New York Journal of Mathematics

New York J. Math. **30** (2024) 1566–1584.

Hopf Galois structures, skew braces for groups of size $p^n q$: The cyclic Sylow subgroup case

Namrata Arvind and Saikat Panja

ABSTRACT. Let $n \geq 1$ be an integer, p, q be distinct odd primes. Let G, N be two groups of order $p^n q$ with their Sylow-p-subgroups being cyclic. We enumerate the Hopf-Galois structures on a Galois G-extension, with type N. This also computes the number of skew braces with additive group isomorphic to N and multiplicative group isomorphic to G. Further when $q < p$, we give a complete classification of the Hopf-Galois structures on Galois-G-extensions.

CONTENTS

1. Introduction

The study of Hopf-Galois structures comes under the realm of group theory and number theory. These structures were first studied by S. Chase and M. Sweedler in 1969, in their work [CS69]. Subsequently in [GP87], C. Greither and B. Pareigis studied and described Hopf-Galois structures for separable extensions. In recent times, algebraic objects called skew braces were introduced in the PhD thesis of D. Bachiller. They have been studied by various mathematicians like W. Rump, D. Bachiller, F. Cedo in [CJO14], [BCJ16] *etc.*. Skew braces are known to give set-theoretic solutions to the Yang-Baxter equations. First observed by D. Bachiller in his PhD thesis and subsequently made precise by A. Smoktunowicz, L. Vendramin and N. Byott in their work [SV18] is a connection between the study of skew braces and that of Hopf-Galois structures.

Received March 12, 2024.

2020 *Mathematics Subject Classification.* 12F10, 16T05.

Key words and phrases. Hopf-Galois structures; field extensions; holomorph.

The first named author is partially supported by the IMSc postdoctoral fellowship and the second author has been partially supported by the HRI postdoctoral fellowship.

ISSN 1076-9803/2024

For more details, and background on this topic, and the interplay between skew braces and Hopf-Galois structures, one can refer to the book [Chi+21] of L. N. Childs *et. al.* and the Ph.D. thesis of K. N. Zenouz [Zen18].

A Hopf-Galois structure on a finite field extension is defined in the following way. Assume K/F is a finite Galois field extension. An F-Hopf algebra $\mathcal{H}$, with an action on K such that K is an H-module algebra and the action makes K into an $\mathcal{H}$-Galois extension, will be called a *Hopf-Galois structure* on K/F. A *left skew brace* is a triple $(\Gamma, +, \times)$ where $(\Gamma, +), (\Gamma, \times)$ are groups and satisfy $a \times (b + c) = (a \times b) + a^{-1} + (a \times c)$, for all $a, b, c \in \Gamma$.

Given a group T, the *holomorph* of T is defined as the semidirect product

$$T \rtimes \mathrm{Aut}(T) = \{(t, \zeta) \,:\, t \in T, \zeta \in \mathrm{Aut}(T)\},$$

via the identity map. It is denoted by $\mathrm{Hol}(G)$. Let G and N be two finite groups of the same order. By $e(G, N)$ we mean the number of Hopf-Galois structures on a finite Galois field extension L/K with Galois group isomorphic to G, and the type isomorphic to N. In [GP87], the authors gave a bijection between Hopf-Galois structures on a finite Galois extension with Galois group G and regular subgroups in $\mathrm{Perm}(G)$, which are normalised by $\lambda(G)$. Further in [Byo96], N. Byott showed that

$$e(G, N) = \frac{|\mathrm{Aut}(G)|}{|\mathrm{Aut}(N)|} \cdot e'(G, N), \tag{1.1}$$

where $e'(G, N)$ is the number of regular subgroups of $\mathrm{Hol}(N)$ isomorphic to G. A subgroup Γ of the holomorph $\mathrm{Hol}(T)$ of a group T is said to be regular if and only if the projection π_1 on the the first component (which is just a map, not a homomorphism) $(t, \zeta) \mapsto t$ is a bijection. Note that such a subgroup Γ of $\mathrm{Hol}(T)$ satisfies that it has exactly one element $(e_T, \zeta) \in \Gamma$ with $\zeta = I$, the identity automorphism. We will also use this condition to check regular embeddings of the concerned groups in the article.

It turns out that $e'(G, N)$ also gives the number of skew braces with the additive group isomorphic to N and the multiplicative group isomorphic to G. The number $e(G, N)$ has been computed for several groups. For example, N. Byott determined $e(G, G)$ when G is isomorphic to a cyclic group [Byo13]; C. Tsang determined $e(G, N)$ when G is a quasisimple group [Tsa21]; N. K. Zenouz consider the groups of order p^3 [Zen18] to determine $e(G, N)$; T. Kohl determined $e(G, G)$ when G is a dihedral group [Koh20].

Previously in [AP22], the authors computed $e(G, N)$ whenever G and N are isomorphic to $\mathbb{Z}_n \rtimes \mathbb{Z}_2$, where n is odd and its radical is a Burnside number. Groups of order p^2q with cyclic Sylow subgroups have been considered in [CCD20]. We can show that any group of order p^nq with cyclic Sylow subgroups, when p and q are distinct primes, is a semidirect product of two cyclic groups (see Section 2). In this article, we compute $e(G, N)$ (and $e'(G, N)$), whenever G and N are groups of order p^nq with cyclic Sylow-p subgroup, where p and q are distinct odd primes. We do this by looking at the number of regular subgroups of $\mathrm{Hol}(N)$ which are isomorphic to G. Finally whenever $q < p$ we give a necessary and sufficient condition on when the pair (G, N) is realizable.

We now fix some notations. For a ring R, we will use $R^\times$ to denote the set of multiplicative units of R. For a group G, the identity element will be sometimes denoted by e_G and mostly by 1, when the context is clear. The automorphism group of a group G will be denoted by $\mathrm{Aut}(G)$, and the holomorph $G \rtimes_{\mathrm{id}} \mathrm{Aut}(G)$ will be denoted by $\mathrm{Hol}(G)$. The binomial coefficients will be denoted by $\binom{l}{m}$. The Euler totient function will be denoted by φ. We will use $\mathbb{Z}_m$ to denote the cyclic group of order m. We will use $\mathbb{Z}_m$ as a group as well as a ring, which will be clear from the context. Equality sign will be used for congruence as well. Now, we state the two main results of this article. To state the second result we use notations from Section 2.

Theorem 1.1. *Let $p > q$ be odd primes and $q \mid p-1$. Let G denote the nonabelian group of the form $\mathbb{Z}_{p^n} \rtimes \mathbb{Z}_q$ and C denote the cyclic group of order $p^n q$. Then the following are true:*

(1) $e'(G, G) = e(G, G) = 2 + 2p^n(q-2)$,
(2) $e'(G, C) = q - 1$, *and* $e(G, C) = p^n$,
(3) $e'(C, G) = p^{2n-1}$, *and* $e(C, G) = 2p^{n-1}(q-1)$.

Theorem 1.2. *Let $p < q$ be odd primes and $p^a || q - 1$. For $1 \leq b \leq \min\{n, a\}$, let G_b denote the unique nonabelian group of the form $\mathbb{Z}_q \rtimes \mathbb{Z}_{p^n}$ determined by b, and C denote the cyclic group of $p^n q$. Then the following results hold;*

(1) $e'(G_b, G_b) = e(G_b, G_b) = 2\left(p^{n-b} + q\left(\varphi(p^n) - p^{n-b}\right)\right)$,
(2) $e'(G_{b_1}, G_{b_2}) = 2qp^{n+b_1-b_2-1}(p-1)$, *and* $e(G_{b_1}, G_{b_2}) = 2qp^{n-1}(p-1)$ *for* $b_1 \neq b_2$,
(3) $e'(C, G_b) = 2p^{n-b}q$, *and* $e(C, G_b) = 2(p-1)p^{n-1}$,
(4) $e'(G_b, C) = p^{n+b-2}(p-1)$, *and* $e(G_b, C) = p^{n-1}b$.

The rest of the article is organised as follows. In Section 2, we give a detailed description of the groups under consideration and determine their automorphism groups. Next, in Section 3 and Section 4 we will prove Theorem 1.1 and Theorem 1.2 respectively. Lastly, in Section 5 we discuss the realizability problem and solve them for some of the groups mentioned in this article.

Acknowledgement. We are immensely thankful to the reviewer for suggesting numerous changes to clarify arguments and shortening the proofs, especially that of Lemma 2.5.

2. Preliminaries

2.1. The groups under consideration. In this subsection, we will describe the groups under consideration and fix some notations. Let p and q be distinct odd primes. We look at groups of order $p^n q$ whose Sylow-p-subgroups are cyclic. These come under two families, depending on whether $p > q$ or $p < q$. We start by recalling the following result of Burnside.

Lemma 2.1. [Bur55, Theorem II, Section 243] *Let G be a finite group. Let r be the smallest prime dividing $|G|$, the order of the group G. Further let $|G| = r^k t$, $r \nmid t$. If a Sylow r-subgroup of G is cyclic, then G has a normal r-complement, that is a normal subgroup H such that $|G : H| = r^k$.*

Let now G be a group of order $p^n q$, where $n \geq 1$, and p, q are distinct primes. Assume that all Sylow p-subgroups are cyclic (of course, any Sylow q-subgroup is cyclic). Then by Lemma 2.1, when $p > q$ a Sylow p-subgroup is normal in G, and when $q > p$ a Sylow q-subgroup is normal in G. Clearly, all Sylow subgroups of G are normal precisely when G is cyclic.

