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Existence and UH-Rassias stability for fractional quantum Duffing problem with sequential q-fractional derivatives

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ABSTRACT. The present manuscript is concerned with the existence and uniqueness of solutions along with the UH-Rassias stability for fractional q-differential Duffing problem having sequential fractional q-derivatives. We make use Banach's and Schaefer's fixed point theorems to prove the uniqueness and the existence of at least one solution for the introduced problem. Also, we discuss the UH-Rassias stability for the mentioned problem. Finally, we give some examples to illustrate the proposed main results.

CONTENTS

1. Introduction and q-fractional calculus

In recent years, difference equations with q-fractional (**qF**) derivatives have aroused great interest, these classes of equations have many applications in different fields and thus have evolved into multidisciplinary subjects as can be seen in [4, 12, 13, 14, 15, 16]. Also, the differential equations involving fractional q-difference calculus have been investigated by several scientific researchers, see for instance [2, 22, 23, 26, 27, 28, 30, 40]. Many researchers have

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considered the existence, uniqueness and stability of solutions, fractional q-differential equations, see for example [1, 6, 17, 18, 24, 29, 35]. Recently, considerable attention has been given to the existence of solutions for sequential fractional q-differential equations, the reader can consult [3, 11, 20, 32], In this work, the existence and stability of solutions for sequential Caputo fractional Duffing q-Difference (**qD**) problem have been discussed. The Duffing problem is considered to be an excellent example of a dynamical system that is used to model certain driven-damped oscillators; for more details and applications on Duffing problem, we refer the works [7, 8, 9, 25, 38]. The classical form of the Duffing equation [7], is given by

$$\begin{cases} D^2 u(t) + dD^1 u(t) + \varphi(t, u(t)) - \phi(t) = 0, 0 \le t \le 1,\ d > 0, \\ u(0) - B_1 = 0, D^1 u(0) - B_2 = 0, B_i \in \mathbb{R}, i = 1, 2, \end{cases}$$

where φ and ϕ are given continuous functions. Many scientific researchers have discussed the fractional types of the above equation, for instance see [5, 10, 19, 31, 33, 39] and references therein. In [10] the authors considered the fractional version of the Duffing problem

$$\begin{cases} D^{\vartheta} u(t) + dD^{\delta} u(t) + \zeta u(t) + \xi u^3(t) - \sin(\mu t) = 0, \\ u(0) - B_1 = 0, D^{\delta} u(0) - B_2 = 0,\ B_i \in \mathbb{R}, i = 1, 2, \\ 0 \le t \le 1, 1 < \vartheta < 2,\ 0 < \delta < 1,\ d, \zeta, \xi, \mu > 0, \end{cases}$$

where D^{ϑ} and D^{δ} are the Caputo fractional derivatives. As in [33], the fractional Duffing problem is given as

$$\begin{cases} D^{\vartheta} u(t) + dD^{\delta} u(t) + \varphi(t, u(t)) - \phi(t) = 0, \\ u(v_0) - u_0 = 0,\ u'(v_0) - u_1 = 0, \\ 0 \le t \le 1, 1 < \vartheta < 2,\ 0 < \delta < 1,\ d > 0, \end{cases}$$

where D^{ϑ} and D^{δ} are the Caputo fractional derivatives. The notion here is to consider with the existence, uniqueness and Ulam-stability (**US**) of solutions for the following sequential Caputo fractional Duffing **qD** problem

$$\begin{cases} D_q^{\omega}\left[D_q^{\gamma} u(t)\right] + \theta\varphi\left(t, u(t), D_q^{\delta} u(t)\right) + \psi\left(t, u(t), I_q^{\alpha} u(t)\right) - \phi(t) = 0, \\ u(0) = 0, \beta_1 u(1) - \beta_2 u(\eta) = 0, D_q^{\gamma} u(0) - D_q^{\gamma} u(1) = 0, \eta \in (0, 1), \\ 0 \le t \le 1, 1 < \omega < 2, 0 < \gamma, q < 1, \delta < \gamma, \theta > 0, \alpha > 0, \beta_i \in \mathbb{R}, i = 1, 2, \end{cases} \tag{1}$$

where D_q^{ϑ} is the Caputo **qF** derivative of order $\vartheta \in \{\omega, \gamma\}, \varphi, \psi : [0, 1] \times \mathbb{R} \to \mathbb{R}$ and $\phi : [0, 1] \to \mathbb{R}$ are continuous maps. The operator D_q^{ϑ} is the **qF** derivative in the sense of Caputo [4, 34], defined by

$$D_q^{\vartheta} u(t) = I_q^{n-\vartheta} D_q^n u(t), \ \vartheta > 0,$$

$$D_q^0 u(t) = u(t),$$

where smallest integer n is such that $n \geq \vartheta$. As in [4, 34], define **qF** integral of the Riemann-Liouville type as

$$I_q^{\alpha}[u(t)] = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} u(s)\, d_q s, \alpha > 0,$$

$$I_q^0 [u(t)] = u(t),$$

where the q-gamma function is given by $\Gamma_q(\vartheta) = \frac{(1-q)^{(\vartheta-1)}}{(1-q)^{\vartheta-1}}$, and satisfies

$$\Gamma_q(\vartheta+1) = [\vartheta]_q \Gamma_q(\vartheta), \ [a]_q = \frac{1-q^a}{1-q}, a \in \mathbb{R}.$$

Using [4, 34], we have.

Lemma 1.1. *Let $\vartheta, \varkappa \geq 0$ and define a function u in $[0,1]$, then*

$$I_q^{\vartheta} I_q^{\varkappa} u(t) = I_q^{\vartheta+\varkappa} u(t) \ \textit{and} \ D_q^{\vartheta} I_q^{\vartheta} u(t) = u(t).$$

Lemma 1.2. *For a positive integer $\varkappa$ and $\vartheta > 0$, we have*

$$I_q^{\vartheta} D_q^{\varkappa} u(t) = D_q^{\varkappa} I_q^{\vartheta} u(t) - \sum_{j=0}^{\varkappa-1} \frac{t^{\vartheta-\varkappa+j}}{\Gamma_q(\vartheta+j-\varkappa+1)} D_q^j u(0).$$

Lemma 1.3. *If $\vartheta \in \mathbb{R}^+ \backslash \mathbb{N}$, then*

$$I_q^{\vartheta} D_q^{\vartheta} u(t) = u(t) - \sum_{j=0}^{n-1} \frac{t^j}{\Gamma_q(j+1)} D_q^j u(0).$$

Lemma 1.4. *For $\vartheta \in \mathbb{R}_+$ and $\varpi > -1$, we have*

$$I_q^{\alpha}\left[(t-u)^{(\varpi)}\right] = \frac{\Gamma_q(\varpi+1)}{\Gamma_q(\vartheta+\varpi+1)} (t-u)^{(\vartheta+\varpi)}.$$

By choosing $u = 0$ and $\varpi = 0$, it yields

$$I_q^{\vartheta}[1] = \frac{1}{\Gamma_q(\vartheta+1)} t^{(\vartheta)}.$$

Lemma 1.5. [36] *Let $Z : S \to S$ be a completely continuous operator and define the bounded set*

$$\{u \in S : u = \zeta Z(u), \ 0 < \zeta < 1\},$$

where S is a Banach space. Assuming this set is bounded, Z has a fixed point in S.

Now, we introduce the following space

$$U=\left\{u: u\in C\left([0,1],\mathbb{R}\right) \text{ and } D_q^{\delta}u\in C\left([0,1],\mathbb{R}\right)\right\},$$

endowed with the norm

$$\|u\|_U=\|u\|+\left\|D_q^{\delta}u\right\|=\sup_{t\in[0,1]}|u(t)|+\sup_{t\in[0,1]}\left|D_q^{\delta}u(t)\right|.$$

Then it is well known that $\left(U,\|.\|_U\right)$ is a Banach space [37].

