $K_0 = \text{GL}_n(\mathfrak{O}_E)$. The representation τ is cuspidal by (a very special case of) A.1 Appendix [ML93]. Set $\rho_0 = \tau\chi$ which is a cuspidal representation of $K_0 \times \text{U}_1(\mathfrak{O}_E)$. Further, we can view $\rho_0 = \tau\chi$ as a cuspidal representation of $\text{GL}_n(k_E) \times \text{U}_1(k_E)$ inflated to $K_0 \times \text{U}_1(\mathfrak{O}_E)$.

By the work of Green [GJA55] or as a very special case of the Deligne-Lusztig construction, irreducible cuspidal representations of $\text{GL}_n(k_E)$ are parametrized by the regular characters of degree n extensions of k_E . We write τ_θ for the irreducible cuspidal representation τ that corresponds to a regular character θ . Let l/k_E be a field extension of degree n . We set $\Gamma = \text{Gal}(l/k_E)$.

Let

$$(l^\times)^\vee = \text{Hom}(l^\times, \mathbb{C}^\times).$$

Clearly, Γ acts on $(l^\times)^\vee$ via

$$\theta^\gamma(x) = \theta(\gamma x), \quad \theta \in (l^\times)^\vee, \quad \gamma \in \Gamma, \quad x \in l^\times.$$

We write $(l^\times)_{\text{reg}}^\vee$ for the group of regular characters of l^\times with respect to this action, that is, characters θ such that $\text{Stab}_\Gamma(\theta) = \{1\}$. We also write l_{reg}^\times for the regular elements in l^\times , that is, elements x such that $\text{Stab}_\Gamma(x) = \{1\}$. The set of

Γ -orbits on $(l^\times)_{\text{reg}}^\vee$ is then in canonical bijection with the set $\text{Irr}_{\text{cusp}} \text{GL}_n(k_E)$ of equivalence classes of irreducible cuspidal representations of $\text{GL}_n(k_E)$:

$$\begin{aligned} \Gamma \backslash (l^\times)_{\text{reg}}^\vee &\longleftrightarrow \text{Irr}_{\text{cusp}} \text{GL}_n(k_E) \\ \theta &\longleftrightarrow \tau_\theta. \end{aligned}$$

The bijection is specified by a character relation

$$\tau_\theta(x) = c \sum_{\gamma \in \Gamma} \theta^\gamma(x), \quad x \in l_{\text{reg}}^\times,$$

for a certain constant c that is independent of θ and x . We denote τ by τ_θ .

Note that we have $k_E = \mathbb{F}_{q^2}$. So $l = \mathbb{F}_{q^{2n}}$.

As $\Gamma = \text{Gal}(l/k_E)$, Γ is generated by the Frobenius map Φ given by $\Phi(\lambda) = \lambda^{q^2}$ for $\lambda \in l$. Note that here $\theta^\Phi = \theta^{q^2}$. Also observe that $\Phi^n(\lambda) = \lambda^{q^{2n}} = \lambda$ (since l^\times is a cyclic group of order $q^{2n} - 1$) $\implies \Phi^n = 1$.

Note that for two regular characters θ and θ' we have $\tau_\theta \simeq \tau_{\theta'} \iff$ there exists $\gamma \in \Gamma$ such that $\theta^\gamma = \theta'$.

We now define the Siegel parahoric subgroup \mathfrak{P} of G which is given by:

$$\mathfrak{P} = \left[\begin{array}{ccc} \text{GL}_n(\mathfrak{O}_E) & M_{n \times 1}(\mathfrak{O}_E) & M_n(\mathfrak{O}_E) \\ M_{1 \times n}(\mathfrak{p}_E) & U_1(\mathfrak{O}_E) & M_{1 \times n}(\mathfrak{O}_E) \\ M_n(\mathfrak{p}_E) & M_{n \times 1}(\mathfrak{p}_E) & \text{GL}_n(\mathfrak{O}_E) \end{array} \right] \cap U(n, n + 1).$$

We have $\mathfrak{P} = (\mathfrak{P} \cap \bar{U})(\mathfrak{P} \cap L)(\mathfrak{P} \cap U)$ (Iwahori factorization of \mathfrak{P}). Let us denote $(\mathfrak{P} \cap \bar{U})$ by \mathfrak{P}_- , $(\mathfrak{P} \cap U)$ by \mathfrak{P}_+ , $(\mathfrak{P} \cap L)$ by \mathfrak{P}_0 . Thus

$$\mathfrak{P}_0 = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & {}^t\bar{a}^{-1} \end{bmatrix} \mid a \in \text{GL}_n(\mathfrak{O}_E), \lambda \in \mathfrak{O}_E^\times, \lambda\bar{\lambda} = 1 \right\},$$

$$\mathfrak{P}_+ = \left\{ \begin{bmatrix} Id_n & u & X \\ 0 & 1 & -{}^t\bar{u} \\ 0 & 0 & Id_n \end{bmatrix} \mid X \in M_n(\mathfrak{O}_E), u \in M_{n \times 1}(\mathfrak{O}_E), X + {}^t\bar{X} + u{}^t\bar{u} = 0 \right\},$$

$$\mathfrak{P}_- = \left\{ \begin{bmatrix} Id_n & 0 & 0 \\ -{}^t\bar{u} & 1 & 0 \\ X & u & Id_n \end{bmatrix} \mid X \in M_n(\mathfrak{p}_E), u \in M_{n \times 1}(\mathfrak{p}_E), X + {}^t\bar{X} + u{}^t\bar{u} = 0 \right\}.$$

By Iwahori factorization of \mathfrak{P} we have $\mathfrak{P} = (\mathfrak{P} \cap \bar{U})(\mathfrak{P} \cap L)(\mathfrak{P} \cap U) = \mathfrak{P}_- \mathfrak{P}_0 \mathfrak{P}_+$. As ρ_0 is a representation of $K_0 \times U_1(\mathfrak{O}_E)$, it can also be viewed as a representation of \mathfrak{P}_0 . This is because $\mathfrak{P}_0 \cong K_0 \times U_1(\mathfrak{O}_E)$. We shall see that, ρ_0 can be extended to a representation ρ of \mathfrak{P} which is given by $\rho(p) = \rho_0(p_0)$ where $p \in \mathfrak{P}$ can be factorized as $p_- p_0 p_+$ where $p_- \in \mathfrak{P}_-, p_0 \in \mathfrak{P}_0, p_+ \in \mathfrak{P}_+$.

Let K be a compact open subgroup of G . Let (ρ, W) be an irreducible smooth representation of K . The Hecke algebra $\mathcal{H}(G, \rho)$ is given by:

$$\mathcal{H}(G, \rho) = \left\{ f : G \rightarrow \text{End}_{\mathbb{C}}(\rho^{\vee}) \mid \begin{array}{l} \text{supp}(f) \text{ is compact and} \\ f(k_1 g k_2) = \rho^{\vee}(k_1) f(g) \rho^{\vee}(k_2) \\ \text{where } k_1, k_2 \in K, g \in G \end{array} \right\}.$$

Then $\mathcal{H}(G, \rho)$ is a \mathbb{C} -algebra with multiplication given by convolution $*$ with respect to some fixed Haar measure μ on G . So for elements $f, g \in \mathcal{H}(G, \rho)$ we have

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)d\mu(y).$$

Let $Z(L)$ denote the center of L . Hence

$$Z(L) = \left\{ \begin{bmatrix} aId_n & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \bar{a}^{-1}Id_n \end{bmatrix} \mid a \in E^{\times}, \lambda \in E^{\times}, \lambda\bar{\lambda} = 1 \right\}.$$

Let us set

$$\zeta = \begin{bmatrix} \varpi_E Id_n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \overline{\varpi_E}^{-1} Id_n \end{bmatrix}.$$

Note that $Z(L)\mathfrak{P}_0 = \coprod_{n \in \mathbb{Z}} \mathfrak{P}_0 \zeta^n$, so we can extend ρ_0 to a representation $\tilde{\rho}_0$ of $Z(L)\mathfrak{P}_0$ via $\tilde{\rho}_0(\zeta^k j) = \rho_0(j)$ for $j \in \mathfrak{P}_0, k \in \mathbb{Z}$. By standard Mackey theory arguments, we show in the paper that $\pi = c\text{-Ind}_{Z(L)\mathfrak{P}_0}^L \tilde{\rho}_0$ is a smooth irreducible supercuspidal depth zero representation of L . Also note that any arbitrary depth zero irreducible supercuspidal representation of L is an unramified twist of π . To that end, we will answer the question which we posed earlier in this paper and prove the following result.

Theorem 1.1. *Let $G = \text{U}(n, n + 1)$. Let P be the Siegel parabolic subgroup of G and L be the Siegel Levi component of P . Let $\pi = c\text{-Ind}_{Z(L)\mathfrak{P}_0}^L \tilde{\rho}_0$ be a smooth irreducible supercuspidal depth zero representation of $L \cong \text{GL}_n(E) \times \text{U}_1(E)$ where $\tilde{\rho}_0(\zeta^k j) = \rho_0(j)$ for $j \in \mathfrak{P}_0, k \in \mathbb{Z}$ and $\rho_0 = \tau_{\theta}$ for some regular character θ of l^{\times} with $[l : k_E] = n$ and $|k_F| = q$. Consider the family $i_p^G(\pi\nu)$ for $\nu \in X_{nr}(L)$.*

- (1) *For E/F unramified, $i_p^G(\pi\nu)$ is reducible $\iff n$ is odd, $\theta^{q^{n+1}} = \theta^{-q}$ and $\nu(\zeta) \in \{q^n, q^{-n}, -1\}$.*
- (2) *For E/F ramified, $i_p^G(\pi\nu)$ is reducible $\iff n$ is even, $\theta^{q^{n/2}} = \theta^{-1}$ and $\nu(\zeta) \in \{q^{n/2}, q^{-n/2}, -1\}$.*

In this paper we solve a similar problem as the one which we did in [RSS21]. In [RSS21], we solved the problem for $\text{U}(n, n)$ over non-Archimedean local fields where as in this paper we are solving the same problem for $\text{U}(n, n + 1)$ over non-Archimedean local fields. Refer to the section 1 in [RSS21] for a better understanding of what we are doing in this paper. All the representations in this paper are smooth and complex representations.

In [GD94], Goldberg computes the reducibility points of $\iota_P^G(\pi)$ by computing the poles of certain L -functions attached to the representations of $\mathrm{GL}_n(E)$. Note however that the base field F is assumed to be of characteristic 0 in [GD94], whereas we assumed characteristic of $F \neq 2$. In [GD94] there is no restriction on the depth of the representation π , while in this paper we have assumed depth of the representation π to be of zero. The final results obtained in [GD94] are in terms of matrix coefficients of π whereas our results are in terms of the unramified characters of L .

In [HV11] and [HV17], Heiermann computed the structure of the Hecke algebras which we look at and makes a connection with Langlands parameters. But his results are not explicit. They do not give the precise values of the parameters in the relevant Hecke algebras.

In [LS20], Stevens and Lust have calculated the parameters of the affine Hecke algebras for all the classical groups, so in particular they have also calculated the parameters of affine Hecke algebras of odd unitary groups in the depth zero setting for both ramified and unramified extensions. However, the approach taken by them is quite different from our approach.

1.2. Organization of the paper. In section 2, we introduce the preliminaries required to solve the question posed in the Introduction section. In section 3 we perform the calculations required to understand the structure of the Hecke algebra $\mathcal{H}(G, \rho)$ in both the ramified and unramified cases. In section 4, the structure of the Hecke algebra $\mathcal{H}(L, \rho_0)$ and that of simple $\mathcal{H}(L, \rho_0)$ -modules are determined and also in this section, further calculations which are required to prove Theorem 1 are performed. Finally, in section 5 proof of Theorem 1 is given.

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2. Preliminaries

2.1. Bernstein Decomposition. Let G be the F -rational points of a reductive algebraic group defined over a non-Archimedean local field F . Let (π, V) be an irreducible smooth representation of G . According to Theorem 3.3 in [KP98], there exists unique conjugacy class of cuspidal pairs (L, σ) with the property that π is isomorphic to a composition factor of $\iota_P^G \sigma$ for some parabolic subgroup P of G . We call this conjugacy class of cuspidal pairs, the cuspidal support of (π, V) .

Given two cuspidal supports (L_1, σ_1) and (L_2, σ_2) of (π, V) , we say they are inertially equivalent if there exists $g \in G$ and $\chi \in X_{nr}(L_2)$ such that $L_2 = L_1^g$ and $\sigma_1^g \simeq \sigma_2 \otimes \chi$. We write $[L, \sigma]_G$ for the inertial equivalence class or inertial support of (π, V) . Let $\mathfrak{B}(G)$ denote the set of inertial equivalence classes $[L, \sigma]_G$.

Let $\mathfrak{R}(G)$ denote the category of smooth representations of G . Let $\mathfrak{R}^s(G)$ be the full sub-category of smooth representations of G with the property that

$(\pi, V) \in \text{ob}(\mathfrak{R}^s(G)) \iff$ every irreducible sub-quotient of π has inertial support $s = [L, \sigma]_G$.

We can now state the Bernstein decomposition:

$$\mathfrak{R}(G) = \prod_{s \in \mathfrak{B}(G)} \mathfrak{R}^s(G).$$

2.2. Types. Let G be the F -rational points of a reductive algebraic group defined over a non-Archimedean local field F . Let K be a compact open subgroup of G . Let (ρ, W) be an irreducible smooth representation of K and (π, V) be a smooth representation of G . Let V^ρ be the ρ -isotypic subspace of V .

$$V^\rho = \sum_{W'} W'$$

where the sum is over all W' such that $(\pi|_K, W') \simeq (\rho, W)$.

Let $\mathcal{H}(G)$ be the space of all locally constant compactly supported functions $f : G \rightarrow \mathbb{C}$. This is a \mathbb{C} -algebra under convolution $*$. So for elements $f, g \in \mathcal{H}(G)$ we have

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)d\mu(y).$$

Here we have fixed a Haar measure μ on G . Let (π, V) be a representation of G . Then $\mathcal{H}(G)$ acts on V via

$$hv = \int_G h(x)\pi(x)v d\mu(x)$$

for $h \in \mathcal{H}(G), v \in V$. Let e_ρ be the element in $\mathcal{H}(G)$ with support K such that

$$e_\rho(x) = \frac{\dim \rho}{\mu(K)} \text{tr}_W(\rho(x^{-1})), x \in K.$$

We have $e_\rho * e_\rho = e_\rho$ and $e_\rho V = V^\rho$ for any smooth representation (π, V) of G . Let $\mathfrak{R}_\rho(G)$ be the full sub-category of $\mathfrak{R}(G)$ consisting of all representations (π, V) where V is generated by ρ -isotypic vectors. So $(\pi, V) \in \mathfrak{R}_\rho(G) \iff V = \mathcal{H}(G) * e_\rho V$. We now state the definition of a type.

Definition 2.1. Let $s \in \mathfrak{B}(G)$. We say that (K, ρ) is an s -type in G if $\mathfrak{R}_\rho(G) = \mathfrak{R}^s(G)$.

2.3. Hecke algebras. Let G be the F -rational points of a reductive algebraic group defined over a non-Archimedean local field F . Let K be a compact open subgroup of G . Let (ρ, W) be an irreducible smooth representation of K . Here we introduce the Hecke algebra $\mathcal{H}(G, \rho)$.

$$\mathcal{H}(G, \rho) = \left\{ f : G \rightarrow \text{End}_{\mathbb{C}}(\rho^\vee) \left| \begin{array}{l} \text{supp}(f) \text{ is compact and} \\ f(k_1 g k_2) = \rho^\vee(k_1) f(g) \rho^\vee(k_2) \\ \text{where } k_1, k_2 \in K, g \in G \end{array} \right. \right\}.$$

Then $\mathcal{H}(G, \rho)$ is a \mathbb{C} -algebra with multiplication given by convolution $*$ with respect to some fixed Haar measure μ on G . So for elements $f, g \in \mathcal{H}(G, \rho)$ we have

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)d\mu(y).$$

The importance of types is seen from the following result. Let π be a smooth representation in $\mathfrak{R}^s(G)$. Let $\mathcal{H}(G, \rho) - Mod$ denote the category of $\mathcal{H}(G, \rho)$ -modules. If (K, ρ) is an s -type then $m_G : \mathfrak{R}^s(G) \rightarrow \mathcal{H}(G, \rho) - Mod$ given by $m_G(\pi) = \text{Hom}_K(\rho, \pi)$ is an equivalence of categories.

2.4. Covers. Let G be the F -rational points of a reductive algebraic group defined over a non-Archimedean local field F . Let K be a compact open subgroup of G . Let (ρ, W) be an irreducible representation of K . Then we say (K, ρ) is decomposed with respect to (L, P) if the following hold:

- (1) $K = (K \cap \bar{U})(K \cap L)(K \cap U)$.
- (2) $(K \cap \bar{U}), (K \cap U) \leq \ker \rho$.

Suppose (K, ρ) is decomposed with respect to (L, P) . We set $K_L = K \cap L$ and $\rho_L = \rho|_{K_L}$. We say an element $g \in G$ intertwines ρ if $\text{Hom}_{K^g \cap K}(\rho^g, \rho) \neq 0$. Let $\mathfrak{I}_G(\rho) = \{x \in G \mid x \text{ intertwines } \rho\}$. We have the Hecke algebras $\mathcal{H}(G, \rho)$ and $\mathcal{H}(L, \rho_L)$. We write

$$\mathcal{H}(G, \rho)_L = \{f \in \mathcal{H}(G, \rho) \mid \text{supp}(f) \subseteq K L K\}.$$

We recall some results and constructions from pages 606-612 in [BK98]. These allow us to transfer questions about parabolic induction into questions concerning the module theory of appropriate Hecke algebras.

Proposition 2.2 (Bushnell and Kutzko, Proposition 6.3 [BK98]). *Let (K, ρ) decompose with respect to (L, P) . Then*

- (1) ρ_L is irreducible.
- (2) $\mathfrak{I}_L(\rho_L) = \mathfrak{I}_G(\rho) \cap L$.
- (3) *There is an embedding $T : \mathcal{H}(L, \rho_L) \rightarrow \mathcal{H}(G, \rho)$ such that if $f \in \mathcal{H}(L, \rho_L)$ has support $K_L z K_L$ for some $z \in L$, then $T(f)$ has support $K z K$.*
- (4) *The map T induces an isomorphism of vector spaces:*

$$\mathcal{H}(L, \rho_L) \xrightarrow{\cong} \mathcal{H}(G, \rho)_L.$$

Definition 2.3. An element $z \in L$ is called (K, P) -positive element if:

- (1) $z(K \cap \bar{U})z^{-1} \subseteq K \cap \bar{U}$.
- (2) $z^{-1}(K \cap U)z \subseteq K \cap U$.

Definition 2.4. An element $z \in L$ is called strongly (K, P) -positive element if:

- (1) z is (K, P) positive.
- (2) z lies in center of L .
- (3) For any compact open subgroups K and K' of U there exists $m \geq 0$ and $m \in \mathbb{Z}$ such that $z^m K z^{-m} \subseteq K'$.

- (4) For any compact open subgroups K and K' of \bar{U} there exists $m \geq 0$ and $m \in \mathbb{Z}$ such that $z^{-m}Kz \subseteq K'$.