In case $p > q$, the group G is isomorphic to a semidirect product of $\mathbb{Z}_{p^n}$ and $\mathbb{Z}_q$. Since $\mathrm{Aut}(\mathbb{Z}_{p^n})$ is cyclic, the semidirect product is either trivial (in this case the group is cyclic) or *uniquely* nontrivial. Let $G \cong \mathbb{Z}_{p^n} \rtimes \mathbb{Z}_q$. If $q \nmid p-1$ then G is cyclic. In case $q|p-1$, let $\phi : \mathbb{Z}_q \to \mathrm{Aut}(\mathbb{Z}_{p^n})$ be the homomorphism defined as $\phi(1) = k$. Here k is an element of $\mathrm{Aut}(\mathbb{Z}_{p^n})$ of order q. Hereafter, we denote $\mathbb{Z}_{p^n} \rtimes_\phi \mathbb{Z}_q$ by $\mathbb{Z}_{p^n} \rtimes_k \mathbb{Z}_q$. Let

$$\langle x, y | x^{p^n} = y^q = 1, yxy^{-1} = x^k \rangle$$

be a presentation of $\mathbb{Z}_{p^n} \rtimes_k \mathbb{Z}_q$. Note that since $e(G, G)$ is already known whenever G is cyclic, we will assume $q|p-1$ for our calculations.

Now if $p < q$, using Lemma 2.1 we get that G is either a cyclic group or a non-trivial semidirect product of $\mathbb{Z}_q$ and $\mathbb{Z}_{p^n}$. Next, we elaborate on different possible semidirect products in this regard. Once again in this case we assume that $p \mid q-1$. Let $p^a || q-1$ and for $b \leq \min\{n, a\}$ fix $\psi_b : \mathbb{Z}_{p^n} \longrightarrow \mathrm{Aut}(\mathbb{Z}_q)$ to be a homomorphism, such that $|\mathrm{Im}\ \psi_b| = p^b$. Take $G_b = \mathbb{Z}_q \rtimes_{\psi_b} \mathbb{Z}_{p^n}$. The group G_b is unique up to isomorphism. The presentation of this group can be taken to be

$$\langle x, y | x^{p^n} = 1, y^q = 1, xyx^{-1} = y^k \rangle,$$

where k is an element of order p^b in $\mathrm{Aut}(\mathbb{Z}_q) = \mathbb{Z}_q^\times$. From now on we denote $\mathbb{Z}_q \rtimes_{\psi_b} \mathbb{Z}_{p^n}$ by $\mathbb{Z}_q \rtimes_k \mathbb{Z}_{p^n}$.

2.2. The basic lemmas.

In this subsection we note down the basic group-theoretic results, which will be used throughout the article.

Lemma 2.2. *Let p be a positive odd integer. Take $a = bp^c$ where $p \nmid b$. Then we have that* $(1+p)^a \equiv 1 + dp^{c+1} \pmod{p^{c+2}}$ *for some $p \nmid d$, for all integer $c \geq 0$.*

Proof. We prove it by induction on c. If $c = 0$, then $(1+p)^a = 1 + ap + a'p^2$, for some $a' \in \mathbb{Z}$. Hence $(1+p)^a \equiv 1 + ap \pmod{p^2}$ with $d = a$. Next, assume it to be true for all $l \leq c$ and in particular for $l = c$. Hence $(1+p)^{bp^c} = 1 + dp^{c+1} + d'p^{c+2}$ for some $d' \in \mathbb{Z}$. Then we have

$$(1+p)^{bp^{c+1}} = \left(1 + dp^{c+1} + d'p^{c+2}\right)^p = (1 + d''p^{c+1})^p,$$

for some $d'' \in Z$ and $(d'', p) = 1$. Hence it follows that $(1+p)^{bp^{c+1}} \equiv 1 + d''p^{c+2} \pmod{p^{c+3}}$, which also finishes the induction, and hence the proof. □

We will use the following matrix form of the elements of the automorphism group of $\mathbb{Z}_{p^n} \rtimes_k \mathbb{Z}_q$.

Lemma 2.3. *Let G be the non-abelian group isomorphic to $\mathbb{Z}_{p^n} \rtimes_k \mathbb{Z}_q$. We have*

$$\mathrm{Aut}(G) \cong \left\{ \begin{pmatrix} \beta & \alpha \\ 0 & 1 \end{pmatrix} : \beta \in \mathbb{Z}_{p^n}^{\times}, \alpha \in \mathbb{Z}_{p^n} \right\}$$

.

Proof. We first embed G as a normal subgroup of $\mathrm{Hol}(\mathbb{Z}_{p^n})$. Take the homomorphism ψ defined as

$$\psi(x) = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}, \; \psi(y) = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}.$$

This embedding is injective, as k is a unit and $\psi(x^i y^j) = \begin{pmatrix} k^j & ki \\ 0 & 1 \end{pmatrix}$. Now consider the following map

$$\Phi : \mathrm{Hol}(\mathbb{Z}_{p^n}) \longrightarrow \mathrm{Aut}(G) \text{ defined as } \Phi(z)(w) = zwz^{-1}$$

for all $z \in \mathrm{Hol}(\mathbb{Z}_{p^n})$ and $y \in G$ is an injective group homomorphism, since $\ker \Phi$ consists only of the identity matrix. From [Wal86, Theorem B] we have $|\mathrm{Aut}(G)| = |\mathrm{Hol}(\mathbb{Z}_{p^n})|$. Thus Φ is an isomorphism. □

The following lemma is easy to prove, but we state it here for our reference.

Lemma 2.4. *Let p, q be primes such that $(p, q) = 1$ and $q \mid p - 1$. Let k be a multiplicative unit in $\mathbb{Z}_{p^n}$, of multiplicative order q. Then $k - 1$ is a multiplicative unit in $\mathbb{Z}_{p^n}$.*

Lemma 2.5. *Let $G_b \cong \mathbb{Z}_q \rtimes_k \mathbb{Z}_{p^n}$, where k is an element of order p^b in $\mathbb{Z}_q^{\times}$. Assume $p \mid q - 1$, then for $b > 0$, we have that $\mathrm{Aut}(G_b) \cong \mathbb{Z}_{p^{n-b}} \times \mathrm{Hol}(\mathbb{Z}_q)$.*

Proof. The proof will be divided into two steps. First, we calculate the size of the automorphism group. In the next step, we will determine the group's description in terms of generators and relations, from which the result will follow.

Let us take an automorphism Ψ of G_b. Assume that $\Psi(x) = y^{\alpha} x^{\gamma}$ and $\Psi(y) = y^{\beta} x^{\delta}$, where $0 \le \alpha, \beta \le q - 1$ and $0 \le \gamma, \delta \le p^n - 1$. Note that we have $\Psi(y)^q = y^{\beta(1+k^{\delta}+k^{2\delta}+k^{(q-1)\delta})} x^{q\delta}$. Since $\Psi(y)^q = 1$, we must have $\delta = 0$. Thus β should be a unit in $\mathbb{Z}_q$. Now consider the equation $\Psi(x)\Psi(y) = \Psi(y)^k \Psi(x)$. This imposes the condition that $y^{\alpha+\beta k^{\gamma}} x^{\gamma} = y^{\beta k+\alpha} x^{\gamma}$. Hence we should have $\beta k^{\gamma} \equiv \beta k \pmod{q}$, whence $k^{\gamma-1} \equiv 1 \pmod{q}$, as β is a unit in $\mathbb{Z}_q$. Since k is an element of order p^b, we get that $\gamma \in \{Rp^b+1 : 0 \le R < p^{n-b}\}$. Next considering the equation $\Psi(x)^{p^n} = 1$, we have that $y^{\alpha(1+k^{\gamma}+k^{2\gamma}+\ldots+k^{(p^n-1)\gamma})} x^{p^n\gamma} = 1$. Since $x^{p^n\gamma} = 1$, we have that $\alpha(1 + k^{\gamma} + k^{2\gamma} + \ldots + k^{(p^n-1)\gamma}) = 0 \pmod{q}$. Regardless of the value of k, any $0 \le \alpha \le q$ satisfies the last congruence. Hence the group is of order $p^{n-b} q(q - 1)$.

We can easily check that $Z(G_b) \cong \langle x^{p^b} \rangle$. Since $Z(G_b/Z(G_b))$ is trivial, using [Wal86, Theorem B] we get that $\mathrm{Aut}(G_b/Z(G_b)) \cong \mathrm{Hol}(\mathbb{Z}_q)$. Now we consider the following homomorphism

$$\xi : \mathrm{Aut}(G_b) \to \mathrm{Aut}(G_b/Z(G_b)),$$

which is well defined since $Z(G_b)$ is a characteristic subgroup of G_b. Then the following statements hold true;

(1) For an element $\begin{pmatrix} u & v \\ 0 & 1 \end{pmatrix} \in \mathrm{Hol}(\mathbb{Z}_q)$, the map $x \mapsto y^v x$ and $y \mapsto y^u$ is an automorphism of G_b and hence $\mathrm{Hol}(\mathbb{Z}_q) \cong \mathrm{Aut}(G_b/Z(G_b)) \subseteq \mathrm{Aut}(G_b)$

(2) $\ker \xi$ acts trivially on $\langle y \rangle$, and $\langle \vartheta \rangle \subseteq \ker \xi$, where $\vartheta(x) = x^{1+p^b}$. Also, $\langle \vartheta \rangle$ has order p^{n-b}, which is the size of $\ker \xi$.

