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Full intrinsic quadrics of dimension two

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ABSTRACT. A full intrinsic quadric is a normal complete variety with a finitely generated Cox ring defined by a single quadratic relation of full rank. We describe all surfaces of this type explicitly via local Gorenstein indices. As applications, we present upper and lower bounds in terms of the Gorenstein index for the degree, the log canonicity and the Picard index. Moreover, we determine all full intrinsic quadric surfaces admitting a Kähler-Einstein metric.

CONTENTS

1. Introduction

The purpose of this article is to structure and explore the (infinite) playground of full intrinsic quadric (algebraic) surfaces X, defined over an algebraically closed field $\mathbb{K}$ of characteristic zero. The name *full intrinsic quadric* refers to the property that the Cox ring of X is defined by a single homogeneous quadratic relation of full rank; see [5]. Intrinsic quadrics exist as well in higher dimensions and form an explicit example class closely beneath the toric varieties which are characterized by having a polynomial ring as Cox ring; see [6, 11, 22] for sample work.

As we will see in Theorem 2.3, every full intrinsic quadric surface X is projective, normal, $\mathbb{Q}$-factorial, rational and allows a non-trivial action of the multiplicative group $\mathbb{K}^*$. This allows us to realize X as a surface in a specific toric threefold Z. More precisely, consider integral $3 \times n$ matrices P of the form

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$$\begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ a & b & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ a & b & 0 & c & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ a & b & 0 & c & 0 & d \end{bmatrix}.$$

Given such P, fix a complete fan Σ in $\mathbb{Z}^3$ having the columns of P as its primitive ray generators, let Z be the associated toric threefold and set

$$X \ := \ \overline{\{t \in \mathbb{T}^3;\ 1 + z_1 + z_2 = 0\}} \ \subseteq \ Z,$$

where $\mathbb{T}^3 \subseteq Z$ is the standard 3-torus with the coordinates z_1, z_2, z_3. Then X is a full intrinsic quadric surface. The Picard number of X is $\rho(X) = n - 3$ and X comes with the effective $\mathbb{K}^*$-action given on $X \cap \mathbb{T}^3$ by

$$t \cdot z \ = \ (z_1, z_2, t z_3).$$

Our main results, Theorems 3.5, 4.5 and 5.5, provide an *explicit and redundance free presentation of all full intrinsic quadric surfaces* via their defining matrices P in terms of local Gorenstein indices and local class group orders of the possibly singular points: for each of the possible Picard numbers $\rho(X) = 1, 2, 3$, we find four infinite series, each depending on two local Gorenstein indices, ι^+, ι^- and on $\rho(X) - 1$ local class group orders, bounded by ι^+, ι^-.

All full intrinsic quadric surfaces X turn out to be log del Pezzo surfaces; see Proposition 6.1. We use our main results to study their geometry. For instance, Corollaries 6.3, 6.6 and 6.8 deliver upper and lower bounds on the degree $\mathcal{K}_X^2$, the log canonicity ε_X and the Picard index $\mathfrak{p}_X$, all in terms of the Gorenstein index ι_X; in particular, we obtain

$$\frac{2}{\iota_X} \le \mathcal{K}_X^2 \le \frac{9}{2} + \frac{9}{2\iota_X}, \qquad \frac{2}{\iota_X} \le \varepsilon_X \le \frac{3}{\sqrt{\iota_X}},$$

and

$$\iota_X \le \mathfrak{p}_X \le \frac{32}{3}\iota_X^3(2\iota_X - 1)^2 + \frac{49}{3}.$$

Another outcome of Theorems 3.5, 4.5 and 5.5 is the following explicit (infinite) list of all full intrinsic quadric complex surfaces admitting a Kähler-Einstein metric; see Corollary 6.9 for the precise formulation and more background.

Corollary 1.1. *The full intrinsic quadric complex surfaces admitting a Kähler-Einstein metric are precisely those constructed from a matrix P of the shape*

$\rho = 1,\ 2 \nmid \iota$:

$$\begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ \iota-1 & -\iota-1 & 1 & 1 \end{bmatrix},$$

$\rho = 3, \quad 2 \nmid \iota, \quad -\iota+1 \le c \le -2,$
$\max(c, -2\iota-2c) \le d \le -\iota-1-c$:

$$\begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ \iota & -\iota-c-d & 0 & c & 0 & d \end{bmatrix},$$

$\rho = 1,\ 4 \mid \iota$:

$$\begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ \frac{\iota}{2}-1 & -\frac{\iota}{2}-1 & 1 & 1 \end{bmatrix},$$

$\rho = 3, \quad -2\iota+1 \le c \le -2,$
$\max(c, -4\iota-2c) \le d \le -2\iota-1-c$:

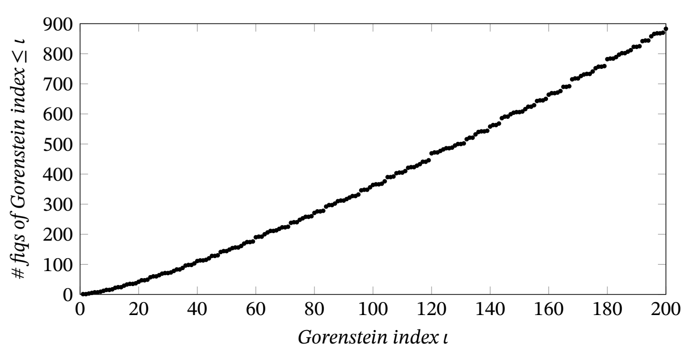
$$\begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 2\iota & -2\iota-c-d & 0 & c & 0 & d \end{bmatrix}.$$

Here, $\rho = 1, 3$ is the Picard number and $\iota \in \mathbb{Z}_{\ge 1}$ the Gorenstein index of the resulting full intrinsic quadric complex surface X arising from the matrix P.

Finally, our description yields a filtration of the whole infinite class of full intrinsic quadric surfaces into finite subclasses by bounding the Gorenstein index. This allows, for instance, counting results as the following.

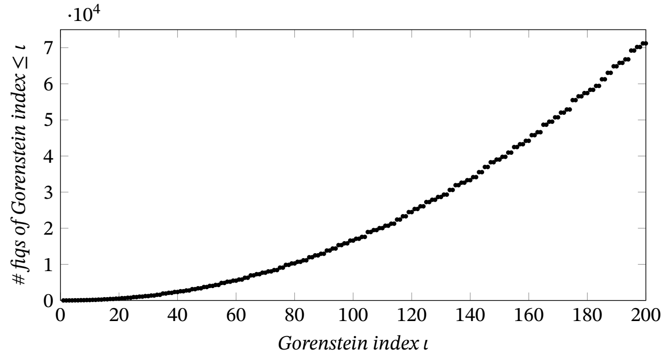
Corollary 1.2. *Up to isomorphy, there are precisely 15 538 339 full intrinsic quadric surfaces of Gorenstein index at most 200.*

(1) *In Picard number one, we find in total 883 full intrinsic quadric surfaces of Gorenstein index at most 200, filtered as follows:*



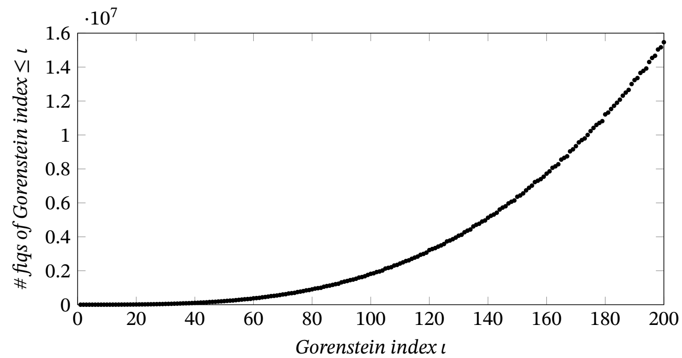
Exactly 150 full intrinsic quadric complex surfaces of Picard number one and Gorenstein index at most 200 admit a Kähler-Einstein metric.

(2) *In Picard number two, we find in total 71 198 full intrinsic quadric surfaces of Gorenstein index at most 200, filtered as follows:*



In Picard number two, there are no full intrinsic quadric complex surfaces at all admitting a Kähler-Einstein metric.

(3) *In Picard number three, we find in total 15 466 258 full intrinsic quadric surfaces of Gorenstein index at most 200, filtered as follows:*



Exactly 1 006 633 full intrinsic quadric complex surfaces of Picard number three and Gorenstein index at most 200 admit a Kähler-Einstein metric.

We assume the reader to be familiar with the very basics of toric geometry, in particular the construction of a toric variety from its defining fan, the orbit decomposition, toric divisors and homogeneous coordinates; see for instance [10, 12, 9]. Theorems 3.5, 4.5 and 5.5 are formulated and proven in Sections 3, 4 and 5, respectively. The considerations follow a common pattern but differ in the details; for convenience, we present the complete arguments in each case. The geometric applications are given in Section 6. The defining matrices for the full intrinsic quadrics are available under [14] for Gorenstein index up to 200 in Picard numbers one, two and for Gorenstein index up to 40 in Picard number three.

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2. Full intrinsic quadric surfaces allow a $\mathbb{K}^*$-action

In this section, we show that every full intrinsic quadric surface is rational, $\mathbb{Q}$-factorial, projective and admits a non-trivial $\mathbb{K}^*$-action. This will allow us to work with the approach to $\mathbb{K}^*$-surfaces provided by [16, 21]; see also [3, Sec. 5.4]. First, let us give a precise definition of a full intrinsic quadric; see also [5, Sec. 9].

Definition 2.1. A *full intrinsic quadric* is a normal complete variety X with finitely generated divisor class group $\mathrm{Cl}(X)$ and Cox ring of the form

$$\mathcal{R}(X) \;=\; \bigoplus_{\mathrm{Cl}(X)} \Gamma(X, \mathcal{O}(D)) \;=\; \mathbb{K}[T_1, \dots, T_n]/\langle g\rangle$$

with $\mathrm{Cl}(X)$-homogeneous generators $T_1, \dots, T_n \in \mathcal{R}(X)$ and a $\mathrm{Cl}(X)$-homogeneous quadric $g \in \mathbb{K}[T_1, \dots, T_n]$ of full rank.

Remark 2.2. In the setting of Definition 2.1, the divisor class group is generated by any $n-1$ of the degrees $w_i = \deg(T_i) \in \mathrm{Cl}(X)$, see [3, Def. 3.2.1.1 and Cor. 3.2.1.11]. Moreover, if X is $\mathbb{Q}$-factorial, then the Picard number $\rho(X)$ equals the dimension of the rational vector space $\mathrm{Cl}_{\mathbb{Q}}(X) = \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Cl}(X)$ and as well the dimension of the convex cone $\mathrm{Mov}(X) \subseteq \mathrm{Cl}_{\mathbb{Q}}(X)$ generated by the movable divisor classes, see for instance [3, Lemma 4.3.3.2].

Theorem 2.3. *Let X be a full intrinsic quadric surface. Then X is $\mathbb{Q}$-factorial, rational, projective and admits an effective $\mathbb{K}^*$-action. Moreover, the Picard number of X satisfies $\rho(X) \le 3$ and its Cox ring allows a $\mathrm{Cl}(X)$-graded presentation as*

$$\mathcal{R}(X) \;\cong\; \begin{cases} \mathbb{K}[T_1, \dots, T_4]/\langle T_1T_2 + T_3^2 + T_4^2\rangle, & \rho(X) = 1, \\ \mathbb{K}[T_1, \dots, T_5]/\langle T_1T_2 + T_3T_4 + T_5^2\rangle, & \rho(X) = 2, \\ \mathbb{K}[T_1, \dots, T_6]/\langle T_1T_2 + T_3T_4 + T_5T_6\rangle, & \rho(X) = 3. \end{cases}$$

Proof. By definition, X is a normal complete surface with finitely generated Cox ring. From [3, Thm. 4.3.3.5] we infer that X is $\mathbb{Q}$-factorial and projective. Moreover, by [11, Prop. 2.1], we have a $\mathrm{Cl}(X)$-graded presentation

$$\mathcal{R}(X) \cong \mathbb{K}[T_1, \dots, T_{n+m}]/\langle g\rangle, \quad g = T_1T_2 + \dots + T_{n-1}T_n + T_{n+1}^2 + \dots + T_{n+m}^2.$$

We use this to show $\rho(X) \le 3$. Recall from [3, Cor. 1.6.2.7 and Constr. 1.6.3.1] that X is the geometric quotient of an open subset of the total coordinate space

$$\bar{X} \;=\; \mathrm{Spec}\ \mathcal{R}(X) \;=\; V(g) \;\subseteq\; \mathbb{K}^{n+m}$$

by the quasitorus $H = \mathrm{Spec}\ \mathbb{K}[\mathrm{Cl}(X)]$, which, due to $\mathbb{Q}$-factoriality of X, is of dimension $\rho(X)$. Consequently, we have

$$2 \;=\; \dim(X) \;=\; \dim(\bar{X}) - \dim(H) \;=\; n + m - 1 - \rho(X).$$

The degrees $w_1, \dots, w_{n+m}$ of $T_1, \dots, T_{n+m}$ generate $\mathrm{Cl}(X)$. Moreover, the degree $\mu = \deg(g) \in \mathrm{Cl}(X)$ satisfies $\mu = w_i + w_{i+1}$ for $i = 1, 3, \dots, n-1$ and $\mu = 2w_{n+j}$ for $j = 1, \dots, m$. Thus, the divis class group $\mathrm{Cl}(X)$ is generated by $w_1, w_2, w_3, w_5, \dots, w_{n-1}$ and we see

$$n + m - 3 \;=\; \rho(X) \;\le\; 2 + \frac{n-2}{2}.$$

We conclude $n/2 + m \leq 4$ and thus $n \leq 8$. Assume $n = 8$. Then $m = 0$ and $\rho(X) = 5$ hold. Consequently, $(w_1, w_2, w_3, w_5, w_7)$ is a basis for the rational vector space $\mathrm{Cl}_{\mathbb{Q}}(X)$. Let u be a linear form on $\mathrm{Cl}_{\mathbb{Q}}(X)$ such that

$$\langle u, w_1 \rangle = \langle u, w_2 \rangle = \langle u, w_3 \rangle = \langle u, w_5 \rangle = 0, \qquad \langle u, w_7 \rangle < 0.$$

Then u annihilates as well $\mu = w_1 + w_2$ and thus also $w_4 = \mu - w_3$ and $w_6 = \mu - w_5$. Moreover, u evaluates positively on $w_8 = \mu - w_7$. Consequently, computing the cone of movable divisor classes according to [3, Prop. 3.3.2.3], we obtain

$$\mathrm{Mov}(X) \ = \ \bigcap_{i=1}^{8} \mathrm{cone}(w_j;\ j \neq i) \ \subseteq \ \ker(u) \ \subseteq \ \mathrm{Cl}_{\mathbb{Q}}(X).$$

This contradicts to Remark 2.2, telling us that $\mathrm{Mov}(X)$ is a cone of full dimension in $\mathrm{Cl}_{\mathbb{Q}}(X)$. We conclude, $n \leq 6$. The case $n = 6$, $m = 1$ and $\rho(X) = 4$ is excluded by the same arguments as used for $n = 8$ and $\rho(X) = 5$.