Lemma 1.6. *Let $\beta_1\neq\beta_2\eta^{\gamma}$ and $g\in C([0,1],\mathbb{R})$. Then the unique solution of the problem*

$$\begin{cases} D_q^{\omega}\left[D_q^{\gamma}u(t)\right]=g(t), 0\leq t\leq 1, 1<\omega<2, 0<\gamma,q<1, \\ u(0)=0,\ \beta_1 u(1)=\beta_2 u(\eta),\ D_q^{\gamma}u(0)=D_q^{\gamma}u(1), \\ 0<\eta<1,\ \lambda\geq 0,\ \beta_i\in\mathbb{R},\ i=1,2, \end{cases} \tag{2}$$

is given by

$$\begin{aligned} u(t) &= \frac{1}{\Gamma_q(\omega+\gamma)}\int_0^t (t-qs)^{(\omega+\gamma-1)}g(s)\,d_qs \\ &+\frac{\beta_2 t^{\gamma}}{(\beta_1-\beta_2\eta^{\gamma})\Gamma_q(\omega+\gamma)}\int_0^{\eta}(\eta-qs)^{(\omega+\gamma-1)}g(s)\,d_qs \\ &-\frac{\beta_1 t^{\gamma}}{(\beta_1-\beta_2\eta^{\gamma})\Gamma_q(\omega+\gamma)}\int_0^1(1-qs)^{(\omega+\gamma-1)}g(s)\,d_qs \\ &+\frac{\left(\beta_1-\beta_2\eta^{\gamma+1}\right)t^{\gamma}}{(\beta_1-\beta_2\eta^{\gamma})[\gamma+1]_q\Gamma_q(\omega)}\int_0^1(1-qs)^{(\omega-1)}g(s)\,d_qs \\ &-\frac{t^{\gamma+1}}{\Gamma_q(\gamma+2)\Gamma_q(\omega)}\int_0^1(1-qs)^{(\omega-1)}g(s)\,d_qs. \end{aligned} \tag{3}$$

where $\beta_1-\beta_2\eta^{\gamma}\neq 0$.

Proof. On taking the operator I_q^{ω} to both sides of (2), and using Lemma 4, it yields

$$D_q^{\gamma}u(t)=I_q^{\omega}[g(t)]+d_0I_q^0[1]+d_1I_q^0[t], d_i\in\mathbb{R}, i=0,1. \tag{4}$$

Next, applying the operator I_q^{γ} to both sides (4), we get

$$u(t)=I_q^{\omega+\gamma}[g(t)]+d_0I_q^{\gamma}[1]+d_1I_q^{\gamma}[t]+d_2, d_i\in\mathbb{R}, i=0,1,2. \tag{5}$$

The condition $D_q^{\gamma}u(0) = D_q^{\gamma}u(1)$, imply that

$$d_1 = -I_q^{\omega}[g(1)]. \tag{6}$$

Now, by using $u(0) = 0$ and $\beta_1 u(1) = \beta_2 u(\eta)$, we obtain

$$\begin{aligned} d_2 &= 0, \\ &\text{and} \\ d_0 &= \frac{\Gamma_q(\gamma+1)}{\beta_1 - \beta_2\eta^{\gamma}}\Big[\beta_2 I_q^{\omega+\gamma}[g(\eta)] - \beta_1 I_q^{\omega+\gamma}[g(1)] + \beta_2 d_1 I_q^{\gamma}[\eta] - \beta_1 d_1 I_q^{\gamma}[1]\Big]. \end{aligned} \tag{7}$$

Hence, we obtain (3). □

2. Existence of solutions for fractional Duffing qD problem

The determination of the existence and uniqueness of the solution of fractional Duffing **qD** problem (1) will be determined in this section. Using Lemma 6, we define operator $Z : U \to U$ as

$$\begin{aligned} Zu(t) = {} & \frac{1}{\Gamma_q(\omega+\gamma)}\int_0^t (t-qs)^{(\omega+\gamma-1)}[\phi(s) - \theta\varphi_u^*(s) - \psi_u^*(s)]\,d_q s \\ & + \frac{\beta_2 t^{(\gamma)}}{(\beta_1-\beta_2\eta^{\gamma})\Gamma_q(\omega+\gamma)}\int_0^{\eta} (\eta-qs)^{(\omega+\gamma-1)}[\phi(s) - \theta\varphi_u^*(s) - \psi_u^*(s)]\,d_q s \\ & - \frac{\beta_1 t^{(\gamma)}}{(\beta_1-\beta_2\eta^{\gamma})\Gamma_q(\omega+\gamma)}\int_0^1 (1-qs)^{(\omega+\gamma-1)}[\phi(s) - \theta\varphi_u^*(s) - \psi_u^*(s)]\,d_q s \\ & + \frac{\left(\beta_1-\beta_2\eta^{(\gamma+1)}\right)t^{(\gamma)}}{(\beta_1-\beta_2\eta^{\gamma})[\gamma+1]_q\Gamma_q(\omega)}\int_0^1 (1-qs)^{(\omega-1)}[\phi(s) - \theta\varphi_u^*(s) - \psi_u^*(s)]\,d_q s \\ & - \frac{t^{(\gamma+1)}}{\Gamma_q(\gamma+2)\Gamma_q(\omega)}\int_0^1 (1-qs)^{(\omega-1)}[\phi(s) - \theta\varphi_u^*(s) - \psi_u^*(s)]\,d_q s. \end{aligned} \tag{8}$$

For simplicity, we use following notations:

$$\begin{aligned}
&\Lambda_1 = \frac{1}{\Gamma_q(\omega+\gamma+1)} + \frac{1}{|\beta_1-\beta_2\eta^{\gamma}|}\left[\frac{|\beta_2|\,\eta^{(\omega+\gamma)}+|\beta_1|}{\Gamma_q(\omega+\gamma+1)} + \frac{\left|\beta_1-\beta_2\eta^{(\gamma+1)}\right|}{[\gamma+1]_q\Gamma_q(\omega+1)}\right] \\
&+\frac{1}{\Gamma_q(\gamma+2)\,\Gamma_q(\omega+1)}, \\
&\Lambda_2 = \frac{1}{\Gamma_q(\omega+\gamma-\delta+1)} + \frac{1}{\Gamma_q(\gamma-\delta+1)\,|\beta_1-\beta_2\eta^{\gamma}|}\left[\frac{|\beta_2|\,\eta^{(\omega+\gamma)}+|\beta_1|}{\Gamma_q(\omega+\gamma+1)}\right. \quad (9) \\
&\left.+\frac{\left|\beta_1-\beta_2\eta^{(\gamma+1)}\right|}{[\gamma+1]_q\Gamma_q(\omega+1)}\right] + \frac{1}{\Gamma_q(\gamma-\delta+2)\,\Gamma_q(\gamma+2)\,\Gamma_q(\omega+1)}.
\end{aligned}$$

The first result is concerned with the existence and uniqueness of the solution for the problem (1) and is based on Banach's fixed point theorem.

Theorem 2.1. *Let $\varphi,\psi : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ and $\phi : \mathbb{R}\to\mathbb{R}$ be continuous functions. Further, we assume that:*

(C_1) : There exists constant $\vartheta_1>0, \vartheta_2>0$ such that for all $t\in J$ and $u_i, v_i \in \mathbb{R}^2, i=1,2,$ we have

$$\left|\varphi(t,u_1,v_1)-\varphi(t,u_2,v_2)\right| \le \vartheta_1\left(|u_1-u_2|+|v_1-v_2|\right),$$

and

$$\left|\psi(t,u_1,v_1)-\psi(t,u_1,v_1)\right| \le \vartheta_2\left(|u_1-u_2|+|v_1-v_2|\right).$$

If

$$\left(\theta\vartheta_1+\vartheta_2\left[1+\frac{1}{\Gamma_q(\alpha+1)}\right]\right)(\Lambda_1+\Lambda_2)<1, \tag{10}$$

where $\Lambda_i, i=1,2,$ are defined in (9), then there exists a unique solution of the problem (1).