(3) $\ker \xi \cap \mathrm{Hol}(Z_q) = \{\mathrm{id}\}$ and for $\Psi_1 \in \ker \xi$, $\Psi_2 \in \mathrm{Hol}(\mathbb{Z}_q)$ one has $\Psi_1 \Psi_2 = \Psi_2 \Psi_1$.

This finishes the proof. □

We denote the elements of $\mathrm{Aut}(G_b)$ by $\left(\gamma, \begin{pmatrix} \beta & \alpha \\ 0 & 1 \end{pmatrix}\right) \in \mathbb{Z}_{p^n}^{\times} \times \mathrm{Hol}(\mathbb{Z}_q)$, such that $\gamma^{p^{n-b}} = 1$.

Remark 2.6. We note down the action of the automorphism group of G_b on the group G_b, by means of generators. This will be necessary for counting the Hopf-Galois structures concerning G_b's. For $b > 0$, the action is as follows.

$$\left(\gamma, \begin{pmatrix} \beta & \alpha \\ 0 & 1 \end{pmatrix}\right) \cdot x = y^{\alpha} x^{\gamma} \text{ and, } \left(\gamma, \begin{pmatrix} \beta & \alpha \\ 0 & 1 \end{pmatrix}\right) \cdot y = y^{\beta}.$$

Remark 2.7. For $b = 0$, the group $G_b \cong \mathbb{Z}_{p^n} \times \mathbb{Z}_q$. Since $(p, q) = 1$ and both are abelian groups, it follows from [BCM06, Theorem 3.2] that $\mathrm{Aut}(G_b) \cong \mathbb{Z}_{p^{n-1}(p-1)} \times \mathbb{Z}_{q-1}$ in this case. The action is defined to be component-wise.

3. The case $p > q$

This section is devoted to the proof of Theorem 1.1. As discussed in Section 2, up to isomorphism there are two groups of order $p^n q$ whenever their Sylow subgroups are cyclic. Counting the number of skew braces with multiplicative group G and additive group N is equivalent to (up to multiplication by a constant; see [AP22, Proof of Proposition 3.2]) counting the number of regular embeddings of G in $\mathrm{Hol}(N)$. Then using Eq. (1.1), we are able to conclude about the number of Hopf-Galois structures on G-extensions of type N. We will use the regularity criterion given in Section 1. This section will be divided into three subsections, depending on the isomorphism types of G and N. From Lemma 2.3, we have that $\mathrm{Aut}(\mathbb{Z}_{p^n} \rtimes_k \mathbb{Z}_q) \cong \mathrm{Hol}(\mathbb{Z}_{p^n})$, where the action is given by,

$$\begin{pmatrix} \beta & \alpha \\ 0 & 1 \end{pmatrix} \cdot x^i y^j = x^{\beta i + \alpha k^{-1} - \alpha k^{j-1}} y^j.$$

3.1. Embedding of $\mathbb{Z}_{p^n} \rtimes_k \mathbb{Z}_q$ into Hol($\mathbb{Z}_{p^n} \rtimes_k \mathbb{Z}_q$). Let $\Phi : \mathbb{Z}_{p^n} \rtimes_k \mathbb{Z}_q \longrightarrow \mathrm{Hol}(\mathbb{Z}_{p^n} \rtimes_k \mathbb{Z}_q)$ be a regular embedding. Let

$$\Phi(x) = \left(x^{i_1}y^{j_1}, \begin{pmatrix} \beta_1 & \alpha_1 \\ 0 & 1 \end{pmatrix}\right), \Phi(y) = \left(x^{i_2}y^{j_2}, \begin{pmatrix} \beta_2 & \alpha_2 \\ 0 & 1 \end{pmatrix}\right).$$

From $(\Phi(x))^{p^n} = 1$ we get $y^{p^n j_1} = 1$ which will imply,

$$j_1 = 0 \pmod{q}, \tag{3.1}$$

since $p^n j_1 = 0 \pmod{q}$ and $(p, q) = 1$,

$$\beta_1^{p^n} = 1 \pmod{p^n}, \tag{3.2}$$

$$i_1(1 + \beta_1 + \beta_1^2 + ... + \beta_1^{p^n-1}) = 0 \pmod{p^n}, \tag{3.3}$$

$$\alpha_1(1 + \beta_1 + \beta_1^2 + ... + \beta_1^{p^n-1}) = 0 \pmod{p^n}. \tag{3.4}$$

Similarly from $\Phi(yxy^{-1}) = \Phi(x^k)$ we get

$$\beta_1^{k-1} = 1 \pmod{p^n}, \tag{3.5}$$

which implies $\beta_1 = 1 \pmod{p^n}$ from Eq. (3.2), Eq. (3.5) and using Lemma 2.4; furthermore,

$$\alpha_1 \cdot (k - \beta_2) = 0 \pmod{p^n}, \tag{3.6}$$

$$ki_1 \cdot (k^{j_2-1}\beta_2 - 1) = \alpha_1 \cdot (1 - k^{j_2}) \pmod{p^n}. \tag{3.7}$$

We note that in general,

$$\Phi(y)^\delta = \left(x^{\ell_\delta}y^{\delta j_2}, \begin{pmatrix} \beta_2^\delta & \alpha_2(1 + \beta_2 + \beta_2^2 + \cdots + \beta_2^{\delta-1}) \\ & 1 \end{pmatrix}\right),$$

where

$$\ell_\delta = i_2\left(\sum_{t=0}^{\delta-1} \left(\beta_2 k^{j_2}\right)^t\right) + \left(\alpha_2 k^{j_2-1} - \alpha_2 k^{2j_2-1}\right)\left(1 + \sum_{u=1}^{\delta-2}\left(\sum_{v=0}^{u} \beta_2^v\right)k^{uj_2}\right). \tag{3.8}$$

Using $\Phi(y)^q = 1$ we get

$$\beta_2^q = 1 \pmod{p^n}, \tag{3.9}$$

$$\alpha_2(1 + \beta_2 + \beta_2^2 + ... + \beta_2^{q-1}) = 0 \pmod{p^n} \qquad j_2 \neq 0, \tag{3.10}$$

$$\ell_q = 0 \pmod{p^n}. \tag{3.11}$$

From Eq. (3.9) we get $\beta_2 = k^a$, for some $0 \leq a \leq q - 1$, since $\mathbb{Z}_{p^n}^*$ has a unique subgroup of order q and is generated by k. First let us show that, in any regular embedding $j_2 \neq 0 \pmod{q}$. If possible let $j_2 = 0$. Then we get that $\beta_2 = k$ from Eq. (3.7). This forces that for any $0 \leq \omega_1 \leq p^n - 1$ and $0 \leq \omega_2 \leq q - 1$

$$\Phi(x)^{\omega_1}\Phi(y)^{\omega_2} = \left(x^{\omega_1 i_1 + i_2\left(1+k+\cdots+k^{\omega_2-1}\right)}, \begin{pmatrix} k^{\omega_2} & \star \\ 0 & 1 \end{pmatrix}\right). \tag{3.12}$$

Thus we see that $\langle \pi_1(\Phi(x)^{\omega_1}\Phi(y)^{\omega_2})\rangle \neq \mathbb{Z}_{p^n} \rtimes_k \mathbb{Z}_q$, which will imply the map Φ is not regular. Hence $j_2 \neq 0$. We now divide the possibilities of a into 3 cases.

3.1.1. Case I: $\boldsymbol{a} = \mathbf{0}$. Using Eq. (3.6) and Eq. (3.7), we conclude that $\alpha_1 = 0 \pmod{p^n}$, $j_2 = 1 \pmod{q}$ and, $\alpha_2 = 0 \pmod{p^n}$. Since i_1 is a unit in $\mathbb{Z}_{p^n}$ and $i_2 \in \mathbb{Z}_{p^n}$ can take any value, the total number of embeddings in this case is given by $p^n\varphi(p^n)$. Moreover, all of these embeddings are regular. We remark that all the above embedding corresponds to the canonical Hopf-Galois structure.

3.1.2. Case II: $\boldsymbol{a} = \mathbf{1}$. Note that using Eq. (3.7) we get that $ki_1 = -\alpha_1 \pmod{p^n}$. We deal with this in two subcases depending on the value of j_2. First, we consider the case j_2 being equal to $q-1$. In this case using $\ell_q = 0$, we get that i_2 gets determined by the value of α_2 since $\left(\sum_{t=0}^{k-1} \left(\beta_2 k^{j_2}\right)^t\right) = q$ is a unit in $\mathbb{Z}_{p^n}$. Hence the number of embedding in this subcase is given by $p^n\varphi(p^n)$.

For the other case, since the element $k^{j_2}(1-k^a)$ is a unit and $j_2 + a \neq 0 \pmod{q}$ we get

$$\begin{aligned}
&1 + \sum_{s=1}^{q-2}\left(\sum_{t=0}^{s} k^{ta}\right)k^{sj_2}\\
=&\frac{1}{k^{j_2}(1-k^a)}\left\{\sum_{t=1}^{q-1}\left(1-k^{ta}\right)k^{tj_2}\right\} = \frac{1}{k^{j_2}(1-k^a)}\cdot(1-1) = 0,
\end{aligned}$$

Thus $\Phi(y)^q = 1$ does not impose any conditions on i_2 and α_2. Hence, in this subcase, the total number of possibilities is $p^{2n}\varphi(p^n)(q-2)$. Since $j_2 \neq 0$, we conclude that all the embeddings are regular.