Thus, we have $\rho(X) \leq 3$. If $\rho(X) = 3$, then $n = 6$ and $m = 0$, which leads to third case in the assertion. For $\rho(X) = 2$, we are left with the choices $n = 4$ with $m = 1$ and $n = 0, 2$. The first one gives the second case of the assertion and the other two would produce, similarly as before, a cone of movable divisors of dimension less than that of $\mathrm{Cl}_{\mathbb{Q}}(X)$ and hence can't occur.

For $\rho(X) = 1$, we find the possibility $n = m = 2$, which is the first case of the assertion. Also, $n = 0, 4$ might happen. We first exclude the case $n = 4$. There, the prospective total coordinate space $\bar{X} = \mathrm{Spec}\ \mathbb{K}[\mathcal{R}(X)]$ is explicitly given as

$$\bar{X} \ = \ V(T_1T_2 + T_3T_4) \ \subseteq \ \mathbb{K}^4.$$

In this setting, we find a diagonal action of a three-dimensional torus $\mathbb{T}$ on $\mathbb{K}^4$ turning $\bar{X}$ into a toric variety. Thus, X as a GIT-quotient of $\bar{X}$ by a one-dimensional subgroup of $\mathbb{T}$ is as well a toric variety and must have a polynomial ring as its Cox ring; a contradiction to $\bar{X}$ being singular.

Finally, we treat the case $\rho(X) = 1$ and $n = 0$. If any two of the degrees $w_i = \deg(T_i)$ coincide, say $w_1 = w_2$ then we may substitute $T_1' = T_1 + IT_2$ and $T_2' = T_1 - IT_2$ with $I = \sqrt{-1}$, which brings us into the setting $n > 0$ just discussed. Thus, we are left with discussing the situation

$$\mathcal{R}(X) \ = \ \mathbb{K}[T_1, T_2, T_3, T_4]/\langle T_1^2 + T_2^2 + T_3^2 + T_4^2 \rangle, \qquad w_i \neq w_j \text{ for } i \neq j.$$

Due to $\mathbb{Q}$-factoriality of X, the divisor class group $\mathrm{Cl}(X)$ is of rank one and hence of the form $\mathbb{Z} \times \Gamma$ with a finite abelian group Γ. We claim that up to renumbering the variables and an automorphism of $\mathrm{Cl}(X)$, we have

$$\mathrm{Cl}(X) = \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \qquad Q = [w_1, \ldots, w_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \bar{0} & \bar{1} & \bar{1} & \bar{0} \\ \bar{0} & \bar{1} & \bar{0} & \bar{1} \end{bmatrix}.$$

Since $w_1, \ldots, w_4$ are non-torsion elements generating a pointed cone in the rational divisor class group $\mathrm{Cl}_{\mathbb{Q}}(X) = \mathbb{Q}$, we may assume $w_i = (s_i, \eta_i) \in \mathbb{Z} \oplus \Gamma$ with $s_i > 0$. By $\mathrm{Cl}(X)$-homogeneity of the relation, all s_i coincide. Hence, as the s_i generate $\mathbb{Z}$, they are all equal to one. Write Γ as a direct product of finite cyclic groups. Then, subtracting suitable multiples of the first row of Q from the last ones, we can achieve

$$Q = [w_1, \dots, w_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \eta_2 & \eta_3 & \eta_4 \end{bmatrix}, \qquad \eta_2, \eta_3, \eta_4 \in \Gamma.$$

Note that this adjusting process is realized by an automorphism of $\mathrm{Cl}(X)$. By Remark 2.2, any two of η_2, η_3, η_4 generate Γ as a group. Thus, Γ is in fact either cyclic or a sum of two cyclic groups. Moreover, $2\eta_i = 0$ by homogeneity of the relation. Hence, any element of Γ is of order two and we are left with the cases

$$\Gamma = \mathbb{Z}/2\mathbb{Z}, \qquad \Gamma = \mathbb{Z}/2 \times \mathbb{Z}/2\mathbb{Z}.$$

The first case cannot occur as it will not allow a choice of pairwise different $w_1, \dots, w_4$. Thus, suitable renumbering of the variables and applying a suitable automorphism of $\mathbb{Z} \oplus \Gamma$ to the w_i leads to $\mathrm{Cl}(X)$ and Q as claimed.

The task is to show that the above $\mathrm{Cl}(X)$-graded algebra $\mathcal{R}(X)$ can't be a Cox ring. Assume that $\mathcal{R}(X)$ is a Cox ring. Then, since the variables $T_1, \dots, T_4$ define pairwisse non-associated primes in $\mathcal{R}(X)$, we are in the setting of [20, Constr. 3.2.1.3] and can apply the theory developed thereafter. In particular, as $\bar{X}$ is smooth apart from the origin, X would be quasismooth [11, Prop. 2.8], hence log terminal. Moreover, we can apply [3, Cor. 3.3.3.3] to see that X is a del Pezzo surface of Picard number one and Gorenstein index one; we used the software package [19] for the computation. The Cox rings of all log del Pezzo surfaces of Picard number one and Gorenstein index one without torus action have been computed in [20, Thm. 4.1] and for those with torus action the Cox rings are listed in [3, 5.4.4.2]; none of these Cox rings is isomorphic to $\mathcal{R}(X)$ from above.

We verified, that the Cox ring of any full intrinsic quadric surface X is as in the assertion, in particular it is defined by trinomial relations. Consequently, the associated total coordinate space $\bar{X}$ allows a diagonal torus action of complexity one. This action induces a non-trivial $\mathbb{K}^*$-action on X. Since $\mathrm{Cl}(X)$ is finitely generated by assumption, this forces X to be rational. $\square$

3. Picard number one

The main result of this section, Theorem 3.5, provides the description of all full intrinsic quadric surfaces of Picard number one in terms of the local Gorenstein indices of two of their possibly singular points.

Construction 3.1 (Full intrinsic quadric surfaces X of Picard number one as $\mathbb{K}^*$-surfaces)**.** Consider an integral matrix of the form

$$P := [v_1, v_2, v_3, v_4] := \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ a & b & 1 & 1 \end{bmatrix}, \quad b \le -2,\ 0 \le a \le -b-2.$$

Let Z be the toric variety arising from the fan Σ in $\mathbb{Z}^3$ with *generator matrix* P, i.e. $v_1, \dots, v_4$ are the primitive ray generators of Σ, and the maximal cones

$$\sigma^+ := \mathrm{cone}(v_1, v_3, v_4), \qquad \sigma^- := \mathrm{cone}(v_2, v_3, v_4), \qquad \tau_0 := \mathrm{cone}(v_1, v_2).$$

Denote by U_1, U_2, U_3 the coordinate functions on the standard 3-torus $\mathbb{T}^3 \subseteq Z$. Then we obtain a normal, non-toric, rational, projective surface

$$X \ := \ X(P) \ := \ \overline{V(h)} \ \subseteq \ Z, \qquad h \ := \ 1 + U_1 + U_2 \ \in \mathcal{O}(\mathbb{T}^3).$$

Moreover, the $\mathbb{K}^*$-action on $\mathbb{T}^3$ given by $t \cdot x = (x_1, x_2, tx_3)$ extends to an action on Z, it leaves $V(h) \subseteq \mathbb{T}^3$ invariant and hence induces a $\mathbb{K}^*$-action on X.

Proposition 3.2. *Consider P and $X \subseteq Z$ as in Construction* 3.1*, let P^* be the transpose of P and set $K := \mathbb{Z}^4 / \operatorname{im}(P^*)$. For the divisor class group of X, we have*

$$\operatorname{Cl}(X) \ \cong \ \operatorname{Cl}(Z) \ \cong \ K \ \cong \ \mathbb{Z} \times \mathbb{Z} \,/\, 2\gcd(2a+2,\ a-b)\mathbb{Z}.$$

Moreover, denoting by $Q\colon\ \mathbb{Z}^4 \to K$ the projection, we obtain the following description of the Cox ring of X as a graded algebra:

$$\mathcal{R}(X) \ \cong \ \mathbb{K}[T_1, \dots, T_4]/\langle T_1T_2 + T_3^2 + T_4^2 \rangle, \qquad \deg(T_i) = Q(e_i) = [D_i],$$

where $D_i \subseteq X$ is the prime divisor on X obtained by intersecting X with the toric prime divisor of Z given by the ray through v_i and $[D_i] \in \operatorname{Cl}(X)$ denotes its class.

Proof of Construction 3.1 and Proposition 3.2. Due to their definition, the columns $v_1, \dots, v_4$ of P are pairwise different primitive integral vectors. Moreover, they generate $\mathbb{Q}^3$ as a convex cone, as we have

$$2v_1 + v_3 + v_4 = [0, 0, 2a+2],\ a \geq 0, \qquad 2v_2 + v_3 + v_4 = [0, 0, 2b+2],\ b \leq -2.$$

Thus, P is a defining matrix of a normal, rational, projective $\mathbb{K}^*$-surface X' in the sense of [3, Constr. 5.4.1.3 and 5.4.1.6 (e-e)]. Both, X' and X from Construction 3.1 share the same ambient toric variety Z and are given in homogeneous coordinates of Z by

$$X' \ = \ V(T_1T_2 + T_3^2 + T_4^2) \ = \ X.$$

Now [3, Thm. 3.4.3.7] tells us that the divisor class group of $X = X'$ is given as $\operatorname{Cl}(X) = \operatorname{Cl}(Z) = K$, that the Cox ring $\mathcal{R}(X)$ of X is as claimed and that the generator degrees satisfy $\deg(T_i) = [D_i]$. Note that X is non-toric as its Cox ring is not a polynomial ring. □

The *local class group* $\operatorname{Cl}(X, x)$ of a point $x \in X$ is the group of Weil divisors of X modulo those being principal near x, and by $\operatorname{cl}(X, x)$ the order of $\operatorname{Cl}(X, x)$.

Proposition 3.3. *Let $X = X(P)$ arise from Construction* 3.1*. The fixed points of the $\mathbb{K}^*$-action on X are given in Cox coordinates by*

$$x^+ \ := \ [0, 1, 0, 0], \qquad x^- \ := \ [1, 0, 0, 0], \qquad x_0 \ := \ [0, 0, 1, I].$$

Moreover, for the orders of the local class groups of the fixed points of the $\mathbb{K}^*$-action we obtain

$$\operatorname{cl}(X, x^+) = 4a + 4, \qquad \operatorname{cl}(X, x^-) = -4b - 4, \qquad \operatorname{cl}(X, x_0) = a - b.$$

Finally, the ordered pair $(4a+4,\ -4-4b)$ is an isomorphy invariant of the algebraic surface X.

Proof. For the first statement, we refer to [13, Rem. 5.6]. For the second one, we use the description [3, Prop. 3.3.1.5] of the local class groups and its Gale dual representation provided by [3, Lemma 2.1.4.1]. Concretely, for the fixed points x^+ and x^- this means

$$\begin{aligned} \mathrm{cl}(X, x^+) &= |K/Q(\mathrm{lin}_{\mathbb{Z}}(e_2))| = |\mathbb{Z}^3/\,\mathrm{lin}_{\mathbb{Z}}(v_1, v_3, v_4)| = \det[v_1, v_3, v_4], \\ \mathrm{cl}(X, x^-) &= |K/Q(\mathrm{lin}_{\mathbb{Z}}(e_1))| = |\mathbb{Z}^3/\,\mathrm{lin}_{\mathbb{Z}}(v_2, v_3, v_4)| = \det[v_2, v_3, v_4]. \end{aligned}$$

Similarly, we obtain that the local class group order $\mathrm{cl}(x_0)$ of the fixed point x_0 is given by

$$|K/Q(\mathrm{lin}_{\mathbb{Z}}(e_3, e_4))| = |\,\mathrm{lin}_{\mathbb{Z}}(-e_1 - e_2, e_3)/\,\mathrm{lin}_{\mathbb{Z}}(v_1, v_2)| = \det \begin{bmatrix} -1 & -1 \\ a & b \end{bmatrix}.$$

For the last statement, recall from [3, Prop. 5.4.1.9] that x^+, x^- are the only $\mathbb{K}^*$-fixed points lying in the closure of infinitely many orbits. Thus,

$$\{\mathrm{cl}(X, x^+), \mathrm{cl}(X, x^-)\}$$

and $\mathrm{cl}(X, x_0)$ are invariants of the $\mathbb{K}^*$-surface X. Since on a non-toric, rational, projective surface any two $\mathbb{K}^*$-actions are conjugate in the automorphism group, the assertion follows. □

Proposition 3.4. *Every full intrinsic quadric surface X of Picard number one is isomorphic to an $X(P)$ for precisely one matrix P from Construction* 3.1.

Proof. According to Theorem 2.3 and [18, Ex. 7.1], the defining matrix P is of the format 3×4 and the first two rows are as in the assertion:

$$P = \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix}.$$

Note that d_3 and d_4 are odd by primitivity of the columns. Thus, subtracting the $(d_3 - 1)/2$-fold of the first and the $(d_4 - 1)/2$-fold of the second row from the last one turns our matrix into a defining matrix

$$P = \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ a & b & 1 & 1 \end{bmatrix}.$$

These are *admissible operations* in the sense of [13, Def. 6.3] which do not affect the resulting $\mathbb{K}^*$-surface due to [13, Prop. 6.7]. Moreover, swapping the first two columns if necessary, we achieve that P is slope-ordered, meaning

$$a > b.$$

Again, this is an admissible operation. As for any defining matrix of a rational $\mathbb{K}^*$-surface with two elliptic fixed points, slope orderedness implies

$$a + \frac{1}{2} + \frac{1}{2} =: m^+ > 0, \qquad b + \frac{1}{2} + \frac{1}{2} =: m^- < 0,$$

see [13, Rem. 7.5]. Multiplying the last row by -1 is another admissible operation and turns $m^{\pm}$ into $m^{\mp}$. Doing so, if necessary, and re-arranging via the first two admissible operation steps yields

$$a+1 \;\le\; -b-1.$$

If $X(P) \cong X(P')$ holds with P, P' as in Construction 3.1, then we have $P = P'$, as due to Proposition 3.3, the entries a, b of P and a', b' of P' satisfy

$$(4a+4,\ -4b-4) \;=\; (4a'+4,\ -4b'-4).$$

□

Recall that the *Gorenstein index* of a $\mathbb{Q}$-factorial variety X is the smallest positive integer ι_X such that the ι_X-fold of a canonical divisor of X is Cartier. The *local Gorenstein index* ι_x of a point $x \in X$ is the smallest positive integer such that the ι_x-fold of a canonical divisor of X is Cartier near x.