Proof. Let us define $A=\max\{A_i, i=1,2,3\}$, where A_i are finite numbers given by $A_1=\sup_{t\in[0,1]}|\varphi(t,0,0,0)|, A_2=\sup_{t\in[0,1]}|\psi(t,0,0,0)|$ and $A_3=\sup_{t\in[0,1]}|\phi(t)|$. Setting

$$\frac{[\Lambda_2+\Lambda_2]\,A(\theta+2)}{1-[\Lambda_2+\Lambda_2]\left(\theta\vartheta_1+\vartheta_2\left[1+\frac{1}{\Gamma_q(\alpha+1)}\right]\right)}\le\epsilon,$$

we show that $ZB_\epsilon\subset B_\epsilon$, where Z defined by (8) and $B_\epsilon=\left\{u\in U : \|u\|_U\le\epsilon\right\}$. For $u\in B_\epsilon$ and by (C_1), we can write

$$\begin{aligned}
\left|\varphi_u^*(t)\right| &= \left|\varphi\left(t,u(t),D_q^{\delta}u(t)\right)\right| \le \left|\varphi\left(t,u(t),D_q^{\delta}u(t)\right)-\varphi(t,0,0)\right|+|\varphi(t,0,0)| \quad (11) \\
&\le \vartheta_1\left(|u(t)|+\left|D_q^{\delta}u(t)\right|\right)+A_1\le\vartheta_1\|u\|_U+A_1\le\vartheta_1\epsilon+A,
\end{aligned}$$

and

$$\begin{aligned}\left|\psi_u^*(t)\right| &= \left|\psi\left(t,u(t),I_q^\alpha u(t)\right)\right| \leq \left|\psi\left(t,u(t),I_q^\alpha u(t)\right)-\psi(t,0,0)\right|+\left|\psi(t,0,0)\right| \quad (12)\\ &\leq \vartheta_2\left(|u(t)|+\left|I_q^\alpha u(t)\right|\right)+A_2 \leq \vartheta_2\left(\|u\|_U+\frac{1}{\Gamma_q(\alpha+1)}\|u\|\right)+A_2\\ &\leq \vartheta_2\left[1+\frac{1}{\Gamma_q(\alpha+1)}\right]\|u\|_U+A_2 \leq \vartheta_2\left[1+\frac{1}{\Gamma_q(\alpha+1)}\right]\epsilon+A.\end{aligned}$$

By (11) and (12), we get

$$\begin{aligned}|Zu(t)| \leq \sup_{t\in[0,1]}&\left\{\frac{1}{\Gamma_q(\omega+\gamma)}\int_0^t (t-qs)^{(\omega+\gamma-1)}\left[\phi(s)-\theta\varphi_u^*(s)-\psi_u^*(s)\right]d_qs\right.\\ &+\frac{|\beta_2|\,t^{(\gamma)}}{(|\beta_1-\beta_2\eta^\gamma|)\,\Gamma_q(\omega+\gamma)}\int_0^\eta (\eta-qs)^{(\omega+\gamma-1)}\left[\phi(s)-\theta\varphi_u^*(s)-\psi_u^*(s)\right]d_qs\\ &+\frac{|\beta_1|\,t^{(\gamma)}}{(|\beta_1-\beta_2\eta^\gamma|)\,\Gamma_q(\omega+\gamma)}\int_0^1 (1-qs)^{(\omega+\gamma-1)}\left[\phi(s)-\theta\varphi_u^*(s)-\psi_u^*(s)\right]d_qs\\ &+\frac{\left|\beta_1-\beta_2\eta^{(\gamma+1)}\right|t^{(\gamma)}}{|\beta_1-\beta_2\eta^\gamma|\,[\gamma+1]_q\Gamma_q(\omega)}\int_0^1 (1-qs)^{(\omega-1)}\left[\phi(s)-\theta\varphi_u^*(s)-\psi_u^*(s)\right]d_qs\\ &\left.+\frac{t^{(\gamma+1)}}{\Gamma_q(\gamma+2)\,\Gamma_q(\omega)}\int_0^1 (1-qs)^{(\omega-1)}\left[\phi(s)-\theta\varphi_u^*(s)-\psi_u^*(s)\right]d_qs\right\},\end{aligned}$$

which implies that

$$\begin{aligned}&\|Zu\|\\ \leq\ &\left(\frac{1}{\Gamma_q(\omega+\gamma+1)}+\frac{1}{|\beta_1-\beta_2\eta^\gamma|}\left[\frac{|\beta_2|\,\eta^{(\omega+\gamma)}+|\beta_1|}{\Gamma_q(\omega+\gamma+1)}\right.\right.\\ &\left.\left.+\frac{\left|\beta_1-\beta_2\eta^{\gamma+1}\right|}{[\gamma+1]_q\Gamma_q(\omega+1)}\right]+\frac{1}{\Gamma_q(\gamma+2)\,\Gamma_q(\omega+1)}\right)\\ &\times\left[\left(\theta\vartheta_1+\vartheta_2\left[1+\frac{1}{\Gamma_q(\alpha+1)}\right]\right)\epsilon+A(\theta+2)\right]\\ =\ &\Lambda_1\left[\left(\theta\vartheta_1+\vartheta_2\left[1+\frac{1}{\Gamma_q(\alpha+1)}\right]\right)\epsilon+A(\theta+2)\right].\end{aligned}$$

Also, we have

$$
\begin{aligned}
&\left|D_q^{\delta} Zu(t)\right| \\
&\leq \sup_{t\in[0,1]} \left\{ \frac{1}{\Gamma_q(\omega+\gamma-\delta)} \int_0^t (t-qs)^{(\omega+\gamma-\delta-1)} \left[\phi(s) - \theta\varphi_u^*(s) - \psi_u^*(s)\right] d_q s \right. \\
&+ \frac{|\beta_2|\, t^{(\gamma-\delta)}}{\Gamma_q(\gamma-\delta+1)\left|\beta_1-\beta_2\eta^{\gamma}\right|\Gamma_q(\omega+\gamma)} \int_0^{\eta} (\eta-qs)^{(\omega+\gamma-1)} \left[\phi(s) - \theta\varphi_u^*(s) - \psi_u^*(s)\right] d_q s \\
&+ \frac{|\beta_1|\, t^{(\gamma-\delta)}}{\Gamma_q(\gamma-\delta+1)\left|\beta_1-\beta_2\eta^{\gamma}\right|\Gamma_q(\omega+\gamma)} \int_0^{1} (1-qs)^{(\omega+\gamma-1)} \left[\phi(s) - \theta\varphi_u^*(s) - \psi_u^*(s)\right] d_q s \\
&+ \frac{\left|\beta_1-\beta_2\eta^{(\gamma+1)}\right| t^{(\gamma-\delta)}}{\Gamma_q(\gamma-\delta+1)\left|\beta_1-\beta_2\eta^{\gamma}\right| [\gamma+1]_q \Gamma_q(\omega)} \int_0^{1} (1-qs)^{(\omega-1)} \left[\phi(s) - \theta\varphi_u^*(s) - \psi_u^*(s)\right] d_q s \\
&\left. + \frac{t^{(\gamma-\delta+1)}}{\Gamma_q(\gamma-\delta+2)\Gamma_q(\gamma+2)\Gamma_q(\omega)} \int_0^{1} (1-qs)^{(\omega-1)} \left[\phi(s) - \theta\varphi_u^*(s) - \psi_u^*(s)\right] d_q s \right\}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\left\|D_q^{\delta} Z(u)\right\| \\
\leq\ & \left(\frac{1}{\Gamma_q(\omega+\gamma-\delta+1)} + \frac{1}{\Gamma_q(\gamma-\delta+1)\left|\beta_1-\beta_2\eta^{\gamma}\right|} \left[\frac{|\beta_2|\,\eta^{(\omega+\gamma)} + |\beta_1|}{\Gamma_q(\omega+\gamma+1)} \right.\right. \\
&\left.\left. + \frac{\left|\beta_1-\beta_2\eta^{(\gamma+1)}\right|}{[\gamma+1]_q \Gamma_q(\omega+1)} \right] + \frac{1}{\Gamma_q(\gamma-\delta+2)\Gamma_q(\gamma+2)\Gamma_q(\omega+1)} \right) \\
&\times \left[\left(\theta\vartheta_1 + \vartheta_2 \left[1 + \frac{1}{\Gamma_q(\alpha+1)} \right] \right) \epsilon + A(\theta+2) \right] \\
&\Lambda_2 \left[\left(\theta\vartheta_1 + \vartheta_2 \left[1 + \frac{1}{\Gamma_q(\alpha+1)} \right] \right) \epsilon + A(\theta+2) \right].
\end{aligned}
$$

