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Finite height subgroups of extended admissible groups

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ABSTRACT. We give a characterization for finite height subgroups in relatively hyperbolic groups by showing that a finitely generated, undistorted subgroup H of a relatively hyperbolic group $(G, \mathbb{P})$ has finite height if and only $H \cap gPg^{-1}$ has finite height in gPg^{-1} for each conjugate of peripheral subgroup in $\mathbb{P}$. Additionally, we prove that the concepts of finite height and strongly quasiconvexity are equivalent within the class of extended admissible groups. This class includes both the fundamental groups of non-geometric 3-manifolds and Croke-Kleiner admissible groups.

CONTENTS

1. Introduction

In the field of geometric group theory, one way to better understand the structure of a group G is to examine its subgroups. One can gain insight into the ambient group by studying subgroups $H \leq G$ whose geometry reflects that of G. One successful example of this approach is the study of quasiconvex subgroups of hyperbolic groups. It has been proven by Gitik, Mitra, Rips, and Sageev in [12] that quasiconvex subgroups of hyperbolic groups have finite height. Roughly speaking, finite height is a measure of how far a subgroup is from being malnormal. It is useful in studying residual finiteness, cubulability of hyperbolic groups, and relatively hyperbolic groups [1, 2, 15].

Definition 1.1 (Height)**.** Let H be a finitely generated subgroup of a finitely generated group G. If H is finite the height of H is 0. Otherwise the height of H in G is the largest n so that there are distinct cosets $\{g_1H, \dots, g_nH\}$ so that the

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intersection $\bigcap g_i H g_i^{-1}$ is infinite. If there is no largest n we say the height is infinite.

Definition 1.2. Let $H \leq G$ be a pair of finitely generated groups, and let $\mathcal{S}$ and $\mathcal{A}$ be finite generating sets of G and H respectively. H is called *undistorted* in G if the inclusion map $H \to G$ induces a quasi-isometric embedding from the Cayley graph $\Gamma(H, \mathcal{A})$ into the Cayley graph $\Gamma(G, \mathcal{S})$.

We remark that undistorted subgroups are independent of the choice of finite generating sets.

Our first result characterizes finite height subgroups in relatively hyperbolic groups.

Theorem 1.3. *Let $(G, \mathbb{P})$ be a finitely generated relatively hyperbolic group and H undistorted subgroup of G. Then H has finite height in G if and only if $H \cap gPg^{-1}$ has finite height in gPg^{-1} for each conjugate gPg^{-1} of peripheral subgroup in $\mathbb{P}$.*

Outside hyperbolic settings, quasiconvexity is not preserved under quasi-isometry. This means that we can not define quasiconvex subgroups of a non-hyperbolic group G which are independent of the choice of finite generating set for G. In [20], Tran introduces a theory regarding strongly quasiconvex subgroups of any finitely generated group. Strong quasiconvexity is independent of the chosen finite generating set of the group, and it coincides with quasiconvexity when the group is hyperbolic.

Definition 1.4. Let G be a finitely generated group and H a subgroup of G. We say H is *strongly quasiconvex* in G if for any $L \geq 1$, $C \geq 0$ there exists $M = M(L, C)$ such that every (L, C)–quasi-geodesic in G with endpoints in H is contained in the M–neighborhood of H

We remark that the term "Morse subgroup" often refers to the same concept as strongly quasiconvex. For instance, the concept of strongly quasiconvex has also been independently introduced by Genevois in [11] under the name Morse subgroup. In many specific contexts, it is known that many subgroups are strongly quasiconvex (or Morse). For example, in the context of relatively hyperbolic groups, it has been shown that peripheral subgroups of relatively hyperbolic groups [10] are Morse, and hyperbolically embedded subgroups are Morse [18].

Tran in [20] generalizes the result of [12] by showing that strongly quasiconvex subgroups in any finitely generated group have finite height. A natural and reasonable question arises: is the converse true? That is, does finite height imply strong quasiconvexity? If the answer is affirmative, then it would be possible to characterize the strongly quasi-convex subgroups of a finitely generated group using solely group theoretic notions. A counter-example to the above question is easy to find, for example, the fundamental group of a torus bundle M over the circle with Anosov monodromy contains finite height subgroups that are not strongly quasiconvex [17]. The main result in [17] shows that having finite height and strong quasiconvexity are equivalent for non-geometric 3-manifold groups. In this paper, we extend this result to a newly introduced class of groups called *extended admissible groups* [16]. Extended admissible groups possess a graph of groups decomposition that generalizes any non-geometric 3-manifold and Croke–Kleiner admissible groups [6].

Here we provide a brief discussion of extended admissible groups. For a more detailed definition, please refer to Definition 4.1. Let M be a non-geometric 3-manifold. The torus decomposition of M yields a nonempty minimal union $\mathcal{T} \subset M$ of disjoint essential tori, unique up to isotopy, such that each component M_v of $M \backslash \mathcal{T}$, called a *piece*, is either Seifert fibered or hyperbolic. There is an induced graph of groups decomposition $\mathcal{G}$ of $\pi_1(M)$ with underlying graph Γ as follows. For each piece M_v, there is a vertex v of Γ with vertex group $\pi_1(M_v)$. For each torus $T_e \in \mathcal{T}$ contained in the closure of pieces M_v and M_w, there is an edge e of Γ between vertices v and w. The associated edge group is $\pi_1(T_e) \cong \mathbb{Z}^2$ and the edge monomorphisms are the maps induced by inclusion. Croke–Kleiner defined the class of admissible groups, which have a graph of groups decomposition generalizing that of graph manifolds [6]. In the extended admissible group, we allow any $\mathbb{Z}$-by-hyperbolic group instead of a Seifert fibered piece, and any toral relatively hyperbolic group instead of a hyperbolic 3-manifold piece.