Theorem 3.5. *For $\iota \in \mathbb{Z}_{\ge 1}$, consider the set M_ι of pairs $\eta = (\iota^+, \iota^-) \in \mathbb{Z}^2_{\ge 1}$ with $\mathrm{lcm}(\iota^+, \iota^-) = \iota$. Define subsets*

$$\begin{aligned}
S_{11}(1,\iota) &:= \{\eta \in M_\iota;\ \iota^+ \text{ odd},\ \iota^- \text{ odd},\ \iota^+ \le \iota^-\},\\
S_{12}(1,\iota) &:= \{\eta \in M_\iota;\ \iota^+ \text{ odd},\ \iota^- \text{ even},\ 4 \mid \iota^-,\ 2\iota^+ \le \iota^-\},\\
S_{21}(1,\iota) &:= \{\eta \in M_\iota;\ \iota^+ \text{ even},\ \iota^- \text{ odd},\ 4 \mid \iota^+,\ \iota^+ \le 2\iota^-\},\\
S_{22}(1,\iota) &:= \{\eta \in M_\iota;\ \iota^+ \text{ even},\ \iota^- \text{ even},\ 4 \mid \iota^+,\ 4 \mid \iota^-,\ \iota^+ \le \iota^-\}.
\end{aligned}$$

Then each set $S_{ij}(1,\iota)$ provides us with a series of defining matrices P_η of full intrinsic quadric surfaces:

$\eta = (\iota^+, \iota^-) \in S_{11}(1,\iota)$:

$$P_\eta = \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ \iota^+ - 1 & -\iota^- - 1 & 1 & 1 \end{bmatrix},$$

$\eta = (\iota^+, \iota^-) \in S_{12}(1,\iota)$:

$$P_\eta = \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ \iota^+ - 1 & -\frac{\iota^-}{2} - 1 & 1 & 1 \end{bmatrix},$$

$\eta = (\iota^+, \iota^-) \in S_{21}(1,\iota)$:

$$P_\eta = \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ \frac{\iota^+}{2} - 1 & -\iota^- - 1 & 1 & 1 \end{bmatrix},$$

$\eta = (\iota^+, \iota^-) \in S_{22}(1,\iota)$:

$$P_\eta = \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ \frac{\iota^+}{2} - 1 & -\frac{\iota^-}{2} - 1 & 1 & 1 \end{bmatrix}.$$

Each of the surfaces $X(P_\eta)$ is of Picard number 1, Gorenstein index $\iota = \mathrm{lcm}(\iota^+, \iota^-)$ and $\iota^\pm$ are the local Gorenstein indices of the points

$$x^+ \;=\; [0,1,0,0], \qquad x^- \;=\; [1,0,0,0].$$

Every full intrinsic quadric surface of Picard number 1 and Gorenstein index ι is isomorphic to $X(P_\eta)$ for precisely one P_η from the above list.

Proof. Let X be a full intrinsic quadric surface of Picard number one. We first show $X \cong X(P_\eta)$ with P_η from the above list and check the local Gorenstein indices. Proposition 3.4 allows us to assume $X = X(P)$ with a unique P of the form

$$P = \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ a & b & 1 & 1 \end{bmatrix}, \qquad b \leq -2,\ 0 \leq a \leq -b-2.$$

According to [3, Prop. 3.3.3.2], we have the anticanonical divisor $-\mathcal{K} = D_3 + D_4$ on X and [13, Prop. 8.9] tells us that the linear forms $u^\pm$ representing the $\iota^\pm$-fold of $-\mathcal{K}$ near $x^\pm$ are given by

$$u^+ = \left[\frac{a\iota^+}{2a+2}, \frac{a\iota^+}{2a+2}, \frac{\iota^+}{a+1},\right], \qquad u^- = \left[\frac{b\iota^-}{2b+2}, \frac{b\iota^-}{2b+2}, \frac{\iota^-}{b+1}\right].$$

By the definition of the local Gorenstein index, these are primitive integral vectors. Consequently, the local Gorenstein indices $\iota^\pm$ of $x^\pm$ are

$$\iota^+ = \begin{cases} a+1, & a \text{ even}, \\ 2a+2, & a \text{ odd}, \end{cases} \qquad \iota^- = \begin{cases} -b-1, & b \text{ even}, \\ -2b-2, & b \text{ odd}. \end{cases}$$

In particular, ι^+, ι^- is even (odd) if and only if a, b is odd (even), respectively. Moreover, if $\iota^\pm$ is even, then it is divisible by four. Thus, P is one of the matrices P_η with $\eta = (\iota^+, \iota^-)$ listed in the assertion and $\iota^\pm$ is the local Gorenstein index of $x^\pm$.

Conversely, all the matrices P_η listed in the assertion fit into the shape of Construction 3.1 and thus deliver full intrinsic quadric surfaces $X = X(P_\eta)$. By [13, Prop. 8.8], the point $x_0 = [0,0,1,I] \in X$ has local Gorenstein index one, hence the resulting X is of Gorenstein index $\iota = \mathrm{lcm}(\iota^+, \iota^-)$.

Finally, we ensure that the matrices P_η listed in the assertion define pairwise non-isomorphic $X(P_\eta)$. By Proposition 3.4, this means to show that any two matrices arising from different $S_{ij}(1,\iota)$ differ from each other. This is done by comparing the parity vectors $(\bar{a}, \bar{b})$ in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ of the first two entries a, b of the third row of P_η for the $\eta \in S_{ij}(1,\iota)$:

	$S_{11}(1,\iota)$	$S_{12}(1,\iota)$	$S_{21}(1,\iota)$	$S_{22}(1,\iota)$
$(\bar{a}, \bar{b})$	$(\bar{0}, \bar{0})$	$(\bar{0}, \bar{1})$	$(\bar{1}, \bar{0})$	$(\bar{1}, \bar{1})$

□

Example 3.6. Consider the full intrinsic quadric surface $X = X(P)$ of Picard number one given by the defining matrix

$$P = \begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ 0 & -2 & 1 & 1 \end{bmatrix}.$$

Then X stems from the series $S_{11}(1,\iota)$ and we have $\iota = \iota^+ = \iota^- = 1$. Theorem 3.5 also says that X is the only Gorenstein full intrinsic quadric surface with $\rho(X) = 1$.

4. Picard number two

The main result of this section, Theorem 4.5, provides the description of all full intrinsic quadric surfaces of Picard number two in terms of the local Gorenstein indices of two of their possibly singular points and the local class group order of another possibly singular point.

Construction 4.1 (Full intrinsic quadric surfaces X of Picard number two as $\mathbb{K}^*$-surfaces)**.** Consider an integral matrix of the form

$$P \;:=\; [v_1, v_2, v_3, v_4, v_5] \;:=\; \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ a & b & 0 & c & 1 \end{bmatrix}, \qquad \begin{array}{l} b<a,\ c<0,\ a\geq 0, \\ b+c\leq -1,\ a-b\leq -c, \\ a\leq -b-c-1. \end{array}$$

Let Z be the toric variety arising from the fan Σ in $\mathbb{Z}^3$ with generator matrix P and the maximal cones

$$\sigma^+ := \mathrm{cone}(v_1, v_3, v_5), \qquad \sigma^- := \mathrm{cone}(v_2, v_4, v_5),$$

$$\tau_0 := \mathrm{cone}(v_1, v_2), \qquad \tau_1 := \mathrm{cone}(v_3, v_4).$$

Denote by U_1, U_2, U_3 the coordinate functions on the standard 3-torus $\mathbb{T}^3 \subseteq Z$. Then we obtain a normal, non-toric, rational, projective surface

$$X \;:=\; X(P) \;:=\; \overline{V(h)} \;\subseteq\; Z, \qquad h \;:=\; 1 + U_1 + U_2 \;\in \mathcal{O}(\mathbb{T}^3).$$

Moreover, the $\mathbb{K}^*$-action on $\mathbb{T}^3$ given by $t \cdot x = (x_1, x_2, tx_3)$ extends to an action on Z, it leaves $V(h) \subseteq \mathbb{T}^3$ invariant and hence induces a $\mathbb{K}^*$-action on X.

Proposition 4.2. *Consider P and $X \subseteq Z$ as in Construction* 4.1*, let P^* be the transpose of P and set $K := \mathbb{Z}^5 / \mathrm{im}(P^*)$. For the divisor class group of X, we have*

$$\mathrm{Cl}(X) \;\cong\; K \;\cong\; \mathrm{Cl}(Z) \;\cong\; \mathbb{Z}^2 \times \mathbb{Z} \,/\, \gcd(2a+1, a-b, -c)\mathbb{Z}.$$

Moreover, denoting by $Q\colon\ \mathbb{Z}^5 \to K$ the projection, we obtain the following description of Cox ring of X as graded algebra:

$$\mathcal{R}(X) \;\cong\; \mathbb{K}[T_1, \dots, T_5]/\langle T_1T_2 + T_3T_4 + T_5^2\rangle, \qquad \deg(T_i) = Q(e_i) = [D_i],$$

where $D_i \subseteq X$ is the prime divisor on X obtained by intersecting X with the toric prime divisor of Z given by the ray through v_i and $[D_i] \in \mathrm{Cl}(X)$ denotes its class.

Proof of Construction 4.1 and Proposition 4.2. Due to their definition, the columns $v_1, \dots, v_5$ of P are pairwise different primitive integral vectors. Moreover, they generate $\mathbb{Q}^3$ as a convex cone, as we have

$$2v_1 + 2v_3 + v_5 = [0, 0, 2a+1],\ a \geq 0,$$

$$2v_2 + 2v_4 + v_5 = [0, 0, 2b+2c+1],\ b + c \leq -1.$$

Consequently, P is a defining matrix of a rational projective $\mathbb{K}^*$-surface X' in the sense of [3, Constr. 5.4.1.3 and 5.4.1.6 (e-e)]. One shows $X' = X$ exactly as for Picard number one and infers the desired statements on the divisor class group and the Cox ring from the same reference. $\square$

Proposition 4.3. *Let $X = X(P)$ arise from Construction* 4.1. *The fixed points of the $\mathbb{K}^*$-action on X are given in Cox coordinates by*

$$x^+ := [0,1,0,1,0], \qquad x^- := [1,0,1,0,0],$$
$$x_0 := [0,0,1,1,I], \qquad x_1 := [1,1,0,0,I].$$

Moreover, the orders of the local class groups of the fixed points of the $\mathbb{K}^*$-action on X are given by

$$\mathrm{cl}(X,x^+) = 1+2a, \qquad \mathrm{cl}(X,x^-) = -1-2b-2c,$$
$$\mathrm{cl}(X,x_0) = a-b, \qquad \mathrm{cl}(X,x_1) = -c.$$

Finally, the ordered pairs $(1+2a,\ -1-2b-2c)$ and $(a-b,\ -c)$ are isomorphy invariants of the algebraic surface X.

Proof. The same references and arguing as in the proof of Proposition 3.3, give us the fixed points and show that the local class group orders of x^+, x^-, x_0 and x_1 compute as

$$\det[v_1,v_3,v_5], \qquad \det[v_2,v_4,v_5], \qquad \det\begin{bmatrix} -1 & -1 \\ a & b \end{bmatrix}, \qquad -\det\begin{bmatrix} 1 & 1 \\ 0 & c \end{bmatrix}.$$

As mentioned in the proof of Proposition 3.3, the fixed points x^+, x^- are the only ones lying in the closure of infinitely many orbits. Moreover, each of the remaining two fixed points x_0, x_1 lies in the closure of precisely two non-trivial orbits. Thus, $\{\mathrm{cl}(X,x^+),\mathrm{cl}(X,x^-)\}$ as well as $\{\mathrm{cl}(X,x_0),\mathrm{cl}(X,x_1)\}$ are invariants of the $\mathbb{K}^*$-surface X. As before, the assertion follows from the fact that on a non-toric, rational, projective surface any two $\mathbb{K}^*$-actions are conjugate in the automorphism group. □

Proposition 4.4. *Every full intrinsic quadric surface X of Picard number two is isomorphic to an $X(P)$ for precisely one matrix P from Construction* 4.1.