Proposition 2.5 (Bushnell and Kutzko, Lemma 6.14 [BK98], Proposition 7.1, [BK98]). *Strongly (K, P) -positive elements exist and given a strongly positive element $z \in L$, there exists a unique function $\phi_z \in \mathcal{H}(L, \rho_L)$ with support $K_L z K_L$ such that $\phi_z(z)$ is identity function in $\text{End}_{\mathbb{C}}(\rho_L^{\vee})$.*

$$\mathcal{H}^+(L, \rho_L) = \left\{ f : L \rightarrow \text{End}_{\mathbb{C}}(\rho_L^{\vee}) \left| \begin{array}{l} \text{supp}(f) \text{ is compact and consists} \\ \text{of strongly } (K, P)\text{-positive elements} \\ \text{and } f(k_1 l k_2) = \rho_L^{\vee}(k_1) f(l) \rho_L^{\vee}(k_2) \\ \text{where } k_1, k_2 \in K_L, l \in L \end{array} \right. \right\}.$$

The isomorphism of vector spaces $T : \mathcal{H}(L, \rho_L) \rightarrow \mathcal{H}(G, \rho)_L$ restricts to an embedding of algebras:

$$T^+ : \mathcal{H}^+(L, \rho_L) \rightarrow \mathcal{H}(G, \rho)_L \hookrightarrow \mathcal{H}(G, \rho).$$

Proposition 2.6 (Bushnell and Kutzko, Theorem 7.2.i [BK98]). *The embedding T^+ extends to an embedding of algebras $t : \mathcal{H}(L, \rho_L) \rightarrow \mathcal{H}(G, \rho) \iff T^+(\phi_z)$ is invertible for some strongly (K, P) -positive element z , where $\phi_z \in \mathcal{H}(L, \rho_L)$ has support $K_L z K_L$ with $\phi_z(z) = 1$.*

Definition 2.7. Let L be a proper Levi subgroup of G . Let K_L be a compact open subgroup of L and ρ_L be an irreducible smooth representation of K_L . Let K be a compact open subgroup of G and ρ be an irreducible, smooth representation of K . Then we say (K, ρ) is a G -cover of (K_L, ρ_L) if

- (1) The pair (K, ρ) is decomposed with respect to (L, P) for every parabolic subgroup P of G with Levi component L .
- (2) $K \cap L = K_L$ and $\rho|_{K_L} \simeq \rho_L$.
- (3) The embedding $T^+ : \mathcal{H}^+(L, \rho_L) \rightarrow \mathcal{H}(G, \rho)$ extends to an embedding of algebras $t : \mathcal{H}(L, \rho_L) \rightarrow \mathcal{H}(G, \rho)$.

Proposition 2.8 (Bushnell and Kutzko, Theorem 8.3 [BK98]). *Let $s_L = [L, \pi]_L \in \mathfrak{B}(L)$ and $s = [L, \pi]_G \in \mathfrak{B}(G)$. Say (K_L, ρ_L) is an s_L -type and (K, ρ) is a G -cover of (K_L, ρ_L) . Then (K, ρ) is an s -type.*

Note that in this paper $K = \mathfrak{P}, K_L = K \cap L = \mathfrak{P} \cap L = \mathfrak{P}_0$ and $\rho_L = \rho_0$. Also note that in this paper, ρ is defined as $\rho(p) = \rho_0(p_0)$ for $p \in \mathfrak{P}$ where by Iwahori Factorization $p = p_+ p_0 p_-, p_+ \in \mathfrak{P} \cap U, p_0 \in \mathfrak{P}_0, p_- \in \mathfrak{P} \cup \bar{U}$. Observe that from definition 2.7, we can conclude that (\mathfrak{P}, ρ) is a cover of (\mathfrak{P}_0, ρ_0) . Also observe that as $(\mathfrak{P}_\zeta, \rho_0)$ is s_L -type and as (\mathfrak{P}, ρ) is a cover of (\mathfrak{P}_0, ρ_0) , so from proposition 2.8, it follows that (\mathfrak{P}, ρ) is a s -type.

Recall the categories $\mathfrak{R}^{s_L}(L), \mathfrak{R}^s(G)$ where $s_L = [L, \pi]_L$ and $s = [L, \pi]_G$. Note that $\pi\nu$ lies in the category $\mathfrak{R}^{s_L}(L)$ and $i_P^G(\pi\nu)$ lies in $\mathfrak{R}^s(G)$.

Note that $\mathcal{H}(G, \rho) - \text{Mod}$ is the category of $\mathcal{H}(G, \rho)$ -modules and $\mathcal{H}(L, \rho_L) - \text{Mod}$ is the category of $\mathcal{H}(L, \rho_L)$ -modules.

The functor ι_p^G was defined earlier. The functor

$$m_L : \mathfrak{R}^{s_L}(L) \longrightarrow \mathcal{H}(L, \rho_L) - Mod$$

is given by $m_L(\pi\nu) = \text{Hom}_{K_L}(\rho_L, \pi\nu)$. The representation $\pi\nu \in \mathfrak{R}^{s_L}(L)$ being irreducible, it corresponds to a simple $\mathcal{H}(L, \rho_0)$ -module under the functor m_L . Let $f \in m_L(\pi\nu), \gamma \in \mathcal{H}(L, \rho_0)$ and $w \in V$. The action of $\mathcal{H}(L, \rho_0)$ on $m_L(\pi\nu)$ is given by $(\gamma.f)(w) = \int_L \pi(l)\nu(l)f(\gamma^\vee(l^{-1})w)dl$. Here γ^\vee is defined on L by $\gamma^\vee(l^{-1}) = \gamma(l)^\vee$ for $l \in L$.

The functor $m_G : \mathfrak{R}^s(G) \longrightarrow \mathcal{H}(G, \rho) - Mod$ is given by:

$$m_G(\iota_p^G(\pi\nu)) = \text{Hom}_K(\rho, \iota_p^G(\pi\nu)).$$

Further the functor $(T_p)_* : \mathcal{H}(L, \rho_L) - Mod \longrightarrow \mathcal{H}(G, \rho) - Mod$ is given by, for M an $\mathcal{H}(L, \rho_0)$ -module,

$$(T_p)_*(M) = \text{Hom}_{\mathcal{H}(L, \rho_0)}(\mathcal{H}(G, \rho), M)$$

where $\mathcal{H}(G, \rho)$ is viewed as a $\mathcal{H}(L, \rho_0)$ -module via T_p . The action of $\mathcal{H}(G, \rho)$ on $(T_p)_*(M)$ is given by

$$h'\psi(h_1) = \psi(h_1h')$$

where $\psi \in (T_p)_*(M), h_1, h' \in \mathcal{H}(G, \rho)$.

The importance of covers is seen from the following commutative diagram which we will use in answering the question which we posed earlier in this paper:

$$\begin{array}{ccc} \mathfrak{R}^s(G) & \xrightarrow{m_G} & \mathcal{H}(G, \rho) - Mod \\ \iota_p^G \uparrow & & (T_p)_* \uparrow \\ \mathfrak{R}^{s_L}(L) & \xrightarrow{m_L} & \mathcal{H}(L, \rho_0) - Mod. \end{array}$$

Let us denote the set of strongly (\mathfrak{P}, P) -positive elements by \mathcal{J}^+ . Thus

$$\mathcal{J}^+ = \{x \in L \mid x\mathfrak{P}_+x^{-1} \subseteq \mathfrak{P}_+, x^{-1}\mathfrak{P}_-x \subseteq \mathfrak{P}_-\},$$

where $\mathfrak{P}_+ = \mathfrak{P} \cap U, \mathfrak{P}_- = \mathfrak{P} \cap \bar{U}$. Let V be the vector space corresponding to ρ_0 . We shall show in section 4 that $\mathcal{H}(L, \rho_0) = \mathbb{C}[\alpha, \alpha^{-1}]$ where $\alpha \in \mathcal{H}(L, \rho_0)$ has support $\mathfrak{P}_0\zeta\mathfrak{P}_0$ and $\alpha(\zeta) = 1_{V^\vee}$. We will also show that $\alpha^n(\zeta^n) = (\alpha(\zeta))^n$ for $n \in \mathbb{Z}$ and $\text{supp}(\alpha^n) = \mathfrak{P}_0\zeta^n\mathfrak{P}_0 = \mathfrak{P}_0\zeta^n = \zeta^n\mathfrak{P}_0$ for $n \in \mathbb{Z}$.

We have

$$\mathcal{H}^+(L, \rho_0) = \{f \in \mathcal{H}(L, \rho_0) \mid \text{supp} f \subseteq \mathfrak{P}_0\mathcal{J}^+\mathfrak{P}_0\}.$$

Note $\zeta \in \mathcal{J}^+$, so $\mathcal{H}^+(L, \rho_0) = \mathbb{C}[\alpha]$. The following discussion is taken from pages 612-619 in [BK98]. Let W be space of ρ_0 . Let $f \in \mathcal{H}^+(L, \rho_0)$ with support

of f being $\mathfrak{P}_0 x \mathfrak{P}_0$ for $x \in \mathcal{I}^+$. The map $F \in \mathcal{H}(G, \rho)$ is supported on $\mathfrak{P} x \mathfrak{P}$ and $f(x) = F(x)$. The algebra embedding

$$T^+ : \mathcal{H}^+(L, \rho_0) \longrightarrow \mathcal{H}(G, \rho)$$

is given by $T^+(f) = F$, where F is invertible.

Recall support of $\alpha \in \mathcal{H}^+(L, \rho_0)$ is $\mathfrak{P}_0 \zeta$. Let $T^+(\alpha) = \psi$, where $\psi \in \mathcal{H}(G, \rho)$ has support $\mathfrak{P} \zeta \mathfrak{P}$ and $\alpha(\zeta) = \psi(\zeta) = 1_{W^\vee}$. As $T^+(\alpha) = \psi$ is invertible, so from Proposition 2.6 we can conclude that T^+ extends to an embedding of algebras

$$t : \mathcal{H}(L, \rho_0) \longrightarrow \mathcal{H}(G, \rho).$$

Let $\phi \in \mathcal{H}(L, \rho_0)$ and $m \in \mathbb{N}$ is chosen such that $\alpha^m \phi \in \mathcal{H}^+(L, \rho_0)$. The map t is then given by $t(\phi) = \psi^{-m} T^+(\alpha^m \phi)$. For $\phi \in \mathcal{H}(L, \rho_0)$, the map

$$t_P : \mathcal{H}(L, \rho_0) \longrightarrow \mathcal{H}(G, \rho)$$

is given by $t_P(\phi) = t(\phi \delta_P)$, where $\phi \delta_P \in \mathcal{H}(L, \rho_0)$ and is the map

$$\phi \delta_P : L \longrightarrow \text{End}_{\mathbb{C}}(\rho_0^\vee)$$

given by $(\phi \delta_P)(l) = \phi(l) \delta_P(l)$ for $l \in L$. As $\alpha \in \mathcal{H}(L, \rho_0)$ we have

$$\begin{aligned} t_P(\alpha)(\zeta) &= t(\alpha \delta_P)(\zeta) \\ &= T^+(\alpha \delta_P)(\zeta) \\ &= \delta_P(\zeta) T^+(\alpha)(\zeta) \\ &= \delta_P(\zeta) \psi(\zeta) \\ &= \delta_P(\zeta) 1_{W^\vee}. \end{aligned}$$

Let $\mathcal{H}(L, \rho_0)\text{-Mod}$ denote the category of $\mathcal{H}(L, \rho_0)$ -modules and $\mathcal{H}(G, \rho)\text{-Mod}$ denote the category of $\mathcal{H}(G, \rho)$ -modules. The map t_P induces a functor $(t_P)_*$ given by

$$(t_P)_* : \mathcal{H}(L, \rho_0)\text{-Mod} \longrightarrow \mathcal{H}(G, \rho)\text{-Mod}.$$

For M an $\mathcal{H}(L, \rho_0)$ -module,

$$(t_P)_*(M) = \text{Hom}_{\mathcal{H}(L, \rho_0)}(\mathcal{H}(G, \rho), M)$$

where $\mathcal{H}(G, \rho)$ is viewed as a $\mathcal{H}(L, \rho_0)$ -module via t_P . The action of $\mathcal{H}(G, \rho)$ on $(t_P)_*(M)$ is given by

$$h' \psi(h_1) = \psi(h_1 h')$$

where $\psi \in (t_P)_*(M)$, $h_1, h' \in \mathcal{H}(G, \rho)$.

Let $\tau \in \mathfrak{R}^{[L, \pi]_L}(L)$ then functor $m_L : \mathfrak{R}^{[L, \pi]_L}(L) \longrightarrow \mathcal{H}(L, \rho_0)\text{-Mod}$ is given by $m_L(\tau) = \text{Hom}_{\mathfrak{P}_0}(\rho_0, \tau)$. The functor m_L is an equivalence of categories. Let $f \in m_L(\tau)$, $\gamma \in \mathcal{H}(L, \rho_0)$ and $w \in W$. The action of $\mathcal{H}(L, \rho_0)$ on $m_L(\tau)$ is given by $(\gamma.f)(w) = \int_L \tau(l) f(\gamma^\vee(l^{-1})w) dl$. Here γ^\vee is defined on L by $\gamma^\vee(l^{-1}) = \gamma(l)^\vee$ for $l \in L$. Let $\tau' \in \mathfrak{R}^{[L, \pi]_G}(G)$ then the functor $m_G : \mathfrak{R}^{[L, \pi]_G}(G) \longrightarrow \mathcal{H}(G, \rho)\text{-Mod}$ is given by $m_G(\tau') = \text{Hom}_{\mathfrak{P}}(\rho, \tau')$. The functor m_G is an equivalence of

categories. From Corollary 8.4 in [BK98], the functors $m_L, m_G, \text{Ind}_P^G, (t_P)_*$ fit into the following commutative diagram:

$$\begin{CD} \mathfrak{R}^{[L,\pi]_G}(G) @>{m_G}>> \mathcal{H}(G, \rho) - \text{Mod} \\ @V{\text{Ind}_P^G}VV @VV{(t_P)_*}V \\ \mathfrak{R}^{[L,\pi]_L}(L) @>{m_L}>> \mathcal{H}(L, \rho_0) - \text{Mod} \end{CD}$$

If $\tau \in \mathfrak{R}^{[L,\pi]_L}(L)$ then from the above commutative diagram, we see that $(t_P)_*(m_L(\tau)) \cong m_G(\text{Ind}_P^G \tau)$ as $\mathcal{H}(G, \rho)$ -modules. Replacing τ by $(\tau \otimes \delta_P^{1/2})$ in the above expression, $(t_P)_*(m_L(\tau \otimes \delta_P^{1/2})) \cong m_G(\text{Ind}_P^G(\tau \otimes \delta_P^{1/2}))$ as $\mathcal{H}(G, \rho)$ -modules. As $\text{Ind}_P^G(\tau \otimes \delta_P^{1/2}) = t_P^G(\tau)$, we have $(t_P)_*(m_L(\tau \otimes \delta_P^{1/2})) \cong m_G(t_P^G(\tau))$ as $\mathcal{H}(G, \rho)$ -modules.

Our aim is to find an algebra embedding $T_P : \mathcal{H}(L, \rho_0) \rightarrow \mathcal{H}(G, \rho)$ such that the following diagram commutes:

$$\begin{CD} \mathfrak{R}^{[L,\pi]_G}(G) @>{m_G}>> \mathcal{H}(G, \rho) - \text{Mod} \\ @V{t_P^G}VV @VV{(T_P)_*}V \\ \mathfrak{R}^{[L,\pi]_L}(L) @>{m_L}>> \mathcal{H}(L, \rho_0) - \text{Mod} \end{CD}$$

Let $\tau \in \mathfrak{R}^{[L,\pi]_L}(L)$ then $m_L(\tau) \in \mathcal{H}(L, \rho_0) - \text{Mod}$. The functor $(T_P)_*$ is defined as below:

$$(T_P)_*(m_L(\tau)) = \left\{ \psi : \mathcal{H}(G, \rho) \rightarrow m_L(\tau) \mid \begin{array}{l} h\psi(h_1) = \psi(T_P(h)h_1) \text{ where} \\ h \in \mathcal{H}(L, \rho_0), h_1 \in \mathcal{H}(G, \rho) \end{array} \right\}.$$

From the above commutative diagram, we see that

$$(T_P)_*(m_L(\tau)) \cong m_G(t_P^G(\tau))$$

as $\mathcal{H}(G, \rho)$ -modules. Recall that

$$(t_P)_*(m_L(\tau \otimes \delta_P^{1/2})) \cong m_G(t_P^G(\tau))$$

as $\mathcal{H}(G, \rho)$ -modules. Hence we have to find an algebra embedding

$$T_P : \mathcal{H}(L, \rho_0) \rightarrow \mathcal{H}(G, \rho)$$

such that

$$(T_P)_*(m_L(\tau)) \cong (t_P)_*(m_L(\tau \otimes \delta_P^{1/2}))$$

as $\mathcal{H}(G, \rho)$ -modules.

Proposition 2.9. *The map T_P is given by*

$$T_P(\phi) = t_P(\phi \delta_P^{-1/2}), \quad \phi \in \mathcal{H}(L, \rho_0),$$

so that we have

$$(T_P)_*(m_L(\tau)) = (t_P)_*(m_L(\tau \otimes \delta_P^{1/2}))$$

as $\mathcal{H}(G, \rho)$ -modules.

Proof. Let W be space of ρ_0 . The vector spaces for $m_L(\tau\delta_p^{1/2})$ and $m_L(\tau)$ are the same. Let $f \in m_L(\tau) = \text{Hom}_{\mathfrak{P}_0}(\rho_0, \tau)$, $\gamma \in \mathcal{H}(L, \rho_0)$ and $w \in W$. Recall the action of $\mathcal{H}(L, \rho_0)$ on $m_L(\tau)$ is given by

$$(\gamma.f)(w) = \int_L \tau(l)f(\gamma^\vee(l^{-1})w)dl.$$

Let $f' \in m_L(\tau\delta_p^{1/2}) = \text{Hom}_{\mathfrak{P}_0}(\rho_0, \tau\delta_p^{1/2})$, $\gamma \in \mathcal{H}(L, \rho_0)$ and $w \in W$. Recall the action of $\mathcal{H}(L, \rho_0)$ on $m_L(\tau\delta_p^{1/2})$ is given by

$$\begin{aligned} (\gamma.f')(w) &= \int_L (\tau\delta_p^{1/2})(l)f'(\gamma^\vee(l^{-1})w)dl \\ &= \int_L \tau(l)\delta_p^{1/2}(l)f'(\gamma^\vee(l^{-1})w)dl. \end{aligned}$$

Now f' is a linear transformation from space of ρ_0 to space of $\tau\delta_p^{1/2}$. As $\delta_p^{1/2}(l) \in \mathbb{C}^\times$, so $\delta_p^{1/2}(l)f'(\gamma^\vee(l^{-1})w) = f'(\delta_p^{1/2}(l)\gamma^\vee(l^{-1})w)$. Hence we have

$$\begin{aligned} (\gamma.f')(w) &= \int_L \tau(l)f'(\delta_p^{1/2}(l)\gamma^\vee(l^{-1})w)dl \\ &= \int_L \tau(l)f'(\delta_p^{1/2}(l)\gamma(l)^\vee w)dl. \end{aligned}$$

Further as $\delta_p^{1/2}(l) \in \mathbb{C}^\times$, so $\delta_p^{1/2}(l)(\gamma(l))^\vee = (\delta_p^{1/2}\gamma)(l)^\vee$. Therefore

$$(\gamma.f')(w) = \int_L \tau(l)f'((\delta_p^{1/2}\gamma)(l)^\vee w)dl = (\delta_p^{1/2}\gamma).f'(w).$$

Hence we can conclude that the action of $\gamma \in \mathcal{H}(L, \rho_0)$ on $f' \in m_L(\tau\delta_p^{1/2})$ is same as the action of $\delta_p^{1/2}\gamma \in \mathcal{H}(L, \rho_0)$ on $f' \in m_L(\tau)$. So we have $(T_p)_*(m_L(\tau)) = (t_p)_*(m_L(\tau \otimes \delta_p^{1/2}))$ as $\mathcal{H}(G, \rho)$ -modules. \square

From Proposition 2.9, $T_p(\alpha) = t_p(\alpha\delta_p^{-1/2})$. So we have,

$$\begin{aligned} T_p(\alpha) &= t_p(\alpha\delta_p^{-1/2}) \\ &= t(\alpha\delta_p^{-1/2}\delta_p) \\ &= t(\alpha\delta_p^{1/2}) \\ &= T^+(\alpha\delta_p^{1/2}). \end{aligned}$$

Hence

$$\begin{aligned} T_p(\alpha)(\zeta) &= T^+(\alpha\delta_p^{1/2})(\zeta) \\ &= \delta_p^{1/2}(\zeta)T^+(\alpha)(\zeta) \\ &= \delta_p^{1/2}(\zeta)\alpha(\zeta) \end{aligned}$$

$$= \delta_p^{1/2}(\zeta)1_{W^\vee}.$$

Thus $T_P(\alpha)(\zeta) = \delta_p^{1/2}(\zeta)1_{W^\vee}$ with $\text{supp}(T_P(\alpha)) = \text{supp}(t_P(\alpha)) = \mathfrak{P}\zeta\mathfrak{P}$.