3.1.3. Case III: $\boldsymbol{a} \geq \mathbf{2}$. This condition together with Eq. (3.6) and Eq. (3.7), imply that $\alpha_1 = 0$ and $j_2 = a-1 \pmod{q}$. Since $a + j_2 \neq 0 \pmod{q}$, a mutatis mutandis of Case II gives that i_2 and α_2 can be chosen independently, whence each of them has p^n possibilities. Thus, in this case, the total number of possibilities is given by $p^{2n}\varphi(p^n)(q-2)$. Similar to the previous case, all the embeddings are regular.

Summarizing the above cases we get the following result.

Lemma 3.1. *The total number of regular embeddings of $\mathbb{Z}_{p^n} \rtimes \mathbb{Z}_q$ inside* $\mathrm{Hol}(\mathbb{Z}_{p^n} \rtimes \mathbb{Z}_q)$ *is given by* $2p^n\varphi(p^n) + 2p^{2n}\varphi(p^n)(q-2)$.

Proposition 3.2. *Let G be a non-abelian groups of the form $\mathbb{Z}_{p^n} \rtimes \mathbb{Z}_q$, where p and q are primes satisfying $q|p-1$. Then $e(G,G)$ is given by* $2 + 2p^n(q-2)$.

Proof. From Lemma 3.1 we get the total number of regular embeddings. Dividing this number by the Automorphism of G will give us the total number of Hopf-Galois structures. □

3.2. Embedding of $\boldsymbol{G} = \mathbb{Z}_{\boldsymbol{p^n}} \rtimes \mathbb{Z}_{\boldsymbol{q}}$ in the $\mathbf{Hol}(\mathbb{Z}_{\boldsymbol{p^n}} \times \mathbb{Z}_{\boldsymbol{q}})$. Next we consider the case of regular embeddings of $G = \mathbb{Z}_{p^n} \rtimes \mathbb{Z}_q$ in the $\mathrm{Hol}(\mathbb{Z}_{p^n} \times \mathbb{Z}_q)$. Let us fix the presentation of $C = \mathbb{Z}_{p^n} \times \mathbb{Z}_q$ to be $\langle r, s | r^{p^n} = s^q = 1, rs = sr\rangle$. Then it can be shown that $\mathrm{Hol}(C) \equiv \mathrm{Hol}(\mathbb{Z}_{p^n}) \times \mathrm{Hol}(\mathbb{Z}_q)$. We take a typical element of $\mathrm{Hol}(C)$ to be $\left(\begin{pmatrix} b & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} d & c \\ 0 & 1 \end{pmatrix}\right)$, where a, c are elements of $\mathbb{Z}_{p^n}, \mathbb{Z}_q$ respectively and b, d are elements of $\mathbb{Z}_{p^n}^\times, \mathbb{Z}_q^\times$ respectively. Starting with an embedding Φ of G inside $\mathrm{Hol}(C)$ and assuming that

$$\Phi(x) = \left(\begin{pmatrix} b_1 & a_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} d_1 & c_1 \\ 0 & 1 \end{pmatrix}\right), \Phi(y) = \left(\begin{pmatrix} b_2 & a_2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} d_2 & c_2 \\ 0 & 1 \end{pmatrix}\right).$$

From $\Phi(x)^{p^n} = e_{\mathrm{Hol}(C)}$ we get the equations

$$b_1^{p^n} = 1 \pmod{p^n}, \tag{3.13}$$

$$a_1\left(1 + b_1 + \cdots + b_1^{p^n-1}\right) = 0 \pmod{p^n}, \tag{3.14}$$

$$d_1^{p^n} = 1 \pmod{q}, \tag{3.15}$$

$$c_1\left(1 + d_1 + \cdots + d_1^{p^n-1}\right) = 0 \pmod{q}. \tag{3.16}$$

Note that $d_1^{q-1} = 1 \pmod{q}$ and $(q-1, p^n) = 1$. Combining this with Eq. (3.15), we get that $d_1 = 1$. Then plugging $d_1 = 1$ in Eq. (3.16), conclude that $c_1 = 0$. For ensuring regularity, we need to take a_1 is a unit in $\mathbb{Z}_{p^n}$. Using the equation $\Phi(y)^q = 1$ we get the equations

$$b_2^q = 1 \pmod{p^n}, \tag{3.17}$$

$$a_2\left(1 + b_2 + \cdots + b_2^{q-1}\right) = 0 \pmod{p^n}, \tag{3.18}$$

$$d_2^q = 1 \pmod{q}, \tag{3.19}$$

$$c_2\left(1 + d_2 + \cdots + d_2^{q-1}\right) = 0 \pmod{q}. \tag{3.20}$$

Since the order of d_2 divides $q - 1$, we get $d_2 = 1$ from Eq. (3.19). Finally comparing both sides of the equation $\Phi(x)^k\Phi(y) = \Phi(y)\Phi(x)$ we get (using the conclusions of the preceding discussions)

$$b_1^{k-1} = 1 \pmod{p^n} \tag{3.21}$$

$$b_2a_1 + a_2 = b_1^k a_2 + a_1\left(1 + b_1 + \cdots + b_1^{k-1}\right) \pmod{p^n}. \tag{3.22}$$

Using Lemma 2.4, Eq. (3.13) and Eq. (3.21) we conclude that $b_1 = 1$. Putting the value of b_1 in Eq. (3.22) we get that $b_2 = k$. Further to ensure regularity we need to impose $c_2 \neq 0$ (using a similar argument in the discussion after Eq. (3.12)). Thus the total number of regular embeddings in this case is given by $\varphi(p^n)p^n(q-1)$.

Proposition 3.3. *Let C be the cyclic group of order p^nq and G be the nonabelian group isomorphic to $\mathbb{Z}_{p^n} \rtimes \mathbb{Z}_q$, where p and q are primes. Then $e(G, C) = p^n$ and $e'(G, C) = q - 1$.*

3.3. Embedding of $C = \mathbb{Z}_{p^n} \times \mathbb{Z}_q$ in the Hol$(\mathbb{Z}_{p^n} \rtimes \mathbb{Z}_q)$.

Recall the description of Hol(G) from Section 3.1 and the presentation for C from Section 3.2. Consider a homomorphism $\Phi : C \longrightarrow \mathrm{Hol}(G)$ determined by

$$\Phi(r) = \left(x^{i_1}y^{j_1}, \begin{pmatrix} \beta_1 & \alpha_1 \\ 0 & 1 \end{pmatrix}\right), \Phi(s) = \left(x^{i_2}y^{j_2}, \begin{pmatrix} \beta_2 & \alpha_2 \\ 0 & 1 \end{pmatrix}\right).$$

Given that $\Phi(r)$ has to be an element of order p^n and the embedding is regular, using a similar argument as in Section 3.1 we conclude that $j_1 = 0$, i_1 is a unit in $\mathbb{Z}_{p_n}$ and, j_2 is a unit in $\mathbb{Z}_q$. From $\Phi(r)^{p^n} = 1$, we get that

$$\begin{aligned} i_1\left(1 + \beta_1 + \cdots + \beta^{p^n-1}\right) &= 0 \pmod{p^n}, \\ \alpha_1\left(1 + \beta_1 + \cdots + \beta^{p^n-1}\right) &= 0 \pmod{p^n}, \\ \beta_1^{p^n} &= 1 \pmod{p^n}. \end{aligned}$$

From the last equation above and [AP22, Corollary 2.2] we get that $\beta_1 = 1 \pmod{p}$. Hence the first two equations will always be satisfied irrespective of choices of i_1 and α_1. From the equation $\Phi(s)^q = 1$, we get

$$\beta_2^q = 1 \pmod{p^n}, \tag{3.23}$$

$$\alpha_2(1 + \beta_2 + \beta_2^2 + \ldots + \beta_2^{q-1}) = 0 \pmod{p^n} \quad , \tag{3.24}$$

$$\ell_q = 0 \pmod{p^n}, \tag{3.25}$$

where ℓ_q is as defined in Section 3.1. Furthermore $\Phi(r)\Phi(s) = \Phi(s)\Phi(r)$ gives that

$$\alpha_2(\beta_1 - 1) = \alpha_1(\beta_2 - 1), \quad \pmod{p^n} \tag{3.26}$$

$$i_1 + \beta_1 i_2 + \alpha_1 k^{-1}\left(1 - k^{j_2}\right) = i_2 + k^{j_2}\beta_2 i_1 \pmod{p^n}. \tag{3.27}$$

Let $\beta_2 = k^a$ for some $a \geq 0$. We divide this into two cases $a = 0$ and $a \neq 0$.

3.3.1. Case I: a=0.

In this case we get $\alpha_2 = 0$ from Eq. (3.24). Hence Eq. (3.26) is always satisfied. Note that Eq. (3.25) holds true, since $j_2 + a \neq q$ by using similar arguments as of Section 3.1. Putting $\beta_2 = 1$ in Eq. (3.27) we get $\left(i_1 + \alpha_1 k^{-1}\right)\left(1 - k^{j_2}\right) = i_2\left(1 - \beta_2\right) \pmod{p^n}$. Hence the choice of α_1 gets determined by those of i_1, i_2, β_1 and, j_2. Hence the total number of embeddings in this case becomes $\varphi(p^n)p^{2n-1}(q-1)$.