Proof. Using again Theorem 2.3 and [18, Ex. 7.1], we see that the defining matrix P is of the format 3×5 and the first two rows look as in the assertion:

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ d_1 & d_2 & d_3 & d_4 & d_5 \end{bmatrix}.$$

As in the proof of Proposition 3.4, we achieve the desired shape of P via admissible operations [13, Def. 6.3]. First, adding suitable multiples of the first two rows to the last one yields

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ a & b & 0 & c & 1 \end{bmatrix}.$$

Second, swapping the columns v_1 and v_2 as well as v_3 and v_4 if neccesary and re-arranging via the first step, we achieve that P is slope-ordered, meaning

$$a > b, \qquad 0 > c.$$

Third, swapping the first two columns blocks, that means $[v_1, v_2]$ and $[v_3, v_4]$, if neccesary and re-adjusting the entries, we can ensure

$$a - b \leq -c.$$

As for any defining matrix of a rational $\mathbb{K}^*$-surface with two elliptic fixed points, slope orderedness implies

$$a + \frac{1}{2} =: m^+ > 0, \qquad b + c + \frac{1}{2} =: m^- < 0.$$

Multiplying the last row by -1 turns $m^\pm$ into $m^\mp$. Doing so, if necessary, and re-arranging via the first two steps yields

$$a \ \leq \ -b - c - 1.$$

We show that $X(P) \cong X(P')$ with matrices P and P' as in Construction 4.1 implies $P = P'$. Proposition 4.3 yields equality of the ordered tuples

$$(1+2a, \ -1-2b-2c) = (1+2a', \ -1-2b'-2c'), \quad (a-b, \ -c) = (a'-b', -c')$$

built from the entries of the third row of P and P' respectively. From this we directly derive $P = P'$. $\square$

Theorem 4.5. *For $\iota \in \mathbb{Z}_{\geq 1}$, consider the set M_ι of triples $\eta = (\iota^+, \iota^-, c)$, where $\iota^+, \iota^- \in \mathbb{Z}_{\geq 1}$ with $\operatorname{lcm}(\iota^+, \iota^-) = \iota$ and $c \in \mathbb{Z}_{\leq -1}$. Define subsets*

$$\begin{aligned}
S_{11}(2,\iota) &:= \left\{ \eta \in M_\iota; \ \begin{array}{l} 2 \nmid \iota^+, \iota^-, \ 3 \nmid \iota^+, \iota^-, \ \iota^+ \leq \iota^-, \\ 1 - \frac{\iota^+ + \iota^-}{2} \leq c \leq -\frac{\iota^+ + \iota^-}{4} \end{array} \right\}, \\
S_{12}(2,\iota) &:= \left\{ \eta \in M_\iota; \ \begin{array}{l} 2 \nmid \iota^+, \iota^-, \ 3 \nmid \iota^+, \ \iota^+ \leq 3\iota^-, \\ 1 - \frac{\iota^+ + 3\iota^-}{2} \leq c \leq -\frac{\iota^+ + 3\iota^-}{4} \end{array} \right\}, \\
S_{21}(2,\iota) &:= \left\{ \eta \in M_\iota; \ \begin{array}{l} 2 \nmid \iota^+, \iota^-, \ 3 \nmid \iota^-, \ 3\iota^+ \leq \iota^-, \\ 1 - \frac{3\iota^+ + \iota^-}{2} \leq c \leq -\frac{3\iota^+ + \iota^-}{4} \end{array} \right\}, \\
S_{22}(2,\iota) &:= \left\{ \eta \in M_\iota; \ \begin{array}{l} 2 \nmid \iota^+, \iota^-, \ \iota^+ \leq \iota^-, \\ 1 - \frac{3\iota^+ + 3\iota^-}{2} \leq c \leq -\frac{3\iota^+ + 3\iota^-}{4} \end{array} \right\}.
\end{aligned}$$

Then each set $S_{ij}(2,\iota)$ provides us with a series of defining matrices P_η of full intrinsic quadric surfaces:

$\eta = (\iota^+, \iota^-, c) \in S_{11}(2, \iota)$:

$$P_\eta = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ \frac{\iota^+-1}{2} & -\frac{\iota^-+1}{2} - c & 0 & c & 1 \end{bmatrix},$$

$\eta = (\iota^+, \iota^-, c) \in S_{12}(2, \iota)$:

$$P_\eta = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ \frac{\iota^+-1}{2} & -\frac{3\iota^-+1}{2} - c & 0 & c & 1 \end{bmatrix},$$

$\eta = (\iota^+, \iota^-, c) \in S_{21}(2, \iota)$:

$$P_\eta = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ \frac{3\iota^+-1}{2} & -\frac{\iota^-+1}{2} - c & 0 & c & 1 \end{bmatrix},$$

$\eta = (\iota^+, \iota^-, c) \in S_{22}(2, \iota)$:

$$P_\eta = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ \frac{3\iota^+-1}{2} & -\frac{3\iota^-+1}{2} - c & 0 & c & 1 \end{bmatrix}.$$

Each $X(P_\eta)$ is of Picard number 2, Gorenstein index $\iota = \mathrm{lcm}(\iota^+, \iota^-)$, and ι^+, ι^- are the local Gorenstein indices and $-c$ local class group order of

$$x^+ \;=\; [0,1,0,1,0], \qquad x^- \;=\; [1,0,1,0,0], \qquad x_1 \;=\; [1,1,0,0,I].$$

Every full intrinsic quadric surface of Picard number 2 and Gorenstein index ι is isomorphic to $X(P_\eta)$ for precisely one P_η from the above list.

Proof. Let X be a full intrinsic quadric surface of Picard number two. Then Proposition 4.4 allows us to assume $X = X(P)$ with

$$P \;=\; \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ a & b & 0 & c & 1 \end{bmatrix}, \qquad \begin{array}{l} b<a,\ c<0,\ a\geq 0, \\ b+c\leq -1,\ a-b\leq -c, \\ a\leq -b-c-1. \end{array}$$

Consider the anticanonical divisor $-\mathcal{K} = D_3 + D_4 + D_5$ on $X(P)$. The linear forms $u^\pm$ representing the $\iota^\pm$-fold of $-\mathcal{K}$ near $x^\pm$ are given by

$$u^+ = \left[\iota^+,\ \frac{(a-1)\iota^+}{1+2a},\ \frac{3\iota^+}{1+2a}\right],$$

and

$$u^- = \left[\frac{(2b-c+1)\iota^-}{2b+2c+1},\ \frac{(b+c-1)\iota^-}{2b+2c+1},\ -\frac{3\iota^-}{2b+2c+1}\right].$$

By the definition of the local Gorenstein index, these are primitive integral vectors. Together with the fact that $\iota^\pm$ divides $\mathrm{cl}(X, x^\pm)$, we obtain

$$3\iota^+ = y^+(1+2a), \qquad 1+2a = z^+\iota^+,$$

$$3\iota^- = -y^-(2b+2c+1), \qquad -(2b+2c+1) = z^-\iota^-$$

with positive integers $y^\pm$ and $z^\pm$. We conclude $y^+z^+ = 3$ and $y^-z^- = 3$. This leaves us with the following four cases:

Case 1.1: $y^+ = 3$, $y^- = 3$. Then we have $\iota^+ = 1+2a$ and $\iota^- = -2b-2c-1$. Solving for a in the first equation, for b in the second one and substituting gives

$$u^+ = \left[\iota^+,\ \frac{\iota^+-3}{2},\ 3\right], \qquad u^- = \left[\iota^- + 3c,\ \frac{\iota^-+3}{2},\ -3\right].$$

We conclude that ι^+ as well as ι^- is odd and none of them is divisible by three. Substituting also in P and the conditions on its entries leads to setting $S_{11}(2,\iota)$.

Case 1.2: $y^+ = 3$, $y^- = 1$. Then we have $\iota^+ = 1 + 2a$ and $3\iota^- = -2b - 2c - 1$. Solving for a in the first equation, for b in the second one and substituting gives

$$u^+ = \left[\iota^+, \frac{\iota^+ - 3}{2}, 3\right], \qquad u^- = \left[\iota^- + c, \frac{\iota^- + 1}{2}, -1\right].$$

We conclude that ι^+ as well as ι^- is odd and ι^+ is not divisible by three. Substituting also in P and the conditions on its entries leads to setting $S_{12}(2,\iota)$.

Case 2.1: $y^+ = 1$, $y^- = 3$. Then we have $3\iota^+ = 1 + 2a$ and $\iota^- = -2b - 2c - 1$. Solving for a in the first equation, for b in the second one and substituting gives

$$u^+ = \left[\iota^+, \frac{\iota^+ - 1}{2} 1,\right], \qquad u^- = \left[\iota^- + 3c, \frac{\iota^- + 3}{2}, -3\right].$$

We conclude that ι^+ as well as ι^- is odd and ι^- is not divisible by three. Substituting also in P and the conditions on its entries leads to setting $S_{21}(2,\iota)$.

Case 2.2: $y^+ = 1$, $y^- = 1$. Then we have $3\iota^+ = 1 + 2a$ and $3\iota^- = -2b - 2c - 1$. Solving for a in the first equation, for b in the second one and substituting gives

$$u^+ = \left[\iota^+, \frac{\iota^+ - 1}{2}, 1\right], \qquad u^- = \left[\iota^- + c, \frac{\iota^- + 1}{2}, -1\right].$$

We conclude that ι^+ as well as ι^- is odd. Substituting also in P and the conditions on its entries leads to setting $S_{22}(2,\iota)$.

We showed that every full intrinsic quadric surface of Picard number two is isomorphic to some $X(P_\eta)$ with P_η as in the assertion. Moreover, x_0 and x_1 are of local Gorenstein index one, see [13, Prop. 8.8 (iii)], we obtain that $X(P_\eta)$ has Gorenstein index $\iota = \operatorname{lcm}(\iota^+,\iota^-)$. Conversely, one directly checks that every matrix P from the assertion defines a full intrinsic quadric surface of Picard number two and Gorenstein index $\iota = \operatorname{lcm}(\iota^+,\iota^-)$.

Finally, we want to see that the matrices P_η listed in the assertion define pairwise non-isomorphic $X(P_\eta)$. Due to Proposition 4.4, this means to show that the sets $S_{ij}(2,\iota)$ are pairwise disjoint. With the aid of Proposition 4.3, we compare the local Gorenstein indices $\iota^\pm$ and the local class group orders $\operatorname{cl}(X,x^\pm)$:

	$S_{11}(2,\iota)$	$S_{12}(2,\iota)$	$S_{21}(2,\iota)$	$S_{22}(2,\iota)$
$(\iota^+,\operatorname{cl}(X,x^+))$	(ι^+,ι^+)	(ι^+,ι^+)	$(\iota^+,3\iota^+)$	$(\iota^+,3\iota^+)$
$(\iota^-,\operatorname{cl}(X,x^-))$	(ι^-,ι^-)	$(\iota^-,3\iota^-)$	(ι^-,ι^-)	$(\iota^-,3\iota^-)$

The listed pairs are invariants of the surface $X(P_\eta)$ up to switching x^+ and x^-. Thus, we see that $S_{11}(2,\iota)$ as well as $S_{22}(2,\iota)$ has trivial intersection with any other $S_{ij}(2,\iota)$. For $S_{12}(2,\iota)$, observe $\iota^+ < 3\iota^-$, as we have $3 \nmid \iota^+$. Similarly, $3\iota^+ < \iota^-$ holds for $S_{21}(3,\iota)$. Thus, in both cases, $\operatorname{cl}(X,x^+)$ is the strictly smallest of $\operatorname{cl}(X,x^\pm)$. Consequently, $S_{12}(2,\iota)$ and $S_{21}(2,\iota)$ intersect trivially. $\square$

Example 4.6. Consider the full intrinsic quadric surfaces X and X' of Picard number two given by the defining matrices

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ 0 & -1 & 0 & -1 & 1 \end{bmatrix}, \qquad P' = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ 1 & 0 & 0 & -2 & 1 \end{bmatrix}.$$

Then X stems from the series $S_{12}(2,\iota)$ and X' from $S_{22}(2,\iota)$. Theorem 4.5 yields that X and X' are the only Gorenstein full intrinsic quadric surfaces with $\rho(X) = 2$.

We conclude the section by taking a look at the possible contractions of the two-dimensional full intrinsic quadrics. Recall that a *contraction* of a prime divisor D on a normal variety X is a proper birational morphism $\pi : X \to X'$ such that the image $\pi(D)$ is of codimension at least two in X' and $X \setminus D$ maps isomorphically onto $X' \setminus \pi(D)$.

Proposition 4.7. *Let $X = X(P)$ arise from Construction* 4.1. *At most the prime divisors $D_1, \dots, D_4 \subseteq X$ are contractible and all possible contractions are projective toric surfaces of Picard number one. More precisely,*

$$D_1: \ b \ge 0 \qquad D_2: \ a+c \le -1 \qquad D_3: \ a+c \ge 0 \qquad D_4: \ b \le -1$$

$$\begin{bmatrix} -1 & -1 & 2 \\ b & b+c & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & -1 & 2 \\ a & a+c & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & -1 & 2 \\ a+c & b+c & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & -1 & 2 \\ a & b & 1 \end{bmatrix}$$

gives us for each D_i the characterizing property of contractibility in terms of the entries a, b, c of P and, for the case that D_i is contractible, also the generator matrix of the contracted surface.

Proof. By [13, Rem. 10.4 (i) and Prop. 10.8], the contractible prime divisors are among the $D_i = V(T_i) \subseteq X$, where $i = 1, \dots, 5$. The same references show that the divisor D_5 is not contractible. Recall that the matrix P is given as

$$P = [v_1, v_2, v_3, v_4, v_5] = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 & 2 \\ a & b & 0 & c & 1 \end{bmatrix}, \qquad \begin{array}{l} b<a,\ c<0,\ a\ge 0, \\ b+c\le -1,\ a-b\le -c, \\ a\le -b-c-1. \end{array}$$

The task is to characterize contractibility for each of $D_1, \dots, D_4$ and to determine the possible contraction in terms of the entries of P. We exemplarily perform this for the divisor D_1. Consider the matrix

$$P_1 := [v_2, v_3, v_4, v_5] = \begin{bmatrix} -1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 2 \\ b & 0 & c & 1 \end{bmatrix},$$

obtained from P by removing the colmun v_1, which corresponds the prime divisor $D_1 \subseteq X$. Then D_1 is contractible if and only if P_1 is a defining matrix of a $\mathbb{K}^*$-surface. The latter in turn holds if and only if

$$m^+ = b + \frac{1}{2} > 0,$$

as P_1 inherits all the other properties from P. Thus, D_1 is contractible if and only if $b \geq 0$ holds. If so, then contracing D_1 gives the $\mathbb{K}^*$-surface X_1 defined by P_1. Via admissible operations, we can turn P_1 into the shape

$$P_1 = \begin{bmatrix} -1 & -1 & 1 & 0 \\ -1 & -1 & 0 & 2 \\ b & b+c & 0 & 1 \end{bmatrix}.$$

Indeed, we swap the first two column blocks, re-arrange the shape and then subtract the b-fold of the first row from the last one. The makes the third column *erasable* [13, Def. 6.2] and we obtain a defining matrix

$$P_1' = \begin{bmatrix} -1 & -1 & 2 \\ b & b+c & 1 \end{bmatrix}$$

by *erasing* the third column [13, Def. 6.3, Prop. 6.7]. This process reflects removing the redundant Cox ring generator $T_3 = T_1T_2 - T_4^2$ in the first presentation of X_1. We conclude that X_1 is the toric surface defined by the generator matrix P_1'. □

Remark 4.8. Consider the two Gorenstein full intrinsic quadric surfaces X and X' of Picard number two from Example 4.6.

(1) In the surface X, the contractible divisors are D_2 and D_4. In each case, the contracted surface is the projective plane $\mathbb{P}^2$.
(2) In the surface X', the contractible divisors are D_1 and D_2. In each case, the contracted surface is the weighted projective plane $\mathbb{P}(1, 2, 3)$.

5. Picard number three

The main result of this section, Theorem 5.5, provides the description of all full intrinsic quadric surfaces of Picard number three in terms of the local Gorenstein indices of two of their possibly singular points and the local class group orders of two further possibly singular points.

