Gauss circle problem over smooth integers

Ankush Goswami

ABSTRACT. For a positive integer n, let $r_2(n)$ be the number of representations of n as sums of two squares (of integers), where the convention is that different signs and different orders of the summands yield distinct representations. A famous result of Gauss shows that $R(x) := \sum_{n \leq x} r_2(n) \sim \pi x$. Let $P(n)$ denote the largest prime factor of n and let $S(x, y) := \{n \leq x : P(n) \leq y\}$. In this paper, we study the asymptotic behavior of $R(x, y) := \sum_{n \in S(x,y)} r_2(n)$ for various ranges of $2 \leq y \leq x$. For y in a certain large range, we show that $R(x, y) \sim \rho(\alpha) \cdot \pi x$ where $\rho(\alpha)$ is the Dickman function and $\alpha = \log x / \log y$. We also obtain the asymptotic behavior of the partial sum of a generalized representation function following a method of Selberg.

CONTENTS

1. Introduction

For a positive integer n, we denote by $r_2(n)$ the number of representations of n as sums of two squares (of integers), where representations that differ only in the order of the summands or in the signs of the numbers being squared are counted as different. The following result of Fermat characterizes primes p which are sums of two squares:

Theorem 1.1 (Fermat). *An odd prime p is a sum of two squares if and only if $p \equiv 1 \pmod 4$.*

Theorem 1.1 can be used to completely characterize the composite numbers that are sums of two squares.

Theorem 1.2 (Two Squares Theorem). *A positive integer n is the sum of two squares if and only if each prime factor p of n such that $p \equiv 3 \pmod 4$ occurs to an even power in the prime factorization of n.*

Going beyond this result, Gauss [8, §182] by use of quadratic forms and Jacobi [11] by use of elliptic functions, proved the following stronger result:

Theorem 1.3 (Gauss, Jacobi). *Denote the number of divisors of n by d(n), and write $d_a(n)$ for the number of those divisors with $d \equiv a \pmod 4$. Let $n = 2^f n_1 n_2$, where $n_1 = \prod_{p \equiv 1 \pmod 4} p^r, n_2 = \prod_{q \equiv 3 \pmod 4} q^s$; then $r_2(n) = 0$ if any of the exponents s is odd. If all s are even, then $r_2(n) = 4 \cdot d(n_1) = 4(d_1(n) - d_3(n))$.*

In about 1800 C.E., Gauss attempted to estimate the following sum:

$$R(x) := \sideset{}{'}\sum_{n \leq x} r_2(n), \quad x > 0, \tag{1.1}$$

which counts the number of integral points inside a circle of radius $\sqrt{x}$. Here $'$ means that when x is an integer, $\frac{1}{2}r_2(x)$ is counted. Then observing that $\pi(\sqrt{x} - \sqrt{2})^2 < R(x) < \pi(\sqrt{x} + \sqrt{2})^2$, he showed that

Theorem 1.4 (Gauss). *We have*

$$R(x) = \pi x + O(\sqrt{x}).$$

No further improvements in the error term in Theorem 1.4 were made until 1906 when W. Sierpiński showed that the error in Theorem 1.4 is $O(x^{1/3})$. In fact, Landau [12], [13] simplified Sierpiński's approach but in doing so, obtained the weaker result that the error is $O(x^{1/3+\epsilon})$, $\epsilon > 0$.

Subsequent attempts to further improve the error term have rested upon the following identity involving the ordinary Bessel function $J_\nu(z)$:

$$\sideset{}{'}\sum_{n \leq x} r_2(n) = \pi x + \sum_{n=1}^{\infty} r_2(n) \left(\frac{x}{n}\right)^{1/2} J_1(2\pi\sqrt{nx}) \tag{1.2}$$

where

$$J_\nu(z) := \sum_{n=1}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n+1)} \left(\frac{z}{2}\right)^{\nu+2n}, \quad 0 < |z| < \infty, \quad \nu \in \mathbb{C}.$$

It turns out that an improvement in the error term in the asymptotic estimate in (1.2) amounts to studying the behavior of certain trigonometric sums. Finding the correct order of magnitude of this error term is the famous *Gauss circle problem* (see [3]).