The following theorem generalizes the main result of [17], but with a completely different proof. It's worth noting that the strategy employed in [17] is not applicable in this more general setting.

Theorem 1.5 (Finite height $\iff$ strongly quasiconvex)**.** *Let G be an extended admissible group. Suppose that H is a finitely generated, undistorted subroup of G. Then H has finite height in G if and only if H is strongly quasiconvex in G.*

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2. Preliminaries

In this section, we review some concepts in geometric group theory that will be used throughout the paper.

We assume familiarity with Bass-Serre theory; see [19] for details. However, to fix notation and terminology, we give some brief definitions.

We first establish some terminology regarding graphs. A *graph* Γ consists of a set $V\Gamma$ of vertices, a set $E\Gamma$ of oriented edges, and maps $\iota, \tau : E\Gamma \to V\Gamma$. There is a fixed-point free involution $E\Gamma \to E\Gamma$, taking an edge $e \in E\Gamma$ such that $\iota e = v$ and $\tau e = w$ to an edge $\bar{e}$ satisfying $\iota \bar{e} = w$ and $\tau \bar{e} = v$. We also write

e_+ and e_- to denote τe and ιe respectively. An *unoriented edge* of Γ is the pair $\{e, \bar{e}\}$.

Each connected graph can be identified with a metric space by equipping its topological realization with the path metric in which each edge has length one. A *combinatorial path* in X is a path $p : [0, n] \to X$ for some $n \in \mathbb{N}$ such that for every integer i, $p(i)$ is a vertex, and $p|_{[i,i+1]}$ is either constant or traverses an edge of X at unit speed. Every geodesic between vertices of X is necessarily a combinatorial path.

Definition 2.1. A *graph of groups* $\mathcal{G} = (\Gamma, \{G_v\}, \{G_e\}, \{\tau_e\})$ consists of the following data:

(1) a graph Γ, called the *underlying graph*,
(2) a group G_v for each vertex $v \in V\Gamma$, called a *vertex group*,
(3) a subgroup $G_e \leq G_{e_-}$ for each edge $e \in E\Gamma$, called an *edge group*,
(4) an isomorphism $\tau_e : G_e \to G_{\bar{e}}$ for each $e \in E\Gamma$ such that $\tau_e^{-1} = \tau_{\bar{e}}$, called an *edge map*.

The *fundamental group* $\pi_1(\mathcal{G})$ of a graph of groups $\mathcal{G}$ is as defined in [19].

We use the following notation for trees of spaces as in [7].

Definition 2.2. *A tree of spaces* $X := X\left(T, \{X_v\}_{v \in VT}, \{X_e\}_{e \in ET}, \{\alpha_e\}_{e \in ET}\right)$ consists of:

(1) a simplicial tree T, called the *base tree*;
(2) a metric space X_v for each vertex v of T, called a *vertex space*;
(3) a subspace $X_e \subseteq X_{e_-}$ for each oriented edge e (with the initial vertex denoted by e_-) of T, called an *edge space*;
(4) maps $\alpha_e : X_e \to X_{\bar{e}}$ for each edge $e \in ET$, such that $\alpha_{\bar{e}} \circ \alpha_e = \mathrm{id}_{X_e}$ and $\alpha_e \circ \alpha_{\bar{e}} = \mathrm{id}_{X_{\bar{e}}}$.

We consider X as a metric space as follows: we take the disjoint union of all the X_v and then, for all unoriented edges $\{e, \bar{e}\}$ and every $x \in X_e$, we attach a unit interval between $x \in X_e$ and $\alpha_e(x) \in X_{\bar{e}}$. Each edge and vertex space can be naturally identified with a subspace of X.

We typically omit the data X_v, X_e and α_e from the notation and write a tree of spaces as the pair (X, T), or simply as a space X. We consider X as a metric space by equipping it with the induced path metric. We now explain how to associate a tree of spaces with a graph of finitely generated groups.

Let $\mathcal{G} = (\Gamma, \{G_v\}, \{G_e\}, \{\tau_e\})$ be a graph of finitely generated groups. We recall the associated Bass–Serre tree T is constructed so that vertices (resp. edges) of T correspond to left cosets of vertex (resp. edge) groups of $\mathcal{G}$.

We now describe a tree of spaces X. For each $x \in V\Gamma \sqcup E\Gamma$, we fix a finite generating set S_x of G_x, chosen such that $\tau_e(S_e) = S_{\bar{e}}$. We now define a graph W with vertex set $V\Gamma \times G$ and edge set

$$\{((v, g), (v, gs)) \mid g \in G, s \in S_v\}.$$

The components of W are in bijective correspondence with left cosets of vertex groups of $\mathcal{G}$, and hence with vertices of T. If $\tilde{v} \in VT$ corresponds to the coset

gG_v, we define $X_{\tilde{v}}$ to be the component of W with vertex set $\{(v,h) \mid h \in gG_v\}$. We note that the component of W corresponding to a coset gG_v is isometric to the Cayley graph of G_v with respect to S_v.

Suppose $\tilde{e} \in ET$ corresponds to a coset gG_e. By the definition of T, if $v = e_-$ and $w = e_+$, then $\tilde{v} := \tilde{e}_-$ and $\tilde{w} := \tilde{e}_+$ correspond to the cosets gG_v and gG_w. We define the edge space $X_{\tilde{e}}$ to be

$$\{(v,h) \mid h \in gG_e\} \subseteq X_{\tilde{v}}.$$

The attaching map $\alpha_{\tilde{e}} : X_{\tilde{e}} \to X_{\tilde{w}}$ is defined by $\alpha_{\tilde{e}} : (v,h) \mapsto (w, g\tau_e(g^{-1}h))$, where $\tau_e : G_e \to G_{\bar{e}} \le G_w$ is the edge map of $\mathcal{G}$. Finally, we equip each $X_{\tilde{e}}$ with the word metric with respect to S_e. (More precisely, we require that the map $X_{\tilde{e}} \xrightarrow{(v,h)\mapsto g^{-1}h} G_e$ is an isometry when G_e is equipped with the word metric with respect to S_e.)