Construction 5.1 (Full intrinsic quadric surfaces X of Picard number three as $\mathbb{K}^*$-surfaces)**.** Consider an integral matrix of the form

$$P := [v_1, v_2, v_3, v_4, v_5, v_6] := \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ a & b & 0 & c & 0 & d \end{bmatrix}, \qquad \begin{array}{l} a>b,\ 0>c,\ 0>d, \\ a-b\geq -c\geq -d, \\ b+c+d<0<a, \\ a\leq -b-c-d. \end{array}$$

Let Z be the toric variety arising from the fan Σ in $\mathbb{Z}^3$ with generator matrix P and the maximal cones

$$\sigma^+ := \mathrm{cone}(v_1, v_3, v_5), \qquad \sigma^- := \mathrm{cone}(v_2, v_4, v_6),$$
$$\tau_0 := \mathrm{cone}(v_1, v_2), \quad \tau_1 := \mathrm{cone}(v_3, v_4), \quad \tau_2 := \mathrm{cone}(v_5, v_6).$$

Denote by U_1, U_2, U_3 the coordinate functions on the standard 3-torus $\mathbb{T}^3 \subseteq Z$. Then we obtain a normal, non-toric, rational, projective surface

$$X := X(P) := \overline{V(h)} \subseteq Z, \qquad h := 1 + U_1 + U_2 \in \mathcal{O}(\mathbb{T}^3).$$

Moreover, the $\mathbb{K}^*$-action on $\mathbb{T}^3$ given by $t \cdot x = (x_1, x_2, tx_3)$ extends to an action on Z, it leaves $V(h) \subseteq \mathbb{T}^3$ invariant and hence induces a $\mathbb{K}^*$-action on X.

Proposition 5.2. *Consider P and $X \subseteq Z$ as in Construction* 5.1*, let P^* be the transpose of P and set $K := \mathbb{Z}^6 / \operatorname{im}(P^*)$. For the divisor class group of X, we have Then the divisor class group of X equals that of Z and is given by*

$$\mathrm{Cl}(X) \cong \mathrm{Cl}(Z) \cong K \cong \mathbb{Z}^3 \times \mathbb{Z} / \gcd(a,b,c,d)\mathbb{Z}.$$

Moreover, denoting by $Q\colon \mathbb{Z}^6 \to K$ the projection, we obtain the following description of the Cox ring of X as a graded algebra:

$$\mathcal{R}(X) \cong \mathbb{K}[T_1, \dots, T_6]/\langle T_1T_2 + T_3T_4 + T_5T_6 \rangle, \qquad \deg(T_i) = Q(e_i) = [D_i],$$

where $D_i \subseteq X$ is the prime divisor on X obtained by intersecting X with the toric prime divisor of Z given by the ray through v_i and $[D_i] \in \mathrm{Cl}(X)$ denotes its class.

Proof of Construction 3.1 and Proposition 3.2. Due to their definition, the columns $v_1, \dots, v_6$ of P are pairwise different primitive integral vectors. Moreover, they generate $\mathbb{Q}^3$ as a convex cone, as we have

$$v_1 + v_3 + v_5 = [0,0,a],\ a > 0, \qquad v_2 + v_4 + v_6 = [0,0,b+c+d],\ b+c+d < 0.$$

Consequently, P is a defining matrix of a rational projective $\mathbb{K}^*$-surface X' in the sense of [3, Constr. 5.4.1.3 and 5.4.1.6 (e-e)]. Again, one shows $X' = X$ exactly as in the case of Picard number one, and the same reference gives the desired statements on the divisor class group and the Cox ring. □

Proposition 5.3. *Let $X = X(P)$ arise from Construction* 5.1*. The fixed points of the $\mathbb{K}^*$-action on X are given in Cox coordinates by*

$$x^+ := [0,1,0,1,0,1], \qquad x^- := [1,0,1,0,1,0],$$

$$x_0 := [0,0,1,1,1,-1], \quad x_1 := [1,1,0,0,1,-1], \quad x_2 := [1,1,1,-1,0,0].$$

Moreover, the orders of the local class groups of the fixed points of the $\mathbb{K}^*$-action are given by

$$\mathrm{cl}(X, x^+) = a, \qquad \mathrm{cl}(X, x^-) = -b-c-d,$$

$$\mathrm{cl}(X, x_0) = a - b, \quad \mathrm{cl}(X, x_1) = -c, \quad \mathrm{cl}(X, x_2) = -d.$$

Finally, the ordered tuples $(a, -b-c-d)$ and $(a-b, -c, -d)$ are isomorphy invariants of the algebraic surface X.

Proof. As in the previous section, the references from the proof of Proposition 3.3 deliver the description of the fixed points and show that the local class group orders of x^+, x^-, x_0, x_1 and x_2 are

$$\det[v_1, v_3, v_5], \qquad \det[v_2, v_4, v_6],$$

$$\det\begin{bmatrix} -1 & -1 \\ a & b \end{bmatrix}, \qquad -\det\begin{bmatrix} 1 & 1 \\ 0 & c \end{bmatrix}, \qquad -\det\begin{bmatrix} 1 & 1 \\ 0 & d \end{bmatrix}.$$

Similarly as in the corresponding earlier proofs, x^+, x^- are the only fixed points lying in the closure of infinitely many orbits and each of x_0, x_1, x_2 lies in the closure of precisely two non-trivial orbits. Thus, the sets $\{\mathrm{cl}(X, x^+), \mathrm{cl}(X, x^-)\}$

and $\{\mathrm{cl}(X,x_0),\mathrm{cl}(X,x_1),\mathrm{cl}(X,x_2)\}$ are invariants of the $\mathbb{K}^*$-surface X. Again, the assertion follows from the fact that on a non-toric, rational, projective, surface any two $\mathbb{K}^*$-actions are conjugate in the automorphism group. □

Proposition 5.4. *Every full intrinsic quadric surface X of Picard number three is isomorphic to an $X(P)$ for precisely one matrix P from Construction* 5.1.

Proof. Applying once more Theorem 2.3 and [18, Ex. 7.1], yields that the defining matrix P is of the format 3×6 and the first two rows look as wanted:

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \end{bmatrix}.$$

Again, suitable admissible operations [13, Def. 6.3] bring us to the setting of Construction 5.1. First, adding suitable multiples of the first two rows to the last one, we achieve

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ a & b & 0 & c & 0 & d \end{bmatrix}.$$

Second, swapping columns inside the pairs (v_1,v_2), (v_3,v_4) and (v_5,v_6) and re-arranging via the first step, we achieve that P is slope-ordered, meaning

$$a > b, \quad 0 > c, \quad 0 > d.$$

Third, suitable swapping the columns blocks $[v_1,v_2]$, $[v_3,v_4]$ and $[v_5,v_6]$ and re-adjusting the entries, we can ensure

$$a - b \geq -c \geq -d.$$

As for any defining matrix of a rational $\mathbb{K}^*$-surface with two elliptic fixed points, slope orderedness implies

$$a =: m^+ > 0, \qquad\qquad b + c + d =: m^- < 0.$$

Multiplying the last row by -1 turns $m^\pm$ into $m^\mp$. Doing so, if necessary, and re-arranging via the first two steps yields

$$a \;\leq\; -b - c - d.$$

We show that $X(P) \cong X(P')$ with matrices P and P' as in Construction 5.1 implies $P = P'$. Proposition 5.3 yields equality of the ordered tuples

$$(a,\, -b-c-d) \;=\; (a',\, -b'-c'-d'), \qquad (a-b,\, -c,\, -d) \;=\; (a'-b',\, -c',\, -d')$$

built from the entries of the third row of P and P' respectively. From this, we directly derive $P = P'$. □

Theorem 5.5. *For $\iota \in \mathbb{Z}_{\geq 1}$, consider the set M_ι of 4-tuples $\eta = (\iota^+, \iota^-, c, d)$, where $\iota^+, \iota^- \in \mathbb{Z}_{\geq 1}$ with $\mathrm{lcm}(\iota^+, \iota^-) = \iota$ and $c, d \in \mathbb{Z}_{\leq -1}$. Define subsets*

$$
\begin{aligned}
S_{11}(3,\iota) &:= \left\{ \eta \in M_\iota;\ \begin{array}{l} 2 \nmid \iota^+, \iota^-,\ \iota^+ \leq \iota^-,\ c \leq d \leq -1, \\ -\iota^+ - \iota^- \leq 2c + d \end{array} \right\}, \\
S_{12}(3,\iota) &:= \left\{ \eta \in M_\iota;\ \begin{array}{l} 2 \nmid \iota^+,\ \iota^+ \leq 2\iota^-,\ c \leq d \leq -1, \\ -\iota^+ - 2\iota^- \leq 2c + d \end{array} \right\}, \\
S_{21}(3,\iota) &:= \left\{ \eta \in M_\iota;\ \begin{array}{l} 2 \nmid \iota^-,\ 2\iota^+ \leq \iota^-,\ c \leq d \leq -1, \\ -2\iota^+ - \iota^- \leq 2c + d \end{array} \right\}, \\
S_{22}(3,\iota) &:= \left\{ \eta \in M_\iota;\ \begin{array}{l} \iota^+ \leq \iota^-,\ c \leq d \leq -1, \\ -2\iota^+ - 2\iota^- \leq 2c + d \end{array} \right\}.
\end{aligned}
$$

Then each set $S_{ij}(3,\iota)$ provides us with a series of defining matrices P_η of full intrinsic quadric surfaces:

$\eta = (\iota^+, \iota^-, c, d) \in S_{11}(3,\iota)$:

$$
P_\eta = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ \iota^+ & -\iota^- - c - d & 0 & c & 0 & d \end{bmatrix},
$$

$\eta = (\iota^+, \iota^-, c, d) \in S_{12}(3,\iota)$:

$$
P_\eta = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ \iota^+ & -2\iota^- - c - d & 0 & c & 0 & d \end{bmatrix},
$$

$\eta = (\iota^+, \iota^-, c, d) \in S_{21}(3,\iota)$:

$$
P_\eta = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 2\iota^+ & -\iota^- - c - d & 0 & c & 0 & d \end{bmatrix},
$$

$\eta = (\iota^+, \iota^-, c, d) \in S_{22}(3,\iota)$:

$$
P_\eta = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 2\iota^+ & -2\iota^- - c - d & 0 & c & 0 & d \end{bmatrix}.
$$

Each $X(P_\eta)$ is of Picard number 3, Gorenstein index $\iota = \mathrm{lcm}(\iota^+, \iota^-)$, and ι^+, ι^- are the local Gorenstein indices, $-c, -d$ the local class group orders of

$$
x^+ = [0,1,0,1,0,1], \qquad x^- = [1,0,1,0,1,0],
$$

$$
x_1 = [1,1,0,0,1,-1], \qquad x_2 = [1,1,1,-1,0,0].
$$

Every full intrinsic quadric surface of Picard number 3 and Gorenstein index ι is isomorphic to $X(P_\eta)$ for precisely one P_η from the above list.

Proof. Let X be a full intrinsic quadric surface of Picard number three. Then Construction 5.1 and Proposition 5.4 allow us to assume $X = X(P)$ with

$$
P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ a & b & 0 & c & 0 & d \end{bmatrix}, \qquad \begin{array}{l} a>b,\ 0>c,\ 0>d, \\ a-b \geq -c \geq -d, \\ b+c+d<0<a, \\ a \leq -b-c-d. \end{array}
$$

Consider anticanonical divisor $-\mathcal{K} = D_3 + D_4 + D_5 + D_6$ on $X(P)$. The linear forms $u^\pm$ representing the $\iota^\pm$-fold of $-\mathcal{K}$ near $x^\pm$ are given as

$$u^+ = \left[\iota^+,\, \iota^+,\, \frac{2\iota^+}{a}\right], \qquad u^- = \left[\frac{(b-c+d)\iota^-}{b+c+d},\, \frac{(b+c-d)\iota^-}{b+c+d},\, -\frac{2\iota^-}{b+c+d}\right].$$

By the definition of the local Gorenstein index, these are primitive integral vectors. Together with the fact that $\iota^\pm$ divides $\mathrm{cl}(X, x^\pm)$, we obtain

$$2\iota^+ = y^+ a, \qquad a = z^+\iota^+,$$

$$2\iota^- = -y^-(b+c+d), \qquad -(b+c+d) = z^-\iota^-$$

with positive integers $y^\pm$ and $z^\pm$. We conclude $y^+z^+ = 2$ and $y^-z^- = 2$. The possible constellations of (y^+, y^-) yield the following four cases:

Case 1.1: $a = \iota^+$, $b = -\iota^- - c - d$. Inserting this, we see that P arises from $S_{11}(3, \iota)$ and its entries satisfy the required estimates. Moreover, $u^\pm$ become

$$u^+ = \left[\iota^+,\, \iota^+,\, 2\right], \qquad u^- = \left[2c + \iota^-,\, 2d + \iota^-,\, -2\right].$$

As these are integral primitive vectors, we see that ι^+ as well as ι^- are odd and that $\iota^\pm$ are indeed the local Gorenstein indices of $x^\pm$.

Case 1.2: $a = \iota^+$, $b = -2\iota^- - c - d$. As in the previous subcase, inserting shows that P stems from $S_{12}(3, \iota)$. Note that this time we have

$$u^+ = \left[\iota^+,\, \iota^+,\, 2\right], \qquad u^- = \left[c + \iota^-,\, d + \iota^-,\, -1\right].$$

Thus, ι^+ is odd and we have no divisibility condition on ι^-. As before, we obtain that $\iota^\pm$ are indeed the local Gorenstein indices of $x^\pm$.

Case 2.1: $a = 2\iota^+$, $b = -\iota^- - c - d$. Inserting shows that P is given by $S_{21}(3, \iota)$ and its entries satisfy the required estimates. Moreover, we have

$$u^+ = \left[\iota^+,\, \iota^+,\, 1\right], \qquad u^- = \left[2c + \iota^-,\, 2d + 2\iota^+ + \iota^-,\, -2\right].$$

These must be integral primitive vectors. Consequently, ι^- is odd and we obtain that $\iota^\pm$ are indeed the local Gorenstein indices of $x^\pm$.

Case 2.2: $a = 2\iota^+$, $b = -2\iota^- - c - d$. Inserting shows that the matrix P arises from $S_{22}(3, \iota)$. Moreover, the linear forms $u^\pm$ are given by

$$u^+ = \left[\iota^+,\, \iota^+,\, 1\right], \qquad u^- = \left[c + \iota^-,\, d + \iota^+ + \iota^-,\, -1\right].$$

Thus, there are no divisibility conditions on $\iota^\pm$ and we see that $\iota^\pm$ are indeed the local Gorenstein indices of $x^\pm$.