In consequence, we get

$$
\begin{aligned}
\|Z(u)\|_U &= \|Z(u)\| + \left\|D_q^{\delta} Z(u)\right\| \\
&\leq [\Lambda_2+\Lambda_2] \left(\theta\vartheta_1 + \vartheta_2 \left[1 + \frac{1}{\Gamma_q(\alpha+1)} \right] \right) \epsilon \\
+ [\Lambda_2+\Lambda_2]\, A(\theta+2) &\leq \epsilon,
\end{aligned}
$$

which means that $ZB_\epsilon \subset B_\epsilon$. For $u, v \in B_\epsilon$ and for each $t \in [0,1]$, we have

$$\begin{aligned}
&|Zu(t) - Zv(t)| \\
&\leq \sup_{t\in[0,1]} \left\{ \frac{1}{\Gamma_q(\omega+\gamma)} \int_0^t (t-qs)^{(\omega+\gamma-1)} \left[\theta \left|\varphi_u^*(s) - \varphi_v^*(s)\right| + \left|\psi_u^*(s) - \psi_v^*(s)\right|\right] d_q s \right. \\
&+ \frac{|\beta_2|\, t^{(\gamma)}}{|\beta_1 - \beta_2\eta^\gamma|\, \Gamma_q(\omega+\gamma)} \int_0^\eta (\eta-qs)^{(\omega+\gamma-1)} \left[\theta \left|\varphi_u^*(s) - \varphi_v^*(s)\right| + \left|\psi_u^*(s) - \psi_v^*(s)\right|\right] d_q s \\
&+ \frac{|\beta_1|\, t^{(\gamma)}}{(|\beta_1 - \beta_2\eta^\gamma|)\, \Gamma_q(\omega+\gamma)} \int_0^1 (1-qs)^{(\omega+\gamma-1)} \left[\theta \left|\varphi_u^*(s) - \varphi_v^*(s)\right| + \left|\psi_u^*(s) - \psi_v^*(s)\right|\right] d_q s \\
&+ \frac{\left|\beta_1 - \beta_2\eta^{(\gamma+1)}\right| t^{(\gamma)}}{|\beta_1 - \beta_2\eta^\gamma|\, [\gamma+1]_q \Gamma_q(\omega)} \int_0^1 (1-qs)^{(\omega-1)} \left[\theta \left|\varphi_u^*(s) - \varphi_v^*(s)\right| + \left|\psi_u^*(s) - \psi_v^*(s)\right|\right] d_q s \\
&\left. + \frac{t^{(\gamma+1)}}{\Gamma_q(\gamma+2)\, \Gamma_q(\omega)} \int_0^1 (1-qs)^{(\omega-1)} \left[\theta \left|\varphi_u^*(s) - \varphi_v^*(s)\right| + \left|\psi_u^*(s) - \psi_v^*(s)\right|\right] d_q s \right\}.
\end{aligned}$$

By (C_1), we can write

$$\begin{aligned}
&\|Z(u) - Z(v)\| \\
\leq\ & \left(\theta\vartheta_1 + \vartheta_2\left[1 + \frac{1}{\Gamma_q(\alpha+1)}\right]\right)\left[\frac{1}{\Gamma_q(\omega+\gamma+1)} + \frac{1}{|\beta_1 - \beta_2\eta^\gamma|}\left(\frac{|\beta_2|\,\eta^{(\omega+\gamma)} + |\beta_1|}{\Gamma_q(\omega+\gamma+1)}\right.\right. \\
& \left.\left. + \frac{\left|\beta_1 - \beta_2\eta^{\gamma+1}\right|}{[\gamma+1]_q \Gamma_q(\omega+1)}\right) + \frac{1}{\Gamma_q(\gamma+2)\,\Gamma_q(\omega+1)}\right] \|u - v\|_U \\
=\ & \left(\theta\vartheta_1 + \vartheta_2\left[1 + \frac{1}{\Gamma_q(\alpha+1)}\right]\right)\Lambda_1 \|u - v\|_U .
\end{aligned}$$

Hence,

$$\|Z(u) - Z(v)\| \leq \left(\theta\vartheta_1 + \vartheta_2\left[1 + \frac{1}{\Gamma_q(\alpha+1)}\right]\right)\Lambda_1 \|u - v\|_U . \tag{13}$$

On the other hand, for each $t \in [0,1]$, we have

$$\begin{aligned}
&\left|D_q^{\delta} Zu(t) - D_q^{\delta} Zv(t)\right| \\
&\leq \sup_{t\in[0,1]} \left\{ \frac{1}{\Gamma_q(\omega+\gamma-\delta)} \int_0^t (t-qs)^{(\omega+\gamma-\delta-1)} \left[\theta\left|\varphi_u^*(s)-\varphi_v^*(s)\right| + \left|\psi_u^*(s)-\psi_v^*(s)\right|\right] d_q s \right. \\
&+ \frac{|\beta_2|\, t^{(\gamma-\delta)}}{\Gamma_q(\gamma-\delta+1)\left|\beta_1-\beta_2\eta^{\gamma}\right|\Gamma_q(\omega+\gamma)} \int_0^{\eta} (\eta-qs)^{(\omega+\gamma-1)} \\
&\times \left[\theta\left|\varphi_u^*(s)-\varphi_v^*(s)\right| + \left|\psi_u^*(s)-\psi_v^*(s)\right|\right] d_q s \\
&+ \frac{|\beta_1|\, t^{(\gamma-\delta)}}{\Gamma_q(\gamma-\delta+1)\left|\beta_1-\beta_2\eta^{\gamma}\right|\Gamma_q(\omega+\gamma)} \int_0^{1} (1-qs)^{(\omega+\gamma-1)} \\
&\times \left[\theta\left|\varphi_u^*(s)-\varphi_v^*(s)\right| + \left|\psi_u^*(s)-\psi_v^*(s)\right|\right] d_q s \\
&+ \frac{\left|\beta_1-\beta_2\eta^{(\gamma+1)}\right| t^{(\gamma-\delta)}}{\Gamma_q(\gamma-\delta+1)\left|\beta_1-\beta_2\eta^{\gamma}\right|[\gamma+1]_q\Gamma_q(\omega)} \int_0^{1} (1-qs)^{(\omega-1)} \\
&\times \left[\theta\left|\varphi_u^*(s)-\varphi_v^*(s)\right| + \left|\psi_u^*(s)-\psi_v^*(s)\right|\right] d_q s \\
&+ \frac{t^{(\gamma-\delta+1)}}{\Gamma_q(\gamma-\delta+2)\Gamma_q(\gamma+2)\Gamma_q(\omega)} \int_0^{1} (1-qs)^{(\omega-1)} \\
&\left. \left[\theta\left|\varphi_u^*(s)-\varphi_v^*(s)\right| + \left|\psi_u^*(s)-\psi_v^*(s)\right|\right] d_q s \right\}.
\end{aligned}$$

Thanks to (C_1), we have

$$\begin{aligned}
&\left\|D_q^{\delta} Z(u) - D_q^{\delta} Z(v)\right\| \\
\leq\ & \left(\theta\vartheta_1 + \vartheta_2\left[1+\frac{1}{\Gamma_q(\alpha+1)}\right]\right)\left[\frac{1}{\Gamma_q(\omega+\gamma-\delta+1)}\right. \\
&+\frac{1}{\Gamma_q(\gamma-\delta+1)\left|\beta_1-\beta_2\eta^{\gamma}\right|}\left(\frac{|\beta_2|\,\eta^{(\omega+\gamma)}+|\beta_1|}{\Gamma_q(\omega+\gamma+1)} + \frac{\left|\beta_1-\beta_2\eta^{(\gamma+1)}\right|}{[\gamma+1]_q\Gamma_q(\omega+1)}\right) \\
&\left.+\frac{1}{\Gamma_q(\gamma-\delta+2)\Gamma_q(\gamma+2)\Gamma_q(\omega+1)}\right]\|u-v\|_U .
\end{aligned}$$

Therefore,

$$\left\|D_q^{\delta} Z(u) - D_q^{\delta} Z(v)\right\| \leq \left(\theta\vartheta_1 + \vartheta_2\left[1+\frac{1}{\Gamma_q(\alpha+1)}\right]\right)\Lambda_2\|u-v\|_U . \tag{14}$$

Then, thanks to (13) and (14), we conclude that

$$\|Z(u)-Z(v)\|_U \leq \left(\theta\vartheta_1 + \vartheta_2\left[1+\frac{1}{\Gamma_q(\alpha+1)}\right]\right)(\Lambda_1+\Lambda_2)\|u-v\|_U .$$

By (10), it is obvious that Z is contractive operator. Consequently, Z has a fixed point which is a solution of (1), using Banach fixed point theorem. □

Now, we prove existence of at least one solutions for the sequential Caputo fractional Duffing **qD** problem (1) by using lemma 5.