2.5. Depth zero supercuspidal representations. Suppose τ is an irreducible cuspidal representation of $\text{GL}_n(k_E)$ inflated to a representation of $\text{GL}_n(\mathfrak{O}_E) = K_0$. Then let $\widetilde{K}_0 = ZK_0$ where $Z = Z(\text{GL}_n(E)) = \{\lambda 1_n \mid \lambda \in E^\times\}$. As any element of E^\times can be written as $u\varpi_E^n$ for some $u \in \mathfrak{O}_E^\times$ and $m \in \mathbb{Z}$. So in fact, $\widetilde{K}_0 = \langle \varpi_E 1_n \rangle K_0$.

Let (λ, V) be a smooth irreducible supercuspidal representation of $\text{GL}_n(E)$ such that $\lambda|_{K_0} = \tau$. Set 1_V to be the identity linear transformation of V . As $\varpi_E 1_n \in Z$, so $\lambda(\varpi_E 1_n) = \omega_\lambda(\varpi_E 1_n)1_V$ where $\omega_\lambda : Z \rightarrow \mathbb{C}^\times$ is the central character of λ .

Let $\widetilde{\tau}$ be a representation of \widetilde{K}_0 such that:

- (1) $\widetilde{\tau}(\varpi_E 1_n) = \omega_\lambda(\varpi_E 1_n)1_V$,
- (2) $\widetilde{\tau}|_{K_0} = \tau$.

Say $\omega_\lambda(\varpi_E 1_n) = z$ where $z \in \mathbb{C}^\times$. Now call $\widetilde{\tau} = \widetilde{\tau}_z$. We have extended τ to $\widetilde{\tau}_z$ which is a representation of \widetilde{K}_0 , so that Z acts by ω_λ . Hence $\lambda|_{\widetilde{K}_0} \supseteq \widetilde{\tau}_z$ which implies that $\text{Hom}_{\widetilde{K}_0}(\widetilde{\tau}_z, \lambda|_{\widetilde{K}_0}) \neq 0$.

By Frobenius reciprocity for induction from open subgroups,

$$\text{Hom}_{\widetilde{K}_0}(\widetilde{\tau}_z, \lambda|_{\widetilde{K}_0}) \simeq \text{Hom}_{\text{GL}_n(E)}(c\text{-Ind}_{\widetilde{K}_0}^{\text{GL}_n(E)} \widetilde{\tau}_z, \lambda).$$

Thus $\text{Hom}_{\text{GL}_n(E)}(c\text{-Ind}_{\widetilde{K}_0}^{\text{GL}_n(E)} \widetilde{\tau}_z, \lambda) \neq 0$. So there exists a non-zero $\text{GL}_n(E)$ -map from $c\text{-Ind}_{\widetilde{K}_0}^{\text{GL}_n(E)} \widetilde{\tau}_z$ to λ . As τ is cuspidal representation, using Cartan decomposition and Mackey's criteria we can show that $c\text{-Ind}_{\widetilde{K}_0}^{\text{GL}_n(E)} \widetilde{\tau}_z$ is irreducible. So $\lambda \simeq c\text{-Ind}_{\widetilde{K}_0}^{\text{GL}_n(E)} \widetilde{\tau}_z$. As $c\text{-Ind}_{\widetilde{K}_0}^{\text{GL}_n(E)} \widetilde{\tau}_z$ is irreducible supercuspidal representation of $\text{GL}_n(E)$ of depth zero, so λ is irreducible supercuspidal representation of $\text{GL}_n(E)$ of depth zero.

Conversely, let λ is a irreducible, supercuspidal, depth zero representation of $\text{GL}_n(E)$. So $\lambda^{K_1} \neq \{0\}$. Hence $\lambda|_{K_1} \supseteq 1_{K_1}$, where 1_{K_1} is trivial representation of K_1 . This means $\lambda|_{K_0} \supseteq \tau$, where τ is an irreducible representation of K_0 such that $\tau|_{K_1} \supseteq 1_{K_1}$. So τ is trivial on K_1 . So $\lambda|_{K_0}$ contains an irreducible representation τ of K_0 such that $\tau|_{K_1}$ is trivial. So τ can be viewed as an irreducible representation of $K_0/K_1 \cong \text{GL}_n(k_E)$ inflated to $K_0 = \text{GL}_n(\mathfrak{O}_E)$. The representation τ is cuspidal by (a very special case of) A.1 Appendix [ML93].

So we have the following bijection of sets:

$$\left\{ \text{Isomorphism classes of irreducible cuspidal representations of } \text{GL}_n(k_E) \right\} \times \mathbb{C}^\times \longleftrightarrow \left\{ \begin{array}{l} \text{Isomorphism classes} \\ \text{of irreducible} \\ \text{supercuspidal} \\ \text{representations of} \\ \text{GL}_n(E) \text{ of depth zero} \end{array} \right\}.$$

$$\begin{array}{ccc}
 (\tau, z) & \xrightarrow{\hspace{15em}} & c - \text{Ind}_{\tilde{K}_0}^{\text{GL}_n(E)} \tilde{\tau}_z \\
 & & \longleftarrow \hspace{15em} \lambda \\
 & & (\tau, \omega_\lambda(\varpi_E 1_n))
 \end{array}$$

Recall that π is an irreducible supercuspidal depth zero representation of $L \cong \text{GL}_n(E) \times \text{U}_1(E)$. So $\pi = \lambda\chi$ where λ is an irreducible supercuspidal depth zero representation of $\text{GL}_n(E)$ and χ is an irreducible supercuspidal depth zero character of $\text{U}_1(E)$. From now on we denote the representation $\tau\chi$ by ρ_0 . So ρ_0 is an irreducible cuspidal representation of $\text{GL}_n(k_E) \times \text{U}_1(k_E)$ inflated to $K_0 \times \text{U}_1(\mathfrak{O}_E)$ where $K_0 = \text{GL}_n(\mathfrak{O}_E)$. Recall that we can extend ρ_0 to a representation $\tilde{\rho}_0$ of $Z(L)\mathfrak{P}_0 = \coprod_{n \in \mathbb{Z}} \mathfrak{P}_0 \zeta^n$ via $\tilde{\rho}_0(\zeta^k j) = \rho_0(j)$ for $j \in \mathfrak{P}_0, k \in \mathbb{Z}$. Also observe that as $\lambda = c\text{-Ind}_{\tilde{K}_0}^{\text{GL}_n(E)} \tilde{\tau}$, so $\pi = \lambda\chi \simeq c\text{-Ind}_{Z(L)\mathfrak{P}_0}^L \tilde{\rho}_0$.

3. Structure of $\mathcal{H}(G, \rho)$

3.1. Representation ρ of \mathfrak{P} . Let V be the vector space associated with ρ_0 . Now ρ_0 is extended to a map ρ from \mathfrak{P} to $GL(V)$ as follows. By Iwahori factorization, if $j \in \mathfrak{P}$ then j can be written as $j_- j_0 j_+$, where $j_- \in \mathfrak{P}_-, j_+ \in \mathfrak{P}_+, j_0 \in \mathfrak{P}_0$. Now the map ρ on \mathfrak{P} is defined as $\rho(j) = \rho_0(j_0)$.

Proposition 3.1. ρ is a homomorphism from \mathfrak{P} to $GL(V)$. So ρ becomes a representation of \mathfrak{P} .

Proof. The proof goes in similar lines as Proposition 5 in [RSS21]. □

3.2. Calculation of $N_G(\mathfrak{P}_0)$. We set $G = \text{U}(n, n + 1)$. To describe $\mathcal{H}(G, \rho)$ we need to determine $N_G(\rho_0)$ which is given by

$$N_G(\rho_0) = \{m \in N_G(\mathfrak{P}_0) \mid \rho_0 \simeq \rho_0^m\}.$$

Further, to find out $N_G(\rho_0)$ we need to determine $N_G(\mathfrak{P}_0)$. To that end we shall calculate $N_{\text{GL}_n(E)}(K_0)$. Let $Z = Z(\text{GL}_n(E))$. So $Z = \{\lambda 1_n \mid \lambda \in E^\times\}$.

Lemma 3.2. $N_{\text{GL}_n(E)}(K_0) = K_0 Z$.

Proof. This follows from the Cartan decomposition by a direct matrix calculation. □

Now let us calculate $N_G(\mathfrak{P}_0)$. Note that $J = \begin{bmatrix} 0 & 0 & Id_n \\ 0 & 1 & 0 \\ Id_n & 0 & 0 \end{bmatrix} \in G$. Indeed, $J \in N_G(\mathfrak{P}_0)$. The center $Z(\mathfrak{P}_0)$ of \mathfrak{P}_0 is given by

$$Z(\mathfrak{P}_0) = \left\{ \begin{bmatrix} u Id_n & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \bar{u}^{-1} Id_n \end{bmatrix} \mid u \in \mathfrak{O}_E^\times, \lambda \in \mathfrak{O}_E^\times, \lambda \bar{\lambda} = 1 \right\}.$$

Recall the center $Z(L)$ of L is given by

$$Z(L) = \left\{ \begin{bmatrix} aId_n & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \bar{a}^{-1}Id_n \end{bmatrix} \mid a \in E^\times, \lambda \in E^\times, \lambda\bar{\lambda} = 1 \right\}.$$

Proposition 3.3. $N_G(\mathfrak{P}_0) = \langle \mathfrak{P}_0 Z(L), J \rangle = \mathfrak{P}_0 Z(L) \rtimes \langle J \rangle$.

Proof. We use Lemma 3.2 to prove this Proposition. The proof goes in the similar lines as Proposition 6 in [RSS21]. \square

3.3. Calculation of $N_G(\rho_0)$.

3.4. Unramified case. We have the following conclusion about $N_G(\rho_0)$ for the unramified case:

If n is even, then $N_G(\rho_0) = Z(L)\mathfrak{P}_0$. If n is odd, then $N_G(\rho_0) = Z(L)\mathfrak{P}_0 \rtimes \langle J \rangle$. For details refer to section 5.1 in [RSS21].

3.5. Ramified case: We have the following conclusion about $N_G(\rho_0)$ for ramified case:

If n is odd, then $N_G(\rho_0) = Z(L)\mathfrak{P}_0$. If n is even, then $N_G(\rho_0) = Z(L)\mathfrak{P}_0 \rtimes \langle J \rangle$. For details refer to section 5.2 in [RSS21].

Lemma 3.4. *When n is odd in the unramified case or when n is even in the ramified case, we have $N_G(\rho_0) = \langle \mathfrak{P}_0, w_0, w_1 \rangle$, where $w_0 = J$ and*

$$w_1 = \begin{bmatrix} 0 & 0 & \overline{\varpi}_E^{-1} Id_n \\ 0 & 1 & 0 \\ \varpi_E Id_n & 0 & 0 \end{bmatrix}.$$

Proof. The proof goes in the similar lines as Lemma 2 in [RSS21]. \square

3.6. Calculation of $\mathcal{H}(G, \rho)$.

3.6.1. Unramified case: In this section, we will determine the structure of $\mathcal{H}(G, \rho)$ for the unramified case when n is odd. Using cuspidality of ρ_0 , it can be shown by Theorem 4.15 in [ML93], that $\mathfrak{S}_G(\rho) = \mathfrak{P} N_G(\rho_0) \mathfrak{P}$. But from Lemma 3.4, $N_G(\rho_0) = \langle \mathfrak{P}_0, w_0, w_1 \rangle$. So $\mathfrak{S}_G(\rho) = \mathfrak{P} \langle \mathfrak{P}_0, w_0, w_1 \rangle \mathfrak{P} = \mathfrak{P} \langle w_0, w_1 \rangle \mathfrak{P}$, as \mathfrak{P}_0 is a subgroup of \mathfrak{P} . Let V be the vector space corresponding to ρ . Let us recall that $\mathcal{H}(G, \rho)$ consists of maps $f : G \rightarrow \text{End}_{\mathbb{C}}(V^\vee)$ such that support of f is compact and $f(pgp') = \rho^\vee(p)f(g)\rho^\vee(p')$ for $p, p' \in \mathfrak{P}, g \in G$. In fact $\mathcal{H}(G, \rho)$ consists of \mathbb{C} -linear combinations of maps $f : G \rightarrow \text{End}_{\mathbb{C}}(V^\vee)$ such that f is supported on $\mathfrak{P}x\mathfrak{P}$ where $x \in \mathfrak{S}_G(\rho)$ and $f(pxp') = \rho^\vee(p)f(x)\rho^\vee(p')$ for $p, p' \in \mathfrak{P}$. We shall now show there exists $\phi_0 \in \mathcal{H}(G, \rho)$ with support $\mathfrak{P}w_0\mathfrak{P}$ and satisfies $\phi_0^2 = q^n + (q^n - 1)\phi_0$. Let

$$\begin{aligned} K(0) &= \text{U}(n, n + 1) \cap \text{GL}_{2n+1}(\mathfrak{O}_E) = \{g \in \text{GL}_{2n+1}(\mathfrak{O}_E) \mid {}^t \bar{g} J g = J\}, \\ K_1(0) &= \{g \in \text{Id}_{n+1} + \varpi_E \text{M}_{2n+1}(\mathfrak{O}_E) \mid {}^t \bar{g} J g = J\}, \\ G &= \{g \in \text{GL}_{2n+1}(k_E) \mid {}^t \bar{g} J g = J\}. \end{aligned}$$

The map r from $K(0)$ to G given by $r : K(0) \xrightarrow{\text{mod } p_E} G$ is a surjective group homomorphism with kernel $K_1(0)$. So by the first isomorphism theorem of groups we have:

$$\frac{K(0)}{K_1(0)} \cong G.$$

$$r(\mathfrak{P}) = P = \left[\begin{array}{ccc} \text{GL}_n(k_E) & M_{n \times 1}(k_E) & M_n(k_E) \\ 0 & \text{U}_1(k_E) & M_{1 \times n}(k_E) \\ 0 & 0 & \text{GL}_n(k_E) \end{array} \right] \cap G = \text{Siegel parabolic sub-}$$

group of G .

Now $P = L \ltimes U$, where L is the Siegel Levi component of P and U is the unipotent radical of P . Here

$$L = \left\{ \left[\begin{array}{ccc} a & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & {}^t \bar{a}^{-1} \end{array} \right] \mid a \in \text{GL}_n(k_E), \lambda \in k_E^\times, \lambda \bar{\lambda} = 1 \right\},$$

$$U = \left\{ \left[\begin{array}{ccc} Id_n & u & X \\ 0 & 1 & -{}^t \bar{u} \\ 0 & 0 & Id_n \end{array} \right] \mid X \in M_n(k_E), u \in M_{n \times 1}(k_E), X + {}^t \bar{X} + u {}^t \bar{u} = 0 \right\}.$$

Let V be the vector space corresponding to ρ . Then the Hecke algebra $\mathcal{H}(K(0), \rho)$ is a sub-algebra of $\mathcal{H}(G, \rho)$.

Let $\bar{\rho}$ be the representation of P which when inflated to \mathfrak{P} is given by ρ and V is also the vector space corresponding to $\bar{\rho}$. The Hecke algebra $\mathcal{H}(G, \bar{\rho})$ looks as follows:

$$\mathcal{H}(G, \bar{\rho}) = \left\{ f : G \rightarrow \text{End}_{\mathbb{C}}(V^\vee) \mid \begin{array}{l} f(pgp') = \bar{\rho}^\vee(p)f(g)\bar{\rho}^\vee(p') \\ \text{where } p, p' \in P, g \in G \end{array} \right\}.$$

Now the homomorphism $r : K(0) \rightarrow G$ extends to a map from $\mathcal{H}(K(0), \rho)$ to $\mathcal{H}(G, \bar{\rho})$ which we again denote by r . Thus

$$r : \mathcal{H}(K(0), \rho) \rightarrow \mathcal{H}(G, \bar{\rho})$$

is given by $r(\phi)(r(x)) = \phi(x)$ for $\phi \in \mathcal{H}(K(0), \rho)$ and $x \in K(0)$.

Proposition 3.5. *The map $r : \mathcal{H}(K(0), \rho) \rightarrow \mathcal{H}(G, \bar{\rho})$ is an algebra isomorphism.*

Proof. Refer to Proposition 17 in [RSS21] □

Let

$$w = r(w_0) = r\left(\left[\begin{array}{ccc} 0 & 0 & Id_n \\ 0 & 1 & 0 \\ Id_n & 0 & 0 \end{array} \right] \right) = \left[\begin{array}{ccc} 0 & 0 & Id_n \\ 0 & 1 & 0 \\ Id_n & 0 & 0 \end{array} \right] \in G.$$

Observe that $K(0) \supseteq \mathfrak{P} \amalg \mathfrak{P}w_0\mathfrak{P}$ and $G \supseteq P \amalg Pw_0P$.

The induced representation $Ind_{\mathbb{P}}^G \bar{\rho}$ is a sum of two irreducible subrepresentations (by general theory). The ratio of the dimensions of these subrepresentations gives a parameter in the Hecke algebra. This is part of Howlett-Lehrer’s general theory. Kutzko-Morris reworks this key observation. Hence we have $Ind_{\mathbb{P}}^G \bar{\rho} = \pi_1 \oplus \pi_2$, where π_1, π_2 are distinct irreducible representations of G with $\dim \pi_2 \geq \dim \pi_1$. Let $\lambda = \frac{\dim \pi_2}{\dim \pi_1}$. By Proposition 3.2 in [KM06], there exists a unique ϕ in $\mathcal{H}(G, \bar{\rho})$ with support $P\omega P$ such that $\phi^2 = \lambda + (\lambda - 1)\phi$. By Proposition 3.5, there is a unique element ϕ_0 in $\mathcal{H}(K(0), \rho)$ such that $r(\phi_0) = \phi$. Thus $\text{supp}(\phi_0) = \mathfrak{P}\omega_0\mathfrak{P}$ and $\phi_0^2 = \lambda + (\lambda - 1)\phi_0$. As $\text{support of } \phi_0 = \mathfrak{P}\omega_0\mathfrak{P} \subseteq K(0) \subseteq G$, so ϕ_0 can be extended to G and viewed as an element of $\mathcal{H}(G, \rho)$. Thus ϕ_0 satisfies the following relation in $\mathcal{H}(G, \rho)$:

$$\phi_0^2 = \lambda + (\lambda - 1)\phi_0.$$

We shall now show that $\lambda = q^n$. Recall that ρ_0 is an irreducible cuspidal representation of $GL_n(k_E) \times U_1(k_E)$, so $\rho_0 = \tau_\theta \chi$, where τ_θ is an irreducible cuspidal representation of $GL_n(k_E)$ and χ is a cuspidal representation of $U_1(k_E)$. Note that here θ is a regular character of l^\times where $[l : k_E] = n$ and $k_E = \mathbb{F}_{q^2}$ so that $l = \mathbb{F}_{q^{2n}}$. Recall that $\theta^\Phi = \theta^{q^2}$. Hence, from Proposition 8 in [RSS21] we have, $\theta^{q^n} = \theta^{-1}$.

As $G = U(n, n + 1)(k_E)$, so the dual group G^* is given by $G^* \cong U(n, n + 1)(k_E)$ (i.e $G^* \cong G$). Note that θ corresponds to a semi-simple element $s^* \in L^*$ in G^* . Then by Theorems 8.4.8 and 8.4.9 in [CR92], we have $\lambda = |c_{G^*}(s^*)|_p$.

Note that $L^* \cong L$. So s^* corresponds to s in L . Hence, we have $\lambda = |c_G(s)|_p$.