We showed that every full intrinsic quadric surface of Picard number three is isomorphic to some $X(P)$ with P as in the assertion. Moreover, as the points $x_0, x_1, x_2 \in X(P)$ are all of local Gorenstein index one, see [13, Prop. 8.9 (iii)], we obtain that $X(P)$ has Gorenstein index $\iota = \mathrm{lcm}(\iota^+, \iota^-)$. Conversely, one directly checks that every matrix P from the assertion defines a full intrinsic quadric surface of Picard number three and Gorenstein index $\iota = \mathrm{lcm}(\iota^+, \iota^-)$.

Finally, we want to see that the matrices P listed in the assertion define pairwise non-isomorphic $X(P)$. According to Proposition 5.4, this amounts to showing that the sets $S_{ij}(3,\iota)$ are pairwise disjoint. We use Proposition 5.3 to compare the local Gorenstein indices $\iota^{\pm}$ and the local class group orders $\mathrm{cl}(X, x^{\pm})$:

	$S_{11}(3,\iota)$	$S_{12}(3,\iota)$	$S_{21}(3,\iota)$	$S_{22}(3,\iota)$
$(\iota^+, \mathrm{cl}(X,x^+))$	(ι^+,ι^+)	(ι^+,ι^+)	$(\iota^+,2\iota^+)$	$(\iota^+,2\iota^+)$
$(\iota^-, \mathrm{cl}(X,x^-))$	(ι^-,ι^-)	$(\iota^-,2\iota^-)$	(ι^-,ι^-)	$(\iota^-,2\iota^-)$

The listed pairs are invariants of the surface up to switching x^+ and x^-. Thus, we see that $S_{11}(3,\iota)$ as well as $S_{22}(3,\iota)$ has trivial intersection with any other $S_{ij}(3,\iota)$. For $S_{12}(3,\iota)$, observe $\iota^+ < 2\iota^-$ as ι^+ is odd. Similarly, for $S_{21}(3,\iota)$, we have $2\iota^+ < \iota^-$. Thus, in both cases, $\mathrm{cl}(X,x^+)$ is the strictly smallest of $\mathrm{cl}(X,x^{\pm})$. It follows that $S_{12}(3,\iota)$ and $S_{21}(3,\iota)$ intersect trivially. □

Example 5.6. Consider the full intrinsic quadric surfaces X and X' of Picard number three given by the defining matrices

$$P = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 & 0 & -1 \end{bmatrix}, \quad P' = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 2 & 0 & 0 & -1 & 0 & -1 \end{bmatrix}.$$

Then X stems from the series $S_{12}(3,\iota)$ and X' from $S_{22}(3,\iota)$. Theorem 5.5 yields that X and X' are the only Gorenstein full intrinsic quadric surfaces with $\rho(X) = 3$.

Proposition 5.7. *Let $X = X(P)$ arise from Construction* 5.1. *At most the prime divisors $D_1, \dots, D_6 \subseteq X$ are contractible and all possible contractions are projective toric surfaces of Picard number two. More precisely,*

$$D_1:\ b \ge 1 \qquad D_2:\ a+c+d \le -1 \qquad D_3:\ a+c \ge 1$$

$$\begin{bmatrix} -1 & -1 & 1 & 1 \\ b & b+c & 0 & d \end{bmatrix} \qquad \begin{bmatrix} -1 & -1 & 1 & 1 \\ a & a+c & 0 & d \end{bmatrix} \qquad \begin{bmatrix} -1 & -1 & 1 & 1 \\ a+c & b+c & 0 & d \end{bmatrix}$$

$$D_4:\ b+d \le -1 \qquad D_5:\ a+d \ge 1 \qquad D_6:\ b+c \le -1$$

$$\begin{bmatrix} -1 & -1 & 1 & 1 \\ a & b & 0 & d \end{bmatrix} \qquad \begin{bmatrix} -1 & -1 & 1 & 1 \\ a+d & b+d & 0 & c \end{bmatrix} \qquad \begin{bmatrix} -1 & -1 & 1 & 1 \\ a & b & 0 & c \end{bmatrix}$$

gives us for each D_i the characterizing property of contractibility in terms of the entries a, b, c, d of P and, for the case that D_i is contractible, also the generator matrix of the contracted surface.

Proof. One succeeds by the same arguments as in the proof of Proposition 4.7. □

A normal surface singularity is of *type A_n* if the exceptional divisor of its minimal resolution is a string of n smooth rational curves, each of self intersection -2.

Remark 5.8. Consider the two Gorenstein full intrinsic quadric surfaces X and X' of Picard number three from Example 5.6.

(1) On X, the contractible divisors are D_2, D_4 and D_6. In each case, the contracted surface is $\mathbb{P}^1 \times \mathbb{P}^1$.
(2) On X', the contractible divisors are D_3, D_4, D_5 and D_6. In each case, the contraction is the toric del Pezzo surface of Picard number 2 with two singularities, both of type A_1.

6. Geometry of full intrinsic quadric surfaces

We present direct applications of Theorems 3.5, 4.5 and 5.5, exploring the geometry of full intrinsic quadric surfaces. In Corollary 6.4, we determine the weighted resolution graphs for the canonical resolution of singularities. Moreover, Corollaries 6.3, 6.6 and 6.8 give explicit upper and lower bounds on the degree, the log canonicity and the Picard index in terms of the Gorenstein index. Finally, Corollary 6.9 characterizes the existence of Kähler-Einstein metrics in terms of the Gorenstein index.

First, recall that a *del Pezzo surface* is a normal projective surface X admitting an ample anticanonical divisor $-\mathcal{K}_X$. Moreover, a del Pezzo surface X is *log terminal* if all the exceptional divisors of its minimal resolution of singularities have discrepancies strictly bigger than -1; if so then one refers to X also as a *log Pezzo surface*.

Proposition 6.1. *Every full intrinsic quadric surface X is a log del Pezzo surface.*

Proof. We may assume $X = X(P)$. Then log terminality is a direct consequence of [13, Cor. 8.12]. According to the possible values of the Picard number $\rho = \rho(X)$, the degree $\mu \in \mathrm{Cl}(X)$ of the defining quadric of X is given as

$$\mu \;=\; \begin{cases} w_1 + w_2 = 2w_3 = 2w_4, & \rho = 1, \\ w_1 + w_2 = w_3 + w_4 = 2w_5, & \rho = 2, \\ w_1 + w_2 = w_3 + w_4 = w_5 + w_6, & \rho = 3, \end{cases}$$

where $w_i = \deg(T_i) \in \mathrm{Cl}(X)$. Due to [3, Prop. 3.3.3.2], the anticanonical class of X equals $w_1 + \ldots + w_{\rho+3} - \mu$ and thus is a positive multiple of μ. From [3, Prop. 3.3.2.9] we infer that the cone of movable divisor classes of X is given by

$$\mathrm{Mov}(X) \;=\; \bigcap_{i=1}^{\varrho+3} \tau_i, \qquad \tau_i \;:=\; \mathrm{cone}(w_j;\ j \neq i) \;\subseteq\; \mathrm{Cl}_{\mathbb{Q}}(X).$$

Observe that μ is an interior point of each τ_i. As all involved cones are of full dimension, we obtain that μ is an interior point of $\mathrm{Mov}(X)$. Thus, [3, Prop. 3.3.2.9, Thm. 4.3.3.5] show that μ, and hence the anticanonical class of X, is ample. $\square$

Remark 6.2. The surfaces from Examples 3.6, 4.6, 5.6 are the only Gorenstein two-dimensional full intrinsic quadrics. By Proposition 6.1 they are all log del Pezzo and thus we recover them as well as the only full intrinsic quadrics in the classification of all rational Gorenstein log del Pezzo $\mathbb{K}^*$-surfaces given in [3, Thms. 5.4.4.2 to 5.4.4.5].

The *(anticanonical) degree* of a del Pezzo surface is the self intersection number $\mathcal{K}_X^2$ of an anticanonical divisor of X. For the full intrinsic quadric surfaces, we obtain the following relations between the degree and the local Gorenstein indices.

Corollary 6.3. *Consider a full intrinsic quadric surface $X = X(P_\eta)$ with P_η as in Theorem 3.5, 4.5 or 5.5. Then the degree $\mathcal{K}_X^2$ of X is given as*

$$
\begin{array}{lllll}
\rho = 1: & \mathcal{K}_X^2 = \frac{1}{\iota^+} + \frac{1}{\iota^-}, & \eta \in S_{11}(1,\iota), & \mathcal{K}_X^2 = \frac{1}{\iota^+} + \frac{2}{\iota^-}, & \eta \in S_{12}(1,\iota),\\
& \mathcal{K}_X^2 = \frac{2}{\iota^+} + \frac{1}{\iota^-}, & \eta \in S_{21}(1,\iota), & \mathcal{K}_X^2 = \frac{2}{\iota^+} + \frac{2}{\iota^-}, & \eta \in S_{22}(1,\iota),\\
\rho = 2: & \mathcal{K}_X^2 = \frac{9}{2\iota^+} + \frac{9}{2\iota^-}, & \eta \in S_{11}(2,\iota), & \mathcal{K}_X^2 = \frac{9}{2\iota^+} + \frac{3}{2\iota^-}, & \eta \in S_{12}(2,\iota),\\
& \mathcal{K}_X^2 = \frac{3}{2\iota^+} + \frac{9}{2\iota^-}, & \eta \in S_{21}(2,\iota), & \mathcal{K}_X^2 = \frac{3}{2\iota^+} + \frac{3}{2\iota^-}, & \eta \in S_{22}(2,\iota),\\
\rho = 3: & \mathcal{K}_X^2 = \frac{4}{\iota^+} + \frac{4}{\iota^-}, & \eta \in S_{11}(3,\iota), & \mathcal{K}_X^2 = \frac{4}{\iota^+} + \frac{2}{\iota^-}, & \eta \in S_{12}(3,\iota),\\
& \mathcal{K}_X^2 = \frac{2}{\iota^+} + \frac{4}{\iota^-}, & \eta \in S_{21}(3,\iota), & \mathcal{K}_X^2 = \frac{2}{\iota^+} + \frac{2}{\iota^-}, & \eta \in S_{22}(3,\iota).
\end{array}
$$

Here, ρ is the Picard number, ι the Gorenstein index of X and $\iota^\pm$ the local Gorenstein index of $x^\pm \in X$. Moreover, we obtain the following upper and lower bounds:

$$
\begin{array}{ll}
\rho = 1: & \frac{2}{\iota} \le \mathcal{K}_X^2 \le 1 + \frac{4}{\iota},\\
\rho = 2: & \frac{3}{\iota} \le \mathcal{K}_X^2 \le \frac{9}{2} + \frac{9}{2\iota},\\
\rho = 3: & \frac{4}{\iota} \le \mathcal{K}_X^2 \le 4 + \frac{4}{\iota}.
\end{array}
$$

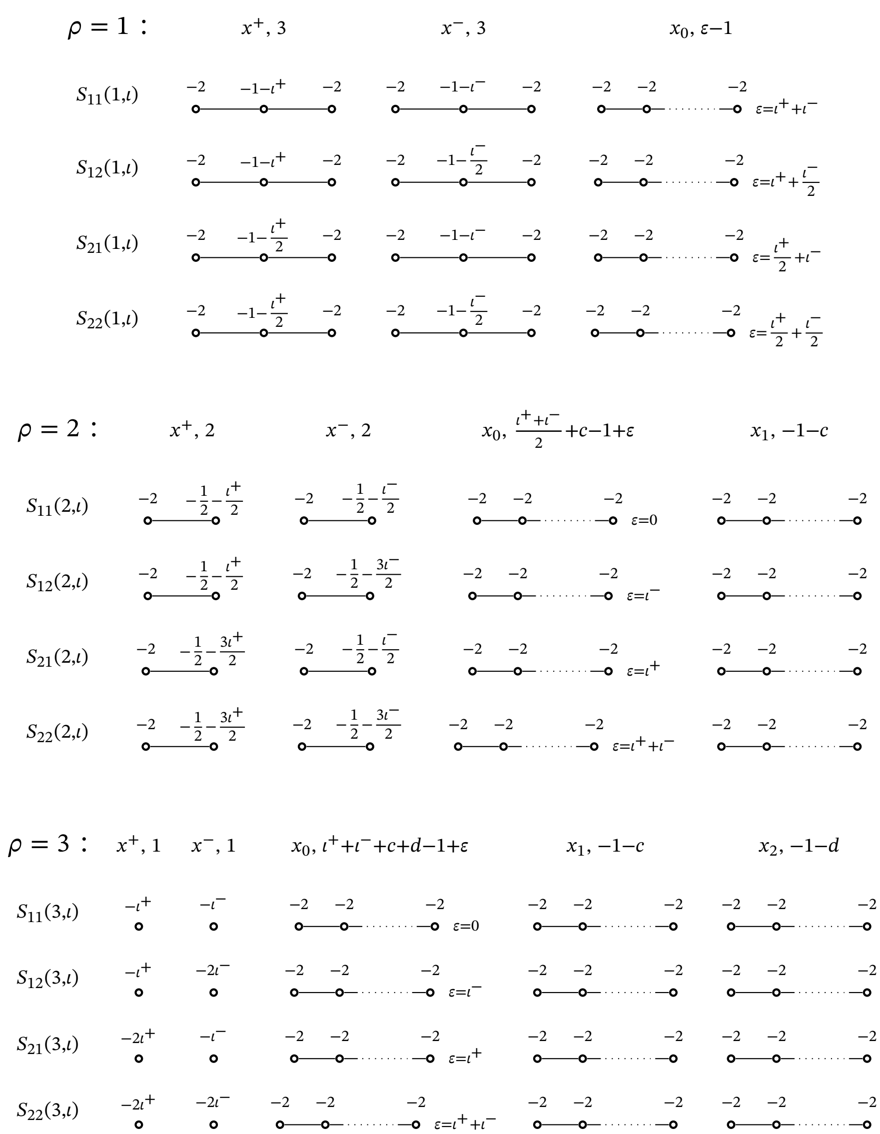
Proof. First, assume $X = X(P)$ with P from Construction 3.1, 4.1 or 5.1. Taking any representative of the anticanonical class as listed in the proof of the preceding proposition, we use the intersection numbers provided by [13, Summary 7.7], and compute according to the cases $\rho = 1, 2, 3$:

$$
\mathcal{K}_X^2 = \frac{1}{a+1} - \frac{1}{b+1}, \qquad \mathcal{K}_X^2 = \frac{9}{4a+2} - \frac{9}{2+4b+4c}, \qquad \mathcal{K}_X^2 = \frac{4}{a} - \frac{4}{b+c+d}.
$$

Inserting the values for a, b, c, d from Theorems 3.5, 4.5 and 5.5 accordingly, we obtain the desired presentations of the anticanonical self intersection number. The estimates are then directly verified. □

We turn to the singularities of full intrinsic quadrics and consider the canonical resolution of singularities in the sense of [25]; see also [3, Sec. 5.4.3].