Theorem 2.2. *Let $\varphi, \psi : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ and $\phi : \mathbb{R} \to \mathbb{R}$ be continuous functions. Assume that:*

(C_2) : There exist a positive constants $B_i, i = 1,2,3$ in such a way that for all $t \in [0,1]$ and $u, v \in \mathbb{R}$.

$$\left|\varphi(t,u,v)\right| \leq B_1, \left|\psi(t,u,v)\right| \leq B_2 \text{ and } \left|\phi(t)\right| \leq B_3.$$

Then the sequential Caputo fractional Duffing **qD** *problem (1) has at least one solution.*

Proof. By continuity of functions of φ, ψ and ϕ, the operator Z is continuous.

Now, we show that the operator Z is completely continuous.

(a_1) : Firstly, we show that Z maps bounded sets of U into bounded sets of U. Let us tak $\sigma > 0$ and $B_\sigma = \{u \in U : \|u\|_U \leq \sigma\}$. Then for $u \in B_\sigma$, we have

$$\begin{aligned} &\|Z(u)\| \qquad (15)\\ \leq &\left[\frac{1}{\Gamma_q(\omega+\gamma+1)} + \frac{1}{|\beta_1-\beta_2\eta^\gamma|}\left(\frac{|\beta_2|\,\eta^{(\omega+\gamma)}}{\Gamma_q(\omega+\gamma+1)} + \frac{|\beta_1|}{\Gamma_q(\omega+\gamma+1)}\right.\right.\\ &\left.\left.+\frac{\left|\beta_1-\beta_2\eta^{\gamma+1}\right|}{[\gamma+1]_q\Gamma_q(\omega+1)}\right) + \frac{1}{\Gamma_q(\gamma+2)\,\Gamma_q(\omega+1)}\right]\left(\theta B_1 + \sum_{i=2}^{3} B_i\right)\\ = &\Lambda_1\left(\theta B_1 + \sum_{i=2}^{3} B_i\right) < \infty. \end{aligned}$$

and

$$\begin{aligned} &\left\|D_q^\delta Z(u)\right\| \qquad (16)\\ \leq &\left[\frac{1}{\Gamma_q(\omega+\gamma-\delta+1)} + \frac{1}{\Gamma_q(\gamma-\delta+1)\,|\beta_1-\beta_2\eta^\gamma|}\left(\frac{|\beta_2|\,\eta^{(\omega+\gamma)}+|\beta_1|}{\Gamma_q(\omega+\gamma+1)}\right.\right.\\ &\left.\left.+\frac{\left|\beta_1-\beta_2\eta^{(\gamma+1)}\right|}{[\gamma+1]_q\Gamma_q(\omega+1)}\right) + \frac{1}{\Gamma_q(\gamma-\delta+2)\,\Gamma_q(\gamma+2)\,\Gamma_q(\omega+1)}\right]\left(\theta B_1 + \sum_{i=2}^{3} B_i\right).\\ = &\Lambda_2\left(\theta B_1 + \sum_{i=2}^{3} B_i\right) < \infty. \end{aligned}$$

It follows from (15) and (16) that $\|Z(u)\|_W < \infty$.

(a_2) : Next, we show that Q is equicontinuous. Let $u \in B_\sigma$ and $t_1, t_2 \in [0,1]$, with $t_1 < t_2$, we have

$$\begin{aligned} & |Qu(t_2) - Qu(t_1)| \qquad (17) \\ \leq & \left(\frac{1}{\Gamma_q(\omega+\gamma+1)} \left[(t_2 - t_1)^{(\omega+\gamma)} + \left| t_2^{(\omega+\gamma)} - t_1^{(\omega+\gamma)} \right| \right] \right. \\ & + \frac{\left(|\beta_2| \eta^{(\omega+\gamma)} - |\beta_1| \right) \left| t_2^{(\gamma)} - t_1^{(\gamma)} \right|}{|\beta_1 - \beta_2 \eta^\gamma| \Gamma_q(\omega+\gamma+1)} + \frac{\left(\beta_1 - \beta_2 \eta^{(\gamma+1)} \right) \left| t_2^{(\gamma)} - t_1^{(\gamma)} \right|}{(\beta_1 - \beta_2 \eta^\gamma) [\gamma+1]_q \Gamma_q(\omega+1)} \\ & \left. + \frac{\left| t_1^{(\gamma+1)} - t_2^{(\gamma+1)} \right|}{\Gamma_q(\gamma+2) \Gamma_q(\omega+1)} \right) \left(\theta B_1 + \sum_{i=2}^{3} B_i \right), \end{aligned}$$

and

$$\begin{aligned} & \left| D_q^\delta Zu(t_2) - D_q^\delta Zu(t_1) \right| \qquad (18) \\ & \leq \left(\frac{1}{\Gamma_q(\omega+\gamma-\delta+1)} \left[(t_2 - t_1)^{\omega+\gamma-\delta} + \left| t_2^{\omega+\gamma-\delta} - t_1^{\omega+\gamma-\delta} \right| \right] \right. \\ & + \frac{(|\beta_2| + |\beta_1|) \left| t_2^{(\gamma-\delta)} - t_1^{(\gamma-\delta)} \right|}{\Gamma_q(\gamma-\delta+1) |\beta_1 - \beta_2 \eta^\gamma| \Gamma_q(\omega+\gamma)} + \frac{\left| \beta_1 - \beta_2 \eta^{(\gamma+1)} \right| \left| t_2^{(\gamma-\delta)} - t_1^{(\gamma-\delta)} \right|}{\Gamma_q(\gamma-\delta+1) |\beta_1 - \beta_2 \eta^\gamma| [\gamma+1]_q \Gamma_q(\omega+1)} \\ & \left. + \frac{\left| t_1^{(\gamma-\delta+1)} - t_2^{(\gamma-\delta+1)} \right|}{\Gamma_q(\gamma-\delta+2) \Gamma_q(\gamma+2) \Gamma_q(\omega+1)} \right) \left(\theta B_1 + \sum_{i=2}^{3} B_i \right). \end{aligned}$$

Thanks to (17) and (18), we can state that $\|Zu(t_2) - Zu(t_1)\|_U \to 0$ as $t_2 \to t_1$. Combining (a_1) and (a_2) and using the Arzelà-Ascoli theorem, we conclude that Z is a completely continuous operator.

(a_3) : Finally, we show that the set Φ, defined by

$$\Phi = \{ u \in U : u = \rho Z(u),\ 0 < \rho < 1 \},$$

is bounded. Let $u \in \Phi$, then $u = \rho Z(u)$ for some $0 < \rho < 1$. Hence, for $t \in [0,1]$, we have

$$u(t) = \rho Zu(t).$$

By (C_2), we have

$$\|u\| \leq \rho \Lambda_1 \left(\theta B_1 + \sum_{i=2}^{3} B_i \right), \qquad (19)$$

and

$$\left\| D_q^\delta u \right\| \leq \rho \Lambda_2 \left(\theta B_1 + \sum_{i=2}^{3} B_i \right). \qquad (20)$$

It follows from (19) and (20), that

$$\|u\|_U \leq \rho (\Lambda_1 + \Lambda_2) \left(\theta B_1 + \sum_{i=2}^{3} B_i \right) \leq (\Lambda_1 + \Lambda_2) \left(\theta B_1 + \sum_{i=2}^{3} B_i \right).$$

Consequently,

$$\|u\|_U < \infty.$$

This shows that the set Φ is bounded.