Definition 2.3. Given a graph of finitely generated groups $\mathcal{G}$, the tree of spaces X constructed above is *the tree of spaces associated with the graph of groups* $\mathcal{G}$.

The tree of spaces X is a proper geodesic metric space (see Lemma 2.13 of [7]). The natural action of G on W (fixing the $V\Gamma$ factor) induces an action of G on X. Applying the Milnor-Schwarz lemma we deduce:

Proposition 2.4 (Section 2.5 of [7])**.** *Suppose G, T, and X are as above. Then there exists a quasi-isometry $f : G \to X$ and $A \ge 0$ such that*

$$d_{\text{Haus}}\left(f\left(gG_x\right), X_{\tilde{x}}\right) \le A$$

for all $\tilde{x} \in VT \sqcup ET$, where $\tilde{x}$ corresponds to the coset gG_x.

3. Characterization of finite height subgroups in relatively hyperbolic groups

In this section, we are going to prove Theorem 1.3.

The notion of relatively hyperbolic groups can be formulated from a number of equivalent ways. Here we shall present a quick definition due to Bowditch [5].

Let G be a finitely generated group with a finite collection of subgroups $\mathbb{P}$. Fixing a finite generating set S for G, we consider the corresponding Cayley graph $\Gamma(G,S)$ equipped with path metric d_S and we denote by $|g|_S = d_S(1,g)$ for the word length.

Denote by $\mathcal{P} = \{gP : g \in G, P \in \mathbb{P}\}$ the collection of peripheral cosets. Let $\hat{G}(\mathcal{P})$ be the coned-off Cayley graph obtained from $\Gamma(G,S)$ as follows. A *cone point* denoted by $c(P)$ is added for each peripheral coset $P \in \mathcal{P}$ and is joined by half edges to each element in P. The union of two half edges at a cone point is called a *peripheral edge*. Denote by $\hat{d}_S$ the induced path metric after coning-off. The pair $(G,\mathbb{P})$ is said to be *relatively hyperbolic* if the coned-off Cayley graph $\hat{G}(\mathcal{P})$ is hyperbolic and *fine*: any edge is contained in finitely many simple circles with uniformly bounded length.

Lemma 3.1. [20, Proposition 3.7] *Assuming P is a finitely generated subgroup of a finitely generated group G. Let S be a finite generating set of G and U a finite generating set of P such that U is a subset of S. If P is strongly quasiconvex in G then for any $K \geq 1, L \geq 0$ there is a constant $C > 0$ depending on K, L such that the following holds. If $\gamma : [0,\infty) \to \Gamma(G,S)$ is a continuous (K,L)–quasigeodesic ray that lies in some finite neighborhood of $\Gamma(P,U)$, then there is $t > 0$ such that $\gamma_{|[t,\infty)}$ lies in the C–neighborhood of $\Gamma(P,U)$.*

In [20, Proposition 3.7], the statement is for geodesic rays γ, but a similar proof also holds when γ is a continuous (K,L)-quasigeodesic ray. We leave the details to the reader.

To prove Theorem 1.3, we also need the following lemma.

Lemma 3.2. *Let H be a finitely generated, undistorted subgroup of a finitely generated group G, and P be a strongly quasiconvex subgroup of G. Let S be a finite generating set of G and U a finite generating set of H such that U is a subset of S. Then there is a constant $C > 0$ such that*

$$\forall g \in G, \text{ if } H \cap gPg^{-1} \text{ is infinite} \Longrightarrow d_S(H, gP) \leq C$$

Proof. Since U is a subset of S, we consider the Cayley graph $\Gamma(G,S)$ of G contains the Cayley graph $\Gamma(H,U)$ of H as a subgraph. Since H is an undistorted subgroup of G, the inclusion map $i\colon \Gamma(H,U) \to \Gamma(G,S)$ is a (K,L)–quasi-isometric embedding for some $K \geq 1$ and $L \geq 0$.

Let $g \in G$ so that $H \cap gPg^{-1}$ is infinite. We choose $\{h_i\}$ as a sequence of distinct group elements in $H \cap gPg^{-1}$. For each i let α_i be a geodesic in $\Gamma(H,U)$ connecting the identity and h_i. Therefore, α_i is a (K,L)–quasi-geodesic in the Cayley graph $\Gamma(G,S)$ since the inclusion $\Gamma(H,U) \to \Gamma(G,S)$ is a (K,L)-quasi-isometric embedding.

Since P is a strongly quasiconvex subgroup of G, there exists $M = M(K,L)$ such that every (K,L)–quasi-geodesic in $\Gamma(G,S)$ with endpoints in P is contained in the M–neighborhood of P. Note that the endpoints of α_i both lie in the $|g|_S$–neighborhood of the left coset gP, and hence the path α_i lies in the C_1–neighborhood of gP with respect to the metric d_S for some constant C_1 only depend on K, L, $M(K,L)$ and $|g|_S$.

Applying Arzela–Ascoli Theorem to the proper metric space $\Gamma(H,U)$ and the collection of geodesics $\{\alpha_i\}$ in $\Gamma(H,U)$, we obtain a sub-sequence of $\{\alpha_i\}$ converges in the compact-open topology to a geodesic ray α in the Cayley graph $\Gamma(H,U)$ which also lies in the C_1–neighborhood of gP with respect to the metric d_S (as α_i does so).