We write $s = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \alpha^{-1} \end{bmatrix}$. Observe that $\lambda \bar{\lambda} = 1, \lambda \in k_E^\times, \alpha \in \mathbb{F}_{q^{2n}}^\times$. More precisely, α is in the image of $\mathbb{F}_{q^{2n}}^\times$ under a fixed embedding $\mathbb{F}_{q^{2n}}^\times \hookrightarrow GL_n(\mathbb{F}_{q^2})$. This embedding arises when we let l act on the basis of l over k_E via multiplication. We can thus embed l in $M_n(k_E)$ and l^\times in $GL_n(k_E)$ which we call the usual embedding. Note that θ is regular implies that $\mathbb{F}_{q^{2n}} = \mathbb{F}_{q^2}(\alpha)$. Our goal is to compute $|c_G(s)|_p$.

By Proposition 3.19 in [DFM91], we have Sylow p -subgroups of $c_G(s)$ are the sets of \mathbb{F}_{q^2} -points of the Unipotent radicals of the Borel subgroups of $c_G(s)$. By Proposition 2.2 in [DFM91], we have Borel subgroups of $c_G(s)$ are of the form $B \cap c_G(s)$, where B is a Borel subgroup of G . As Siegel parabolic subgroup P of G contains a Borel subgroup of G , so $c_P(s) = P \cap c_G(s)$ contains a Sylow p -subgroup of $c_G(s)$.

Lemma 3.6. $c_P(s) = c_L(s) \rtimes c_U(s)$.

Proof. Recall that $P = L \rtimes U$. Hence $L \cap U = \emptyset$ and $U \trianglelefteq P$. As $L \cap U = \emptyset \implies c_L(s) \cap c_U(s) = \emptyset$. Note that $c_U(s) \trianglelefteq (c_L(s) \times c_U(s))$. So it makes sense to talk of $c_L(s) \rtimes c_U(s)$.

Let $x \in P(s) \implies x \in P, sxs^{-1} = x$. Note that as $x \in P$ so $x = lu$ for some $l \in L, u \in U$. Therefore,

$$\begin{aligned} slus^{-1} &= lu \\ \implies sls^{-1}sus^{-1} &= lu. \end{aligned}$$

Let $sls^{-1} = m$ and $sus^{-1} = n$. Now as $s \in L$, so $sls^{-1} = m \in L$. Note that $sus^{-1} = n \in U$ as $U \trianglelefteq P$. Therefore, we have $mn = lu$ or $m^{-1}l = nu^{-1}$. But $m^{-1}l \in L$ and $nu^{-1} \in U$, so we have $m^{-1}l, nu^{-1} \in L \cap U$. Recall that $L \cap U = e$, so $m = l, n = u$. Therefore, $sls^{-1} = l, sus^{-1} = u$. So we have $l \in c_L(s), u \in c_U(s)$. Hence, $x \in c_L(s) \rtimes c_U(s)$. So $c_P(s) \subseteq c_L(s) \rtimes c_U(s)$.

Conversely, let $x \in c_L(s) \rtimes c_U(s)$. So $x = lu$ where $l \in c_L(s)$ and $u \in c_U(s)$. Hence $sls^{-1} = l$ and $sus^{-1} = u$. Therefore, $sxs^{-1} = slus^{-1} = sls^{-1}sus^{-1} = lu = x$. So $x \in c_P(s)$. Hence $c_L(s) \rtimes c_U(s) \subseteq c_P(s)$. Therefore, $c_P(s) = c_L(s) \rtimes c_U(s)$. \square

From Lemma 3.6, we get $|c_P(s)|_p = |c_L(s)|_p |c_U(s)|_p$. Note that $|c_L(s)|_p = 1$. Therefore, $|c_P(s)|_p = |c_L(s)|_p |c_U(s)|_p = |c_U(s)|_p$.

Lemma 3.7. $|c_U(s)| = |c_U(s)|_p = q^n$.

Proof. Recall that the elements of U are of form

$$m = \begin{bmatrix} Id_n & u & X \\ 0 & 1 & -{}^t\bar{u} \\ 0 & 0 & Id_n \end{bmatrix}$$

where $x \in M_n(k_E), u \in M_{n \times 1}(k_E), X + {}^t\bar{X} + u{}^t\bar{u} = 0$. If $m \in c_U(s)$ then $ms = sm$. So we have,

$$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & {}^t\bar{\alpha}^{-1} \end{bmatrix} \begin{bmatrix} Id_n & u & X \\ 0 & 1 & -{}^t\bar{u} \\ 0 & 0 & Id_n \end{bmatrix} = \begin{bmatrix} Id_n & u & X \\ 0 & 1 & -{}^t\bar{u} \\ 0 & 0 & Id_n \end{bmatrix} \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & {}^t\bar{\alpha}^{-1} \end{bmatrix}.$$

From the above matrix relation, it follows that $\alpha u = \lambda u, \alpha X = X {}^t\bar{\alpha}^{-1}, \lambda {}^t\bar{u} = {}^t\bar{u} {}^t\bar{\alpha}^{-1}$. Recall that $X + {}^t\bar{X} + u{}^t\bar{u} = 0, \lambda \bar{\lambda} = 1$. Also recall that $u \in M_{n \times 1}(k_E), \alpha \in \mathbb{F}_{q^{2n}}^\times, k_E(\alpha) = l$. As $\alpha u = \lambda u$, so if $u \neq 0$ then $\lambda \in k_E$ is an eigen value of α . So λ is a root of the minimal polynomial of α over k_E . But as the minimal polynomial is irreducible over $k_E[x]$, so this is a contradiction. So $u = 0$.

So we have to find X such that $X + {}^t\bar{X} = 0, \alpha X = X {}^t\bar{\alpha}^{-1}$. Let $\Xi = M_n(k_E)$ and set $\Xi_\epsilon = \{X \in \Xi \mid {}^t\bar{X} = \epsilon X\}$. Note that $X \in \Xi$ can be written as $\frac{X + {}^t\bar{X}}{2} + \frac{X - {}^t\bar{X}}{2}$, so $\Xi = \Xi_1 \oplus \Xi_{-1}$.

Let us set $\Xi(\alpha) = \{X \in \Xi \mid \alpha X {}^t\bar{\alpha} = X\}$ and $\Xi_\epsilon(\alpha) = \{X \in \Xi_\epsilon \mid \alpha X {}^t\bar{\alpha} = X\}$. Then we have, $\Xi(\alpha) = \Xi_1(\alpha) \oplus \Xi_{-1}(\alpha)$. Let us choose $\gamma \in k_E$ such that $\gamma \neq 0$ and $\bar{\gamma} = -\gamma$. Note that, if $X \in \Xi_1(\alpha)$ then $X = {}^t\bar{X}$ and $\alpha X {}^t\bar{\alpha} = X$. So ${}^t(\gamma \bar{X}) = -(\gamma X)$

and $\alpha(\gamma X)^t \bar{\alpha} = \gamma X$. Therefore, $\gamma X \in \Xi_{-1}(\alpha)$. We also have a bijection from $c_U(s) \longrightarrow \Xi_1(\alpha)$ given by:

$$\begin{bmatrix} Id_n & 0 & X \\ 0 & 1 & 0 \\ 0 & 0 & Id_n \end{bmatrix} \longrightarrow X.$$

Hence we have, $|c_U(s)| = |\Xi_1(\alpha)| = |\Xi_{-1}(\alpha)|$. Let us now compute $|\Xi(\alpha)|$. So we want to find the cardinality of $X \in \Xi$ such that $\alpha X^t \bar{\alpha} = X$ for a fixed $\alpha \in \mathbb{F}_{q^{2n}}^\times$. Let $\phi_1 : \mathbb{F}_{q^{2n}} \hookrightarrow M_n(\mathbb{F}_{q^2})$ be the usual embedding take β to m_β . Let $f(x)$ be the minimal polynomial of α over $k_E = \mathbb{F}_{q^2}$. So we have $\mathbb{F}_{q^{2n}} \cong \frac{\mathbb{F}_{q^2}[x]}{\langle f(x) \rangle}$. Hence, a polynomial $p(\alpha) \in k_E(\alpha)$ is mapped to $p(m_\alpha)$.

Let us consider an another embedding $\phi_2 : \mathbb{F}_{q^{2n}} \cong \frac{\mathbb{F}_{q^2}[x]}{\langle f(x) \rangle} \hookrightarrow M_n(\mathbb{F}_{q^2})$ given by $\phi_2(\alpha) = {}^t \bar{m}_\alpha^{-1}$. We must show that ϕ_2 is well-defined. That is, we have to show that $f({}^t \bar{m}_\alpha^{-1}) = 0$. But observe that, $f({}^t \bar{m}_\alpha^{-1}) = {}^t f(\bar{m}_\alpha^{-1}) = {}^t f(m_\alpha^{-1}) = {}^t f(m_\alpha^{q^n}) = {}^t (f(m_\alpha))^{q^n} = 0^{q^n} = 0$. In the above relations, we have used the fact that $\theta^{-1} = \theta^{q^n}$ which follows from Proposition 8 in [RSS21]. Therefore, ϕ_2 is well-defined.

Hence we have two different embeddings ϕ_1 and ϕ_2 of l in $M_n(q^2)$. Recall that, we want to compute the cardinality of $X \in \Xi$ such that $\alpha X^t \bar{\alpha} = X$ for a fixed $\alpha \in \mathbb{F}_{q^{2n}}^\times$. That is, we want to compute the cardinality of $X \in \Xi$ such that $X\phi_2(\lambda) = \phi_1(\lambda)X$ for $\lambda \in l = \mathbb{F}_{q^{2n}}$.

Note that, we can make $V = k_E^n$ into a l -module in two different ways. Namely, for $\lambda \in l$ and $v \in V$ we have,

$$\begin{aligned} \lambda.v &= \phi_1(\lambda).v \\ \lambda * v &= \phi_2(\lambda).v \end{aligned}$$

Let us denote the two l -modules by ${}_1 k_E^n$ and ${}_2 k_E^n$. So $X\phi_2(\lambda) = \phi_1(\lambda)X \iff X \in \text{Hom}_l({}_1 k_E^n, {}_2 k_E^n) \cong \text{Hom}_l(l, l) \cong l$. Therefore, we have $|\Xi(\alpha)| = |\text{Hom}_l({}_1 k_E^n, {}_2 k_E^n)| = |l| = q^{2n}$.

Note that $|\Xi(\alpha)| = |\Xi_1(\alpha)| \cdot |\Xi_{-1}(\alpha)|$. As $|\Xi_1(\alpha)| = |\Xi_{-1}(\alpha)|$, we have $|\Xi(\alpha)| = |\Xi_{-1}(\alpha)|^2 = q^{2n}$. Thus $|\Xi_{-1}(\alpha)| = q^n$. Therefore, $|c_U(s)|_p = |c_U(s)| = |\Xi_{-1}(\alpha)| = q^n$. \square

From Lemmas 3.7 and 3.6 we have,

$$\lambda = |c_U(s)|_p = |c_L(s)|_p \cdot |c_L(s)|_p = 1 \cdot q^n = q^n.$$

Recall that $\phi_0 \in \mathcal{H}(G, \rho)$ has support $\mathfrak{P}w_0\mathfrak{P}$ and satisfies the relation $\phi_0^2 = \lambda + (\lambda - 1)\phi_0$. So we have $\phi_0^2 = q^n + (q^n - 1)\phi_0$ in $\mathcal{H}(G, \rho)$.

Now we shall now show that there exists $\phi_1 \in \mathcal{H}(G, \rho)$ with support $\mathfrak{P}w_1\mathfrak{P}$ satisfying the same relation as ϕ_0 . Let $\eta \in U(n, n+1)$ be such that $\eta w_0 \eta^{-1} = w_1$ and $\eta \mathfrak{P} \eta^{-1} = \mathfrak{P}$.

As $\mathfrak{P} \subseteq K(0)$ and $w_0 \in K(0)$, so $K(0) \supseteq \mathfrak{P} \amalg \mathfrak{P}w_0\mathfrak{P} \implies \eta K(0)\eta^{-1} \supseteq \eta\mathfrak{P}\eta^{-1} \amalg \eta\mathfrak{P}w_0\mathfrak{P}\eta^{-1}$. But observe that $\eta\mathfrak{P}\eta^{-1} = \mathfrak{P}$ and

$$\eta\mathfrak{P}w_0\mathfrak{P}\eta^{-1} = (\eta\mathfrak{P}\eta^{-1})(\eta w_0\eta^{-1})(\eta\mathfrak{P}\eta^{-1}) = \mathfrak{P}w_1\mathfrak{P}$$

(since $\eta w_0\eta^{-1} = w_1$). So $\eta K(0)\eta^{-1} \supseteq \mathfrak{P} \amalg \mathfrak{P}w_1\mathfrak{P}$.

Let r' be homomorphism of groups given by the map $r' : \eta K(0)\eta^{-1} \longrightarrow G$ such that $r'(x) = (\eta^{-1}x\eta) \bmod p_E$ for $x \in \eta K(0)\eta^{-1}$. Observe that r' is a surjective homomorphism of groups because

$$r'(\eta K(0)\eta^{-1}) = (\eta^{-1}\eta K(0)\eta^{-1}\eta) \bmod p_E = K(0) \bmod p_E = G.$$

The kernel of group homomorphism is $\eta K_1(0)\eta^{-1}$. Now by the first isomorphism theorem of groups we have $\frac{\eta K(0)\eta^{-1}}{\eta K_1(0)\eta^{-1}} \cong \frac{K(0)}{K_1(0)} \cong G$. Also $r'(\eta\mathfrak{P}\eta^{-1}) = (\eta^{-1}\eta\mathfrak{P}\eta^{-1}\eta) \bmod p_E = \mathfrak{P} \bmod p_E = P$. Let $\bar{\rho}$ be representation of P which when inflated to \mathfrak{P} is given by ρ . The Hecke algebra of $\eta K(0)\eta^{-1}$ which we denote by $\mathcal{H}(\eta K(0)\eta^{-1}, \rho)$ is a sub-algebra of $\mathcal{H}(G, \rho)$.

The map $r' : \eta K(0)\eta^{-1} \longrightarrow G$ extends to a map from $\mathcal{H}(\eta K(0)\eta^{-1}, \rho)$ to $\mathcal{H}(G, \bar{\rho})$ which we gain denote by r' . Thus $r' : \mathcal{H}(\eta K(0)\eta^{-1}, \rho) \longrightarrow \mathcal{H}(G, \bar{\rho})$ is given by $r'(\phi)(r'(x)) = \phi(x)$ for $\phi \in \mathcal{H}(\eta K(0)\eta^{-1}, \rho)$ and $x \in \eta K(0)\eta^{-1}$.

The proof that r' is an isomorphism goes in the similar lines as Proposition 3.5. We can observe that $r'(w_1) = w \in G$, where w is defined as before in this section. As we know from our previous discussion in this section, that there exists a unique ϕ in $\mathcal{H}(G, \bar{\rho})$ with support PwP such that $\phi^2 = q^n + (q^n - 1)\phi$. Hence there is a unique element $\phi_1 \in \mathcal{H}(\eta K(0)\eta^{-1}, \rho)$ such that $r'(\phi_1) = \phi$. Thus $\text{supp}(\phi_1) = \mathfrak{P}w_1\mathfrak{P}$ and $\phi_1^2 = q^n + (q^n - 1)\phi_1$. Now ϕ_1 can be extended to G and viewed as an element in $\mathcal{H}(G, \rho)$ as $\mathfrak{P}w_1\mathfrak{P} \subseteq \eta K(0)\eta^{-1} \subseteq G$. Thus ϕ_1 satisfies the following relation in $\mathcal{H}(G, \rho)$:

$$\phi_1^2 = q^n + (q^n - 1)\phi_1.$$

Thus we have shown there exists $\phi_i \in \mathcal{H}(G, \rho)$ with $\text{supp}(\phi_i) = \mathfrak{P}w_i\mathfrak{P}$ satisfying $\phi_i^2 = q^n + (q^n - 1)\phi_i$ for $i = 0, 1$.

Lemma 3.8. ϕ_0 and ϕ_1 are units in $\mathcal{H}(G, \rho)$.

Proof. As $\phi_i^2 = q^n + (q^n - 1)\phi_i$ for $i = 0, 1$. So $\phi_i(\frac{\phi_i + (1 - q^n)}{q^n}) = 1$ for $i=0,1$. Hence ϕ_0 and ϕ_1 are units in $\mathcal{H}(G, \rho)$. \square

Lemma 3.9. Let $\phi, \psi \in \mathcal{H}(G, \rho)$ with support of ϕ, ψ being $\mathfrak{P}x\mathfrak{P}, \mathfrak{P}y\mathfrak{P}$ respectively. Then $\text{supp}(\phi * \psi) = \text{supp}(\phi\psi) \subseteq (\text{supp}(\phi))(\text{supp}(\psi)) = \mathfrak{P}x\mathfrak{P}y\mathfrak{P}$.

Proof. The proof is same as that of Lemma 5 in [RSS21]. \square

Let $\zeta = w_0w_1$, So

$$\zeta = \begin{bmatrix} \varpi_E Id_n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varpi_E^{-1} Id_n \end{bmatrix}.$$

Lemma 3.10. $\text{supp}(\phi_0 * \phi_1) = \mathfrak{P}\zeta\mathfrak{P} = \mathfrak{P}w_0w_1\mathfrak{P}$.

Proof. It follows from Lemma 3.9 that $\text{supp}(\phi_0 * \phi_1) \subseteq \mathfrak{P}w_0\mathfrak{P}w_1\mathfrak{P}$. Now let us recall $\mathfrak{P}_0, \mathfrak{P}_+, \mathfrak{P}_-$.

$$\mathfrak{P}_0 = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & {}^t\bar{a}^{-1} \end{bmatrix} \mid a \in GL_n(\mathfrak{D}_E), \lambda \in \mathfrak{D}_E^\times, \lambda\bar{\lambda} = 1 \right\},$$

$$\mathfrak{P}_+ = \left\{ \begin{bmatrix} Id_n & u & X \\ 0 & 1 & -{}^t\bar{u} \\ 0 & 0 & Id_n \end{bmatrix} \mid X \in M_n(\mathfrak{D}_E), u \in M_{n \times 1}(\mathfrak{D}_E), X + {}^t\bar{X} + u{}^t\bar{u} = 0 \right\},$$

$$\mathfrak{P}_- = \left\{ \begin{bmatrix} Id_n & 0 & 0 \\ -{}^t\bar{u} & 1 & 0 \\ X & u & Id_n \end{bmatrix} \mid X \in M_n(\mathfrak{P}_E), u \in M_{n \times 1}(\mathfrak{P}_E), X + {}^t\bar{X} + u{}^t\bar{u} = 0 \right\}.$$

It is easy observe that $w_0\mathfrak{P}_-w_0^{-1} \subseteq \mathfrak{P}_+, w_0\mathfrak{P}_0w_0^{-1} = \mathfrak{P}_0, w_1^{-1}\mathfrak{P}_+w_1 \subseteq \mathfrak{P}_-$. Now we have

$$\begin{aligned} \mathfrak{P}w_0\mathfrak{P}w_1\mathfrak{P} &= \mathfrak{P}w_0\mathfrak{P}_-\mathfrak{P}_0\mathfrak{P}_+w_1\mathfrak{P} \\ &= \mathfrak{P}w_0\mathfrak{P}_-w_0^{-1}w_0\mathfrak{P}_0w_0^{-1}w_0w_1w_1^{-1}\mathfrak{P}_+w_1\mathfrak{P} \\ &\subseteq \mathfrak{P}\mathfrak{P}_+\mathfrak{P}_0w_0w_1\mathfrak{P}_-\mathfrak{P} \\ &= \mathfrak{P}w_0w_1\mathfrak{P} \\ &= \mathfrak{P}\zeta\mathfrak{P}. \end{aligned}$$