Thanks to $(a_i), i = 1, 2, 3$, and by Lemma 5, we deduce that Q has at least one fixed point, which is a solution of problem (1). □

3. UH Stability of fractional Duffing qD problem

In this part, the **UH** stability and the **UH**-Rassias stability of the Caputo fractional Duffing **qD** problem (1) will be discussed. We consider the **US** for the sequential Caputo fractional Duffing **qD** problem (1). For $b > 0$ and $m : [0,1] \to \mathbb{R}_+$, we give the following inequalities:

$$\left| D_q^{\omega} \left[D_q^{\gamma} u(t) \right] - \left[\phi(t) - \theta \varphi_v^*(t) - \psi_v^*(t) \right] \right| \le b, t \in [0,1], \tag{21}$$

and

$$\left| D_q^{\omega} \left[D_q^{\gamma} u(t) \right] - \left[\phi(t) - \theta \varphi_v^*(t) - \psi_v^*(t) \right] \right| \le bm(t), t \in [0,1], \tag{22}$$

where $\varphi_v^*(t) = \varphi\left(t, u(t), D_q^{\delta} u(t)\right)$ and $\psi_v^*(t) = \psi\left(t, u(t), I_q^{\alpha} u(t)\right)$.

Definition 3.1. *Duffing* **qD** *problem (1) is Ulam-Hyers (***UH***) stable if there exists a real number $\Pi_{\varphi,\psi} > 0$ such that for each $b > 0$ and for each solution v of the inequality (21), there exists a solution u of the Duffing* **qD** *problem (1) with*

$$\|v - u\|_U \le \Pi_{\varphi,\psi} b.$$

Definition 3.2. *Duffing* **qD** *problem (1) is generalized* **UH** *stable if there exists $\Upsilon_{\varphi,\psi} \in C(\mathbb{R}_+, \mathbb{R}_+), \Upsilon_{\varphi,\psi}(0) = 0$, such that for each solution v of the inequality (21), there exists a solution u of the the Duffing* **qD** *problem (1) with*

$$\|v - u\|_U \le \Pi_{\varphi,\psi}(b).$$

Definition 3.3. *Duffing* **qD** *problem (1) is* **UH***-Rassias stable with respect to $m \in C([0,1], \mathbb{R}_+)$ if there exists a real number $\Pi_{\varphi,\psi,\phi} > 0$ such that for each $b > 0$ and for each solution v of the inequality (22), there exists a solution u of the Duffing* **qD** *problem (1) with*

$$\|v - u\|_U \le \Pi_{\varphi,\psi} bm(t).$$

Remark 3.4. *A map $v \in C([0,1], \mathbb{R})$ is a solution of (21) if and only if there exists a map $h : [0,1] \to \mathbb{R}$ (depending on v) such that*

$$|h(t)| \le b, \ t \in [0,1],$$

and

$$D_q^{\omega} \left[D_q^{\gamma} u(t) \right] = \phi(t) - \theta \varphi_v^*(t) - \psi_v^*(t) + h(t), t \in [0,1].$$

Theorem 3.5. *Assume that $\varphi, \psi : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\phi : \mathbb{R} \to \mathbb{R}$ are continuous functions and suppose that (C_1) holds. If*

$$\theta \vartheta_1 + \vartheta_2 \left[1 + \frac{1}{\Gamma_q(\alpha + 1)} \right] < \Gamma_q(\omega + \gamma + 1), \tag{23}$$

then the Caputo fractional Duffing **qD** *problem (1) is* **UH** *stable.*

Proof. Let the solution of the inequality (21) be $\upsilon \in U$ and represent by $u \in U$ as the unique solution of the problem

$$\begin{cases} D_q^{\omega}\left[D_q^{\gamma}u(t)\right] + \theta\varphi_u^*(t) + \psi_u^*(t) - \phi(t) = 0,\ t \in [0,1], 0 < q < 1, \\ u(0) = \upsilon(0), u(1) = \upsilon(1), u(\eta) = \upsilon(\eta), D_q^{\gamma}u(0) = D_q^{\gamma}\upsilon(0), D_q^{\gamma}u(1) = D_q^{\gamma}\upsilon(1), \end{cases}$$

Using Lemma 6, we can write

$$u(t) = I_q^{\omega+\gamma}[g_u(t)] + d_0 I_q^{\gamma}[1] + d_1 I_q^{\gamma}[t] + d_2, d_i \in \mathbb{R}, i = 0, 1, 2.$$

On integrating (21), we see

$$\begin{aligned} &\left|\upsilon(t) - I_q^{\omega+\gamma}[g_\upsilon(t)] - d_0 I_q^{\beta}[1] - d_1 I_q^{\beta}[t] - d_2\right| \\ \leq\ & \frac{b}{\Gamma_q(\omega+\gamma+1)} t^{\omega+\gamma} \leq \frac{b}{\Gamma_q(\omega+\gamma+1)}. \end{aligned}$$

Also, if $u(r) = \upsilon(r), r \in \{0, \eta, 1\}$ and $D_q^{\gamma}u(r) = D_q^{\gamma}\upsilon(r), r \in \{0, 1\}$, then $d_0 = d_3, d_1 = d_4$ and $d_2 = d_5$.

For all $t \in [0,1]$, we have

$$\begin{aligned} & |\upsilon(t) - u(t)| \\ =\ & \left|\upsilon(t) - I_q^{\omega+\gamma}[g_u(t)] - d_3 I_q^{\gamma}[1] - d_3 I_q^{\gamma}[t] - d_5 + I_q^{\omega+\gamma}[g_\upsilon(t) - g_u(t)]\right| \\ \leq\ & \left|\upsilon(t) - I_q^{\omega+\gamma}[g_u(t)] - d_3 I_q^{\gamma}[1] - d_4 I_q^{\gamma}[t] - d_5\right| + \left|I_q^{\omega+\gamma}[g_\upsilon(t) - g_u(t)]\right|, \end{aligned}$$

where

$$g_u(t) = \phi(t) - (\theta\varphi_u^*(t) + \psi_u^*(t)),$$

and

$$g_\upsilon(t) = \phi(t) - (\theta\varphi_\upsilon^*(t) + \psi_\upsilon^*(t)).$$

Then, using (C_1), we get

$$\begin{aligned} |\upsilon(t) - u(t)| &\leq \left|\upsilon(t) - I_q^{\omega+\gamma}[g_u(t)] - d_3 I_q^{\gamma}[1] - d_4 I_q^{\gamma}[t] - d_5\right| \\ &\quad + I_q^{\omega+\gamma}[\theta|\varphi_\upsilon^*(t) - \varphi_u^*(t)|] + I_q^{\omega+\gamma}[|\psi_\upsilon^*(t) - \psi_u^*(t)|] \\ &\leq \frac{b}{\Gamma_q(\omega+\gamma+1)} \\ &\quad + \frac{1}{\Gamma_q(\omega+\gamma+1)}\left(\theta\vartheta_1 + \vartheta_2\left[1 + \frac{1}{\Gamma_q(\alpha+1)}\right]\right)\|\upsilon(s) - u(s)\|_U. \end{aligned}$$

Thus

$$\begin{aligned} & \|\upsilon(s) - u(s)\|_U\left[1 - \frac{1}{\Gamma_q(\omega+\gamma+1)}\left(\theta\vartheta_1 + \vartheta_2\left[1 + \frac{1}{\Gamma_q(\alpha+1)}\right]\right)\right] \\ \leq\ & \frac{b}{\Gamma_q(\omega+\gamma+1)}. \end{aligned}$$

Then, we have

$$\|v-u\|_U \leq \frac{b}{\Gamma_q(\omega+\gamma+1)-\left(\theta\vartheta_1+\vartheta_2\left[1+\frac{1}{\Gamma_q(\alpha+1)}\right]\right)} = \Pi_{\varphi,\psi} b.$$

Hence, the problem (1) is stable in UH sense.