Since $\Gamma(H,U)$ is included in $\Gamma(G,S)$ as a (K,L)-quasi-isometric embedding, we know that α is also a (K,L)-quasi-geodesic ray in $\Gamma(G,S)$. As α belongs to the C_1–neighborhood of gP with respect to the metric d_S, we then can apply Lemma 3.1 to conclude that there exists a value of t such that $\alpha_{|[t,\infty)}$ lies in the C-neighborhood of gP where C is also the constant given by Lemma 3.1 which is independent with C_1. This indicates that $d_S(H, gP) \leq C$. □

Proof of Theorem 1.3. The implication [$\Longrightarrow$] follows from Proposition 2.2 in [17]. For the rest of the proof, we are going to verify the implication [$\Longleftarrow$]. Assume that the subgroup $H \cap gPg^{-1}$ has finite height in gPg^{-1} for each conjugate gPg^{-1} of peripheral subgroup in $\mathbb{P}$. We would like to show that H has finite height in G.

Pick finite generating sets S, T for G and H respectively so that T is a subset of S. [10, Lemma 4.15 and Theorem A.1] implies that each conjugate of a peripheral subgroup in $\mathbb{P}$ is strongly quasiconvex, and hence we can apply Lemma 3.2 to each $P \in \mathbb{P}$ to obtain a constant C as in Lemma 3.2 . Since there are finitely many peripheral subgroups in $\mathbb{P}$, we can also enlarge C so that Lemma 3.2 applies to all peripheral subgroups $P \in \mathbb{P}$.

To show H has finite height in G, we must show that there is a uniform constant N so that whenever there are distinct left cosets $g_1H, g_2H, \cdots, g_nH$ with $\bigcap_{i=1}^n g_iHg_i^{-1}$ is infinite then $n \leq N$. This uniform constant N will be defined explicitly during the proof. We'll consider the following cases.

Case 1: The subgroup $\bigcap_{i=1}^n g_iHg_i^{-1}$ is not contained in any conjugate of a peripheral subgroup.

Using [13, Theorem 1.5] and [14, Corollary 8.5], we can find a positive constant D that is independent of n and the choices of g_i so that the ball of radius D about the identity in the Cayley graph $\Gamma(G,S)$ intersects every left coset g_iH. Therefore, the number n is bounded above by the number of group elements of a ball with a radius of D in the Cayley graph $\Gamma(G,S)$.

Case 2: The subgroup $\bigcap_{i=1}^n g_iHg_i^{-1}$ is contained within a conjugate gPg^{-1} of a peripheral subgroup $P \in \mathbb{P}$.

Consider the closed ball $B(1,C)$ in the Cayley graph $\Gamma(G,S)$. For each $z \in B(1,C) \cap G$, by our assumption, $zHz^{-1} \cap P$ has finite height in P, and hence let us denote the height of $zHz^{-1} \cap P$ in P by n_z. Define

$$N := \sum_{z \in B(1,C) \cap G} n_z$$

This constant is well-defined since $B(1,C) \cap G$ is finite.

Claim: $n \leq N$.

For each $i \in \{1, 2, \dots, n\}$, we let $k_i := g^{-1}g_i$. Then $k_1H, k_2H, \dots, k_nH$ are distinct left cosets and $\bigcap k_iHk_i^{-1}$ is an infinite subgroup in P. Additionally, the subgroup $H \cap k_i^{-1}Pk_i$ is infinite. According to Lemma 3.2, we have

$$d_S(H, k_i^{-1}P) < C$$

Thus, there is a group element z_i with $|z_i|_S < C$ such that $k_i \in Pz_iH$.

For each group element z in $B(1,C) \cap G$, we define

$$I_z = \{u_1, u_2, \cdots, u_m\}$$

be the set of elements in $\{k_1, k_2, \cdots, k_n\}$ such that each u_i is an element in the double coset PzH.

Note that the set I_z may be empty for $z \in B(1,C) \cap G$ but I_{z_i} is non-empty since it contains k_i. We observe that $m \leq n_z$. Indeed, for each $i \in \{1, 2, \dots, m\}$, since $u_i \in PzH$, there are $h_i \in H$ and $p_i \in P$ such that $u_i = p_i z h_i$. Therefore,

$$u_i H u_i^{-1} \cap P = (p_i z h_i) H (p_i z h_i)^{-1} \cap P = p_i (zHz^{-1} \cap P) p_i^{-1}$$

Note that $p_1(zHz^{-1} \cap P), p_2(zHz^{-1} \cap P), \cdots, p_m(zHz^{-1} \cap P)$ are distinct left cosets. To see this, suppose that $p_i(zHz^{-1} \cap P) = p_j(zHz^{-1} \cap P)$, then $p_i^{-1} p_j \in zHz^{-1}$. This means that $u_i^{-1} u_j = h_i^{-1} z^{-1} p_i^{-1} p_j z h_j$ is a group element in H. Thus, $u_i H = u_j H$, which implies that $i = j$ because u_i, u_j are elements in $\{k_1, k_2, \dots, k_n\}$ and $k_1 H, k_2 H, \dots, k_n H$ are distinct left cosets. Additionally, the subgroup

$$\bigcap p_i(zHz^{-1} \cap P) p_i^{-1} = \bigcap (u_i H u_i^{-1} \cap P)$$

contains the subgroup $\bigcap k_i H k_i^{-1}$, and hence $\bigcap p_i(zHz^{-1} \cap P) p_i^{-1}$ is infinite. Since n_z is the height of $zHz^{-1} \cap P$ in P, it follows that

$$|I_z| = m \le n_z$$

For each $x \in \bigcup_{z \in B(1,C) \cap G} I_z$, pick a z so that $x \in I_z$, and hence $x = k_i$ for some $k_i \in I_z$. We then consider

$$\zeta : \bigcup_{z \in B(1,C) \cap G} I_z \to \{1, 2, \dots, n\}$$

by sending x to i. As this map is surjective, it follows that

$$n \le \Big| \bigcup_{z \in B(1,C) \cap G} I_z \Big| \le \sum_{z \in B(1,C) \cap G} \big| I_z \big| \le \sum_{z \in B(1,C) \cap G} n_z = N$$

The Claim is proved; thus, H is a finite height subgroup of G. □

4. Finite height subgroups in extended admissible groups

We will first recall the definition of extended admissible groups introduced in [16].