So $\mathfrak{P}w_0\mathfrak{P}w_1\mathfrak{P} \subseteq \mathfrak{P}w_0w_1\mathfrak{P} = \mathfrak{P}\zeta\mathfrak{P}$. On the contrary, as $1 \in \mathfrak{P}$, so $\mathfrak{P}\zeta\mathfrak{P} = \mathfrak{P}w_0w_1\mathfrak{P} \subseteq \mathfrak{P}w_0\mathfrak{P}w_1\mathfrak{P}$. Hence we have $\mathfrak{P}w_0\mathfrak{P}w_1\mathfrak{P} = \mathfrak{P}w_0w_1\mathfrak{P} = \mathfrak{P}\zeta\mathfrak{P}$. Therefore $\text{supp}(\phi_0 * \phi_1) \subseteq \mathfrak{P}w_0\mathfrak{P}w_1\mathfrak{P} = \mathfrak{P}w_0w_1\mathfrak{P} = \mathfrak{P}\zeta\mathfrak{P}$. This implies $\text{supp}(\phi_0 * \phi_1) = \emptyset$ or $\mathfrak{P}\zeta\mathfrak{P}$. But if $\text{supp}(\phi_0 * \phi_1) = \emptyset$ then $(\phi_0 * \phi_1) = 0$ which is a contradiction. Thus $\text{supp}(\phi_0 * \phi_1) = \mathfrak{P}\zeta\mathfrak{P}$. \square

We shall now show that ϕ_0 and ϕ_1 generate the Hecke algebra $\mathcal{H}(G, \rho)$. To this end, recall that $\mathcal{H}(G, \rho)$ consists of \mathbb{C} -linear combinations of maps $f : G \rightarrow \text{End}_{\mathbb{C}}(V^\vee)$ such that f is supported on $\mathfrak{P}x\mathfrak{P}$ where $x \in \mathfrak{S}_G(\rho)$ and $f(pxp') = \rho^\vee(p)f(x)\rho^\vee(p')$ for $p, p' \in \mathfrak{P}$. Also note that $\mathfrak{S}_G(\rho) = \mathfrak{P} \langle \mathfrak{P}_0, w_0, w_1 \rangle \mathfrak{P}$. Observe that as w_0 normalizes \mathfrak{P}_0 and as \mathfrak{P}_0 being a subgroup of \mathfrak{P} , so $\mathcal{H}(G, \rho)$ consists of \mathbb{C} -linear combinations of maps $f : G \rightarrow \text{End}_{\mathbb{C}}(V^\vee)$ such that f is supported on $\mathfrak{P}x\mathfrak{P}$ where x is a word in w_0, w_1 with $w_0^2 = w_1^2 = Id$ and $f(pxp') = \rho^\vee(p)f(x)\rho^\vee(p')$ for $p, p' \in \mathfrak{P}$. Recall that support of ϕ_0 is $\mathfrak{P}w_0\mathfrak{P}$ and support of ϕ_1 is $\mathfrak{P}w_1\mathfrak{P}$. Also note that from Lemma 3.10, we have $\text{supp}(\phi_0 * \phi_1) = \mathfrak{P}w_0w_1\mathfrak{P}$. So any $f : G \rightarrow \text{End}_{\mathbb{C}}(V^\vee)$ such that f is supported on $\mathfrak{P}x\mathfrak{P}$ where x is a word in w_0, w_1 and $f(pxp') = \rho^\vee(p)f(x)\rho^\vee(p')$ for $p, p' \in \mathfrak{P}$ can be written as word in ϕ_0 and ϕ_1 . Therefore, $\mathcal{H}(G, \rho)$ consists

of \mathbb{C} -linear combinations of words in ϕ_0 and ϕ_1 . Hence, ϕ_0 and ϕ_1 generate $\mathcal{H}(G, \rho)$. Let us denote the Hecke algebra $\mathcal{H}(G, \rho)$ by \mathcal{A} . So we have

$$\mathcal{A} = \mathcal{H}(G, \rho) = \left\langle \phi_i : G \rightarrow \text{End}_{\mathbb{C}}(\rho^\vee) \mid \begin{array}{l} \phi_i \text{ is supported on } \mathfrak{P}w_i\mathfrak{P} \\ \text{and } \phi_i(pw_i p') = \rho^\vee(p)\phi_i(w_i)\rho^\vee(p') \\ \text{where } p, p' \in \mathfrak{P}, i = 0, 1 \end{array} \right\rangle$$

where ϕ_i satisfies the relation:

$$\phi_i^2 = q^n + (q^n - 1)\phi_i \text{ for } i = 0, 1.$$

As ϕ_0, ϕ_1 are units in algebra \mathcal{A} , so $\psi = \phi_0\phi_1$ is a unit too in \mathcal{A} and $\psi^{-1} = \phi_1^{-1}\phi_0^{-1}$. Now as we have seen before that

$$\text{supp}(\phi_0\phi_1) \subseteq \mathfrak{P}w_0w_1\mathfrak{P} \implies \text{supp}(\psi) \subseteq \mathfrak{P}\zeta\mathfrak{P} \implies \text{supp}(\psi) = \emptyset \text{ or } \mathfrak{P}\zeta\mathfrak{P}.$$

If $\text{supp}(\psi) = \emptyset \implies \psi = 0$ which is a contradiction as ψ is a unit in \mathcal{A} . So $\text{supp}(\psi) = \mathfrak{P}\zeta\mathfrak{P}$. As ψ is a unit in \mathcal{A} , we can show as before that $\text{supp}(\psi^2) = \mathfrak{P}\zeta^2\mathfrak{P}$. Hence by induction on $n \in \mathbb{N}$, we can further show that that $\text{supp}(\psi^n) = \mathfrak{P}\zeta^n\mathfrak{P}$ for $n \in \mathbb{N}$.

Now \mathcal{A} contains a sub- algebra generated by ψ, ψ^{-1} over \mathbb{C} and we denote this sub-algebra by \mathcal{B} . So $\mathcal{B} = \mathbb{C}[\psi, \psi^{-1}]$ where

$$\mathcal{B} = \mathbb{C}[\psi, \psi^{-1}] = \left\{ c_k\psi^k + \dots + c_l\psi^l \mid \begin{array}{l} c_k, \dots, c_l \in \mathbb{C}; \\ k < l; k, l \in \mathbb{Z} \end{array} \right\}.$$

Proposition 3.11. *The unique algebra homomorphism $\mathbb{C}[x, x^{-1}] \longrightarrow \mathcal{B}$ given by $x \longrightarrow \psi$ is an isomorphism. So $\mathcal{B} \simeq \mathbb{C}[x, x^{-1}]$.*

Proof. The proof is same as that of Proposition 18 in [RSS21]. □

3.6.2. Ramified case: In this section we determine the structure of $\mathcal{H}(G, \rho)$ for the ramified case when n is even. Recall $\mathfrak{S}_G(\rho) = \mathfrak{P}N_G(\rho_0)\mathfrak{P}$. But from lemma 3.4, $N_G(\rho_0) = \langle \mathfrak{P}_0, w_0, w_1 \rangle$. So $\mathfrak{S}_G(\rho) = \mathfrak{P} \langle \mathfrak{P}_0, w_0, w_1 \rangle \mathfrak{P} = \mathfrak{P} \langle w_0, w_1 \rangle \mathfrak{P}$, as \mathfrak{P}_0 is a subgroup of \mathfrak{P} . Let V be the vector space corresponding to ρ . Let us recall that $\mathcal{H}(G, \rho)$ consists of maps $f : G \rightarrow \text{End}_{\mathbb{C}}(V^\vee)$ such that support of f is compact and $f(pgp') = \rho^\vee(p)f(g)\rho^\vee(p')$ for $p, p' \in \mathfrak{P}, g \in G$. In fact $\mathcal{H}(G, \rho)$ consists of \mathbb{C} -linear combinations of maps $f : G \longrightarrow \text{End}_{\mathbb{C}}(V^\vee)$ such that f is supported on $\mathfrak{P}x\mathfrak{P}$ where $x \in \mathfrak{S}_G(\rho)$ and $f(pxp') = \rho^\vee(p)f(x)\rho^\vee(p')$ for $p, p' \in \mathfrak{P}$. We shall now show there exists $\phi_0 \in \mathcal{H}(G, \rho)$ with support $\mathfrak{P}w_0\mathfrak{P}$ and satisfies $\phi_0^2 = q^{n/2} + (q^{n/2} - 1)\phi_0$. Let

$$\begin{aligned} K(0) &= \text{U}(n, n + 1) \cap \text{GL}_{2n+1}(\mathfrak{O}_E) = \{g \in \text{GL}_{2n+1}(\mathfrak{O}_E) \mid {}^t \bar{g}Jg = J\}, \\ K_1(0) &= \{g \in \text{Id}_{2n+1} + \varpi_E \text{M}_{2n+1}(\mathfrak{O}_E) \mid {}^t \bar{g}Jg = J\}, \\ G &= \{g \in \text{GL}_{2n+1}(k_E) \mid {}^t \bar{g}Jg = J\}. \end{aligned}$$

The map r from $K(0)$ to G given by $r : K(0) \xrightarrow{\text{mod } \mathfrak{p}_E} G$ is a surjective group homomorphism with kernel $K_1(0)$. So by the first isomorphism theorem of groups we have:

$$\frac{K(0)}{K_1(0)} \cong G.$$

$r(\mathfrak{P}) = P = \left[\begin{array}{ccc} \text{GL}_n(k_E) & M_{n \times 1}(k_E) & M_n(k_E) \\ 0 & \text{U}_1(k_E) & M_{1 \times n}(k_E) \\ 0 & 0 & \text{GL}_n(k_E) \end{array} \right] \cap G =$ Siegel parabolic sub-
group of G .

Now $P = L \ltimes U$, where L is the Siegel Levi component of P and U is the unipotent radical of P . Here

$$L = \left\{ \left[\begin{array}{ccc} a & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & {}^t\bar{a}^{-1} \end{array} \right] \mid a \in \text{GL}_n(k_E), \lambda \in E^\times, \lambda\bar{\lambda} = 1 \right\},$$

$$U = \left\{ \left[\begin{array}{ccc} Id_n & u & X \\ 0 & 1 & -{}^t\bar{u} \\ 0 & 0 & Id_n \end{array} \right] \mid X \in M_n(k_E), u \in M_{n \times 1}(k_E), X + {}^t\bar{X} + u{}^t\bar{u} = 0 \right\}.$$

If V be the vector space corresponding to ρ , the Hecke algebra $\mathcal{H}(K(0), \rho)$ is a sub-algebra of $\mathcal{H}(G, \rho)$. Let $\bar{\rho}$ be the representation of P which when inflated to \mathfrak{P} is given by ρ and V is also the vector space corresponding to $\bar{\rho}$. Recall the Hecke algebra $\mathcal{H}(G, \bar{\rho})$ has the same structure as was defined earlier in section 3.6.1 for the unramified case.

Now the homomorphism $r : K(0) \rightarrow G$ extends to a map from $\mathcal{H}(K(0), \rho)$ to $\mathcal{H}(G, \bar{\rho})$ which we again denote by r . Thus $r : \mathcal{H}(K(0), \rho) \rightarrow \mathcal{H}(G, \bar{\rho})$ is given by

$$r(\phi)(r(x)) = \phi(x)$$

for $\phi \in \mathcal{H}(K(0), \rho)$ and $x \in K(0)$.

As in the unramified case, when n is odd, we can show that $\mathcal{H}(K(0), \rho)$ is isomorphic to $\mathcal{H}(G, \bar{\rho})$ as algebras via r .

Let $w = r(w_0) = r\left(\begin{bmatrix} 0 & 0 & Id_n \\ 0 & 1 & 0 \\ Id_n & 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & Id_n \\ 0 & 1 & 0 \\ Id_n & 0 & 0 \end{bmatrix} \in G$. Clearly $K(0) \supseteq$

$\mathfrak{P} \amalg \mathfrak{P}w_0\mathfrak{P}$ and $G \supseteq P \amalg PwP$.

Now G is a finite group. In fact, it is the special orthogonal group consisting of matrices of size $(2n+1) \times (2n+1)$ over finite field k_E or \mathbb{F}_q . So $G = SO_{2n+1}(\mathbb{F}_q)$.

According to the Theorem 6.3 in [KM06], there exists a unique ϕ in $\mathcal{H}(G, \bar{\rho})$ with support PwP such that $\phi^2 = q^{n/2} + (q^{n/2} - 1)\phi$. Hence there is a unique element $\phi_0 \in \mathcal{H}(K(0), \rho)$ such that $r(\phi_0) = \phi$. Thus $\text{supp}(\phi_0) = \mathfrak{P}w_0\mathfrak{P}$ and $\phi_0^2 = q^{n/2} + (q^{n/2} - 1)\phi_0$. Now ϕ_0 can be extended to G and viewed as an element in $\mathcal{H}(G, \rho)$ as $\mathfrak{P}w_0\mathfrak{P} \subseteq K(0) \subseteq G$. Thus ϕ_0 satisfies the following relation in $\mathcal{H}(G, \rho)$:

$$\phi_0^2 = q^{n/2} + (q^{n/2} - 1)\phi_0.$$

We shall now show there exists $\phi_1 \in \mathcal{H}(G, \rho)$ with support $\mathfrak{P}\omega_1\mathfrak{P}$ satisfying the same relation as ϕ_0 .

$$\text{Recall that } \omega_1 = \begin{bmatrix} 0 & 0 & \overline{\varpi}_E^{-1} Id_n \\ 0 & 1 & 0 \\ \varpi_E Id_n & 0 & 0 \end{bmatrix}, \overline{\varpi}_E^{-1} = -\varpi_E^{-1}. \text{ So}$$

$$\omega_1 = \begin{bmatrix} 0 & 0 & -\varpi_E^{-1} Id_n \\ 0 & 1 & 0 \\ \varpi_E Id_n & 0 & 0 \end{bmatrix}$$

. Let $\eta \in U(n, n+1)$ be such that $\eta\omega_1\eta^{-1} = J' = \begin{bmatrix} 0 & 0 & -Id_n \\ 0 & 1 & 0 \\ Id_n & 0 & 0 \end{bmatrix}$ and

$$\eta \begin{bmatrix} GL_n(\mathfrak{O}_E) & M_{n \times 1}(\mathfrak{O}_E) & M_n(\mathfrak{O}_E) \\ M_{1 \times n}(\mathfrak{p}_E) & U_1(\mathfrak{O}_E) & M_{1 \times n}(\mathfrak{O}_E) \\ M_n(\mathfrak{p}_E) & M_{n \times 1}(\mathfrak{p}_E) & GL_n(\mathfrak{O}_E) \end{bmatrix} \eta^{-1} = \begin{bmatrix} GL_n(\mathfrak{O}_E) & M_{n \times 1}(\mathfrak{p}_E) & M_n(\mathfrak{p}_E) \\ M_{1 \times n}(\mathfrak{O}_E) & U_1(\mathfrak{O}_E) & M_{1 \times n}(\mathfrak{p}_E) \\ M_n(\mathfrak{O}_E) & M_{n \times 1}(\mathfrak{O}_E) & GL_n(\mathfrak{O}_E) \end{bmatrix}.$$

Recall that \mathfrak{P} looks as follows:

$$\mathfrak{P} = \begin{bmatrix} GL_n(\mathfrak{O}_E) & M_{n \times 1}(\mathfrak{O}_E) & M_n(\mathfrak{O}_E) \\ M_{1 \times n}(\mathfrak{p}_E) & U_1(\mathfrak{O}_E) & M_{1 \times n}(\mathfrak{O}_E) \\ M_n(\mathfrak{p}_E) & M_{n \times 1}(\mathfrak{p}_E) & GL_n(\mathfrak{O}_E) \end{bmatrix} \cap G.$$

Note that

$$\eta G \eta^{-1} = \{g \in GL_{2n+1}(E) \mid {}^t \bar{g} J' g = J'\}.$$

Hence

$$\eta \mathfrak{P} \eta^{-1} = \begin{bmatrix} GL_n(\mathfrak{O}_E) & M_{n \times 1}(\mathfrak{p}_E) & M_n(\mathfrak{p}_E) \\ M_{1 \times n}(\mathfrak{O}_E) & U_1(\mathfrak{O}_E) & M_{1 \times n}(\mathfrak{p}_E) \\ M_n(\mathfrak{O}_E) & M_{n \times 1}(\mathfrak{O}_E) & GL_n(\mathfrak{O}_E) \end{bmatrix} \cap \eta G \eta^{-1}.$$

Therefore $\eta \mathfrak{P} \eta^{-1}$ is the opposite of the Siegel Parahoric subgroup of $\eta G \eta^{-1}$.
Let

$$K'(0) = \langle \mathfrak{P}, \omega_1 \rangle.$$

And let

$$\begin{aligned} G' &= \{g \in GL_{2n+1}(k_E) \mid {}^t \bar{g} J' g = J'\} \\ &= \{g \in GL_{2n+1}(k_E) \mid {}^t g J' g = J'\}. \end{aligned}$$

Let $r' : K'(0) \longrightarrow G'$ be the group homomorphism given by

$$r'(x) = (\eta x \eta^{-1}) \text{ mod } \mathfrak{p}_E \text{ where } x \in K'(0).$$

So we have $r'(K(0)) = (\eta K'(0) \eta^{-1}) \text{ mod } \mathfrak{p}_E = (\eta \langle \mathfrak{P}, \omega_1 \rangle \eta^{-1}) \text{ mod } \mathfrak{p}_E$. Let

$$r'(\mathfrak{P}) = (\eta \mathfrak{P} \eta^{-1}) \text{ mod } \mathfrak{p}_E = \overline{\mathfrak{P}}'.$$

We can see that

$$r'(w_1) = (\eta w_1 \eta^{-1}) \text{mod } p_E = J' \text{mod } p_E = w' = \begin{bmatrix} 0 & 0 & -Id_n \\ 0 & 1 & 0 \\ Id_n & 0 & 0 \end{bmatrix}.$$

So

$$\bar{P}' = r'(\mathfrak{P}) = (\eta \mathfrak{P} \eta^{-1}) \text{mod } p_E = \left[\begin{array}{ccc} GL_n(k_E) & 0 & 0 \\ M_{1 \times n}(k_E) & U_1(k_E) & 0 \\ M_n(k_E) & M_{n \times 1}(k_E) & GL_n(k_E) \end{array} \right] \cap G'.$$

Clearly \bar{P}' is the opposite of Siegel parabolic subgroup of G' . So $r'(K'(0)) = \langle \bar{P}', w' \rangle = G'$, as \bar{P}' is a maximal subgroup of G' and w' does not lie in \bar{P}' . So r' is a surjective homomorphism of groups.

Let V be the vector space corresponding to ρ . Note that the Hecke algebra $\mathcal{H}(K'(0), \rho)$ is a sub-algebra of $\mathcal{H}(G, \rho)$.

Let $\bar{\rho}'$ be the representation of \bar{P}' which when inflated to ${}^n\mathfrak{P}$ is given by ${}^n\rho$ and V is also the vector space corresponding to $\bar{\rho}'$. Note that the Hecke algebra $\mathcal{H}(G', \bar{\rho}')$ has a similar structure as that of $\mathcal{H}(G, \bar{\rho})$ which was defined earlier.

Now the homomorphism $r' : K'(0) \rightarrow G'$ extends to a map

$$r' : \mathcal{H}(K'(0), \rho) \rightarrow \mathcal{H}(G', \bar{\rho}').$$

where $r' : \mathcal{H}(K'(0), \rho) \rightarrow \mathcal{H}(G', \bar{\rho}')$ is given by:

$$r'(\phi)(r'(x)) = \phi(x)$$

for $\phi \in \mathcal{H}(K'(0), \rho)$ and $x \in K'(0)$.

As in the unramified case when n is odd, we can show that $\mathcal{H}(K'(0), \rho)$ is isomorphic to $\mathcal{H}(G', \bar{\rho}')$ as algebras via r' .

Clearly, $K'(0) \supseteq \mathfrak{P} \amalg \mathfrak{P} w_1 \mathfrak{P}$ and $G' \supseteq \bar{P}' \amalg \bar{P}' w' \bar{P}'$.