3.3.2. Case II: $a \neq 0$.

From Eq. (3.26), substituting $\alpha_1 = \alpha_2(\beta_1 - 1)(k^a - 1)^{-1}$ in Eq. (3.27) we get

$$i_1\left(k^a - 1\right)\left(1 - k^{j_2+a}\right) = \left(1 - \beta_1\right)\left(i_2\left(k^a - 1\right) + \alpha_2 k^{-1}\left(1 - k^{j_2}\right)\right) \pmod{p^n}. \tag{3.28}$$

We claim that $j_2+a = q$. Indeed, if $j_2+a \neq q$, we have that the LHS of Eq. (3.28) is a unit in $\mathbb{Z}_{p^n}$, whereas $(1-\beta_1)$ is never a unit (since $\beta_1 = 1 \pmod{p^n}$). Next, putting $j_2 + a = q$ in Eq. (3.28), the LHS becomes 0. Substituting $j_2 + a = q$ in Eq. (3.25) we get $i_2 = -\alpha_2 k^{-1}(1 - k^{j_2})(k^{j_2}q^{-1})(1 + (1 + k^a)k^{j_2} + \cdots + (1 + k^a + \cdots + k^{(q-2)a})k^{(q-2)j_2}) \pmod{p^n}$. Further substituting this value of i_2 to

Eq. (3.28), we get that both sides of the equation become zero. Hence we get that in this case, the total number of regular embeddings of C in $\mathrm{Hol}(G)$ is given by $\varphi(p^n)p^{2n-1}(q-1)$.

Proposition 3.4. *Let C be the cyclic group of order p^nq and G be the nonabelian group isomorphic to $\mathbb{Z}_{p^n} \rtimes \mathbb{Z}_q$. Then $e(C,G) = 2p^{n-1}(q-1)$ and $e'(C,G) = p^{2n-1}$.*

Now Theorem 1.1 follows from Proposition 3.2, Proposition 3.3, and Proposition 3.4.

4. The case $p < q$

In this section, we prove Theorem 1.2. We use methods, described in the beginning of Section 3. In this case, there are exactly $b+1$ types of groups up to isomorphism, where $b = \min\{a, n\}$ with $p^a || q-1$. This section will be divided into four subsections, depending on the isomorphism types of $G = G_{b_1}$ and $N = G_{b_2}$, where $0 \leq b_1, b_2 \leq n$.

4.1. Isomorphic type.

First, we consider the isomorphic case. Let $G = \mathbb{Z}_q \rtimes_k \mathbb{Z}_p^n$, where k is an element of order p^b. We are looking at $e(G,G)$.

4.1.1. The case $b = 0$.

In this case, the groups are cyclic and $e'(G,G)$, $e(G,G)$ have been enumerated in [Byo13, Theorem 2].

4.1.2. The case when $0 < b \leq n$.

Let us take a group homomorphism $\Phi : G_b \longrightarrow \mathrm{Hol}(G_b)$ defined by

$$\Phi(x) = \left(y^{j_1}x^{i_1}, \left(\gamma_1, \begin{pmatrix} \beta_1 & \alpha_1 \\ 0 & 1 \end{pmatrix}\right)\right),$$
$$\text{and } \Phi(y) = \left(y^{j_2}x^{i_2}, \left(\gamma_2, \begin{pmatrix} \beta_2 & \alpha_2 \\ 0 & 1 \end{pmatrix}\right)\right).$$

From $\Phi(y)^q = 1$ and from $\Phi(xy) = \Phi(y^kx)$, we get the relations $i_2 = 0$, $\beta_2 = 1$, $\gamma_2 = 1$ and

$$\alpha_2(k - \beta_1) = 0 \pmod{q}, \tag{4.1}$$
$$j_2(k^{i_1-1}\beta_1 - 1) = \alpha_2(1 + k + k^2 \cdots k^{i_1-1}) \pmod{q}. \tag{4.2}$$

Thus if $\alpha_2 = 0$, then $\beta_1 = k^{1-i_1}$. If $\alpha_2 \neq 0$, then $\beta_1 = k$ and $\alpha_2 = j_2(k-1)$. From $\Phi(x)^{p^n} = 1$, we get the following equivalences in $\mathbb{Z}_q$.

$${\beta_1}^{p^n} = 1 \tag{4.3}$$
$$\alpha_1(1 + \beta_1 + {\beta_1}^2 \cdots \beta_1^{p^n-1}) = 0. \tag{4.4}$$

By explicit calculations, we can show that, the exponent of y in $\Phi(x)^{p^n}$ is given by

$$\begin{aligned}\mathrm{Exp}_y\left(\Phi(x)^{p^n}\right) =& j_1\left(\sum_{u=0}^{p^n-1} m^u\right)\\ &+\frac{\alpha_1}{m(k^{\gamma_1}-1)}\left\{\sum_{v=1}^{p^n-1} m^{p^n-v}\left(k^{i_1\left(1+\gamma_1+\ldots+\gamma_1^{v-1}\right)}-k^{i_1}\right)\right\},\end{aligned}$$

where $m=\beta_1 k^{i_1}$. Using Eq. (4.1) and Eq. (4.2), we can show that $m\in\{k,k^{i_1+1}\}$ First, let us take $m=k$. Then $\sum\limits_{u=0}^{p^n-1} m^u\equiv 0 \pmod{q}$. We aim to show that the other summand is also zero in $\mathbb{Z}_q$. We have in $\mathbb{Z}_q$

$$\sum_{v=1}^{p^n-1} m^{p^n-v}\left(k^{i_1\left(1+\gamma_1+\ldots+\gamma_1^{v-1}\right)}-k^{i}\right)=\sum_{v=1}^{p^n} k^{i_1\left(1+\gamma_1+\ldots+\gamma_1^{v-1}\right)-v}.$$

Note that here i_1 and γ_1 are fixed. Denote by $\Gamma(v)=i_1(1+\gamma_1+\ldots+\gamma_1^{v-1})-v \pmod{p^n}$. Suppose for $1\le v_1\ne v_2\le p^n$ we have $\Gamma(v_1)\equiv\Gamma(v_2) \pmod{p^n}$. Then we have $i(\gamma_1^{v_1}-\gamma_1^{v_2})\equiv(v_1-v_2)(\gamma_1-1) \pmod{p^n}$. Since the Sylow-$p$-subgroup of $\mathbb{Z}_{p^n}^\times$ is generated by $(1+p)$ and γ_1 is an element having p-power order, say an element of order p^g. Then $p^{n-g}||\gamma_1-1$. Thus $v_1-v_2\equiv 0 \pmod{p^g}$, using Lemma 2.2. Conversely if $v_1-v_2\equiv 0 \pmod{\mathrm{ord}\gamma_1}$, then $i(\gamma_1^{v_1}-\gamma_1 v_2)\equiv(v_1-v_2)(\gamma_1-1) \pmod{p^n}$. Thus Γ gives rise to a function from $\mathbb{Z}_{p^n}$ to the subset $\{p^g,2p^g,3p^g,\ldots,p^n\}$. Thus the sum is reduced to $p^g\sum\limits_{t=1}^{p^{n-g}} k^{tp^g}$. If $k^{p^g}=1$, we get the sum to be zero. Otherwise this sum is $p^g\dfrac{k^{p^n}-1}{k^{p^g}-1}=0$. This finishes the proof.

Now, take the case when $m=k^{i_1+1}$. Then again the multiplier of j_1 is zero in $\mathbb{Z}_q$. We claim that the other summand is also zero in the above expression. We have in this case,

$$\begin{aligned}&\sum_{v=1}^{p^n-1} m^{p^n-v}\left(k^{i_1\left(1+\gamma_1+\ldots+\gamma_1^{v-1}\right)}-k^{i_1}\right)\\ &=\sum_{v=1}^{p^n} k^{i_1\left(1+\gamma_1+\ldots+\gamma_1^{v-1}-v\right)-v}-\sum_{v=1}^{p^n} k^{i_1(1-v)-v}\\ &=\begin{cases}\sum\limits_{v=1}^{p^n} k^{i_1\left(1+\gamma_1+\ldots+\gamma_1^{v-1}\right)-(i_1+1)v} & \text{when } i_1+1\ne 0 \pmod{p}\\ \sum\limits_{v=1}^{p^n} k^{i_1\left(1+\gamma_1+\ldots+\gamma_1^{v-1}-v\right)-v}-\sum\limits_{v=1}^{p^n} k^{i_1(1-v)-v} & \text{otherwise.}\end{cases}\end{aligned}$$

We start by considering the first subcase, i.e. $i_1 + 1$ being a unit in $\mathbb{Z}_{p^n}$. Again denote by $\Gamma(v) = i_1\left(1 + \gamma_1 + \ldots + \gamma_1^{v-1} - v\right) - (i_1 + 1)v$. Then $\Gamma(v_1) \equiv \Gamma(v_2) \pmod{p^n}$ implies that $i_1(\gamma_1^{v_1} - \gamma_1^{v_2}) \equiv (i_1 + 1)(\gamma_1 - 1)(v_1 - v_2) \pmod{p^n}$. Then proceeding as before, we get the result. Next, consider the second subcase. In this case, we show that both of the sums are zero. Take $\Gamma_1(v) = i_1\left(1 + \gamma_1 + \ldots + \gamma_1^{v-1} - v\right) - (i_1+1)v$ and $\Gamma_2(v) = i_1(1-v) - v$. Assume $p^h || i_1+1$, then $\Gamma_2(v') = \Gamma_2(v'')$ iff $v' \equiv v'' \pmod{p^{n-h}}$, using Lemma 2.2. Thus Γ_2 determines a function to the subset $\{p^{n-h}, 2p^{n-h}, \ldots, p^n\}$ and hence the second term of the expression above vanishes. An argument similar to the previous cases of $\Gamma(v)$, shows that the first term is 0 as well in $\mathbb{Z}_q$. Thus we have proved the following lemma.