Definition 4.1. A group G is an *extended admissible group* if it is the fundamental group of a graph of groups $\mathcal{G}$ such that:

(1) The underlying graph Γ of $\mathcal{G}$ is a connected finite graph with at least one edge, and every edge group is virtually $\mathbb{Z}^2$.
(2) Each vertex group G_v is one of the following two types:
   (a) Type $\mathcal{S}$: G_v contains an infinite cyclic normal subgroup $Z_v \lhd G_v$, such that the quotient $Q_v := G_v / Z_v$ is a non-elementary hyperbolic group. We call Z_v and Q_v the *kernel* and *hyperbolic quotient* of G_v respectively.
   (b) Type $\mathcal{H}$: G_v is hyperbolic relative to a collection $\mathbb{P}_v$ of virtually $\mathbb{Z}^2$-subgroups, where all edge groups incident to G_v are contained in $\mathbb{P}_v$, and G_v doesn't split relative to $\mathbb{P}_v$ over a subgroup of an element of $\mathbb{P}_v$.
(3) For each vertex group G_v, if $e, e' \in \mathrm{Link}(v)$ and $g \in G_v$, then $g G_e g^{-1}$ is commensurable to $G_{e'}$ if and only if both $e = e'$ and $g \in G_e$.

(4) For every edge group G_e such that G_{e_-} and G_{e_+} are vertex groups of type $\mathcal{S}$, the subgroup generated by $\tau_{\bar{e}}(Z_{e_+} \cap G_{\bar{e}})$ and $Z_{e_-} \cap G_e$ has finite index in G_e.

Definition 4.2. An extended admissible group G is called an *admissible group* if it has no vertex group of type $\mathcal{H}$.

Below are some examples of extended admissible groups.

Example 4.3. (1) (3-manifold groups) The fundamental group of a compact, orientable, irreducible 3-manifold M with empty or toroidal boundary is an extended admissible group. Seifert fibered and hyperbolic pieces correspond to type $\mathcal{S}$ and $\mathcal{H}$ vertex respectively. Fundamental groups of graph manifolds are admissible groups.

(2) (Torus complexes) Let $n \geq 3$ be an integer. Let $T_1, T_2, \dots, T_n$ be a family of flat two-dimensional tori. For each i, we choose a pair of simple closed geodesics a_i and b_i such that length(b_i) = length(a_{i+1}), identifying b_i and a_{i+1} and denote the resulting space by X. The space X is a graph of spaces with $n-1$ vertex spaces $V_i := T_i \cup T_{i+1}/\{b_i = a_{i+1}\}$ (with $i \in \{1, \dots, n-1\}$) and $n-2$ edge spaces $E_i := V_i \cap V_{i+1}$.

The fundamental group $G = \pi_1(X)$ has a graph of groups structure where each vertex group is the fundamental group of the product of a figure eight and S^1. Vertex groups are isomorphic to $F_2 \times \mathbb{Z}$ and edge groups are isomorphic to $\pi_1(E_i) \cong \mathbb{Z}^2$. The generators $[a_i], [b_i]$ of the edge group $\pi_1(E_i)$ each map to a generator of either a $\mathbb{Z}$ or F_2 factor of $F_2 \times \mathbb{Z}$. It is clear that with this graph of groups structure, $\pi_1(X)$ is an admissible group.

(3) Admissible groups in the sense of Croke-Kleiner [6]. Croke–Kleiner defined a more restrictive notion of an admissible group, where they also assume each edge group G_e is isomorphic to $\mathbb{Z}^2$ and each infinite cyclic $Z_v \lhd G_v$ is central.

4.1. Finite height subgroups in admissible groups. In this section, we are going to prove the following.

Proposition 4.4 (Finite height $\iff$ strongly quasiconvex)**.** *A finitely generated subgroup in an admissible group has finite height if and only if it is strongly quasiconvex.*

To prove Proposition 4.4 we need several lemmas. Assume G is an admissible group with its Bass-Serre tree T, and let X be the associated tree of spaces for $\mathcal{G}$.

Lemma 4.5. *Each finite height subgroup H of vertex group G_v of G must be finite or have finite index in G_v.*

Proof. We assume that H is an infinite subgroup of G_v. Recall that G_v is a Z_v-by-hyperbolic group. Let $K \leq G_v$ be the subgroup of index at most two centralizing Z_v. By [17, Proposition 2.2 (1)], $H \cap K$ has finite height in K. As K is a group with infinite center, it follows from [17, Proposition 2.3] that $H \cap K$

is either finite or it has finite index in K, and hence H is either finite or it has finite index in G_v since K has finite index in G_v. □

Lemma 4.5 combines with [17, Proposition 2.6] yield to the following lemma.

Lemma 4.6. *If a finitely generated subgroup H of infinite index of G has finite height then the intersection of H with any conjugate of a vertex group of G must be finite.*

Let $\mathcal{B}$ be the collection of left cosets of vertex groups of G. Let H be a finitely generated subgroup of infinite index of G and has finite height in G. Fix a generating set S of G, and we define a function $f_S \colon [0,\infty) \to [0,\infty)$ as follows. For any $r \geq 0$,

$$f_S(r) := \max\big\{\operatorname{diam}\big(\mathcal{N}_r(H) \cap B\big) \mid B \in \mathcal{B} \text{ and } B \cap B_S(e,r) \neq \emptyset\big)\big\}$$

Lemma 4.7. *The map f_S is well-defined and*

$$\operatorname{diam}(\mathcal{N}_r(H) \cap gG_v) \leq f_S(r)$$

for any left coset of a vertex group G_v in G.