Corollary 6.4. *Consider a full intrinsic quadric surface $X = X(P_\eta)$ with P_η as in Theorem 3.5, 4.5 or 5.5 and its canonical resolution of singularities. Then the possible singularities $x^+, x^-, x_0, x_1, x_2 \in X$ have the following resolution graphs:*



The numbers after x^+, x^-, x_0, x_1, x_2 count their exceptional curves and the weights of the vertices are the corresponding self intersection numbers. The canonical resolution is minimal unless $x^+ \in X$ is smooth; the latter happens if and only if

$$\iota^+ = 1,\ \eta \in S_{11}(2,\iota) \cup S_{12}(2,\iota) \cup S_{11}(3,\iota) \cup S_{12}(3,\iota).$$

Proof. For $X = X(P)$ with P as in Construction 3.1, 4.1 or 5.1, we use [3, Sec. 5.4.3] to determine the canonical resolution of singularities of X; see also [13, Summary 8.2]. Then we compute the self intersection numbers of the exceptional divisors according to [3, Sec. 5.4.2]; see also [13, Summary 7.7] and insert the values of a, b, c, d from Theorems 3.5, 4.5 and 5.5. $\square$

Remark 6.5. Corollary 6.4 tells us in particular the following about the singularities of the Gorenstein full intrinsic quadric surfaces from Example 3.6, 4.6, and 5.6.

(1) On X from Example 3.6, the points x^+, x^- are singularities of type A_3 and x_0 is a singularity of type A_1.
(2) On X from Example 4.6, the points x^+, x_0, x_1 are smooth and x^- is a singularity of type A_2.
(3) On X' from Example 4.6, the points x^+, x^- are singularities of type A_2, the point x_1 is a singularity of type A_1 and x_0 is smooth.
(4) On X from Example 5.6, the points x^- is a singularity of type A_1 and x^+, x_0, x_1, x_2 are smooth.
(5) On X' from Example 5.6, the points x^+, x^-, x_0 are singularities of type A_1 and x_1, x_2 are smooth.

The singularity types of the Gorenstein log del Pezzo surfaces are well known and we find those of our examples just discussed also in classification results, as for instance [3, Thms. 5.4.4.2 to 5.4.4.5].

By the *log canonicity* of a log terminal projective surface X, we mean the number $\varepsilon_X := a_E + 1$, where a_E is the minimal possible discrepancy appearing among the exceptional divisors $E \subseteq \tilde{X}$ of its minimal resolution of singularities. Note that $1/\iota_X$ is bounded by the log canonicity. Alexeev's results [1] show in particular that bounding the log canonicity gives finiteness for log del Pezzo surfaces.

Corollary 6.6. *Consider a full intrinsic quadric surface $X = X(P_\eta)$ with P_η as in Theorem 3.5, 4.5 or 5.5. Then the log canonicity ε_X of X is given by*

$$\begin{aligned}
\rho = 1: &\quad \varepsilon_X = \tfrac{1}{\iota^-},\ \eta \in S_{11}(1,\iota) \cup S_{21}(1,\iota), &\quad \varepsilon_X = \tfrac{2}{\iota^-},\ \eta \in S_{12}(1,\iota) \cup S_{22}(1,\iota),\\
\rho = 2: &\quad \varepsilon_X = \tfrac{3}{\iota^-},\ \eta \in S_{11}(2,\iota) \cup S_{21}(2,\iota), &\quad \varepsilon_X = \tfrac{1}{\iota^-},\ \eta \in S_{12}(2,\iota) \cup S_{22}(2,\iota),\\
\rho = 3: &\quad \varepsilon_X = \tfrac{2}{\iota^-},\ \eta \in S_{11}(3,\iota) \cup S_{21}(3,\iota), &\quad \varepsilon_X = \tfrac{1}{\iota^-},\ \eta \in S_{12}(3,\iota) \cup S_{22}(3,\iota).
\end{aligned}$$

In particular, we obtain the following upper and lower bounds for the log canonicity ε_X of X:

$$\rho = 1:\ \frac{1}{\iota} \le \varepsilon_X \le \frac{2}{\sqrt{\iota}}, \quad \rho = 2:\ \frac{1}{\iota} \le \varepsilon_X \le \frac{3}{\sqrt{\iota}}, \quad \rho = 3:\ \frac{1}{\iota} \le \varepsilon_X \le \frac{2}{\sqrt{\iota}}.$$

In the proof, we make use of the *anticanonical complex* $\mathcal{A}_X$ introduced in [4] for varieties X with a torus action of complexity one. This is a polyhedral complex supported on the *tropical variety*, which in the case of full intrinsic quadric surface $X = X(P) \subseteq Z$ is given by

$$\operatorname{trop}(X) \ = \ \tau_0 \cup \tau_1 \cup \tau_2, \qquad \tau_i \ := \ \mathbb{Q}_{\ge 0} \cdot e_i + \mathbb{Q} \cdot e_3,$$

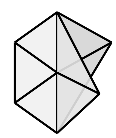
where $e_1, e_2, e_3 \in \mathbb{Q}^3$ are the canonical basis vectors and $e_0 = -e_1 - e_2$. The anticanonical complex $\mathcal{A}_X$ is bounded if and only if X is log terminal and in this case, the discrepancy of a divisor E on the minimal resolution of X is given as

$$a_E = \frac{||v_\varrho||}{||v'_\varrho||} - 1,$$

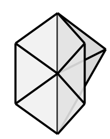
where $\varrho \subseteq \mathbb{Q}^3$ is the ray corresponding to E the vector v_ϱ is the primitive lattice vector in ϱ and v'_ϱ is the intersection point of ϱ and the boundary of $\mathcal{A}_X$. We refer to [13, Sec. 9] for more background.

Proof. For $X = X(P)$ with P as in Construction 3.1, 4.1 or 5.1, we use the anticanonical complex $\mathcal{A}_X$ to determine the minimal discrepancies. According to [13, Thm. 9.17 (i) and (ii)], the maximal cells of $\mathcal{A}_X$ are given in terms of the columns v_i of the matrix P as

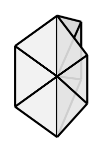
$$\begin{aligned}\rho = 1: \quad & \tilde{v}^+ = (a+1)e_3,\ \tilde{v}^- = (b+1)e_3,\\ & \operatorname{conv}(0, \tilde{v}^+, v_1),\ \operatorname{conv}(0, v_1, v_2),\ \operatorname{conv}(0, v_2, \tilde{v}^-),\\ & \operatorname{conv}(0, \tilde{v}^+, v_3),\ \operatorname{conv}(0, v_3, \tilde{v}^-),\\ & \operatorname{conv}(0, \tilde{v}^+, v_4),\ \operatorname{conv}(0, v_4, \tilde{v}^-),\end{aligned}$$



$$\begin{aligned}\rho = 2: \quad & \tilde{v}^+ = \tfrac{2a+1}{3}e_3,\ \tilde{v}^- = \tfrac{2b+2c+1}{3}e_3,\\ & \operatorname{conv}(0, \tilde{v}^+, v_1),\ \operatorname{conv}(0, v_1, v_2),\ \operatorname{conv}(0, v_2, \tilde{v}^-),\\ & \operatorname{conv}(0, \tilde{v}^+, v_3),\ \operatorname{conv}(0, v_3, v_4),\ \operatorname{conv}(0, v_4, \tilde{v}^-),\\ & \operatorname{conv}(0, \tilde{v}^+, v_5),\ \operatorname{conv}(0, v_5, \tilde{v}^-),\end{aligned}$$



$$\begin{aligned}\rho = 3: \quad & \tilde{v}^+ = \tfrac{a}{2}e_3,\ \tilde{v}^- = \tfrac{b+c+d}{2}e_3,\\ & \operatorname{conv}(0, \tilde{v}^+, v_1),\ \operatorname{conv}(0, v_1, v_2),\ \operatorname{conv}(0, v_2, \tilde{v}^-),\\ & \operatorname{conv}(0, \tilde{v}^+, v_3),\ \operatorname{conv}(0, v_3, v_4),\ \operatorname{conv}(0, v_4, \tilde{v}^-),\\ & \operatorname{conv}(0, \tilde{v}^+, v_5),\ \operatorname{conv}(0, v_5, v_6),\ \operatorname{conv}(0, v_6, \tilde{v}^-).\end{aligned}$$



Now [13, Thm. 9.17 (iii)] tells us that the discrepancies of the exceptional divisors E^+, E^- corresponding to the rays through $e_3, -e_3$ are given for $\rho = 1, 2, 3$ by

$$\frac{1}{a+1} - 1,\ \frac{1}{-b-1} - 1, \qquad \frac{3}{2a+1} - 1,\ -\frac{3}{2b+2c+1} - 1, \qquad \frac{2}{a} - 1,\ -\frac{2}{b+c+d} - 1.$$

Moreover, these are obviously the minimal discrepancies of the canonical resolution. Inserting the values of a, b, c, d from Theorems 3.5, 4.5 and 5.5, we arrive at the assertion. □

Remark 6.7. By Corollary 6.6, the surfaces X from Examples 3.6, 4.6 and 5.6 are all of log canonicity $\varepsilon_X = 1$ in accordance with the fact that they are Gorenstein.

The *Picard index* $\mathfrak{p}_X$ of a normal variety X is the index $[\operatorname{Cl}(X) : \operatorname{Pic}(X)]$ of its Picard group in its divisor class group. Note that the Gorenstein index always divides the Picard index. Bounding the Picard index yields finiteness for del Pezzo surfaces of Picard number one with torus action [26]; see also [17] for a higher dimensional analogue in the special case of divisor class group $\mathbb{Z}$.

Corollary 6.8. *Consider a full intrinsic quadric surface $X = X(P_\eta)$ with P_η as in Theorem 3.5, 4.5 or 5.5. Then, according to the Picard number $\rho = \rho(X)$, the Picard index $\mathfrak{p} = \mathfrak{p}_X$ of X is given by*

$$\rho = 1:$$
$$\mathfrak{p} = \frac{8\iota^+\iota^-(\iota^++\iota^-)}{\gcd(2\iota^+, \iota^++\iota^-)}, \quad \eta \in S_{11}(1,\iota), \qquad \mathfrak{p} = \frac{4\iota^+\iota^-(2\iota^++\iota^-)}{\gcd(4\iota^+, 2\iota^++\iota^-)}, \quad \eta \in S_{12}(1,\iota),$$
$$\mathfrak{p} = \frac{4\iota^+\iota^-(\iota^++2\iota^-)}{\gcd(2\iota^+, \iota^++2\iota^-)}, \quad \eta \in S_{21}(1,\iota), \qquad \mathfrak{p} = \frac{2\iota^+\iota^-(\iota^++\iota^-)}{\gcd(2\iota^+, \iota^++\iota^-)}, \quad \eta \in S_{22}(1,\iota),$$

$$\rho = 2:$$
$$\mathfrak{p} = -\frac{c\iota^+\iota^-(\iota^++\iota^-+2c)}{\gcd(2\iota^+, \iota^++\iota^-, 2c)}, \quad \eta \in S_{11}(2,\iota), \qquad \mathfrak{p} = -\frac{3c\iota^+\iota^-(\iota^++3\iota^-+2c)}{\gcd(2\iota^+, \iota^++3\iota^-, 2c)}, \quad \eta \in S_{12}(2,\iota),$$
$$\mathfrak{p} = -\frac{3c\iota^+\iota^-(3\iota^++\iota^-+2c)}{\gcd(6\iota^+, 3\iota^++\iota^-, 2c)}, \quad \eta \in S_{21}(2,\iota), \qquad \mathfrak{p} = -\frac{9c\iota^+\iota^-(3\iota^++3\iota^-+2c)}{\gcd(6\iota^+, 3\iota^++3\iota^-, 2c)}, \quad \eta \in S_{22}(2,\iota),$$

$$\rho = 3:$$
$$\mathfrak{p} = \frac{cd\iota^+\iota^-(\iota^++\iota^-+c+d)}{\gcd(\iota^+, \iota^-, c, d)}, \quad \eta \in S_{11}(3,\iota), \qquad \mathfrak{p} = \frac{2cd\iota^+\iota^-(\iota^++2\iota^-+c+d)}{\gcd(\iota^+, 2\iota^-, c, d)}, \quad \eta \in S_{12}(3,\iota),$$
$$\mathfrak{p} = \frac{2cd\iota^+\iota^-(2\iota^++\iota^-+c+d)}{\gcd(2\iota^+, \iota^-, c, d)}, \quad \eta \in S_{21}(3,\iota), \qquad \mathfrak{p} = \frac{4cd\iota^+\iota^-(2\iota^++2\iota^-+c+d)}{\gcd(2\iota^+, 2\iota^-, c, d)}, \quad \eta \in S_{22}(3,\iota),$$

where ι is the Gorenstein index of X and $\iota^\pm$ the local Gorenstein index of $x^\pm \in X$ and $-c$ the local class group order of $x_1 \in X$. In particular, we obtain the following upper and lower bounds:

$$\rho = 1: \quad \iota \le \mathfrak{p} \le 8\iota^2,$$
$$\rho = 2: \quad \iota \le \mathfrak{p} \le \frac{27}{2}\iota^3(3\iota - 1),$$
$$\rho = 3: \quad \iota \le \mathfrak{p} \le \frac{32}{3}\iota^3(2\iota - 1)^2.$$

Proof. First, assume $X = X(P)$ with P from Construction 3.1, 4.1 or 5.1. Then Springer's formula [26, Thm. 1.1] gives us the Picard indices

$$-\frac{8(a+1)(b+1)(a-b)}{\gcd(2a+2, a-b)}, \ \frac{c(1+2a)(1+2b+2c)(a-b)}{\gcd(1+2a, a-b, c)}, \ -\frac{acd(b+c+d)(a-b)}{\gcd(a,b,c,d)},$$

according to the possible Picard numbers $\rho = 1, 2, 3$. The assertion is obtained by inserting the values of a, b, c, d from Theorems 3.5, 4.5 and 5.5. □

A *Kähler-Einstein metric* on a rational projective del Pezzo surface is a Kähler orbifold metric g such that the associated Kähler form ω_g equals its Ricci form $\mathrm{Ric}(\omega_g)$. The smooth del Pezzo surfaces with a Kähler-Einstein metric are $\mathbb{P}^2$, its blowing up in $k = 3, \dots, 8$ points in general position and $\mathbb{P}^1 \times \mathbb{P}^1$; see [28, 27]. The case of quasismooth del Pezzo surfaces coming anticanonically embedded into a three-dimensional weighted projective space is understood as well; see [24, 2, 7, 8]. We settle the case of full intrinsic quadric surfaces.