By taking $\Upsilon_{\varphi,\psi}(0) = \Pi_{\varphi,\psi} b, \Upsilon_{\varphi,\psi}(0) = 0$ yields that the fractional Duffing **qD** problem (1) is generalized UH stable. □

Theorem 3.6. *Let $\varphi, \psi : [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $\phi : \mathbb{R} \to \mathbb{R}$ be continuous functions and assume that (C_1), (23) hold. Suppose there exists $\pi_m > 0$ such that*

$$\frac{1}{\Gamma_q(\omega+\gamma)} \int_0^t (t-qs)^{(\omega+\gamma-1)} m(s) d_q s \leq \pi_m m(t), \tag{24}$$

*for all $t \in [0,1]$, where $m \in C([0,1], \mathbb{R}_+)$ is nondecreasing. Then the Caputo fractional Duffing **qD** problem (1) is **UH**-Rassias stable with respect to m.*

Proof. Let $v \in U$ be a solution of the inequality (21) and let us denote by $u \in U$ the unique solution of the problem

$$\begin{cases} D_q^{\omega}\left[D_q^{\gamma} u(t)\right] + \theta\varphi_u^*(t) + \psi_u^*(t) - \phi(t) = 0,\ t \in [0,1], 0<q<1, \\ u(0) = v(0), u(1) = v(1), u(\eta) = v(\eta), D_q^{\gamma} u(0) = D_q^{\gamma} v(0), D_q^{\gamma} u(1) = D_q^{\gamma} v(1), \end{cases}$$

Using Lemma 6, we can write

$$u(t) = I_q^{\omega+\gamma}[g_u(t)] + d_0 I_q^{\gamma}[1] + d_1 I_q^{\gamma}[t] + d_2, d_i \in \mathbb{R}, i = 0,1,2.$$

By integration of the inequality (21), we have

$$\begin{aligned} & \left| v(t) - I_q^{\omega+\gamma}[g_v(t)] - d_0 I_q^{\beta}[1] - d_1 I_q^{\beta}[t] - d_2 \right| \\ \leq\ & \frac{b}{\Gamma_q(\omega+\gamma+1)} t^{\omega+\gamma} \leq \frac{b}{\Gamma_q(\omega+\gamma+1)}. \end{aligned}$$

Also, if $u(r) = v(r), r \in \{0,\eta,1\}$ and $D_q^{\gamma} u(r) = D_q^{\gamma} v(r), r \in \{0,1\}$, then $d_0 = d_3, d_1 = d_4$ and $d_2 = d_5$.

For all $t \in [0,1]$, we have

$$\begin{aligned} & |v(t) - u(t)| \\ = & \left| v(t) - I_q^{\omega+\gamma}[g_u(t)] - d_3 I_q^{\gamma}[1] - d_3 I_q^{\gamma}[t] - d_5 + I_q^{\omega+\gamma}[g_v(t) - g_u(t)] \right| \\ \leq & \left| v(t) - I_q^{\omega+\gamma}[g_u(t)] - d_3 I_q^{\gamma}[1] - d_4 I_q^{\gamma}[t] - d_5 \right| + \left| I_q^{\omega+\gamma}[g_v(t) - g_u(t)] \right|, \end{aligned}$$

where

$$g_u(t) = \phi(t) - (\theta\varphi_u^*(t) + \psi_u^*(t)),$$

and

$$g_v(t) = \phi(t) - (\theta\varphi_v^*(t) + \psi_v^*(t)).$$

Then, using (C_1), we get

$$\begin{aligned}|\upsilon(t)-u(t)| &\leq \left|\upsilon(t)-I_q^{\omega+\gamma}[g_u(t)]-d_3I_q^{\gamma}[1]-d_4I_q^{\gamma}[t]-d_5\right| \\ &+I_q^{\omega+\gamma}[\theta|\varphi_\upsilon^*(t)-\varphi_u^*(t)|]+I_q^{\omega+\gamma}[|\psi_\upsilon^*(t)-\psi_u^*(t)|] \\ &\leq \frac{b}{\Gamma_q(\omega+\gamma+1)} \\ &+\frac{1}{\Gamma_q(\omega+\gamma+1)}\left(\theta\vartheta_1+\vartheta_2\left[1+\frac{1}{\Gamma_q(\alpha+1)}\right]\right)\|\upsilon(s)-u(s)\|_U.\end{aligned}$$

Thus

$$\|\upsilon(s)-u(s)\|_U\left[1-\frac{1}{\Gamma_q(\omega+\gamma+1)}\left(\theta\vartheta_1+\vartheta_2\left[1+\frac{1}{\Gamma_q(\alpha+1)}\right]\right)\right]$$
$$\leq \frac{b}{\Gamma_q(\omega+\gamma+1)}.$$

Then, we have

$$\|\upsilon-u\|_U\leq\frac{b}{\Gamma_q(\omega+\gamma+1)-\left(\theta\vartheta_1+\vartheta_2\left[1+\frac{1}{\Gamma_q(\alpha+1)}\right]\right)}=\Pi_{\varphi,\psi}b.$$

Hence, the problem (1) is stable in UH sense.

By taking $\Upsilon_{\varphi,\psi}(0)=\Pi_{\varphi,\psi}b, \Upsilon_{\varphi,\psi}(0)=0$ yields that the fractional Duffing **qD** problem (1) is generalized UH stable. □

4. Conclusion

One of the interesting differential equations relates to Duffing problem. Some researchers have studied the Duffing problem from different views. In this work, we study its fractional q-differential version. In fact, we study uniqueness of solutions as well as the UH-Rassias stability for the fractional q-differential Duffing problem by considering sequential fractional q-derivatives.

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References

[1] S. Abbas, M. Benchohra, N. Laledj and Y. Zhou. Existence and Ulam stability for implicit fractional q-difference equations, *Adv. Differ. Equ.* **480** (2019), 1–12. MR4035943, Zbl 0939.01007, doi: 10.1186/s13662-019-2411-y. 1480

[2] T. Abdeljawad, M. E. Samei. Applying quantum calculus for the existence of solution of q-integro-differential equations with three criteria, *Discrete Contin. Dyn. Syst. Ser. S.* **14** (2021), no. 10, 3351–3386. MR4296152, Zbl 1469.34103, doi: 10.3934/dcdss.2020440. 1479

[3] R. P. Agarwal, B. Ahmad, A. Alsaedi and H. Al-Hutami. Existence theory for q-antiperiodic boundary value problems of sequential q-fractional integrodifferential equations, *Abstr. Appl. Anal.* **2014**, (2014), 1–12. MR3206773, Zbl 1470.34169, doi: 10.1155/2014/207547. 1480

[4] M. H. Annaby, Z. S. Mansour. q-fractional calculus and equations. Lecture Notes in Mathematics 2056, Springer-Verlag, Berlin. 2012. MR2963764, Zbl 1267.26001, doi: 10.1007/978-3-642-30898-7. 1479, 1481

[5] M. Bezziou, I. Jebril and Z. Dahmani. A new nonlinear duffing system with sequential fractional derivatives, *Chaos Solitons Fractals.* **151** (2021), 1–7. MR4289066, Zbl 1498.34100, doi: 10.1016/j.chaos.2021.111247. 1480

[6] R. I. Butt, T. Abdeljawad, M. A. Alqudah and M. U Rehman. Ulam stability of Caputo q-fractional delay difference equation: q-fractional Gronwall inequality approach, *J. Inequal. Appl.* **30** (2019), 1–13. MR4062051, Zbl 1499.39073, doi: 10.1186/s13660-019-2257-6. 1480

[7] S. Chandrasekhar. An introduction to the study of stellar structure, Chicago: Univ. of Chicago Press a. London: Cambridge Univ. Press 520 p., (1939). MR0092663, Zbl 0022.19207. 1480

[8] H. Chen, Y. Li. Rate of decay of stable periodic solutions of Duffing equations, *J. Differential Equations.* **236** (2007), 493–503. MR2322021, Zbl 1169.34027, doi: 10.1016/j.jde.2007.01.023. 1480

[9] G. Duffing. Forced oscillations with variable natural frequency and their technical significance, Vieweg, Braunschweig, Issue 41/42, 1918. Zbl 46.1168.01, doi: 10.1002/zamm.19210010109. 1480

[10] C. L. Ejikeme, M. O. Oyesanya, D. F. Agbebaku, M. B. Okofu. Solution to nonlinear Duffing oscillator with fractional derivatives using Homotopy Analysis Method (HAM). *Global Journal of Pure and Applied Mathematics.* **14** (2018), no. 10, 1363–1383. `https://www.ripublication.com/gjpam18/gjpamv14n10_05.pdf`. 1480

[11] S. Etemad, S. K. Ntouyas. Application of the fixed point theorems on the existence of solutions for q-fractional boundary value problems. *AIMS Math.* **4** (2019), no. 3, 997–1018. MR4136141, Zbl 1484.39004, doi: 10.3934/math.2019.3.997. 1480