Proof. Firstly, we show that f_S is well-defined. Indeed, if B is an element in the collection $\mathcal{B}$ then B is a left coset gG_v for some vertex group G_v and for some group element g in G. By [13, Proposition 9.4] there exists a constant $r' = r'(r, H, gG_v)$ such that

$$\mathcal{N}_r(H) \cap \mathcal{N}_r(gG_v) \subset \mathcal{N}_{r'}(H \cap gG_v g^{-1})$$

According to Lemma 4.6, the intersection $H \cap gG_v g^{-1}$ is finite, and thus $\mathcal{N}_{r'}(H \cap gG_v g^{-1})$ and $\mathcal{N}_r(H) \cap \mathcal{N}_r(gG_v)$ are finite. As $\mathcal{N}_r(H) \cap B$ is a subset of $\mathcal{N}_r(H) \cap \mathcal{N}_r(gG_v)$, it follows that

$$\operatorname{diam}(\mathcal{N}_r(H) \cap B) < \infty$$

Since there are only finitely many elements in the collection $\mathcal{B}$ that have a nonempty intersection with the closed ball $B_S(e,r)$, it follows that $f_S(r)$ is a constant in $[0,\infty)$.

Now we will prove that

$$\operatorname{diam}\big(\mathcal{N}_r(H) \cap gG_v\big) \leq f_S(r)$$

for any left coset of a vertex group G_v in G.

Indeed, denote $B := gG_v$. If $d_S(H,B) > r$, then $\mathcal{N}_r(H) \cap B$ is empty and then its diameter is zero and less than or equal $f_S(r)$. We now assume $d_S(H,B) \leq r$. Then there are $h \in H$ and $b \in B$ such that $d_S(h,b) \leq r$. Denote $\zeta := h^{-1}g$ and consider the left coset $A := \zeta G_v$. As $\zeta \in B_S(e,r)$, it follows that $A \cap B_S(e,r) \neq \emptyset$. Thus, we have

$$\operatorname{diam}\big(\mathcal{N}_r(H) \cap A\big) \leq f_S(r)$$

Since $b \in B = gG_v = h\zeta G_v = hA$ and $h\mathcal{N}_r(H) = \mathcal{N}_r(H)$, we have that

$$\mathcal{N}_r(H) \cap B = h\big(\mathcal{N}_r(H) \cap A\big)$$

Therefore,

$$\operatorname{diam}\big(\mathcal{N}_r(H)\cap B\big) = \operatorname{diam}(\mathcal{N}_r(H)\cap A) \le f_S(r)$$

The claim is proved. □

We are now ready for the proof of Proposition 4.4.

Proof of Proposition 4.4. If H is strongly quasiconvex in G then H has finite height in G by [20, Theorem 1.2]. For the rest of the proof, we are going to prove the converse implication. We will also assume that H is an infinite index in G, otherwise, it is obvious.

Let S be a finite generating set of G and U a finite generating set of H such that U is a subset of S, and hence the Cayley graph $\Gamma(G,S)$ of G contains the Cayley graph $\Gamma(H,U)$ of H as a subgraph.

By Lemma 4.6, we have that $H \cap gG_v g^{-1}$ is finite for all vertex v in VT. It follows that H acts properly on the Bass-Serre tree T and the stabilizer in H of each vertex in T is finite. Hence, it follows from [9, Theorem 7.51] that there exists a finite index subgroup K of H such that K is a free group.

Fix a vertex v_0 in the Bass-Serre tree T projecting to a vertex v in the underlying graph of G, and considering the vertex space X_{v_0} of the tree of spaces X which we identify with the Cayley graph of G_v. Fix a basepoint o in X_{v_0} which is the identity element of G_v. Recall that G acts on the tree of spaces X, we thus consider the orbit map

$$\xi:\ G \to (X,o)$$

which is a quasi-isometry by Proposition 2.4.

Let $\{\gamma_1,\gamma_2,\dots,\gamma_s\}$ be a finite generating set of the free group K. We note that γ_i is not conjugate into any vertex group of G. For each $i \in \{1,2,\dots,s\}$, let $\gamma_{s+i} = \gamma_i^{-1}$ and let ℓ_i be a combinatorial path in X connecting o to $\xi(\gamma_i) = \gamma_i(o)$, and let $\ell_{s+i} := \ell_i$.

We define Y as the subspace of X which is the union of combinatorial paths $g(\ell_i)$ with $g \in K$, $1 \le i \le 2s$, i.e,

$$Y := \bigcup_{g\in K,\, i\in\{1,\dots,2s\}} g\ell_i$$

Then Y and $K(o)$ are within a finite Hausdorff distance. It also follows from the construction of Y and Lemma 4.7 (for $r = 1$) that there exists a constant $\delta > 0$ such that for each vertex v, edge e in T then

$$\operatorname{diam}\big(Y\cap X_v\big) \le \delta \quad \text{and} \quad \operatorname{diam}\big(Y\cap X_e\big) \le \delta$$

Note that G acts acylindrically on its Bass-Serre tree T by [16, Lemma 2.9]. Let us consider the action of H on T which is induced by the action of G on T. By [3, Theorem 1.5], to see H is strongly quasiconvex in G, it suffices to show that the orbit map

$$\tau:\ H \to T$$

given by $h \mapsto h(v_0)$ is a quasi-isometric embedding.

Let h be a nontrivial element in H. Let $e_1 \cdot e_2 \cdots e_t$ be the combinatorial path in the Bass-Serre tree T connecting v_0 to $h(v_0)$. Let v_i be the endpoint of e_i.