Now G' is a finite group over the field K_E or \mathbb{F}_q . Note that $G' \cong Sp_{2n}(k_E)$. According to the Theorem 6.3 in [KM06], there exists a unique ϕ in $\mathcal{H}(G', \bar{\rho}')$ with support $\bar{P}' w' \bar{P}'$ such that $\phi^2 = q^{n/2} + (q^{n/2} - 1)\phi$. Hence there is a unique element $\phi_1 \in \mathcal{H}(K'(0), \rho)$ such that $r'(\phi_1) = \phi$. Thus $\text{supp}(\phi_1) = \mathfrak{P} w_1 \mathfrak{P}$ and $\phi_1^2 = q^{n/2} + (q^{n/2} - 1)\phi_1$. Now ϕ_1 can be extended to G and viewed as an element in $\mathcal{H}(G, \rho)$ as $\mathfrak{P} w_1 \mathfrak{P} \subseteq K'(0) \subseteq G$. Thus ϕ_1 satisfies the following relation in $\mathcal{H}(G, \rho)$:

$$\phi_1^2 = q^{n/2} + (q^{n/2} - 1)\phi_1.$$

Thus we have shown there exists $\phi_i \in \mathcal{H}(G, \rho)$ with $\text{supp}(\phi_i) = \mathfrak{P} w_i \mathfrak{P}$ satisfying $\phi_i^2 = q^{n/2} + (q^{n/2} - 1)\phi_i$ for $i = 0, 1$. It can be further shown as in

the unramified case that ϕ_0 and ϕ_1 generate the Hecke algebra $\mathcal{H}(G, \rho)$. Let us denote the Hecke algebra $\mathcal{H}(G, \rho)$ by \mathcal{A} . So

$$\mathcal{A} = \mathcal{H}(G, \rho) = \left\langle \phi_i : G \rightarrow \text{End}_{\mathbb{C}}(\rho^\vee) \left| \begin{array}{l} \phi_i \text{ is supported on } \mathfrak{P}w_i\mathfrak{P} \\ \text{and } \phi_i(pw_i p') = \rho^\vee(p)\phi_i(w_i)\rho^\vee(p') \\ \text{where } p, p' \in \mathfrak{P}, i = 0, 1 \end{array} \right. \right\rangle$$

where ϕ_i has support $\mathfrak{P}w_i\mathfrak{P}$ and ϕ_i satisfies the relation:

$$\phi_i^2 = q^{n/2} + (q^{n/2} - 1)\phi_i \text{ for } i = 0, 1.$$

Lemma 3.12. ϕ_0 and ϕ_1 are units in \mathcal{A} .

Proof. As $\phi_i^2 = q^{n/2} + (q^{n/2} - 1)\phi_i$ for $i = 0, 1$. So $\phi_i(\frac{\phi_i + (1 - q^{n/2})1}{q^{n/2}}) = 1$ for $i=0,1$. Hence ϕ_0 and ϕ_1 are units in \mathcal{A} . \square

As ϕ_0, ϕ_1 are units in \mathcal{A} which is an algebra, so $\psi = \phi_0\phi_1$ is a unit too in \mathcal{A} and $\psi^{-1} = \phi_1^{-1}\phi_0^{-1}$. As in the unramified case when n is odd, we can show that \mathcal{A} contains sub-algebra $\mathcal{B} = \mathbb{C}[\psi, \psi^{-1}]$ where

$$\mathcal{B} = \mathbb{C}[\psi, \psi^{-1}] = \left\{ c_k \psi^k + \dots + c_l \psi^l \mid \begin{array}{l} c_k, \dots, c_l \in \mathbb{C}; \\ k < l; k, l \in \mathbb{Z} \end{array} \right\}.$$

Further, as in the unramified case when n is odd, we can show that $\mathbb{C}[\psi, \psi^{-1}] \simeq \mathbb{C}[x, x^{-1}]$ as \mathbb{C} -algebras.

4. Final calculations to answer the question

4.1. Structure of $\mathcal{H}(L, \rho_0)$. In this section we describe the structure of $\mathcal{H}(L, \rho_0)$. Thus we need first to determine

$$N_L(\rho_0) = \{m \in N_L(\mathfrak{P}_0) \mid \rho_0^m \simeq \rho_0\}.$$

We know from lemma 3.2 that $N_{\text{GL}_n(E)}(K_0) = K_0 Z$, so we have $N_L(\mathfrak{P}_0) = Z(L)\mathfrak{P}_0$. Since $Z(L)$ clearly normalizes ρ_0 and ρ_0 is an irreducible cuspidal representation of \mathfrak{P}_0 , so $N_L(\rho_0) = Z(L)\mathfrak{P}_0 = \coprod_{n \in \mathbb{Z}} \mathfrak{P}_0 \zeta^n$.

Define $\alpha \in \mathcal{H}(L, \rho_0)$ by $\text{supp}(\alpha) = \mathfrak{P}_0 \zeta$ and $\alpha(\zeta) = 1_{V^\vee}$. We can show that $\alpha^n(\zeta^n) = (\alpha(\zeta))^n$ for $n \in \mathbb{Z}$ and $\text{supp}(\alpha^n) = \mathfrak{P}_0 \zeta^n \mathfrak{P}_0 = \mathfrak{P}_0 \zeta^n = \zeta^n \mathfrak{P}_0$ for $n \in \mathbb{Z}$. Further we can show that $\mathcal{H}(L, \rho_0) = \mathbb{C}[\alpha, \alpha^{-1}]$. For details refer to section 7 in [RSS21].

Proposition 4.1. *The unique algebra homomorphism $\mathbb{C}[x, x^{-1}] \longrightarrow \mathbb{C}[\alpha, \alpha^{-1}]$ given by $x \longrightarrow \alpha$ is an isomorphism. So $\mathbb{C}[\alpha, \alpha^{-1}] \simeq \mathbb{C}[x, x^{-1}]$.*

We have already shown before in sections 6.1 and 6.2 that $\mathcal{B} = \mathbb{C}[\psi, \psi^{-1}]$ is a sub-algebra of $\mathcal{A} = \mathcal{H}(G, \rho)$, where ψ is supported on $\mathfrak{P}\zeta\mathfrak{P}$ and $\mathcal{B} \cong \mathbb{C}[x, x^{-1}]$. As $\mathcal{H}(L, \rho_0) = \mathbb{C}[\alpha, \alpha^{-1}] \cong \mathbb{C}[x, x^{-1}]$, so $\mathcal{B} \cong \mathcal{H}(L, \rho_0)$ as \mathbb{C} -algebras. Hence $\mathcal{H}(L, \rho_0)$ can be viewed as a sub-algebra of $\mathcal{H}(G, \rho)$.

Now we would like to find out how simple $\mathcal{H}(L, \rho_0)$ -modules look like. Thus to understand them we need to find out how simple $\mathbb{C}[x, x^{-1}]$ -modules look like.

4.2. Calculation of simple $\mathcal{H}(L, \rho_0)$ -modules. Recall that $\mathcal{H}(L, \rho_0) = \mathbb{C}[\alpha, \alpha^{-1}]$. Note that $\mathbb{C}[\alpha, \alpha^{-1}] \cong \mathbb{C}[x, x^{-1}]$ as \mathbb{C} -algebras. It can be shown by direct calculation that the simple $\mathbb{C}[x, x^{-1}]$ -modules are of the form \mathbb{C}_λ for $\lambda \in \mathbb{C}^\times$, where \mathbb{C}_λ is the vector space \mathbb{C} with the $\mathbb{C}[x, x^{-1}]$ -module structure given by $x.z = \lambda z$ for $z \in \mathbb{C}_\lambda$.

So the distinct simple $\mathcal{H}(L, \rho_0)$ -modules (up to isomorphism) are the various \mathbb{C}_λ for $\lambda \in \mathbb{C}^\times$. The module structure is determined by $\alpha.z = \lambda z$ for $z \in \mathbb{C}_\lambda$.

4.3. Calculation of $\delta_P(\zeta)$. Let us recall the modulus character

$$\delta_P : P \longrightarrow \mathbb{R}_{>0}^\times$$

introduced in section 1. The character δ_P is given by

$$\delta_P(p) = \|\det(\text{Ad } p)|_{\text{Lie } U}\|_F$$

for $p \in P$, where $\text{Lie } U$ is the Lie algebra of U . We have

$$U = \left\{ \begin{bmatrix} Id_n & u & X \\ 0 & 1 & -{}^t\bar{u} \\ 0 & 0 & Id_n \end{bmatrix} \mid X \in M_n(E), u \in M_{n \times 1}(E), X + {}^t\bar{X} + u{}^t\bar{u} = 0 \right\},$$

$$\text{Lie } U = \left\{ \begin{bmatrix} 0 & u & X \\ 0 & 0 & -{}^t\bar{u} \\ 0 & 0 & 0 \end{bmatrix} \mid X \in M_n(E), u \in M_{n \times 1}(E), X + {}^t\bar{X} = 0 \right\}.$$

4.3.1. Unramified case: Recall $\zeta = \begin{bmatrix} \varpi_E Id_n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varpi_E^{-1} Id_n \end{bmatrix}$ in the unramified case. So

$$(\text{Ad } \zeta) \begin{bmatrix} Id_n & u & X \\ 0 & 1 & -{}^t\bar{u} \\ 0 & 0 & Id_n \end{bmatrix} = \zeta \begin{bmatrix} Id_n & u & X \\ 0 & 1 & -{}^t\bar{u} \\ 0 & 0 & Id_n \end{bmatrix} \zeta^{-1} = \begin{bmatrix} Id_n & \varpi_E u & \varpi_E^2 X \\ 0 & 1 & -\varpi_E {}^t\bar{u} \\ 0 & 0 & Id_n \end{bmatrix}.$$

Hence

$$\begin{aligned} \delta_P(\zeta) &= \|\det(\text{Ad } \zeta)|_{\text{Lie } U}\|_F \\ &= \|\varpi_E^{2n+2n^2}\|_F \\ &= \|\varpi_F^{2n+2n^2}\|_F \\ &= q^{-2n-2n^2}. \end{aligned}$$

4.3.2. Ramified case: Recall $\zeta = \begin{bmatrix} \varpi_E Id_n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\varpi_E^{-1} Id_n \end{bmatrix}$ in the ramified case. So

$$(Ad \zeta) \begin{bmatrix} Id_n & u & X \\ 0 & 1 & -{}^t \bar{u} \\ 0 & 0 & Id_n \end{bmatrix} = \zeta \begin{bmatrix} Id_n & u & X \\ 0 & 1 & -{}^t \bar{u} \\ 0 & 0 & Id_n \end{bmatrix} \zeta^{-1} = \begin{bmatrix} Id_n & \varpi_E u & -\varpi_E^2 X \\ 0 & 1 & \varpi_E {}^t \bar{u} \\ 0 & 0 & Id_n \end{bmatrix}.$$

Hence

$$\begin{aligned} \delta_P(\zeta) &= \|\det(Ad \zeta)|_{\text{Lie } U}\|_F \\ &= \|\varpi_E^{2n+2n^2}\|_F \\ &= \|\varpi_F^{n+n^2}\|_F \\ &= q^{-n-n^2}. \end{aligned}$$

4.4. Calculation of $(\phi_0 * \phi_1)(\zeta)$. In this section we calculate $(\phi_0 * \phi_1)(\zeta)$. Let $g_i = q^{-n/2} \phi_i$ for $i = 0, 1$ in the unramified case and $g_i = q^{-n/4} \phi_i$ for $i = 0, 1$ in the ramified case. Determining $(\phi_0 * \phi_1)(\zeta)$ would be useful in showing $g_0 * g_1 = T_P(\alpha)$ in both ramified and unramified cases. From now on, we assume without loss of generality that $\text{vol} \mathfrak{P}_0 = \text{vol} \mathfrak{P}_- = \text{vol} \mathfrak{P}_+ = 1$. Thus we have $\text{vol} \mathfrak{P} = 1$.

For $r \in \mathbb{Z}$ let,

$$K_{-,r} = \left\{ \begin{bmatrix} Id_n & 0 & 0 \\ -{}^t \bar{u} & 1 & 0 \\ X & u & Id_n \end{bmatrix} \mid X \in M_n(\mathfrak{p}_E^r), u \in M_{n \times 1}(\mathfrak{p}_E^r), X + {}^t \bar{X} + u {}^t \bar{u} = 0 \right\},$$

$$K_{+,r} = \left\{ \begin{bmatrix} Id_n & u & X \\ 0 & 1 & -{}^t \bar{u} \\ 0 & 0 & Id_n \end{bmatrix} \mid X \in M_n(\mathfrak{O}_E^r), u \in M_{n \times 1}(\mathfrak{O}_E^r), X + {}^t \bar{X} + u {}^t \bar{u} = 0 \right\}.$$

Proposition 4.2. $(\phi_0 * \phi_1)(\zeta) = \phi_0(w_0)\phi_1(w_1)$.

Proof. From Lemma 3.10, $\text{supp}(\phi_0 * \phi_1) = \mathfrak{P}\zeta\mathfrak{P} = \mathfrak{P}w_0w_1\mathfrak{P}$. So now let us consider

$$\begin{aligned} (\phi_0 * \phi_1)(\zeta) &= (\phi_0 * \phi_1)(w_0w_1) \\ &= \int_G \phi_0(y)\phi_1(y^{-1}\zeta)dy \\ &= \int_{\mathfrak{P}w_0\mathfrak{P}} \phi_0(y)\phi_1(y^{-1}\zeta)dy. \end{aligned}$$

We know that $\mathfrak{P}w_0\mathfrak{P} = \coprod_{z \in \mathfrak{P}w_0\mathfrak{P}/\mathfrak{P}} z\mathfrak{P}$. Let $y = zp \in z\mathfrak{P}$. So we have

$$\phi_0(y)\phi_1(y^{-1}\zeta) = \phi_0(zp)\phi_1(p^{-1}z^{-1}\zeta)$$

$$\begin{aligned} &= \phi_0(z)\rho^\vee(p)\rho^\vee(p^{-1})\phi_1(z^{-1}\zeta) \\ &= \phi_0(z)\phi_1(z^{-1}\zeta). \end{aligned}$$

Hence

$$(\phi_0 * \phi_1)(\zeta) = \sum_{z \in \mathfrak{P}w_0\mathfrak{P}/\mathfrak{P}} \phi_0(z)\phi_1(z^{-1}\zeta)\text{Vol}\mathfrak{P} = \sum_{z \in \mathfrak{P}w_0\mathfrak{P}/\mathfrak{P}} \phi_0(z)\phi_1(z^{-1}\zeta)$$

Let $\alpha : \mathfrak{P}/w_0\mathfrak{P}w_0^{-1} \cap \mathfrak{P} \rightarrow \mathfrak{P}w_0\mathfrak{P}/\mathfrak{P}$ be the map given by $\alpha(x(w_0\mathfrak{P}w_0^{-1} \cap \mathfrak{P})) = xw_0\mathfrak{P}$ where $x \in \mathfrak{P}$. We can observe that the map α is bijective. So $\mathfrak{P}/w_0\mathfrak{P}w_0^{-1} \cap \mathfrak{P}$ is in bijection with $\mathfrak{P}w_0\mathfrak{P}/\mathfrak{P}$.

Hence

$$(\phi_0 * \phi_1)(\zeta) = \sum_{x \in \mathfrak{P}/w_0\mathfrak{P}w_0^{-1} \cap \mathfrak{P}} \phi_0(xw_0)\phi_1(w_0^{-1}x^{-1}\zeta).$$

From Iwahori factorization of \mathfrak{P} we have $\mathfrak{P} = \mathfrak{P}_-\mathfrak{P}_0\mathfrak{P}_+ = K_{-,1}\mathfrak{P}_0K_{+,0}$. Therefore $w_0\mathfrak{P}w_0^{-1} = {}^{w_0}\mathfrak{P} = {}^{w_0}K_{-,1}{}^{w_0}\mathfrak{P}_0{}^{w_0}K_{+,0} = K_{+,-1}\mathfrak{P}_0K_{-,0}$. So $\mathfrak{P}_0 \cap w_0\mathfrak{P}w_0^{-1} = \mathfrak{P} \cap {}^{w_0}\mathfrak{P} = K_{+,-1}\mathfrak{P}_0K_{-,1}$. Let $\beta : \mathfrak{P}/w_0\mathfrak{P}w_0^{-1} \cap \mathfrak{P} \rightarrow K_{+,0}/K_{+,-1}$ be the map given by $\beta(x(\mathfrak{P} \cap {}^{w_0}\mathfrak{P})) = x_+K_{+,-1}$ where $x \in \mathfrak{P}$ and $x = x_+px_-, x_+ \in \mathfrak{P}_+, p \in \mathfrak{P}_0, x_- \in \mathfrak{P}_-$. We can observe that the map β is bijective. So $\mathfrak{P}/w_0\mathfrak{P}w_0^{-1} \cap \mathfrak{P}$ is in bijection with $K_{+,0}/K_{+,-1}$.

Therefore

$$\begin{aligned} (\phi_0 * \phi_1)(\zeta) &= \sum_{x_+ \in K_{+,0}/K_{+,-1}} \phi_0(x_+w_0)\phi_1(w_0^{-1}x_+^{-1}\zeta) \\ &= \sum_{x_+ \in K_{+,0}/K_{+,-1}} \rho^\vee(x_+)\phi_0(w_0)\phi_1(w_0^{-1}x_+^{-1}\zeta). \end{aligned}$$

As ρ^\vee is trivial on \mathfrak{P}_+ and $x_+ \in \mathfrak{P}_+$ so we have

$$(\phi_0 * \phi_1)(\zeta) = \sum_{x_+ \in K_{+,0}/K_{+,-1}} \phi_0(w_0)\phi_1(w_0^{-1}x_+^{-1}\zeta).$$

The terms in above summation which do not vanish are the ones for which $w_0^{-1}x_+^{-1}\zeta \in \mathfrak{P}w_1\mathfrak{P} \implies x_+^{-1} \in w_0\mathfrak{P}w_1\mathfrak{P}\zeta^{-1} \implies x_+ \in \zeta\mathfrak{P}w_1^{-1}\mathfrak{P}w_0^{-1} \implies w_0^{-1}x_+w_0 \in w_1\mathfrak{P}w_1^{-1}\mathfrak{P}$. It is clear $w_1\mathfrak{P}w_1^{-1}\mathfrak{P} = ({}^{w_1}\mathfrak{P})(\mathfrak{P})$. As ${}^{w_1}\mathfrak{P} = {}^{w_1}K_{-,1}{}^{w_1}\mathfrak{P}_0{}^{w_1}K_{+,-1} = K_{-,2}\mathfrak{P}_0K_{+,-1}$, so

$$w_1\mathfrak{P}w_1^{-1}\mathfrak{P} = ({}^{w_1}\mathfrak{P})(\mathfrak{P}) = K_{-,2}\mathfrak{P}_0K_{+,-1}\mathfrak{P}_0K_{-,1}$$

Hence we have $w_0^{-1}x_+w_0 \in K_{-,2}\mathfrak{P}_0K_{+,-1}\mathfrak{P}_0K_{-,1}$ which implies that $w_0^{-1}x_+w_0 = k_-p_0k_+k'_-$ where $k_- \in K_{-,2}, k_+ \in K_{+,-1}, k'_- \in K_{-,1}, p_0 \in \mathfrak{P}_0$. Hence we have $p_0k_+ = k_-^{-1}w_0^{-1}x_+w_0k'_-^{-1}$. Now as $w_0^{-1}x_+w_0 \in K_{-,0}, k_-^{-1} \in K_{-,2}, k'_-^{-1} \in K_{-,1}$, so $k_-^{-1}w_0^{-1}x_+w_0k'_-^{-1} \in K_{-,0}$ and $p_0k_+ \in \mathfrak{P}_0K_{+,-1}$. But we know that $K_{-,0} \cap \mathfrak{P}_0K_{+,-1} = 1 \implies p_0k_+ = 1 \implies w_0^{-1}x_+w_0 = k_-k'_- \in K_{-,1} \implies x_+ \in w_0K_{-,1}w_0^{-1} = K_{+,-1}$. As $x_+ \in K_{+,-1}$, so only the trivial coset contributes to the above summation. Hence

$$(\phi_0 * \phi_1)(\zeta) = \phi_0(w_0)\phi_1(w_0^{-1}\zeta) = \phi_0(w_0)\phi_1(w_1).$$

□

4.5. Relation between g_0, g_1 and $T_P(\alpha)$.

4.5.1. Unramified case: Recall that $\mathcal{H}(G, \rho) = \langle \phi_0, \phi_1 \rangle$ where ϕ_0 is supported on $\mathfrak{P}w_0\mathfrak{P}$ and ϕ_1 is supported on $\mathfrak{P}w_1\mathfrak{P}$ respectively with $\phi_i^2 = q^n + (q^n - 1)\phi_i$ for $i = 0, 1$. In this section we show that $g_0 * g_1 = T_P(\alpha)$, where $g_i = q^{-n/2}\phi_i$ for $i = 0, 1$.