Lemma 4.1. *In* $\operatorname{Exp}_y\left(\Phi(x)^{p^n}\right)$*, if the coefficient of* j_1 *is zero in* $\mathbb{Z}_q$*, then so is the coefficient of* α_1*.*

We claim that i_1 is a unit. Suppose i_1 is not a unit. We note that $\Phi(x)^{p^{n-1}} = \left(y^J, \left(1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\right)$, for some J. Note that if $\beta_1 = 0$ then $\alpha_1 = 0$, otherwise $1 + \beta_1 + \ldots + \beta_1^{p^n-1} = 0$, whence the matrix entry is justified. Now, if $J = 0$ then this map is not regular. Otherwise when $J \neq 0$, we get J is a unit in $\mathbb{Z}_q$. Since p is a unit is $\mathbb{Z}_q$, we get that $\Phi(x_1)^{p^n}$ is not identity element. This proves claim. Now we are ready to count the number of Hopf-Galois structures on extensions, whose group is of the form G_b for some $0 < b < n$. This will be divided into four cases. Before proceeding, we note that none of the cases, impose any condition on j_2 and γ_1.

Case 1*: The case* $\beta_1 = 1$. This implies $\alpha_2 = 0$. Since if $\alpha_2 \neq 0$, then $\beta_1 = k \neq 1$. From $\alpha_2 = 0$ we get that $i_1 \equiv 1 \pmod{p^b}$, from which we get that i_1 has p^{n-b} possibilities. Further $\alpha_1 = 0$ from Eq. (4.4). In this case, j_1 has q possibilities since $m \neq 1$, using Lemma 4.1. Thus in this case we get $\varphi(q)qp^{2(n-b)}$ embedding.

Case 2*: The case* $\beta_1 \neq 1$*, and* $\alpha_2 = 0$*.* Note that $\alpha_2 = 0$ implies that $\beta_1 = k^{1-i_1}$. Also, $\beta_1 \neq 1$ imposes the condition that i_1 has $\varphi(p^n) - p^{n-b}$ possibilities. In this case, j_1 and α_1 have q possibilities each. Thus in this case we have $\varphi(q)\left(\varphi(p^n) - p^{n-b}\right)q^2p^{n-b}$ embeddings.

Case 3*: The case* $\beta_1 \neq 1$*,* $\alpha_2 \neq 0$*, and* $1 + i_1 \equiv 0 \pmod{p^b}$*.* Since $1 + i_1 \equiv 0 \pmod{p^b}$, we get $m = 1$. Hence the value of j_1 gets fixed. Thus in this case, we have $\varphi(q)qp^{2(n-b)}$ embeddings.

Case 4*: The case* $\beta_1 \neq 1$*,* $\alpha_2 \neq 0$*, and* $1 + i_1 \not\equiv 0 \pmod{p^b}$*.* In this case i_1 has $\varphi(p^n) - p^{n-b}$ possibilities. Similar to Case 2, j_1 has q possible values. Thus in this case, we have $\varphi(q)\left(\varphi(p^n) - p^{n-b}\right)q^2p^{n-b}$ embeddings.

In all of the cases above the embeddings are regular, which is guaranteed by the conditions that i_1 and j_2 are units. Furthermore, In conclusion, we have proved the following result.

Proposition 4.2. *Let $G_b = \mathbb{Z}_q \rtimes_k \mathbb{Z}_{p^n}$, where $k \in \mathbb{Z}_q$ is of order p^b for some $0 < b \leq n$. Then $e'(G_b, G_b) = e(G_b, G_b) = 2\left(p^{n-b} + q\left(\varphi(p^n) - p^{n-b}\right)\right)$.*

4.2. Non-isomorphic type.

This case will be divided into three cases, depending on the values of b_1 and b_2.

4.2.1. The case $1 \leq b_1 \neq b_2 \leq n$. We will need a variation of Lemma 4.1, for dealing with this case. We start with a presentation of these two groups. For $t = 1$ and 2, let us fix

$$G_{b_t} = \left\langle x_t, y_t \middle| x_t^{p^n} = y_t^q = 1, x_t y_t x_t^{-1} = y_t^{k_t} \right\rangle,$$

where k_t is an element of order p^{b_t}. Now we consider $\Phi : G_{b_1} \longrightarrow \operatorname{Hol}(G_{b_2})$ is an regular embedding and $\Phi(x_1) = \left(y_2^{j_1} x_2^{i_1}, \left(\gamma_1, \begin{pmatrix} \beta_1 & \alpha_1 \\ 0 & 1 \end{pmatrix}\right)\right)$, then it can be proved that,

$$\begin{aligned}\operatorname{Exp}_y\left(\Phi(x)^{p^n}\right) =& j\left(\sum_{u=0}^{p^n-1} m^u\right) \\ &+ \frac{\alpha_1}{m(k_2^{\gamma_1} - 1)}\left\{\sum_{v=1}^{p^n-1} m^{p^n-v}\left(k_2^{i_2(1+\gamma_1+\ldots+\gamma_1^{v-1})} - k_2^{i_2}\right)\right\},\end{aligned}$$

where $m = \beta_1 k_2^{i_1}$. It can be shown that $m \in \left\{k_1, k_1 k_2^{i_1}\right\}$ modulo q, using Eq. (4.1) and Eq. (4.2). Note that in any of the cases $b_1 < b_2$ or $b_2 < b_1$, m is purely a power of k_1 or k_2, since $\mathbb{Z}_{p^n}^{\times}$ is cyclic. Then a variation of the argument before Lemma 4.1, proves the following result.

Lemma 4.3. *In $\operatorname{Exp}_y\left(\Phi(x_1)^{p^n}\right)$ if the coefficient of j_1 is 0 in $\mathbb{Z}_q$, then so is the coefficients of α_1.*

Hoping that the reader is now familiar with the flow of arguments, without loss of generality in this case we will assume that the embedding is given by,

$$\Phi(x_1) = \left(y_2^{j_1} x_2^{i_1}\left(\gamma_1, \begin{pmatrix} \beta_1 & \alpha_1 \\ 0 & 1 \end{pmatrix}\right)\right), \Phi(y_1) = \left(y_2^{j_2}\left(1, \begin{pmatrix} 1 & \alpha_2 \\ 0 & 1 \end{pmatrix}\right)\right),$$

where i_1 is a unit in $\mathbb{Z}_{p^n}$ (using same argument as in Section 4.1), γ_1 is a unit in $\mathbb{Z}_{p^n}$ satisfying $\gamma_1^{p^{n-b_2}} = 1$, and j_2 is a unit in $\mathbb{Z}_q$. Comparing the both sides of the equation $\Phi(x_1)\Phi(y_1) = \Phi(y_1)^{k_1}\Phi(x_1)$, we get

$$\alpha_2(k_1 - \beta_1) = 0 \pmod{q}, \tag{4.5}$$

$$k_2^{i_1}\beta_1 j_2 = j_2 k_1 + j_2\left(1 + k_2 + \ldots + k_2^{i_1-1}\right) \pmod{q}. \tag{4.6}$$

From Eq. (4.5) either $\alpha_2 = 0$ or $\beta_1 = k_1$. Irrespective of the cases $\beta_1 k_1^{i_1} \neq 1$. Thus from Lemma 4.3 j_1 can take any value from $\mathbb{Z}_q$. Now, in the first case, $\beta_1 = k_1 k_2^{-i_1}$ (from Eq. (4.6)). Also γ_1 and α_1 have p^{n-b_2} and q many choices respectively. This gives that total number of embeddings in this case is given by $\varphi(q)\varphi(p^n)q^2p^{n-b_2}$. In the second case, $\alpha_2 = (k_2-1)j_2$ and γ_1, α_1 have p^{n-b_2}, q many choices respectively. Thus the total number of embeddings arising from this case is given by $\varphi(q)\varphi(p^n)q^2p^{n-b_2}$. Given that i_1 and j_2 are units, we get that the constructed map is regular. We now have the following result.