[12] R. Finkelstein, E. Marcus. Transformation theory of the q-oscillator. *J. Math. Phys.* **36** (1995), no. 6, 2652–2672. MR1331280, Zbl 0845.58030, doi: 10.1063/1.531057. 1479

[13] P. G. O. Freund, A.V. Zabrodin. The spectral problem for the q-Knizhnik-Zamolodchikov equation and continuous q-Jacobi polynomials. *Commun. Math. Phys.* **173** (1995), no. 1, 17–42. MR1355617, Zbl 0832.35106, doi: 10.48550/arXiv.hep-th/9309144. 1479

[14] A. H. Ganie, M. Houas, M.M. Albaidani and D. Fathima. Coupled system of three sequential Caputo fractional differential equations: Existence and stability analysis, *Math. Meth. Appl. Sci.* **46** (2023), no. 7, 1–14. MR4631196, Zbl 1538.34023, doi: 10.1002/mma.9278. 1479

[15] A. H. Ganie, M. m. Albaidani and A. Khan. A comparative study of the fractional partial differerential equations via novel transform, *Symmetry* **15** (2023), No. 5. doi: 10.3390/sym15051101. 1479

[16] A. H. Ganie, F. Mofarreh, A. Khan. A Fractional Analysis of Zakharov Kuznetsov equations with the Liouville Caputo Operator, *Axioms* **12** (2023), No. 6. doi: 10.3390/axioms12060609. 1479

[17] F. Jarad, T. Abdeljawad and D. Baleanu. Stability of q-fractional non-autonomous systems, *Nonlinear Anal. Real World Appl.* **14** (2013), no. 1, 780–784. MR2969872, Zbl 1258.34014, doi: 10.1016/j.nonrwa.2012.08.001. 1480

[18] M. Houas, M.E. Samei and S. Rezapour. Solvability and stability for a fractional quantum jerk type problem including Riemann-Liouville-Caputo fractional q-derivatives, *Partial Differential Equations in Applied Mathematics* **7** (2023), 1–11. doi: 10.1016/j.padiff.2023.100514. 1480

[19] M. Houas and M.E. Samei. Existence and stability of solutions for linear and nonlinear damping of q-fractional Duffing-Rayleigh problem, *Mediterr. J. Math.* **20** (2023), no. 3, 1–28. MR4556275, Zbl 1538.34026, doi: 10.21203/rs.3.rs-1843587/v1. 1480

[20] M. Houas, F. Martínez, M.E. Samei, M.K.A. Kaabar. Uniqueness and Ulam-Hyers-Rassias stability results for sequential fractional pantograph q-differential equations. *J. Inequal. Appl.* **93** (2022), 1–24. MR4454399, Zbl 1506.34017, doi: 10.1186/s13660-022-02828-7. 1480

[21] G. A. Hamid, S. Mallik, M. M. AlBaidani, A. Khan, M. A. Shah, Mohd. Novel analysis of nonlinear seventh-order fractional Kaup-Kupershmidt equation via the Caputo operator, *Boundary Value Problems* **87** (2024). MR4771990, Zbl 07897013, doi: 10.1186/s13661-024-01895-7.

[22] A. G. Hamid, F. Mofarreh and A. Khan. On new computations of the time-fractional nonlinear KdV-Burgers equation with exponential memory, *Phys. Scr.* **99** (2024). doi: 10.1088/1402-4896/ad2e60. 1479

[23] A. G. Hamid, S. Noor, M. A. Huwayz, A. Shafee, and S. A. El-Tantawy. Numerical simulations for fractional Hirota–Satsuma coupled Korteweg–de Vries systems, *Open physics* **22** (2024). doi: 10.1515/phys-2024-0008. 1479

[24] M. Jiang, R. Huang. Existence of solutions for q-fractional differential equations with nonlocal Erdélyi-Kober q-fractional integral condition, *AIMS Math.* **5** (2020), no. 6, 6537–6551. Zbl 1484.39005, doi: 10.3934/math.2020421. 1480

[25] A. C. Lazer and P. J. Mckenna. On the existence of stable periodic solutions of differential equations of Duffing type, *Proc. Amer. Math. Soc.* **110** (1990), 125–133. MR1013974, Zbl 0714.34067, doi: 10.2307/2048251. 1480

[26] X. Li, Z. Han, S. Sun. Existence of positive solutions of nonlinear fractional q-difference equation with parameter, *Adv.Difference Equ.* **260** (2013), 1–13. MR3110765, Zbl 1375.39021, doi: 10.1186/1687-1847-2013-260. 1479

[27] S. Liang and M.E. Same. New approach to solutions of a class of singular fractional q-differential problem via quantum calculus, *Adv. Differ. Equ.* (2020), 1–22. MR4048368, Zbl 1487.34022, doi: 10.1186/s13662-019-2489-2. 1479

[28] K. Ma , X. Li and S. Sun. Boundary value problems of fractional q-difference equations on the half-line, *Bound. Value. Probl.* (2019), 1–16. MR3918790, Zbl 1513.39035, doi: 10.1186/s13661-019-1159-3. 1479

[29] K. Ma, Z. Han and Y. Zhang. Stability conditions of a coupled system of fractional q-difference Lotka-Volterra model, *International Journal of Dynamical Systems and Differential Equations* **6** (2016), no. 4, 305-317. MR3607088, Zbl 1442.39016. 1480

[30] J. Ma, J. Yang. Existence of solutions for multi-point boundary value problem of fractional q-difference equation, *Electron. J. Qual. Theory Differ. Equ.* **92** (2011), 1–10. MR2861421, Zbl 1340.39010, doi: 10.14232/ejqtde.2011.1.92. 1479

[31] J. Niu, R. Liu, Y. Shen and S. Yang. Chaos detection of Duffing system with fractional order derivative by Melnikov method, *Chaos* **29** (2019), 123–126. MR4040517, Zbl 1429.34041, doi: 10.1063/1.5124367. 1480

[32] N. D. Phuong, S. Etemad and S. Rezapour. On two structures of the fractional q-sequential integro-differential boundary value problems, *Math. Methods Appl. Sci.* **45** (2022), no. 2, 618–639. MR4361512, Zbl 07787253, doi: 10.1002/mma.7800. 1480

[33] P. Pirmohabbati, A. H. Refahi Sheikhani, H. Saberi Najafi, A. Abdolahzadeh Ziabari. Numerical solution of full fractional Duffing equations with Cubic-Quintic-Heptic nonlinearities, *AIMS Mathematics.* **5** (2020), no. 2, 1621–1641. MR4141935, Zbl 1484.65147, doi: 10.3934/math.2020110. 1480

[34] P. M. Rajkovic, S. D. Marinkovic, M.S. Stankovic. On q-analogues of Caputo derivative and Mittag-Leffer function, *Fract. Calc. Appl. Anal.* **10** (2007), 359–373. MR2378985, Zbl 1157.33325. 1481

[35] D. R. Smart. Fixed point theorems, Cambridge Tracts in Mathematics, No. 66. Cambridge University Press. (1980). MR467717, Zbl 0427.47036. 1480

[36] X. Su. Boundary value problem for a coupled system of nonlinear fractional differential equations, *Appl. Math. Lett.* **22** (2009), 64–69. MR2483163, Zbl 1163.34321, doi: 10.1016/j.aml.2008.03.001. 1481

[37] J. Sunday. The Duffing oscillator: Applications and computational simulations, *Asian Research Journal of Mathematics.* **2** (2017), No. 3, 1–13. doi: 10.9734/ARJOM/2017/31199. 1482

[38] K. TABLENNEHAS, Z. DAHMANI. A three sequential fractional differential problem of Duffing type, *Applied Mathematics E-Notes.* **21** (2021), 587–598. MR4305556, Zbl 1498.30015. 1480

[39] W. YANG. Positive solutions for three-point boundary value problem of nonlinear fractional q-difference equation, *Kyungpook Math. J.* **56** (2016), No. 2, 419–430. MR3520785, Zbl 1379.39009, doi: 10.3390/sym10090358. 1480

[40] A. ZADA, M. ALAM AND U. RIAZ. Analysis of q-fractional implicit boundary value problems having Stieltjes integral conditions, *Math. Methods Appl. Sci.* **44** (2021), No. 6, 4381–4413. MR4235512, Zbl 1471.39007, doi: 10.1002/mma.7038. 1479

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