Proposition 4.3. $g_0g_1 = T_P(\alpha)$.

Proof. Let us choose $\psi_i \in \mathcal{H}(G, \rho)$ for $i = 0, 1$ such that $\text{supp}(\psi_i) = \mathfrak{P}w_i\mathfrak{P}$ for $i = 0, 1$. So ϕ_i is a scalar multiple of ψ_i for $i = 0, 1$. Hence $\phi_i = \lambda_i\psi_i$ where $\lambda_i \in \mathbb{C}^\times$ for $i = 0, 1$. Let $\psi_i(w_i) = A \in \text{Hom}_{\mathfrak{P}w_i\mathfrak{P}}(w_i\rho^\vee, \rho^\vee)$ for $i = 0, 1$ and W be the space of ρ . So $A^2 = 1_{W^\vee}$. From Proposition 4.2, we have $(\psi_0 * \psi_1)(\zeta) = \psi_0(w_0)\psi_1(w_1) = A^2 = 1_{W^\vee}$. Now let ψ_i satisfies the quadratic relation given by $\psi_i^2 = a\psi_i + b$ where $a, b \in \mathbb{R}$ for $i = 0, 1$. As $\psi_i^2 = a\psi_i + b \implies (-\psi_i)^2 = (-a)(-\psi_i) + b$, so a can be arranged such that $a > 0$. We can see that $1 \in \mathcal{H}(G, \rho)$ is defined as below:

$$1(x) = \begin{cases} 0, & \text{if } x \notin \mathfrak{P}; \\ \rho^\vee(x) & \text{if } x \in \mathfrak{P}. \end{cases}$$

Let us consider $\psi_i^2(1) = \int_G \psi_i(y)\psi_i(y^{-1})dy$ for $i = 0, 1$. Now let $y = pw_ip'$ where $p, p' \in \mathfrak{P}$ for $i = 0, 1$. So we have

$$\begin{aligned} \psi_i^2(1) &= \int_{\mathfrak{P}w_i\mathfrak{P}} \psi_i(pw_ip')\psi_i(p'^{-1}w_i^{-1}p^{-1})d(pw_ip') \\ &= \int_{\mathfrak{P}w_i\mathfrak{P}} \rho^\vee(p)\psi_i(w_i)\rho^\vee(p')\rho^\vee(p'^{-1})\psi_i(w_i^{-1})\rho^\vee(p^{-1})d(pw_ip') \\ &= \int_{\mathfrak{P}w_i\mathfrak{P}} \rho^\vee(p)\psi_i(w_i)\psi_i(w_i^{-1})\rho^\vee(p^{-1})d(pw_ip') \\ &= \int_{\mathfrak{P}w_i\mathfrak{P}} \rho^\vee(p)\psi_i(w_i)\psi_i(w_i)\rho^\vee(p^{-1})d(pw_ip') \\ &= \int_{\mathfrak{P}w_i\mathfrak{P}} \rho^\vee(p)A^2\rho^\vee(p^{-1})d(pw_ip') \\ &= \int_{\mathfrak{P}w_i\mathfrak{P}} A^2\rho^\vee(p)\rho^\vee(p^{-1})d(pw_ip') \\ &= A^2\text{vol}(\mathfrak{P}w_i\mathfrak{P}) \\ &= 1_{W^\vee}\text{vol}(\mathfrak{P}w_i\mathfrak{P}). \end{aligned}$$

So $\psi_i^2(1) = 1_{W^\vee}\text{vol}(\mathfrak{P}w_i\mathfrak{P})$ for $i = 0, 1$. We already know that $\psi_i^2 = a\psi_i + b$ where $a, b \in \mathbb{R}$ and for $i = 0, 1$. Now evaluating the expression $\psi_i^2 = a\psi_i + b$

at 1, we have $\psi_i^2(1) = a\psi_i(1) + b1(1)$. We can see that $\psi_i(1) = 0$ as support of ψ_i is $\mathfrak{P}w_i\mathfrak{P}$ for $i = 0, 1$. We have seen before that $\psi_i^2(1) = 1_{W^\vee}\text{vol}(\mathfrak{P}w_i\mathfrak{P})$ for $i = 0, 1$ and as $1 \in \mathfrak{P}, 1(1) = \rho^\vee(1) = 1_{W^\vee}$. So $\psi_i^2(1) = a\psi_i(1) + b1(1) \implies 1_{W^\vee}\text{vol}(\mathfrak{P}w_i\mathfrak{P}) = 1_{W^\vee}b$ for $i = 0, 1$. Comparing coefficients of 1_{W^\vee} on both sides of the equation $1_{W^\vee}\text{vol}(\mathfrak{P}w_i\mathfrak{P}) = 1_{W^\vee}b$ for $i = 0, 1$ we get

$$b = \text{vol}(\mathfrak{P}w_i\mathfrak{P}).$$

As $\phi_i = \lambda_i\psi_i$ for $i = 0, 1$, hence $\phi_i^2 = \lambda_i^2\psi_i^2 = \lambda_i^2(a\psi_i + b) = (\lambda_i a)(\lambda_i\psi_i) + \lambda_i^2 b = (\lambda_i a)\phi_i + \lambda_i^2 b$ for $i = 0, 1$. But $\phi_i^2 = (q^n - 1)\phi_i + q^n$ for $i = 0, 1$. So $\phi_i^2 = (\lambda_i a)\phi_i + \lambda_i^2 b = (q^n - 1)\phi_i + q^n$ for $i = 0, 1$. As ϕ_i and 1 are linearly independent, hence $\lambda_i a = (q^n - 1)$ for $i = 0, 1$. Therefore $\lambda_i = \frac{q^n - 1}{a}$ for $i = 0, 1$. As $a > 0, a \in \mathbb{R}$, so $\lambda_i > 0, \lambda_i \in \mathbb{R}$ for $i = 0, 1$. Similarly, as ϕ_i and 1 are linearly independent, hence $\lambda_i^2 b = q^n \implies \lambda_i^2 = \frac{q^n}{b}$ for $i = 0, 1$.

Now $\mathfrak{P}w_i\mathfrak{P} = \coprod_{x \in \mathfrak{P}/\mathfrak{P} \cap^{w_i}\mathfrak{P}} xw_i\mathfrak{P} \implies \text{vol}(\mathfrak{P}w_i\mathfrak{P}) = [\mathfrak{P}w_i\mathfrak{P} : \mathfrak{P}]\text{vol}\mathfrak{P} = [\mathfrak{P}w_i\mathfrak{P} : \mathfrak{P}] = [\mathfrak{P} : \mathfrak{P} \cap^{w_i}\mathfrak{P}]$ for $i = 0, 1$. Hence $b = \text{vol}(\mathfrak{P}w_i\mathfrak{P}) = [\mathfrak{P} : \mathfrak{P} \cap^{w_i}\mathfrak{P}]$ for $i = 0, 1$. Now as $\lambda_0^2 = \lambda_1^2 = \frac{q^n}{b} \implies \lambda_0 = \lambda_1 = \frac{q^{n/2}}{b^{1/2}} = \frac{q^{n/2}}{[\mathfrak{P} : \mathfrak{P} \cap^{w_0}\mathfrak{P}]^{1/2}}$. Therefore

$$\begin{aligned} \phi_0\phi_1 &= (\lambda_0\psi_0)(\lambda_1\psi_1) \\ &= \lambda_0^2\psi_0\psi_1 \\ &= \frac{q^n\psi_0\psi_1}{[\mathfrak{P} : \mathfrak{P} \cap^{w_0}\mathfrak{P}]} \end{aligned}$$

We have seen before that, $\mathfrak{P} = K_{-,1}\mathfrak{P}_0K_{+,0}$ and $\mathfrak{P} \cap^{w_0}\mathfrak{P} = K_{-,1}\mathfrak{P}_0K_{+,1}$. So

$$\begin{aligned} [\mathfrak{P} : \mathfrak{P} \cap^{w_0}\mathfrak{P}] &= \left| \frac{K_{+,0}}{K_{+,1}} \right| \\ &= |\{X \in M_n(k_E), u \in M_{n \times 1}(k_E) \mid X + {}^t\bar{X} + u {}^t\bar{u} = 0\}| \\ &= (q^{2n})(q)^{\frac{(n)(n-1)}{2}} \\ &= (q^{2n})(q^{n^2-n}) \\ &= q^{n^2+n}. \end{aligned}$$

Hence

$$\begin{aligned} (\phi_0\phi_1)(\zeta) &= \frac{q^n(\psi_0\psi_1)(\zeta)}{[\mathfrak{P} : \mathfrak{P} \cap^{w_0}\mathfrak{P}]} \\ &= \frac{q^n(\psi_0\psi_1)(\zeta)}{q^{n^2+n}} \\ &= q^{-n^2}1_{W^\vee}. \end{aligned}$$

Recall $g_i = q^{-n/2}\phi_i$ for $i = 0, 1$. We know that $\phi_i^2 = (q^n - 1)\phi_i + q^n$ for $i = 0, 1$. So for $i = 0, 1$ we have

$$\begin{aligned} g_i^2 &= q^{-n}\phi_i^2 \\ &= q^{-n}((q^n - 1)\phi_i + q^n) \\ &= (1 - q^{-n})\phi_i + 1 \\ &= (1 - q^{-n})q^{n/2}g_i + 1 \\ &= (q^{n/2} - q^{-n/2})g_i + 1. \end{aligned}$$

So $g_0g_1 = (q^{-n/2}\phi_1)(q^{-n/2}\phi_2) = q^{-n}\phi_1\phi_2$ implies that

$$(g_0g_1)(\zeta) = q^{-n}(\phi_1\phi_2)(\zeta) = q^{-n}q^{-n^2}1_{W^\vee} = q^{-n^2-n}1_{W^\vee}.$$

From the earlier discussion in this section we have $T_P(\alpha)(\zeta) = \delta_P^{1/2}(\zeta)1_{W^\vee}$. From section 4.3, we have $\delta_P(\zeta) = q^{-2n^2-2n}$. Hence $\delta_P^{1/2}(\zeta) = q^{-n^2-n}$. Therefore $(g_0g_1)(\zeta) = T_P(\alpha)(\zeta)$. So $(g_0g_1)(\zeta) = T_P(\alpha)(\zeta)$. We have $\text{supp}(T_P(\alpha)) = \mathfrak{P}\zeta\mathfrak{P}$. As $\text{supp}(g_i) = \mathfrak{P}w_i\mathfrak{P}$, Lemma 3.10 then gives $\text{supp}(g_0g_1) = \mathfrak{P}\zeta\mathfrak{P}$. Therefore $g_0g_1 = T_P(\alpha)$. \square

4.5.2. Ramified case: We know that $\mathcal{H}(G, \rho) = \langle \phi_0, \phi_1 \rangle$ where ϕ_0 is supported on $\mathfrak{P}w_0\mathfrak{P}$ and ϕ_1 is supported on $\mathfrak{P}w_1\mathfrak{P}$ respectively with $\phi_i^2 = q^{n/2} + (q^{n/2} - 1)\phi_i$ for $i = 0, 1$. In this section we show that $g_0 * g_1 = T_P(\alpha)$, where $g_i = q^{-n/4}\phi_i$ for $i = 0, 1$.

Proposition 4.4. $g_0g_1 = T_P(\alpha)$.

Proof. Let us choose $\psi_i \in \mathcal{H}(G, \rho)$ for $i = 0, 1$ such that $\text{supp}(\psi_i) = \mathfrak{P}w_i\mathfrak{P}$ for $i = 0, 1$. So ϕ_i is a scalar multiple of ψ_i for $i = 0, 1$. Hence $\phi_i = \lambda_i\psi_i$ where $\lambda_i \in \mathbb{C}^\times$ for $i = 0, 1$. Let $\psi_i(w_i) = A_i \in \text{Hom}_{\mathfrak{P} \cap w_i\mathfrak{P}}(w_i\rho^\vee, \rho^\vee)$ for $i = 0, 1$ and W be the space of ρ . So $A_i^2 = 1_{W^\vee}$ for $i = 0, 1$. From section 5.1 on page 24 in [KM06], we can say that $A_0 = A_1$. From Proposition 4.2, we have $(\psi_0 * \psi_1)(\zeta) = \psi_0(w_0)\psi_1(w_1) = A_0A_1 = A_0^2 = 1_{W^\vee}$. Now let ψ_i satisfies the quadratic relation given by $\psi_i^2 = a_i\psi_i + b_i$ where $a_i, b_i \in \mathbb{R}$ for $i = 0, 1$. As $\psi_i^2 = a_i\psi_i + b_i \implies (-\psi_i)^2 = (-a_i)(-\psi_i) + b_i$, so a_i can be arranged such that $a_i > 0$ for $i = 0, 1$. We can see that $1 \in \mathcal{H}(G, \rho)$ is defined as below:

$$1(x) = \begin{cases} 0, & \text{if } x \notin \mathfrak{P}; \\ \rho^\vee(x) & \text{if } x \in \mathfrak{P}. \end{cases}$$

Let us consider $\psi_0^2(1) = \int_G \psi_0(y)\psi_0(y^{-1})dy$. Now let $y = pw_0p'$ where $p, p' \in \mathfrak{P}$. So we have

$$\psi_0^2(1) = \int_{\mathfrak{P}w_0\mathfrak{P}} \psi_0(pw_0p')\psi_0(p'^{-1}w_0^{-1}p^{-1})d(pw_0p')$$

$$\begin{aligned}
 &= \int_{\mathfrak{P}w_0\mathfrak{P}} \rho^\vee(p)\psi_0(w_0)\rho^\vee(p')\rho^\vee(p'^{-1})\psi_0(w_0^{-1})\rho^\vee(p^{-1})d(pw_0p') \\
 &= \int_{\mathfrak{P}w_0\mathfrak{P}} \rho^\vee(p)\psi_0(w_0)\psi_0(w_0^{-1})\rho^\vee(p^{-1})d(pw_0p') \\
 &= \int_{\mathfrak{P}w_0\mathfrak{P}} \rho^\vee(p)\psi_0(w_0)\psi_0(w_0)\rho^\vee(p^{-1})d(pw_0p') \\
 &= \int_{\mathfrak{P}w_0\mathfrak{P}} \rho^\vee(p)A_0^2\rho^\vee(p^{-1})d(pw_0p') \\
 &= \int_{\mathfrak{P}w_0\mathfrak{P}} A_0^2\rho^\vee(p)\rho^\vee(p^{-1})d(pw_0p') \\
 &= A_0^2\text{vol}(\mathfrak{P}w_0\mathfrak{P}) \\
 &= 1_{W^\vee}\text{vol}(\mathfrak{P}w_0\mathfrak{P}).
 \end{aligned}$$

So $\psi_0^2(1) = 1_{W^\vee}\text{vol}(\mathfrak{P}w_0\mathfrak{P})$. We already know that $\psi_0^2 = a_0\psi_0 + b_0$ where $a_0, b_0 \in \mathbb{R}$. Now evaluating the expression $\psi_0^2 = a_0\psi_0 + b_0$ at 1, we have $\psi_0^2(1) = a_0\psi_0(1) + b_01(1)$. We can see that $\psi_0(1) = 0$ as support of ψ_0 is $\mathfrak{P}w_0\mathfrak{P}$. We have seen before that $\psi_0^2(1) = 1_{W^\vee}\text{vol}(\mathfrak{P}w_0\mathfrak{P})$ and as $1 \in \mathfrak{P}, 1(1) = \rho^\vee(1) = 1_{W^\vee}$. So

$$\psi_0^2(1) = a_0\psi_0(1) + b_01(1) \implies 1_{W^\vee}\text{vol}(\mathfrak{P}w_0\mathfrak{P}) = 1_{W^\vee}b_0.$$

Comparing coefficients of 1_{W^\vee} on both sides of the equation

$$1_{W^\vee}b_0 = 1_{W^\vee}\text{vol}(\mathfrak{P}w_0\mathfrak{P}),$$

we get $b_0 = \text{vol}(\mathfrak{P}w_0\mathfrak{P})$.

As $\phi_0 = \lambda_0\psi_0$, hence $\phi_0^2 = \lambda_0^2\psi_0^2 = \lambda_0^2(a_0\psi_0 + b_0) = (\lambda_0a_0)(\lambda_0\psi_0) + \lambda_0^2b_0 = (\lambda_0a_0)\phi_0 + \lambda_0^2b_0$. But $\phi_0^2 = (q^{n/2} - 1)\phi_0 + q^{n/2}$. So $\phi_0^2 = (\lambda_0a_0)\phi_0 + \lambda_0^2b_0 = (q^{n/2} - 1)\phi_0 + q^{n/2}$. As ϕ_0 and 1 are linearly independent, hence $\lambda_0a_0 = (q^{n/2} - 1)$. Therefore $\lambda_0 = \frac{q^{n/2}-1}{a_0}$. As $a_0 > 0, a_0 \in \mathbb{R}$, so $\lambda_0 > 0, \lambda_0 \in \mathbb{R}$. Similarly, as ϕ_0

and 1 are linearly independent, hence $\lambda_0^2b_0 = q^{n/2} \implies \lambda_0^2 = \frac{q^{n/2}}{b_0}$.

Now $\mathfrak{P}w_0\mathfrak{P} = \coprod_{x \in \mathfrak{P}/\mathfrak{P} \cap w_0\mathfrak{P}} xw_0\mathfrak{P} \implies \text{vol}(\mathfrak{P}w_0\mathfrak{P}) = [\mathfrak{P}w_0\mathfrak{P} : \mathfrak{P}]\text{vol}\mathfrak{P} = [\mathfrak{P}w_0\mathfrak{P} : \mathfrak{P}] = [\mathfrak{P} : \mathfrak{P} \cap w_0\mathfrak{P}]$. Hence $b_0 = \text{vol}(\mathfrak{P}w_0\mathfrak{P}) = [\mathfrak{P} : \mathfrak{P} \cap w_0\mathfrak{P}]$. Now as $\lambda_0^2 = \frac{q^{n/2}}{b_0} \implies \lambda_0 = \frac{q^{n/4}}{b_0^{1/2}} = \frac{q^{n/4}}{[\mathfrak{P} : \mathfrak{P} \cap w_0\mathfrak{P}]^{1/2}}$.