Proposition 4.4. *Let $G_{b_t} = \mathbb{Z}_q \rtimes_{k_t} \mathbb{Z}_{p^n}$, where k_t is an element of $\mathbb{Z}_{p^n}$ of order p^{b_t}, for $t = 1, 2$. Let $0 < b_1 \neq b_2 \leq n$. Then*

$$e'\left(G_{b_1}, G_{b_2}\right) = 2qp^{n+b_1-b_2-1}(p-1),\ e\left(G_{b_1}, G_{b_2}\right) = 2qp^{n-1}(p-1).$$

4.2.2. The case $0 = b_1 < b_2 \leq n$. In this case G_{b_1} is cyclic and hence the presentations of the groups G_{b_1} and G_{b_2} are chosen to be

$$G_{b_1} = \left\langle x_1, y_1 \middle| x_1^{p^n} = y_1^q = 1, x_1y_1x_1^{-1} = y_1\right\rangle,$$
$$G_{b_2} = \left\langle x_2, y_2 \middle| x_2^{p^n} = y_2^q = 1, x_2y_2x_2^{-1} = y_2^{k_2}\right\rangle,$$

with $k_2 \in \mathbb{Z}_{p^n}$ being an element of multiplicative order p^{b_2}. Fix a homomorphism $\Phi : G_{b_1} \longrightarrow \mathrm{Hol}(G_{b_2})$ given by

$$\Phi(x_1) = \left(y_2^{j_1}x_2^{i_1}\left(\gamma_1, \begin{pmatrix}\beta_1 & \alpha_1\\ 0 & 1\end{pmatrix}\right)\right), \Phi(y_1) = \left(y_2^{j_2}x_2^{i_2}\left(\gamma_2, \begin{pmatrix}\beta_2 & \alpha_2\\ 0 & 1\end{pmatrix}\right)\right).$$

From the condition $\Phi(y_1)^q$, we get the conditions that $i_2 = 0, \gamma_2 = 0$ and $\beta_2 = 1$. To ensure the regularity of the maps, we will need i_1 and j_2 to be units in $\mathbb{Z}_{p^n}$ and $\mathbb{Z}_q$ respectively (see Section 4.1). Equating the two sides of the equality $\Phi(x_1)\Phi(y_1) = \Phi(y_1)\Phi(x_1)$, we get that

$$\alpha_2(1-\beta_1) = 0 \pmod q, \tag{4.7}$$
$$\beta_1k_2^{i_1}j_2 = j_2 + \alpha_2\left(1 + k_2 + ... + k_2^{i_1-1}\right) \pmod q. \tag{4.8}$$

Hence from Eq. (4.7) we have either $\alpha_2 = 0$ or $\beta_1 = 1$. In case $\alpha_2 = 0$, plugging the value in Eq. (4.8) we get that $\beta_1k_2^{i_1} = 1$, whence j_1 has fixed choice, once α_1 is fixed. Furthermore α_1, γ_1 have q, p^{n-b_2} choices. In the case $\beta + 1 = 1$, from Eq. (4.8) we get that $\alpha_2 = j_2(k_2-1)$ and $\beta_1k_2^{i_1} \neq 1$. Hence Lemma 4.3 applies. Thus j_1, and γ_1 have q, and p^{n-b_2} possibilities. We conclude that in both cases the number of regular embedding of the cyclic group of order p^nq in $\mathrm{Hol}\left(G_{b_2}\right)$ is given by $q\varphi(q)p^{n-b_2}\varphi(p^n)$. We have the following result.

Proposition 4.5. *Let C denotes the cyclic group of order p^nq and $G_b \cong \mathbb{Z}_q \rtimes_{k_b} \mathbb{Z}_{p^n}$, where $k_b \in \mathbb{Z}_q$ is an element of multiplicative order p^b. Then*

$$e'(C, G_b) = 2p^{n-b}q,\ \text{and}\ e(C, G_b) = 2(p-1)p^{n-1}$$

4.2.3. The case $\boldsymbol{0 = b_2 < b_1 \leq n}$. Here we count the number $e'(G_{b_1}, G_{b_2})$ (equivalently $e(G_{b_1}, G_{b_2})$). Here G_{b_2} is a cyclic group of order $p^n q$. In this case, we have,

$$\mathrm{Hol}(G_{b_2}) \cong \left\{ \left(y_2^j x_2^i, (\omega, \delta) \right) \middle|_{(\omega,\delta) \in \mathbb{Z}_q^\times \times \mathbb{Z}_{p^n}^\times}^{(j,i) \in \mathbb{Z}_q \times \mathbb{Z}_{p^n}} \right\}.$$

We fix an embedding $\Phi : G_{b_1} \longrightarrow \mathrm{Hol}\left(G_{b_2}\right)$ determined by

$$\Phi(x_1) = \left(y_2^{j_1} x_2^{i_1}, (\omega_1, \delta_1) \right), \ \Phi(x_1) = \left(y_2^{j_2} x_2^{i_2}, (\omega_2, \delta_2) \right).$$

From $\Phi(y_1)^q = 1$, we get $\omega_2 = 1, \delta_2 = 1$ and $i_2 = 0$. Considering $\Phi(x_1)^{p^n} = 1$ we get that $\omega_1^{p^n} = 1$, $\delta_1^{p^n} = 1$, and

$$j_1 \left(1 + \omega_1 + \dots + \omega_1^{p^n - 1} \right) = 0 \pmod{q}, \tag{4.9}$$

$$i_1 \left(1 + \delta_1 + \dots + \delta_1^{p^n - 1} \right) = 0 \pmod{p^n}.. \tag{4.10}$$

Finally comparing both sides of the equation $\Phi(x_1)\Phi(y_1) = \Phi(y_1)^{k_1}\Phi(x_1)$, we get that $\omega_1 = k_1$ and hence Eq. (4.9) gets satisfied automatically. To ensure that the embedding is regular, we will need that i_1 and j_2 are units. Any choice of δ_1 satisfies Eq. (4.10). Thus j_1, j_2, i_1, and δ_1 have $\varphi(q)$, q, $\varphi(p^n)$, and p^{n-1} possibilities respectively. We conclude with the following result.

Proposition 4.6. *Let $G_b \cong \mathbb{Z}_q \rtimes_{k_b} \mathbb{Z}_{p^n}$, where k_b is an element of $\mathbb{Z}_q$ of order p^b, $1 \leq b \leq n$, and C denote the cyclic group of order $p^n q$. Then we have*

$$e'\left(G_b, C\right) = p^{n+b-2}(p-1), \ e\left(G_b, C\right) = p^{n-1} q.$$

The Theorem 1.2 follows from Proposition 4.2, Proposition 4.4, Proposition 4.5, and Proposition 4.6.

5. Realizable pair of groups

Given two finite groups G and N of the same order, we say that the pair (G, N) is *realizable* if there exists a Hopf-Galois structure on a Galois G-extension, of type N. In other words a pair (G, N) is realizable if $e(G, N) \neq 0$. This is equivalent to saying there exists a skew brace with its multiplicative group isomorphic to G and its additive group isomorphic to N. This problem is not well studied since given an integer n, the classification of all the groups of size n is not known. However, they have been studied for a variety of groups. When G is a cyclic group of odd order and the pair (G, N) is realizable then in [Byo13], the author showed that if N is abelian then it is cyclic. If N is a non-abelian simple group and G is a solvable group with the pair (G, N) being realizable, then in [Tsa23] N was completely classified. Whenever N or G is isomorphic to $\mathbb{Z}_n \rtimes \mathbb{Z}_2$ for an odd n then their realizabilities were studied in [AP23].

Among a few available techniques, the notion of bijective crossed homomorphism to study realizability problems for a pair of groups of the same order was introduced by Tsang in the work [Tsa19]. Given an element $\mathfrak{f} \in \mathrm{Hom}(G, \mathrm{Aut}(N))$, a map $\mathfrak{g} \in \mathrm{Map}(G, N)$ is said to be a *crossed homomorphism* with respect to $\mathfrak{f}$ if

$\mathfrak{g}(ab) = \mathfrak{g}(a)\mathfrak{f}(a)(\mathfrak{g}(b))$ for all $a, b \in G$. Setting $Z^1_{\mathfrak{f}}(G, N) = \{\mathfrak{g} : \mathfrak{g}$ is bijective crossed homomorphism with respect to $\mathfrak{f}\}$, we have the following two results.

Proposition 5.1. [Tsa19, Proposition 2.1] *The regular subgroups of* $\mathrm{Hol}(N)$ *which are isomorphic to G are precisely the subsets of* $\mathrm{Hol}(N)$ *of the form* $\{(\mathfrak{g}(a), \mathfrak{f}(a)) : a \in G\}$, *where* $\mathfrak{f} \in \mathrm{Hom}(G, \mathrm{Aut}(N))$, $\mathfrak{g} \in Z^1_{\mathfrak{f}}(G, N)$.

Proposition 5.2. [TQ20, Proposition 3.3] *Let G, N be two groups such that $|G| = |N|$. Let $\mathfrak{f} \in \mathrm{Hom}(G, \mathrm{Aut}(N))$ and $\mathfrak{g} \in Z^1_{\mathfrak{f}}(G, N)$ be a bijective crossed homomorphism (i.e. (G, N) is realizable). Then if M is a characteristic subgroup of N and $H = \mathfrak{g}^{-1}(M)$, we have that the pair (H, M) is realizable.*

We will need the following two results, where the realizability of cyclic groups have been characterized. We will use modifications of these characterizations towards proving the realizability of groups of the form $\mathbb{Z}_{p^n} \rtimes \mathbb{Z}_q$.