Corollary 6.9. *Let X be a complex full intrinsic quadric surface admitting a Kähler-Einstein metric. Then $X \cong X(P)$ for precisely one P from the following:*

$\rho = 1,\ 2 \nmid \iota$:

$$\begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ \iota-1 & -\iota-1 & 1 & 1 \end{bmatrix},$$

$\rho = 1,\ 4 \mid \iota$:

$$\begin{bmatrix} -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \\ \frac{\iota}{2}-1 & -\frac{\iota}{2}-1 & 1 & 1 \end{bmatrix},$$

$\rho = 3,\quad 2 \nmid \iota,\quad -2\iota \le 2c+d,$
$c \le d \le -1,\ c+d \le -\iota-1$:

$$\begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ \iota & -\iota-c-d & 0 & c & 0 & d \end{bmatrix},$$

$\rho = 3,\quad -4\iota \le 2c+d,\ c \le d \le -1,$
$c+d \le -2\iota-1$:

$$\begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 2\iota & -2\iota-c-d & 0 & c & 0 & d \end{bmatrix},$$

where ρ denotes the Picard number and ι the Gorenstein index of $X(P)$. Conversely, each $X(P)$ with P from the above list admits a Kähler-Einstein metric.

We will verify existence or non-existence of Kähler-Einstein metrics via K-stability. Let us give an idea of the approach; we refer to [15] for the complete background. For a full intrinsic quadric $X = X(P) \subseteq Z$ arising from Construction 3.1, 4.1 or 5.1, consider the varieties $\mathcal{X}_\kappa \subseteq Z \times \mathbb{C}$, $\kappa = 0, 1, 2$, given by the equations

	$\kappa = 0$	$\kappa = 1$	$\kappa = 2$
$\rho = 1$:	$ST_1T_2+T_3^2+T_4^2$	$T_1T_2+ST_3^2+T_4^2$	$T_1T_2+T_3^2+ST_4^2$
$\rho = 2$:	$ST_1T_2+T_3T_4+T_5^2$	$T_1T_2+ST_3T_4+T_5^2$	$T_1T_2+T_3T_4+ST_5^2$
$\rho = 3$:	$ST_1T_2+T_3T_4+T_5T_6$	$T_1T_2+ST_3T_4+T_5T_6$	$T_1T_2+T_3T_4+ST_5T_6$

where $T_1, \dots, T_4$ are the homogeneous coordinates on Z and S is the standard coordinate on $\mathbb{C}$. The projection $Z \times \mathbb{C} \to \mathbb{C}$ induces a flat family $\mathcal{X}_\kappa \to \mathbb{C}$. The fiber $\mathcal{X}_{\kappa,1}$ over $1 \in \mathbb{C}$ is our full intrinsic quadric surface and the fiber $\mathcal{X}_{\kappa,0}$ over $0 \in \mathbb{C}$, given by a binomial equation, is a toric surface. Consider the sublattices

$$N_\kappa \ := \ \mathbb{Z} \cdot e_\kappa + \mathbb{Z} \cdot e_3, \qquad \kappa = 0, 1, 2, \quad e_0 := -e_1 - e_2.$$

The fan Δ_κ associated with the toric degeneration $\mathcal{X}_{\kappa,0}$ of X is obtained by restricting the fan of $Z \times \mathbb{C}$ to N_κ. It turns out that $\mathcal{X}_{\kappa,0}$ is normal in all cases except $(\rho, \kappa) = (1, 0)$, where Δ_0 describes the normalization. The families $\mathcal{X}_\kappa \to \mathbb{C}$ are so-called *equivariant test configurations* and [15] provides a combinatorial K-stability criterion, characterizing existence of Kähler-Einstein metrics in terms of the barycenters b_κ of the moment polytopes $\mathcal{B}_\kappa$ associated with the toric degenerations $\mathcal{X}_{\kappa,0}$.

Proof of Corollary 6.9. We may assume $X = X(P)$ with P as in Construction 3.1, 4.1 or 5.1. This allows us to use the combinatorial K-stability criterion for existence of Kähler-Einstein metrics on $X(P)$ provided by [15]; see also [23]. We consider the test configurations $\mathcal{X}_\kappa \to \mathbb{C}$ of $X(P)$, where $\kappa = 0, 1, 2$, provided by [15, Constr. 4.1, Prop. 5.3] and use [15, Prop. 5.3] to figure out the special ones, i.e. those with normal central fiber $\mathcal{X}_{\varkappa,0}$. With the aid of [15, Prop. 5.6], we compute the associated moment polytope $\mathcal{B}_\varkappa$, which in case of special test configuration is just the dual of the Fano polytope $\mathcal{A}_\varkappa$ of $\mathcal{X}_{\varkappa,0}$. Then we determine the barycenter $b_\varkappa \in \mathcal{B}_\varkappa$. Finally, [15, Thm. 6.2] tells us that $X(P)$ admits a Kähler-Einstein metric if and only if the coordinates $b_{\varkappa,j}$ of the barycenter $b_\varkappa$ satisfy $b_{\varkappa,1} = 0$ for all $\varkappa$ and $b_{\varkappa,2} > 0$ for all special $\varkappa$.

For Picard number $\rho = 1$, consider $X = X(P)$ with P as in Construction 3.1. In this setting, we obtain a non-special toric degeneration for $\varkappa = 0$, where we compute

$$\begin{aligned} \mathcal{B}_0 &= \operatorname{conv}\left((0,0),\, (-\frac{1}{1+b}, -\frac{b}{1+b}),\, (-\frac{1}{1+a}, -\frac{a}{1+a})\right), \\ b_0 &= \left(-\frac{2+a+b}{3(1+a)(1+b)}, -\frac{a+b+2ab}{3(1+a)(1+b)}\right). \end{aligned}$$

Thus, $b_{0,1}$ vanishes if and only if $b = -2-a$. Moreover, we obtain special toric degenerations for $\varkappa = 1, 2$. There we compute for both cases

$$\begin{aligned} \mathcal{A}_\varkappa &= \operatorname{conv}\left((1,-2),\, (1+2a, 2),\, (1+2b, 2)\right), \\ \mathcal{B}_\varkappa &= \operatorname{conv}\left((0, -\tfrac{1}{2}),\, (-\tfrac{1}{1+b}, \tfrac{b}{2+2b}),\, (-\tfrac{1}{1+a}, \tfrac{a}{2+2a})\right), \\ b_\varkappa &= \left(-\tfrac{2+a+b}{3(1+a)(1+b)}, \tfrac{ab-1}{6(1+a)(1+b)}\right). \end{aligned}$$

Also here, $b_{\varkappa,1} = 0$ if and only if $b = -2-a$. In this case, we have $b_{\varkappa,2} = 1/6 > 0$. Comparing with Theorem 3.5, we arrive at the shapes given by $S_{11}(1,\iota)$ and $S_{22}(1,\iota)$ with $\iota^+ = \iota^-$, where $S_{12}(1,\iota)$, $S_{21}(1,\iota)$ are ruled out by ι^+, ι^- being odd.

For Picard number $\rho = 2$, take P as in Construction 4.1. Then $\varkappa = 2$ yields a special degeneration and we end up with barycenter $b_2 = (0,0)$, as soon as $b_{2,1} = 0$. Thus, none of the $X(P)$ admits a Kähler-Einstein metric. For completeness, we list the intermediate steps:

$$\begin{aligned} \mathcal{A}_2 &= \operatorname{conv}\left((1,-2),\, (a,1),\, (b+c,1)\right), \\ \mathcal{B}_2 &= \operatorname{conv}\left((-\tfrac{3}{2a+1}, \tfrac{a-1}{2a+1}),\, (0,-1),\, (-\tfrac{3}{2b+2c+1}, \tfrac{b+c-1}{2b+2c+1})\right), \\ b_2 &= \left(-2\tfrac{a+b+c+1}{(2a+1)(2b+2c+1)}, -\tfrac{a+b+c+1}{(2a+1)(2b+2c+1)}\right). \end{aligned}$$

We turn to Picard number $\rho = 3$. Let $X(P)$ arise from Construction 5.1. Then we have special toric degenerations for $\varkappa = 0, 1, 2$. The computation results are

$$
\begin{aligned}
\mathcal{A}_0 &= \operatorname{conv}\left((0,1),\ (c+d,1),\ (b,-1),\ (a,-1)\right),\\
\mathcal{B}_0 &= \operatorname{conv}\left((0,-1),\ \left(-\tfrac{2}{b+c+d}, \tfrac{c+d-b}{b+c+d}\right),\ (0,1),\ \left(-\tfrac{2}{a},-1\right)\right),\\
b_0 &= \left(-\tfrac{2(a+b+c+d)}{3a(b+c+d)}, \tfrac{(b+2c+2d-a)b+(c+a+d)(c+d))}{3(a-b-c-d)(b+c+d)}\right),\\
\mathcal{A}_1 &= \operatorname{conv}\left((a,1),\ (b+d,1),\ (c,-1),\ (0,-1)\right),\\
\mathcal{B}_1 &= \operatorname{conv}\left((0,-1),\ \left(-\tfrac{2}{b+c+d}, \tfrac{b+d-c}{b+c+d}\right),\ (0,1),\ \left(-\tfrac{2}{a},1\right)\right),\\
b_1 &= \left(-\tfrac{2(a+b+c+d)}{3a(b+c+d)}, \tfrac{(a-b-2c-2d)b-(a+c+2d)c+(a-d)d)}{3(a-b-c-d)(b+c+d)}\right),\\
\mathcal{A}_2 &= \operatorname{conv}\left((a,1),\ (b+c,1),\ (d,-1),\ (0,-1)\right),\\
\mathcal{B}_2 &= \operatorname{conv}\left((0,-1),\ \left(-\tfrac{2}{b+c+d}, \tfrac{b+c-d}{b+c+d}\right),\ (0,1),\ \left(-\tfrac{2}{a},1\right)\right),\\
b_2 &= \left(-\tfrac{2(a+b+c+d)}{3a(b+c+d)}, \tfrac{(a-b-2c-2d)b+(a-c-2d)c-(a+d)d}{3(a-b-c-d)(b+c+d)}\right).
\end{aligned}
$$

We conclude that $X(P)$ admits a Kähler-Einstein metric if and only if we have $d = -a-b-c$, reflecting $b_{\varkappa,1} = 0$, and

$$
b > 0, \quad b+d < 0, \quad a+d > 0,
$$

reflecting $b_{\varkappa,2} > 0$. Substituting a, b with the corresponding entries from Theorem 5.5, we arrive at

$$
\iota^+ = \iota^-, \qquad \iota^+ = 2\iota^-, \qquad 2\iota^+ = \iota^-, \qquad \iota^+ = \iota^-,
$$

according to the shapes defined by $S_{11}(3,\iota)$, $S_{12}(3,\iota)$, $S_{21}(3,\iota)$ and $S_{22}(3,\iota)$. Note that $S_{12}(3,\iota)$, $S_{21}(3,\iota)$ are ruled out by ι^+, ι^- being odd, respectively. □

Remark 6.10. Let $X = X(P)$ arise from Construction 3.1, 4.1 or 5.1. Set $\rho = \rho(X)$. Then, for $(\rho,\varkappa) \neq (1,0)$, the toric degeneration $\mathcal{X}_{\varkappa,0}$ is a normal projective toric del Pezzo surface and, according to the constellations $(\rho,\varkappa)$, the generator matrix of its defining complete fan Δ_κ is given by

(1,1) $\begin{bmatrix} 1 & 1{+}2a & 1{+}2b \\ -2 & 2 & 2 \end{bmatrix}$ (1,2) $\begin{bmatrix} 1 & 1{+}2a & 1{+}2b \\ -2 & 2 & 2 \end{bmatrix}$

(2,0) $\begin{bmatrix} a & b & 1 & 2c{+}1 \\ -1 & -1 & 2 & 2 \end{bmatrix}$ (2,1) $\begin{bmatrix} 0 & c & 2a{+}1 & 2b{+}1 \\ -1 & -1 & 2 & 2 \end{bmatrix}$ (2,2) $\begin{bmatrix} 1 & a & b{+}c \\ -2 & 1 & 1 \end{bmatrix}$

(3,0) $\begin{bmatrix} 0 & c{+}d & b & a \\ 1 & 1 & -1 & -1 \end{bmatrix}$ (3,1) $\begin{bmatrix} a & b{+}d & c & 0 \\ 1 & 1 & -1 & -1 \end{bmatrix}$ (3,2) $\begin{bmatrix} a & b{+}c & d & 0 \\ 1 & 1 & -1 & -1 \end{bmatrix}$

The above presentation of the primitive ray generators of Δ_κ in $\mathbb{Z}^2$ is done with respect to the *antitropical coordinates* introduced in [15, Constr. 4.5]. Note that the columns of the above matrices are precisely the vertices of the associated Fano polytopes $\mathcal{A}_\kappa$, as listed in the preceding proof.

Remark 6.11. Among the surfaces discussed in Examples 3.6, 4.6, 5.6 only the surface X from Example 3.6 admits a Kähler-Einstein metric, due to Corollary 6.9. Let $w_X \in \mathrm{Cl}(X)$ be the anticanonical class. Then the anticanoncial ring of X is a Veronese subalgebra of its Cox ring:

$$S_X \;:=\; \bigoplus_{k\in\mathbb{Z}} \Gamma(X, -\mathcal{K}_X) \;\cong\; \bigoplus_{k\in\mathbb{Z}} \mathcal{R}_{kw_X}(X).$$

Using, for instance, the software package [19], we can explicitly compute a minimal homogeneous generator system that has the generator degrees $1, 1, 1, 2$ and a single defining relation in degree 4. This identifies X from Example 3.6 as the second sporadic case in the list of [24, Thm. 8].

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