Let $P(n)$ be the largest prime factor of n and define $S(x, y) := \{n \leq x : P(n) \leq y\}$. In this paper, among other things, we study the asymptotic behavior of

$$R(x, y) := \sum_{n \in S(x,y)} r_2(n) \tag{1.3}$$

for various ranges of $2 \leq y \leq x$. In particular, we show that for a fixed $\alpha = \log x / \log y > 1$, we have

$$\lim_{x \to \infty} \frac{R(x, x^{1/\alpha})}{x} = \pi \cdot \rho(\alpha) \tag{1.4}$$

where $\rho(\alpha)$ is the Dickman function satisfying the delay-differential equation

$$\rho'(\alpha) = -\frac{\rho(\alpha - 1)}{\alpha}, \quad \alpha > 1.$$

Thus, we see from (1.4) that when $y = x$

$$R(x) = R(x, x) \sim \pi x$$

since $\rho(1) = 1$ (see Section 3), thus generalizing Theorem 1.4 as far as the main term is concerned since the error obtained in the asymptotic estimate as a result of the generalization is weaker than Theorem 1.4. However, our results are uniform in y and α (see Section 2), and thus they are interesting.

Our analysis of $R(x, y)$ will involve a careful study of a related function, which we now describe. For a positive integer n, the *core of n* (also called the *radical of n*) is the largest square-free factor of n. Let $\nu(n)$ denote the number of distinct prime factors of n. Define the set $\mathcal{A} := \{n \in \mathbb{N} : p|n \Rightarrow p \equiv 1 \pmod 4\}$ and let $\kappa(n)$ be the characteristic function on $\mathcal{A}$. Then from Theorem 1.3, we see that $4 \cdot \kappa(n)2^{\nu(n)}$ counts the number of representations of the core of the odd-part of n as sums of two squares.

We study the asymptotic behavior of the following partial sums:

$$\tilde{R}(x) := \sum_{n \leq x} \kappa(n)2^{\nu(n)}, \quad x > 0 \tag{1.5}$$

and

$$\tilde{R}(x, y) := \sum_{n \in S(x,y)} \kappa(n)2^{\nu(n)}, \quad 2 \leq y \leq x. \tag{1.6}$$

There is a close connection of $\tilde{R}(x)$ with $R(x)$. The motivation to consider $\tilde{R}(x)$ comes from a problem concerning the *generalized divisor function*, $d_z(n)$, which are coefficients in the Dirichlet series expansion of $\zeta(s)^z$ ($z \in \mathbb{C}$) where $\zeta(s)$ is the Riemann zeta function. In [17], Selberg studied the asymptotic behavior of the partial sum of $d_z(n)$. This yields the asymptotic behavior of $\sum_{n \leq x} z^{\nu(n)}$, which, in turn, leads to an understanding of the distribution of numbers with a prescribed number of (distinct) prime factors. In passing, we note that since $d_z(p) = z = z^{\nu(p)}$, the two functions $d_z(n)$ and $z^{\nu(n)}$ are 'nearby'. Similarly, we see that $\kappa(p)2^{\nu(p)} = r_2(p)/4$ for $p \neq 2$ and thus $\kappa(n)2^{\nu(n)}$ and $r_2(n)/4$ are 'nearby'. Thus, the asymptotic behavior of $R(x, y)$ can be derived from $\tilde{R}(x, y)$ by elementary reasoning.

Another reason for considering the function $\kappa(n)2^{\nu(n)}$ instead of $r_2(n)$ is the fact that the former allows us to conveniently estimate the partial sum over smooth integers.