Recall that $\delta > 0$ is the constant where $\operatorname{diam}(Y \cap X_v) \leq \delta$ and $\operatorname{diam}(Y \cap X_e) \leq \delta$ for each vertex v, edge e in T. Also recall that in the tree of spaces X, for every $x \in X_e$ we attach a unit interval between $x \in X_e$ and $\alpha_e(x) \in X_{\bar{e}}$. Hence in the tree of spaces X, we can find a sequence of points $x_i \in X_{e_i} \cap Y$, $\bar{x}_i \in X_{\bar{e}_i} \cap Y$ on edge spaces $X_{e_i}, X_{\bar{e}_i}$ with $1 \leq i \leq t-1$ such that

$$d_X(x_i, \bar{x}_i) \leq 2\delta + 1$$

For notational purposes, let us denote $\bar{x}_0 := o$ and $x_t := h(o)$. Hence we have

$$d_X(x_0, x_1) \leq \delta \quad \text{and} \quad d_X(\bar{x}_{t-1}, x_t) \leq \delta$$

As Y and $K(o)$ are within a finite Hausdorff distance, let $\Lambda > 0$ be such a finite constant, and thus Y is a subset of the Λ–neighborhood of $K(o)$. It follows that for each $1 \leq i \leq t-1$, there exists a group element $h_i \in K \leq H$ such that

$$d_X(x_i, h_i(o)) \leq \Lambda$$

As the orbit map $\xi : G \to (X, o)$ is a quasi-isometric embedding, routine arguments yield a constant L depending on the quasi-isometric constant of the orbit map ξ, δ, Λ such that

$$d_S(1, h_1) \leq L, \quad d_S(h_i, h_{i+1}) \leq L, \quad d_S(h_{t-1}, h) \leq L$$

As H is a finitely generated subgroup of G, the cardinality $|H \cap B_G(1, L)|$ is finite, and hence there is a constant $C = C(L)$ such that with respect to the metric d_U on H we have

$$d_U(1, h_1) \leq C, \quad d_U(h_i, h_{i+1}) \leq C, \quad d_U(h_{t-1}, h) \leq C$$

As a consequence, we have

$$d_U(1, h) \leq d_U(1, h_1) + \sum_{i=1}^{t-2} d_U(h_i, h_{i+1}) + d_U(h_{t-1}, h)$$

$$\leq tC = C d_T(v_0, h(v_0))$$

On the other hand, since the orbit map of any isometric action is Lipschitz (see [4, Lemma I.8.18]), we have that the orbit map $\tau : H \to T$ is a Lipschitz map. In particular, there exists a uniform constant B such that

$$d_T(v_0, h(v_0)) \leq B + B d_U(1, h)$$

Therefore, we can conclude that the orbit map $\tau : H \to T$ is a quasi-isometric embedding. By [3, Theorem 1.5], H is strongly quasiconvex in G. □

4.2. Finite height subgroups in extended admissible groups. Let G be an extended admissible group with graph of groups $\mathcal{G}$ and underlying graph Γ. By the normal form theorem, for each connected subgraph Γ' of Γ, there is a subgroup $G_{\Gamma'} \leq G$ which is the fundamental group of the graph of groups with underlying graph Γ', and with vertex, edge groups, and edge monomorphisms coming from $\mathcal{G}$. Let Λ be the full subgraph of Γ with vertex set $\{v \in V\Gamma \mid \mathcal{G}_v$ is type $\mathcal{S}\}$. For each component Γ' of Λ, we say that $G_{\Gamma'}$ is

(1) a *maximal admissible component* if Γ' contains an edge;
(2) an *isolated type $\mathcal{S}$ vertex group* if Γ' consists of a single vertex of type $\mathcal{S}$.

is a subgroup $G_{\Gamma'} \leq G$ for some connected component Γ' of Λ.

Proposition 4.8. *Let G be an extended admissible group with the graph of groups structure $\mathcal{G}$ such that it contains at least one vertex group of type $\mathcal{H}$. Suppose H is a undistorted subgroup of G. Then H has finite height in G if and only if H is strongly quasiconvex in G.*

Proof. [$\Longrightarrow$]: Let $G_1, \ldots, G_k$ be the maximal admissible components and isolated vertex pieces of type $\mathcal{S}$ of an extended admissible group G. Let $G_{e_1}, \ldots, G_{e_m}$ be the edge groups so that both its associated vertex groups $G_{(e_i)_\pm}$ are of type $\mathcal{H}$, and let $T_1, \ldots, T_\ell$ be groups in $\bigcup \mathbb{P}_v$ which are not edge groups of G. By Combination Theorem of relatively hyperbolic groups [8, Theorem 0.1] we have G is hyperbolic relative to

$$\mathbb{P} = \{G_i\}_{i=1}^k \cup \{G_{e_s}\}_{s=1}^m \cup \{T_i\}_{i=1}^\ell$$

By Theorem 1.3, $H \cap gPg^{-1}$ has finite height in gPg^{-1} for each conjugate gPg^{-1} of peripheral subgroup in $\mathbb{P}$. If P is either G_{e_s} or T_i which are virtually $\mathbb{Z}^2$ then $H \cap gPg^{-1}$ is either finite or has finite index in gPg^{-1}, and hence $H \cap gPg^{-1}$ is strongly quasiconvex in gPg^{-1}. If P is G_i for some i, then $H \cap gPg^{-1}$ is strongly quasiconvex in gPg^{-1} by Proposition 4.4. In other words, we have shown that $H \cap gPg^{-1}$ is strongly quasiconvex in gPg^{-1} for each conjugate gPg^{-1} of each peripheral subgroup in $\mathbb{P}$. By [20, Theorem 1.9], H is strongly quasiconvex in G.