We have seen before that, $\mathfrak{P} = K_{-,1}\mathfrak{P}_0K_{+,0}$ and $\mathfrak{P} \cap w_0\mathfrak{P} = K_{-,1}\mathfrak{P}_0K_{+,1}$. So

$$\begin{aligned}
 [\mathfrak{P} : \mathfrak{P} \cap w_0\mathfrak{P}] &= \left| \frac{K_{+,0}}{K_{+,1}} \right| \\
 &= |\{X \in M_n(k_E), u \in M_{n \times 1}(k_E) \mid X + {}^t\bar{X} + u^t\bar{u} = 0\}| \\
 &= (q^n)(q^{\binom{n-1}{2}})
 \end{aligned}$$

$$= q^{\frac{n^2+n}{2}}.$$

So

$$\lambda_0 = \frac{q^{n/4}}{[\mathfrak{P} : \mathfrak{P} \cap w_0 \mathfrak{P}]^{1/2}} = \frac{q^{n/4}}{q^{\frac{n^2+n}{4}}}.$$

Let us consider $\psi_1^2(1) = \int_G \psi_1(y)\psi_1(y^{-1})dy$. Now let $y = pw_1p'$ where $p, p' \in \mathfrak{P}$. So we have

$$\begin{aligned} \psi_1^2(1) &= \int_{\mathfrak{P}w_1\mathfrak{P}} \psi_1(pw_1p')\psi_1(p'^{-1}w_1^{-1}p^{-1})d(pw_1p') \\ &= \int_{\mathfrak{P}w_1\mathfrak{P}} \rho^\vee(p)\psi_1(w_1)\rho^\vee(p')\rho^\vee(p'^{-1})\psi_1(w_1^{-1})\rho^\vee(p^{-1})d(pw_1p') \\ &= \int_{\mathfrak{P}w_1\mathfrak{P}} \rho^\vee(p)\psi_1(w_1)\psi_1(w_1^{-1})\rho^\vee(p^{-1})d(pw_1p') \\ &= \int_{\mathfrak{P}w_1\mathfrak{P}} \rho^\vee(p)\psi_1(w_1)\psi_1(-w_1)\rho^\vee(p^{-1})d(pw_1p') \\ &= \int_{\mathfrak{P}w_1\mathfrak{P}} \rho^\vee(p)\psi_1(w_1)\rho^\vee(-1)\psi_1(w_1)\rho^\vee(p^{-1})d(pw_1p') \\ &= \rho^\vee(-1) \int_{\mathfrak{P}w_1\mathfrak{P}} A_1^2 \rho^\vee(p)\rho^\vee(p^{-1})d(pw_1p') \\ &= \rho^\vee(-1)A_1^2 \text{vol}(\mathfrak{P}w_1\mathfrak{P}) \\ &= \rho^\vee(-1)1_{W^\vee} \text{vol}(\mathfrak{P}w_1\mathfrak{P}). \end{aligned}$$

So $\psi_1^2(1) = 1_{W^\vee} \text{vol}(\mathfrak{P}w_1\mathfrak{P})$. We already know that $\psi_1^2 = a_1\psi_1 + b_1$ where $a_1, b_1 \in \mathbb{R}$. Now evaluating the expression $\psi_1^2 = a_1\psi_1 + b_1$ at 1, we have $\psi_1^2(1) = a_1\psi_1(1) + b_11(1)$. We can see that $\psi_1(1) = 0$ as support of ψ_1 is $\mathfrak{P}w_1\mathfrak{P}$. We have seen before that $\psi_1^2(1) = 1_{W^\vee} \text{vol}(\mathfrak{P}w_1\mathfrak{P})$ and as $1 \in \mathfrak{P}$, $1(1) = \rho^\vee(1) = 1_{W^\vee}$. So

$$\psi_1^2(1) = a_1\psi_1(1) + b_11(1) \implies \rho^\vee(-1)1_{W^\vee} \text{vol}(\mathfrak{P}w_1\mathfrak{P}) = 1_{W^\vee}b_1.$$

Comparing coefficients of 1_{W^\vee} on both sides of the equation

$$1_{W^\vee}b_1 = 1_{W^\vee}\rho^\vee(-1)\text{vol}(\mathfrak{P}w_1\mathfrak{P}),$$

we get $b_1 = \rho^\vee(-1)\text{vol}(\mathfrak{P}w_1\mathfrak{P})$.

As $\phi_1 = \lambda_1\psi_1$, hence $\phi_1^2 = \lambda_1^2\psi_1^2 = \lambda_1^2(a_1\psi_1 + b_1) = (\lambda_1a_1)(\lambda_1\psi_1) + \lambda_1^2b_1 = (\lambda_0a_1)\phi_1 + \lambda_1^2b_1$. But $\phi_1^2 = (q^{n/2} - 1)\phi_1 + q^{n/2}$. So $\phi_1^2 = (\lambda_1a_1)\phi_1 + \lambda_1^2b_1 = (q^{n/2} - 1)\phi_1 + q^{n/2}$. As ϕ_1 and 1 are linearly independent, hence $\lambda_1a_1 = (q^{n/2} - 1)$. Therefore $\lambda_1 = \frac{q^{n/2}-1}{a_1}$. As $a_1 > 0, a_1 \in \mathbb{R}$, so $\lambda_1 > 0, \lambda_1 \in \mathbb{R}$. Similarly, as ϕ_1 and 1 are linearly independent, hence $\lambda_1^2b_1 = q^{n/2} \implies \lambda_1^2 = \frac{q^{n/2}}{b_1}$.

Now $\mathfrak{P}w_1\mathfrak{P} = \coprod_{x \in \mathfrak{P}/\mathfrak{P} \cap {}^{w_1}\mathfrak{P}} xw_1\mathfrak{P} \implies \text{vol}(\mathfrak{P}w_1\mathfrak{P}) = [\mathfrak{P}w_1\mathfrak{P} : \mathfrak{P}]\text{vol}\mathfrak{P} = [\mathfrak{P}w_1\mathfrak{P} : \mathfrak{P}] = [\mathfrak{P} : \mathfrak{P} \cap {}^{w_1}\mathfrak{P}]$. Hence $b_1 = \text{vol}(\mathfrak{P}w_1\mathfrak{P}) = [\mathfrak{P} : \mathfrak{P} \cap {}^{w_1}\mathfrak{P}]$. Now as $\lambda_1^2 = \frac{q^{n/2}}{b_1} \implies \lambda_1 = \frac{q^{n/4}}{b_1^{1/2}} = \frac{q^{n/4}}{[\mathfrak{P} : \mathfrak{P} \cap {}^{w_1}\mathfrak{P}]^{1/2}}$.

We have seen before that $\mathfrak{P} = K_{-,1}\mathfrak{P}_0K_{+,0}$, ${}^{w_1}\mathfrak{P} = K_{-,2}\mathfrak{P}_0K_{+,-1}$. So $\mathfrak{P} \cap {}^{w_1}\mathfrak{P} = K_{-,2}\mathfrak{P}_0K_{+,0}$. Hence

$$\begin{aligned} [\mathfrak{P} : \mathfrak{P} \cap {}^{w_1}\mathfrak{P}] &= \left| \frac{K_{-,1}}{K_{-,2}} \right| \\ &= |\{X \in M_n(k_E), u \in M_{n \times 1}(k_E) \mid X^{-t}\bar{X} - u^t\bar{u} = 0\}| \\ &= q^{\frac{n^2+n}{2}}. \end{aligned}$$

So

$$\lambda_1 = \frac{q^{n/4}}{[\mathfrak{P} : \mathfrak{P} \cap {}^{w_1}\mathfrak{P}]^{1/2}} = \frac{q^{n/4}}{q^{\frac{n^2+n}{4}}(\rho(-1))^{1/2}}.$$

Hence

$$\begin{aligned} (\phi_0\phi_1)(\zeta) &= (\lambda_0\psi_0)(\lambda_1\psi_1)(\zeta) \\ &= (\lambda_0\lambda_1)(\psi_0\psi_1)(\zeta) \\ &= \frac{q^{n/4}}{q^{\frac{n^2+n}{4}}} \frac{q^{n/4}}{q^{\frac{n^2+n}{4}}(\rho(-1))^{1/2}} 1_W^\vee \\ &= \frac{q^{-\frac{n^2}{2}} 1_W^\vee}{(\rho(-1))^{1/2}}. \end{aligned}$$

As $-1 \in Z(\mathfrak{P})$ and ρ^\vee is a representation of \mathfrak{P} , so $\rho^\vee(-1) = \omega_{\rho^\vee}(-1)$ where ω_{ρ^\vee} is the central character of \mathfrak{P} . Now $1 = \omega_{\rho^\vee}(1) = (\omega_{\rho^\vee}(-1))^2$, so $\rho^\vee(-1) = \omega_{\rho^\vee}(-1) = \pm 1$. We have seen before that $\lambda_1 = \frac{q^{n/2-1}}{a_1}$ and $a_1 \in \mathbb{R}, a_1 > 0$, so $\lambda_1 > 0$. But we know that $\lambda_1 = \frac{q^{n/4}}{[\mathfrak{P} : \mathfrak{P} \cap {}^{w_1}\mathfrak{P}]^{1/2}} = \frac{q^{n/4}}{q^{\frac{n^2+n}{4}}(\rho(-1))^{1/2}}$, hence $\rho^\vee(-1) = 1$.

Recall $g_i = q^{-n/4}\phi_i$ for $i = 0, 1$. We know that $\phi_i^2 = (q^{n/2} - 1)\phi_i + q^{n/2}$ for $i = 0, 1$. So for $i = 0, 1$ we have

$$\begin{aligned} g_i^2 &= q^{-n/2}\phi_i^2 \\ &= q^{-n/2}((q^{n/2} - 1)\phi_i + q^{n/2}) \\ &= (1 - q^{-n/2})\phi_i + 1 \\ &= (1 - q^{-n/2})q^{n/4}g_i + 1 \\ &= (q^{n/4} - q^{-n/4})g_i + 1. \end{aligned}$$

So $g_0g_1 = q^{-n/2}\phi_1\phi_2$, which implies that

$$(g_0g_1)(\zeta) = q^{-n/2}(\phi_0\phi_1)(\zeta) = q^{-n/2} \frac{q^{-\frac{n^2}{2}} 1_W^\vee}{(\rho(-1))^{1/2}} = q^{-\frac{n^2-n}{2}} 1_W^\vee.$$

From the earlier discussion in this section we have $T_P(\alpha)(\zeta) = \delta_P^{1/2}(\zeta)1_{W^\vee}$.

From Section 4.3, we have $\delta_P(\zeta) = q^{-n^2-n}$. Hence $\delta_P^{1/2}(\zeta) = q^{-\frac{n^2-n}{2}}$. Therefore $(g_0g_1)(\zeta) = T_P(\alpha)(\zeta)$, $(g_0g_1)(\zeta) = T_P(\alpha)(\zeta)$, and $\text{supp}(T_P(\alpha)) = \mathfrak{P}\zeta\mathfrak{P}$. As $\text{supp}(g_i) = \mathfrak{P}w_i\mathfrak{P}$, Lemma 3.10 gives $\text{supp}(g_0g_1) = \mathfrak{P}\zeta\mathfrak{P}$. Therefore $g_0g_1 = T_P(\alpha)$. \square

4.6. Calculation of $m_L(\pi\nu)$. Recall that $\pi = \lambda\chi$ where λ is an irreducible supercuspidal depth zero representation of $GL_n(E)$ and χ is a supercuspidal depth zero character of $U_1(E)$. Note that $\pi\nu$ lies in $\mathfrak{R}^{[L,\pi]_L}(L)$. Recall m_L is an equivalence of categories. As $\pi\nu$ is an irreducible representation of L , it follows that $m_L(\pi\nu)$ is a simple $\mathcal{H}(L, \rho_0)$ -module. In this section, we identify the simple $\mathcal{H}(L, \rho_0)$ -module corresponding to $m_L(\pi\nu)$. Calculating $m_L(\pi\nu)$ will be useful in answering the question in next section.

From Section 2.5, we know that $\pi = c\text{-Ind}_{\mathfrak{P}_0}^L \tilde{\rho}_0$, where

$$\tilde{\rho}_0 = \langle \zeta \rangle \mathfrak{P}_0, \tilde{\rho}_0(\zeta^k j) = \rho_0(j)$$

for $j \in \mathfrak{P}_0, k \in \mathbb{Z}$. Let us recall that ν is unramified character of L from section 1. Let V be space of $\pi\nu$ and W be space of ρ_0 . Recall $m_L(\pi\nu) = \text{Hom}_{\mathfrak{P}_0}(\rho_0, \pi\nu)$. Let $f \in \text{Hom}_{\mathfrak{P}_0}(\rho_0, \pi\nu)$. As \mathfrak{P}_0 is a compact open subgroup of L and ν is an unramified character of L , so $\nu(j) = 1$ for $j \in \mathfrak{P}_0$. We already know that $\alpha \in \mathcal{H}(L, \rho_0)$ with support of α being $\mathfrak{P}_0\zeta$ and $\alpha(\zeta) = 1_{W^\vee}$. Let $w \in W$ and we have seen in section 2.4 that the way $\mathcal{H}(L, \rho_0)$ acts on $\text{Hom}_{\mathfrak{P}_0}(\rho_0, \pi\nu)$ is given by:

$$\begin{aligned} (\alpha.f)(w) &= \int_L (\pi\nu)(l)f(\alpha^\vee(l^{-1})w)dl \\ &= \int_L (\pi\nu)(l)f((\alpha(l))^\vee w)dl \\ &= \int_{\mathfrak{P}_0} (\pi\nu)(p\zeta)f((\alpha(p\zeta))^\vee w)dp \\ &= \int_{\mathfrak{P}_0} (\pi\nu)(p\zeta)f((\rho_0^\vee(p)\alpha(\zeta))^\vee w)dp \\ &= \int_{\mathfrak{P}_0} (\pi\nu)(p\zeta)f((\rho_0^\vee(p)1_{W^\vee})^\vee w)dp \\ &= \int_{\mathfrak{P}_0} (\pi\nu)(p\zeta)f((\rho_0^\vee(p))^\vee w)dp \end{aligned}$$

$$\begin{aligned}
 &= \int_{\mathfrak{P}_0} \pi(p\zeta)\nu(p\zeta)f((\rho_0^\vee(p))^\vee w)dp \\
 &= \int_{\mathfrak{P}_0} \pi(p\zeta)\nu(\zeta)f((\rho_0^\vee(p))^\vee w)dp.
 \end{aligned}$$

Now $\langle, \rangle : W \times W^\vee \longrightarrow \mathbb{C}$ is given by: $\langle w, \rho_0^\vee(p)w^\vee \rangle = \langle \rho_0(p^{-1})w, w^\vee \rangle$ for $p \in \mathfrak{P}_0, w \in W$. So we have $(\rho_0^\vee(p))^\vee = \rho_0(p^{-1})$ for $p \in \mathfrak{P}_0$. Hence

$$(\alpha.f)(w) = \int_{\mathfrak{P}_0} \pi(p\zeta)\nu(\zeta)f(\rho_0(p^{-1})w)dp.$$

As $f \in \text{Hom}_{\mathfrak{P}_0}(\rho_0, \pi\nu)$, so $(\pi\nu)(p)f(w) = f(\rho_0(p)w)$ for $p \in \mathfrak{P}_0, w \in W$. Hence

$$\begin{aligned}
 (\alpha.f)(w) &= \nu(\zeta) \int_{\mathfrak{P}_0} \pi(p\zeta)(\pi\nu)(p^{-1})f(w)dp \\
 &= \nu(\zeta) \int_{\mathfrak{P}_0} \pi(p\zeta)\pi(p^{-1})\nu(p^{-1})f(w)dp \\
 &= \nu(\zeta) \int_{\mathfrak{P}_0} \pi(p\zeta)\pi(p^{-1})f(w)dp.
 \end{aligned}$$

Now as $\pi = c\text{-Ind}_{\mathfrak{P}_0}^L \tilde{\rho}_0$ and $\tilde{\mathfrak{P}}_0 = \langle \zeta \rangle \mathfrak{P}_0, \tilde{\rho}_0(\zeta^k j) = \rho_0(j)$ for $j \in \mathfrak{P}_0, k \in \mathbb{Z}$, so $\pi(p\zeta) = \pi(p)\tilde{\rho}_0(\zeta) = \pi(p)\rho_0(1) = \pi(p)1_{W^\vee}$. Therefore

$$\begin{aligned}
 (\alpha.f)(w) &= \nu(\zeta) \int_{\mathfrak{P}_0} \pi(p)\pi(p^{-1})f(w)dp \\
 &= \nu(\zeta)f(w)\text{Vol}(\mathfrak{P}_0) \\
 &= \nu(\zeta)f(w)
 \end{aligned}$$

So $(\alpha.f)(w) = \nu(\zeta)f(w)$ for $w \in W$. So α acts on f by multiplication by $\nu(\zeta)$. Recall for $\lambda \in \mathbb{C}^\times$, we write \mathbb{C}_λ for the $\mathcal{H}(L, \rho_0)$ -module with underlying abelian group \mathbb{C} such that $\alpha.z = \lambda z$ for $z \in \mathbb{C}_\lambda$. Therefore $m_L(\pi\nu) \cong \mathbb{C}_{\nu(\zeta)}$.

5. Proof of Theorem 1.1

Recall the following commutative diagram which we have described earlier.

$$\begin{array}{ccc}
 \mathfrak{R}^{[L, \pi]_G}(G) & \xrightarrow{m_G} & \mathcal{H}(G, \rho) - \text{Mod} \\
 \uparrow i_P^G & & \uparrow (T_P)_* \\
 \mathfrak{R}^{[L, \pi]_L}(L) & \xrightarrow{m_L} & \mathcal{H}(L, \rho_0) - \text{Mod}
 \end{array} \tag{CD}$$

Recall that in the unramified case when n is even or in the ramified case when n is odd we have $N_G(\rho_0) = Z(L)\mathfrak{P}_0$. Thus $\mathfrak{F}_G(\rho) = \mathfrak{P}(Z(L)\mathfrak{P}_0)\mathfrak{P} = \mathfrak{P}Z(L)\mathfrak{P}$.

From Corollary 6.5 in [KP98] it follows that if $\mathfrak{F}_G(\rho) \subseteq \mathfrak{P}L\mathfrak{P}$ then

$$T_P : \mathcal{H}(L, \rho_0) \longrightarrow \mathcal{H}(G, \rho)$$

is an isomorphism of \mathbb{C} -algebras. As we have $\mathfrak{F}_G(\rho) = \mathfrak{P}Z(L)\mathfrak{P}$ in the unramified case when n is even or in the ramified case when n is odd, so $\mathcal{H}(L, \rho_0) \cong \mathcal{H}(G, \rho)$ as \mathbb{C} -algebras. So from the commutative diagram (CD), we can conclude that $\iota_P^G(\pi\nu)$ is irreducible for any unramified character ν of L .

Recall that $\pi\nu$ lies in $\mathfrak{R}^{[L, \pi]L}(L)$. Note that from the above commutative diagram, it follows that $\iota_P^G(\pi\nu)$ lies in $\mathfrak{R}^{[L, \pi]G}(G)$ and $m_G(\iota_P^G(\pi\nu))$ is an $\mathcal{H}(G, \rho)$ -module. Recall $m_L(\pi\nu) \cong C_{\nu(\zeta)}$ as $\mathcal{H}(L, \rho_0)$ -modules. From the commutative diagram (CD), we have

$$m_G(\iota_P^G(\pi\nu)) \cong (T_P)_*(C_{\nu(\zeta)})$$

as $\mathcal{H}(G, \rho)$ -modules. Thus to determine the unramified characters ν for which $\iota_P^G(\pi\nu)$ is irreducible, we have to understand when $(T_P)_*(C_{\nu(\zeta)})$ is a simple $\mathcal{H}(G, \rho)$ -module.

Using notation on page 438 in [KM09], we have $\gamma_1 = \gamma_2 = q^{n/2}$ for unramified case when n is odd and $\gamma_1 = \gamma_2 = q^{n/4}$ for ramified case when n is even. As in Proposition 1.6 of [KM09], let $\Gamma = \{\gamma_1\gamma_2, -\gamma_1\gamma_2^{-1}, -\gamma_1^{-1}\gamma_2, (\gamma_1\gamma_2)^{-1}\}$. So by Proposition 1.6 in [KM09], $(T_P)_*(C_{\nu(\zeta)})$ is a simple $\mathcal{H}(G, \rho)$ -module $\iff \nu(\zeta) \notin \Gamma$. Recall $\pi = c\text{-Ind}_{Z(L)\mathfrak{P}_0}^L \tilde{\rho}_0$ where $\tilde{\rho}_0(\zeta^k j) = \rho_0(j)$ for $j \in \mathfrak{P}_0, k \in \mathbb{Z}$ and $\rho_0 = \tau_\theta$ for some regular character θ of l^\times with $[l : k_E] = n$. Hence we can conclude that $\iota_P^G(\pi\nu)$ is irreducible for the unramified case when n is odd $\iff \nu(\zeta) \notin \{q^n, q^{-n}, -1\}$, $\theta^{q^{n+1}} = \theta^{-q}$ and $\iota_P^G(\pi\nu)$ is irreducible for the ramified case when n is even $\iff \nu(\zeta) \notin \{q^{n/2}, q^{-n/2}, -1\}$, $\theta^{q^{n/2}} = \theta^{-1}$. That proves Theorem 1.1.

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(Subha Sandeep Repaka) DEPARTMENT OF MATHEMATICS, SRM UNIVERSITY- AP, AMARAVATI
522240, ANDHRA PRADESH, INDIA
sandeep.repaka@gmail.com
subhasandeep.r@srmmap.edu.in

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