We shall use the Buchstab-de Bruijn iteration technique to establish the asymptotic behavior of $\tilde{R}(x, y)$ uniformly in the range $\exp((\log x)^{2/3+\epsilon}) \leq y \leq x$ ($\epsilon > 0$). In particular, our analysis of the function $\tilde{R}(x, y)$ shows that

$$\lim_{x \to \infty} \frac{\tilde{R}(x, x^{1/\alpha})}{x} = \frac{\rho(\alpha)}{\pi}, \quad \alpha = \log x / \log y.$$

In many applications (see [9]), estimates of sums such as $\tilde{R}(x, y)$ are very useful. For example, it is well-known that the partial sum of the Möbius function has no asymptotic formula, but has very beautiful and subtle asymptotic behavior when the sum is taken over smooth integers [1].

Next, for $z \in \mathbb{C}$, let $q_z(n)$ denote the coefficients in the following Dirichlet series expansion:

$$(\zeta(s)\beta(s))^{z-1} := \sum_{n=1}^{\infty} \frac{q_z(n)}{n^s}, \quad \mathrm{Re}(s) > 1. \tag{1.7}$$

When $z = 2$ above, we see that $q_2(n) = r_2(n)/4$. Thus, we call $r_z(n) := 4^{z-1} q_z(n)$ the *generalized sums of two squares function*. Here $\beta(s)$ is the Dirichlet beta function (see Section 3 for definition and properties). Using a method of Selberg, we shall find the asymptotic behavior of

$$R_z(x) := \sum_{n \leq x} r_z(n), \quad |z| < B, \ B > 0. \tag{1.8}$$

2. Main results

The first two main results below yield asymptotic estimates for $\tilde{R}(x, y)$ and $R(x, y)$, respectively, for *y large*.

Theorem 2.1. *Let $\epsilon > 0$ and $\exp((\log x)^{2/3+\epsilon}) \leq y \leq x$. Then there exists a $c_1 > 0$ such that*

$$\tilde{R}(x, y) = \frac{x \cdot \rho(\alpha)}{\pi} + O\left(\frac{x\rho(\alpha)\log(\alpha+1)}{\log y}\right) + O\left(\alpha^2 \log \alpha \ x \log y \exp(-c_1\sqrt{\log y})\right)$$

uniformly for α and y. In particular, for any fixed $\alpha > 1$, we have

$$\lim_{x \to \infty} \frac{\tilde{R}(x, x^{1/\alpha})}{x} = \frac{\rho(\alpha)}{\pi}.$$

Theorem 2.2. *Let $\epsilon > 0$ and $\exp((\log x)^{2/3+\epsilon}) \leq y \leq x$. Then there exists a $c_1 > 0$ such that*

$$R(x, y) = \rho(\alpha) \cdot \pi \cdot x + O\left(\frac{x\rho(\alpha)\log^2(\alpha+1)}{\log y}\right) + O\left(\alpha^2 \log \alpha \ x \log y \exp(-c_1\sqrt{\log y})\right)$$

uniformly in α and y. In particular, for any fixed $\alpha > 1$

$$\lim_{x \to \infty} \frac{R(x, x^{1/\alpha})}{x} = \pi \cdot \rho(\alpha).$$

Our third main result below yields an asymptotic estimate for the partial sum of the generalized sums of two squares function.

Theorem 2.3. *Let $B > 0$. Then uniformly for $|z| < B$, we have*

$$R_z(x) := \sum_{n \leq x} r_z(n) = \frac{\pi^{z-1}}{\Gamma(z-1)} \cdot x(\log x)^{z-2} + O\left(x(\log x)^{\mathrm{Re}(z)-3}\right)$$

where $r_z(n) := 4^{z-1}q_z(n)$ and $q_z(n)$ is defined as in (1.7).

3. Notations and preliminaries

Throughout p will denote a prime, x will denote a non-integral positive real number and $2 \leq y \leq x$. For a positive integer n, we denote by $\nu(n)$ the number of distinct prime factors of n. Let $\gamma(n)$ denote the *core of n*, which is the square-free part of n. Let $P(n)$ denote the largest prime factor of n. We put $P(1) = 1$. An arithmetical function $f : \mathbb{Z} \to \mathbb{C}$ is called *multiplicative* if $f(mn) = f(m)f(n)$ when $(m, n) = 1$. If $f(mn) = f(m)f(n)$ for all $m, \ n \geq 1$, we call f *totally* or *completely multiplicative*. The Dirichlet convolution for two arithmetical functions f and g is defined as follows:

$$(f * g)(n) := \sum_{d|n} f(d)g(n/d) = \sum_{d|n} f(n/d)g(d).$$

For two real or complex valued functions f and g, the Landau notation $f = O(g)$ means that $|f(x)| \leq M|g(x)|, \ x \to \infty$ for some $M > 0$. Equivalently, the Vinogradov notation $f \ll g$ means that $f = O(g)$. If the constant M depends on some additional parameter, say, κ, we write $f \ll_\kappa g$ or $f = O_\kappa(g)$. We denote by $\alpha := \log x / \log y$ and use $\lfloor x \rfloor$ to denote the greatest integer $\leq x$.

Let $\varphi(n)$ denote Euler totient function, which counts the number of positive integers up to n and coprime to n. For a complex number $s = \sigma + it$ ($\sigma = \mathrm{Re}(s), \ t = \mathrm{Im}(s)$), we define the Riemann zeta function $\zeta(s)$ and the Dirichlet beta function $\beta(s)$ as follows:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \ \ \sigma > 1, \qquad \beta(s) := \sum_{n=0}^{\infty} \frac{\chi_{-4}(n)}{n^s}, \ \ \sigma > 0$$

where $\chi_{-4}(\cdot)$ denotes the Kronecker symbol defined as follows:

$$\chi_{-4}(n) := \left(\frac{-4}{n}\right) := \begin{cases} 1, & \text{if } n \equiv 1 \pmod 4 \\ -1, & \text{if } n \equiv 3 \pmod 4 \\ 0, & \text{otherwise.} \end{cases}$$

Define the set $\mathcal{A} := \{n \in \mathbb{N} : p|n \Rightarrow p \equiv 1 \pmod 4\}$, and let $\kappa(n)$ be the characteristic function on $\mathcal{A}$. In view of Theorem 1.3, we see that the quantity $\kappa(n)2^{\nu(n)}$ counts the number of representations of $\gamma(n)$ as a sum of squares for

n odd. In other words, $\kappa(n)2^{\nu(n)} = \frac{1}{4}r_2(\gamma(n))$, $(n,2) = 1$. We thus define

$$\tilde{R}(x) = \sum_{n \leq x} \kappa(n)2^{\nu(n)} = \frac{1}{4} \sum_{\substack{n \leq x \\ (n,2)=1}} r_2(\gamma(n)). \tag{3.1}$$

For $x > 0$ and $2 \leq y \leq x$, we also define the function

$$\tilde{R}(x,y) := \sum_{\substack{n \leq x \\ P(n) \leq y}} \kappa(n)2^{\nu(n)}. \tag{3.2}$$

Let $S(x,y) := \{n \leq x : P(n) \leq y\}$ denote the set of y-smooth integers up to x and $\Psi(x,y) := |S(x,y)|$. Dickman showed that [6]

$$\lim_{x \to \infty} \frac{\Psi(x, x^{1/\alpha})}{x} = \rho(\alpha)$$

where $\rho(\alpha)$ satisfies the integral equation

$$\rho(\alpha) = \begin{cases} 0, & \alpha < 0 \\ 1, & 0 \leq \alpha \leq 1 \\ 1 - \int_1^\alpha \frac{\rho(u-1)}{u}du, & \alpha > 1. \end{cases}$$

We define the following function, which we will require later:

$$\mathcal{R}(x,y) := x \int_0^\infty \rho\left(\frac{\log x - \log t}{\log y}\right) d\left(\frac{\tilde{R}(t)}{t}\right). \tag{3.3}$$

We have the following classical zero-free region for $\zeta(s)$ due to de la Vallée Poussin [5] and estimates for $\zeta(s)$ (see [14, Chapter 6, Theorem 6.6], [4, Chapter 13, page 86]):

Theorem 3.1. *There is an absolute constant $c > 0$ such that $\zeta(s) \neq 0$ for $\sigma > 1 - c/\log(|t| + 1)$. If $\sigma > 1 - c/(2\log(|t| + 4))$ and $|t| > 7/8$, then*

$$|\log\zeta(s)| \ll \log\log(|t| + 4) + O(1) \quad and \quad \frac{1}{\zeta(s)} \ll \log(|t| + 4). \tag{3.4}$$

On the other hand, if $1 - c/(2\log(|t|+4)) < \sigma \leq 2$ and $|t| \leq 7/8$, then $\log(\zeta(s)(s-1)) \ll 1$ and $1/\zeta(s) \ll |s - 1|$.