[$\Longleftarrow$]: Strongly quasiconvex subgroups always have finite height by [20, Theorem 1.2]. □

Proof of Theorem 1.5. The proof is a combination of Propositions 4.4 and 4.8. □

References

[1] Agol, Ian.; Groves, Daniel.; Manning, Jason. Residual finiteness, QCERF and fillings of hyperbolic groups. *Geom. Topol.* **13** (2009), no.2, 1043–1073. MR2470970, Zbl 1229.20037, doi: 10.2140/gt.2009.13.1043. 1220

[2] Agol, Ian. The virtual Haken conjecture. *Doc. Math.* **18** (2013), 1045–1087. with an appendix by Ian Agol, Daniel Groves and Jason Manning. MR3104553, Zbl 1286.57019, doi: 10.4171/DM/421. 1220

[3] ABBOTT, COROLYN; MANNING, JASON. Acylindrically hyperbolic groups and their quasi-isometrically embedded subgroups. *Michigan Math. J.* **74** (2024), no.2, 357–402. MR4739420, Zbl 7855090, doi: 10.1307/mmj/20216112. 1230, 1231

[4] BRIDSON, MARTIN; HAEFLIGER,ANDRÉ.. Metric spaces of non-positive curvature. *Grundlehren der Mathematischen Wissenschaften* **319** (1999) [Fundamental Principles of Mathematical Sciences] Springer-Verlag, Berlin, xxii+643 pp. MR1744486, Zbl 0988.53001, doi: 10.1007/978-3-662-12494-9. 1231

[5] BOWDITCH, BRIAN. Relatively hyperbolic groups. *Internat. J. Algebra Comput.* **22** (2012), no.3, 1250016, 66 pp. MR2922380, Zbl 1259.20052, doi: 10.1142/S0218196712500166. 1224

[6] CROKE, CHRISTOPHER ; KLEINER, BRUCE. The geodesic flow of a nonpositively curved graph manifold. *Geom. Funct. Anal.* **12** (2002), no.3, 479–545. MR1924370, Zbl 1035.53116, doi: 10.1007/s00039-002-8255-7. 1222, 1228

[7] CASHEN, CHRISTOPHER; MARTIN, ALEXANDRE. Quasi-isometries between groups with two-ended splittings. *Math. Proc. Cambridge Philos. Soc.* **162** (2017), no.2, 249–291. MR3604915, Zbl 1476.20041, doi: 10.1017/S0305004116000530. 1223, 1224

[8] DAHMANI, FRANÇOIS. Combination of convergence groups. *Geom. Topol.* **7** (2003), 933–963. MR2026551, Zbl 1037.20042, doi: 10.2140/gt.2003.7.933. 1232

[9] DRUŢU, CORNELIA; KAPOVICH, MICHAEL. Geometric group theory. **63** (2018) American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2018. With an appendix by Bogdan Nica. MR3753580, Zbl 1447.20001, doi: 10.1090/coll/063. 1230

[10] DRUŢU, CORNELIA ; SAPIR, MARK. Tree-graded spaces and asymptotic cones of groups. With an appendix by Denis Osin and Sapir. *Topology* **44**(2005), no.5, 959–1058. MR2153979, Zbl 1101.20025, doi: 10.1016/j.top.2005.03.003. 1221, 1226

[11] GENEVOIS, ANTHONY. Hyperbolicities in CAT(0) cube complexes. *Enseign. Math.* **65** (2019), no.1-2, 33–100. MR4057355, Zbl 1472.20090, doi: 10.4171/lem/65-1/2-2. 1221

[12] GITIK, RITA; MITRA, MAHAN; RIPS, ELIYAHU; SAGEEV, SAGEEV. Widths of subgroups. *Trans. Amer. Math. Soc.* **350** (1998), no.1, 321–329. MR1389776, Zbl 0897.20030, doi: 10.1090/S0002-9947-98-01792-9. 1220, 1221

[13] HRUSKA, CHRISTOPHER. Relative hyperbolicity and relative quasiconvexity for countable groups. *Algebr. Geom. Topol.* **10** (2010), no.3, 1807–1856. MR2684983, Zbl 1202.20046, doi: 10.2140/agt.2010.10.1807. 1226, 1229

[14] HRUSKA, CHRISTOPHER; WISE, DANIEL. Packing subgroups in relatively hyperbolic groups. *Geom. Topol.* **13** (2009), no.4, 1945–1988. MR2497315, Zbl 1188.20042, doi: 10.2140/gt.2009.13.1945. 1226

[15] HRUSKA, CHRISTOPHER; WISE, DANIEL. Finiteness properties of cubulated groups. *Compos. Math.* **150** (2014), no.3, 453–506. MR3187627, Zbl 1335.20043, doi: 10.1112/S0010437X13007112. 1220

[16] MARGOLIS, ALEX; NGUYEN, HOANG. *Quasi-isometric rigidity of extended admissible groups*. `https://arxiv.org/abs/2401.03635`. 1222, 1227, 1230

[17] NGUYEN, HOANG; TRAN, HUNG; YANG, WENYUAN. Quasiconvexity in 3-manifold groups. *Math. Ann.* **381** (2021), no.1-2, 405–437. MR4322616, Zbl 1477.57018, doi: 10.1007/s00208-020-02044-y. 1222, 1226, 1228, 1229

[18] SISTO, ALEXANDRO. Quasi-convexity of hyperbolically embedded subgroups. *Math. Z.* **283** (2016), no.3-4, 649–658. MR3519976, Zbl 1380.20044, doi: 10.1007/s00209-016-1615-z. 1221

[19] SCOTT, PETER; WALL, TERRY. Topological methods in group theory. *Homological group theory, Proc. Symp., Durham 1977, Lond. Math. Soc. Lect. Note Ser.* **36** (1979), 137-203. In Homological group theory (Proc. Sympos., Durham, 1977), volume 36 of London Math. Soc. Lecture Note Ser., pages 137–203. Cambridge Univ. Press, Cambridge-New York, 1979. MR0564422, Zbl 0423.20023, 1222, 1223

[20] Tran, Hung. On strongly quasiconvex subgroups. *Geom. Topol.* **23** (2019), no.3, 1173–1235. MR3956891, Zbl , doi: 10.2140/gt.2019.23.1173. 1221, 1225, 1230, 1232

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