Proposition 5.3. [Tsa22, Theorem 3.1] *Let N be a group of odd order n such that the pair $(\mathbb{Z}_n, N)$ is realizable. Then N is a C-group (i.e. all the Sylow subgroups are cyclic).*

Proposition 5.4. [Rum19, Theorem 1] *Let G be a group of order n such that $(G, \mathbb{Z}_n)$ is realizable. Then G is solvable and almost Sylow-cyclic (i.e. its Sylow subgroups of odd order are cyclic, and every Sylow-2 subgroup of G has a cyclic subgroup of index at most 2).*

Theorem 5.5. *Let N be a group of order qp^n, where q is a prime, $q < p$ and $(q, p) = 1$. Then the pair $(\mathbb{Z}_{p^n} \rtimes \mathbb{Z}_q, N)$ (or $(N, \mathbb{Z}_{p^n} \rtimes \mathbb{Z}_q)$) is realizable if and only if $N \cong \mathbb{Z}_{p^n} \rtimes \mathbb{Z}_q$.*

Proof. Let $(\mathbb{Z}_{p^n} \rtimes \mathbb{Z}_q, N)$ be realizable. By Proposition 5.1 there exists a bijective crossed homomorphism $\mathfrak{g} \in Z^1_{\mathfrak{f}}(\mathbb{Z}_{p^n} \rtimes \mathbb{Z}_q, N)$ for some $\mathfrak{f} \in \mathrm{Hom}(\mathbb{Z}_{p^n} \rtimes \mathbb{Z}_q, \mathrm{Aut}(N))$. Let H_p be the Sylow-p subgroup of N (it is unique since $q < p$). Then using Proposition 5.2 the pair $(\mathfrak{g}^{-1}H_p, H_p)$ is realizable. Note that $\mathbb{Z}_{p^n} \rtimes \mathbb{Z}_q$ has unique subgroup of order p^n, which is cyclic. This implies that $(\mathbb{Z}_{p^n}, H_p)$ is realizable. Hence by Proposition 5.3 we get that H_p is isomorphic to $\mathbb{Z}_{p^n}$ and therefore $N \cong \mathbb{Z}_{p^n} \rtimes \mathbb{Z}_q$. Conversely if $N \cong \mathbb{Z}_{p^n} \rtimes \mathbb{Z}_q$ then the pair $(\mathbb{Z}_{p^n} \rtimes \mathbb{Z}_q, N)$ is realizable since $e(\mathbb{Z}_{p^n} \rtimes \mathbb{Z}_q, N)$ is non-zero from Section 3.

Now if the pair $(N, \mathbb{Z}_{p^n} \rtimes \mathbb{Z}_q)$ is realizable, then by Proposition 5.1 there exists a bijective crossed homomorphism $\mathfrak{g} \in Z^1_{\mathfrak{f}}(N, \mathbb{Z}_{p^n} \rtimes \mathbb{Z}_q)$ for some $\mathfrak{f} \in \mathrm{Hom}(G, \mathrm{Aut}(\mathbb{Z}_n \rtimes \mathbb{Z}_2))$. Since $\mathbb{Z}_{p^n}$ is a characteristic subgroups of $\mathbb{Z}_{p^n} \rtimes \mathbb{Z}_q$, we get that $\mathfrak{g}^{-1}(\mathbb{Z}_{p^n})$ is a subgroup of N and $(\mathfrak{g}^{-1}(\mathbb{Z}_{p^n}), \mathbb{Z}_{p^n})$ is realizable. Then by Proposition 5.4, we have that $\mathfrak{g}^{-1}(\mathbb{Z}_{p^n})$ is almost Sylow-cylic and therefore isomorphic to $\mathbb{Z}_{p^n}$. Hence $N \cong \mathbb{Z}_{p^n} \rtimes \mathbb{Z}_q$. Conversely if $N \cong \mathbb{Z}_{p^n} \rtimes \mathbb{Z}_q$, then by Section 3 we have the pair $(N, \mathbb{Z}_{p^n} \rtimes \mathbb{Z}_q)$ is realizable. □

References

[AP22] ARVIND, N. & PANJA, S. On $\mathbb{Z}_n \rtimes \mathbb{Z}_2$-Hopf-Galois structures. *J. Algebra*. **596** pp. 37-52 (2022), MR4366304, Zbl 1485.12006, doi: 10.1016/j.jalgebra.2021.12.035. 1567, 1571, 1575

[AP23] ARVIND, N. & PANJA, S. Hopf-Galois realizability of $\mathbb{Z}_n \rtimes \mathbb{Z}_2$. *J. Pure Appl. Algebra*. **227**, Paper No. 107261, 6 (2023), MR4510807, Zbl 1508.20005, doi: 10.1016/j.jpaa.2022.107261. 1581

[BCJ16] BACHILLER, D., CEDÓ, F. & JESPERS, E. Solutions of the Yang-Baxter equation associated with a left brace. *J. Algebra*. **463** pp. 80-102 (2016), MR3527540, Zbl 1348.16027, doi: 10.1016/j.jalgebra.2016.05.024. 1566

[BCM06] BIDWELL, J., CURRAN, M. & MCCAUGHAN, D. Automorphisms of direct products of finite groups. *Arch. Math. (Basel)*. **86**, 481-489 (2006), MR2241597, Zbl 1103.20016, doi: 10.1007/s00013-005-1547-z. 1571

[Bur55] BURNSIDE, W. Theory of groups of finite order. *Dover Publications, Inc., New York*. xxiv+512, MR0069818, Zbl 1375.20001. 1569

[Byo13] BYOTT, N. Nilpotent and abelian Hopf-Galois structures on field extensions. *J. Algebra*. **381** pp. 131-139 (2013), MR3030514, Zbl 1345.12002, doi: 10.1016/j.jalgebra.2013.02.008. 1567, 1576, 1581

[Byo96] BYOTT, N. Uniqueness of Hopf Galois structure for separable field extensions. *Comm. Algebra*. **24**, 3217-3228 (1996), MR1402555, Zbl 0878.12001, doi: 10.1080/00927879608825743. 1567

[CCD20] CAMPEDEL, E., CARANTI, A. & DEL CORSO, I. Hopf-Galois structures on extensions of degree p^q and skew braces of order p^2q: the cyclic Sylow p-subgroup case. *J. Algebra*. **556** pp. 1165-1210 (2020), MR4089566, Zbl 1465.12006, doi: 10.1016/j.jalgebra.2020.04.009. 1567

[Chi+21] CHILDS, L., GREITHER, C., KEATING, K., KOCH, A., KOHL, T., TRUMAN, P. & UNDERWOOD, R. Hopf algebras and Galois module theory. (American Mathematical Society, Providence, RI), MR4390798, Zbl 1489.16001, doi: 10.1090/surv/260. 1567

[CJO14] CEDÓ, F., JESPERS, E. & OKNIŃSKI, J. Braces and the Yang-Baxter equation. *Comm. Math. Phys.*. **327**, 101-116 (2014), MR3177933, Zbl 1287.81062, doi: 10.1007/s00220-014-1935-y. 1566

[CS69] CHASE, S. & SWEEDLER, M. Hopf algebras and Galois theory. (Springer-Verlag, Berlin-New York,1969), MR0260724, Zbl 0197.01403. 1566

[GP87] GREITHER, C. & PAREIGIS, B. Hopf Galois theory for separable field extensions. *J. Algebra*. **106**, 239-258 (1987), MR0878476, Zbl 0615.12026, doi: 10.1016/0021-8693(87)90029-9. 1566, 1567

[Koh20] KOHL, T. Enumerating dihedral Hopf-Galois structures acting on dihedral extensions. *J. Algebra*. **542** pp. 93-115 (2020), MR4018326, Zbl 1462.12002, doi: 10.1016/j.jalgebra.2019.08.040. 1567

[Rum19] RUMP, W. Classification of cyclic braces, II. *Trans. Amer. Math. Soc.*. **372**, 305-328 (2019), MR3968770, Zbl 1417.81140, doi: 10.1090/tran/7569. 1582

[SV18] SMOKTUNOWICZ, A. & VENDAMIN, L. On skew brace (with an appendix by N. Byott and L. Vendramin). *J. Comb. Algebra*. **2**, 47-86 (2018), MR3763907, Zbl 1416.16037, doi: 10.4711/JCA/2-1-3. 1566

[TQ20] TSANG, C. & QIN, C. On the solvability of regular subgroups in the holomorph of a finite solvable group. *Internat. J. Algebra Comput.*. **30**, 253-265 (2020), MR4077413, Zbl 1493.20002, doi: 10.1142/S0218196719500735. 1582

[Tsa19] TSANG, C. Non-existence of Hopf-Galois structures and bijective crossed homomorphisms. *J. Pure Appl. Algebra*. **223**, 2804-2821 (2019),MR3912948, Zbl 1472.12001, doi: 10.1016/j.jpaa.2018.09.06. 1581, 1582

[Tsa21] TSANG, C. Hopf-Galois structures on finite extensions with quasisimple Galois group. *Bull. Lond. Math. Soc.*. **53**, 148-160 (2021),MR4224519, Zbl 1509.20009, doi: 10.1112/blms.12407. 1567

[Tsa22] TSANG, C. Hopf-Galois structures on cyclic extensions and skew braces with cyclic multiplicative group. *Proc. Amer. Math. Soc. Ser. B.* **9** pp. 377-392 (2022), MR4500760, Zbl 1509.20009 doi: 10.1090/bproc/138. 1582

[Tsa23] TSANG, C. Non-abelian simple groups which occur as the type of a Hopf–Galois structure on a solvable extension. *Bulletin Of The London Mathematical Society*. (2023),MR4672898,Zbl 1528.20017, doi: 10.1112/blms.12860. 1581

[Wal86] WALLS, G. Automorphism groups. *Amer. Math. Monthly*. **93**, 459-462 (1986), MR843190,doi: 10.1080/00029890.1986.11971854. 1570, 1571

[Zen18] ZENOUZ, N. On Hopf-Galois Structures and Skew Braces of Order p^3. *PhD Thesis, The University Of Exeter, UK*. (2018),MR4639134. 1567

namchey@gmail.com

(Namrata Arvind) THE INSTITUTE OF MATHEMATICAL SCIENCES, 4TH CROSS ST, CIT CAMPUS, THARAMANI, CHENNAI, TAMIL NADU 600113, INDIA

panjasaikat300@gmail.com

(Saikat Panja) HARISH-CHANDRA RESEARCH INSTITUTE- MAIN BUILDING, CHHATNAG ROAD, JHUSI, UTTAR PRADESH 211019, INDIA

This paper is available via http://nyjm.albany.edu/j/2024/30-66.html.