Let $\pi(x; q, a)$ denote the number of primes $p \leq x$ for which $p \equiv a \pmod{q}$. Also, let $\text{li}(x) := \int_2^x \frac{dt}{\log t}$. Then the Siegel–Walfisz theorem (see [14, Chapter 11, Corollary 11.21]) gives

Theorem 3.2. *Given $A > 0$, there is a constant $c_1 > 0$ such that if $q \leq (\log x)^A$ and $(a, q) = 1$, then*

$$\pi(x; q, a) = \frac{\text{li}(x)}{\varphi(q)} + O_A\left(x \exp(-c_1\sqrt{\log x})\right).$$

For our applications later, we have $x \geq 3$ and $q = 4$ in Theorem 3.2. Thus, the inequality $4 \leq (\log x)^A$ is satisfied for some $A \leq 15$, and hence, we can suppress the dependence of A from the O-term in Theorem 3.2. We next obtain an estimate for $\beta(s)$, which we will require later.

Lemma 3.3. *For $\delta > 0$, we have*

$$\beta(s) \ll \zeta(1 + \delta), \quad \text{if } \sigma \geq 1 + \delta$$

and if $0 < \delta < 1$, we have

$$\beta(s) \ll \begin{cases} 4^{\delta-1} \max\left(\dfrac{|t|^{\delta}}{\delta}, \dfrac{|t|^{\delta}}{1-\delta}, 4^{\delta-1}\right), & \text{if } 1 - \delta < \sigma < 2, \ |t| \geq 1 \\ \dfrac{|s|}{\sigma}, & \text{if } 0 < \sigma < 1 + \delta, \ |t| < 1. \end{cases} \tag{3.5}$$

Proof. First, note that

$$\beta(s) = 4^{-s}\left(\zeta\left(s, \frac{1}{4}\right) - \zeta\left(s, \frac{3}{4}\right)\right) \tag{3.6}$$

where $\zeta(s, \alpha)$ is the Hurwitz zeta function defined by

$$\zeta(s, \alpha) := \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s}.$$

If $\sigma \geq 1 + \delta$, trivially from the definition of $\beta(s)$ we have $\beta(s) \ll \zeta(1 + \delta)$. Next, let $1 - \delta \leq \sigma \leq 2$ and $|t| \geq 1$. Then from (3.6), we have

$$|\beta(s)| \leq 4^{\delta-1}\left(|\zeta(s, 1/4) - (1/4)^s| + |\zeta(s, 3/4) - (3/4)^s|\right) + 4^{2(\delta-1)}|1 + 3^s|$$
$$\leq 4^{\delta-1}\left(|\zeta(s, 1/4) - (1/4)^s| + |\zeta(s, 3/4) - (3/4)^s|\right) + 10 \cdot 4^{2(\delta-1)}. \tag{3.7}$$

From the proof in [2, Chapter 12, Theorem 12.23 (b)], it can be seen that

$$|\zeta(s, \alpha) - \alpha^s| \ll \max\left(\frac{|t|^{\delta}}{\delta}, \frac{|t|^{\delta}}{1-\delta}\right), \quad |t| \geq 1, \quad 0 < \delta < 1 \tag{3.8}$$

where the implicit constant is absolute. Thus, (3.7) and (3.8) give

$$|\beta(s)| \ll 4^{\delta-1} \max\left(\frac{|t|^{\delta}}{\delta}, \frac{|t|^{\delta}}{1-\delta}, 4^{\delta-1}\right), \quad |t| \geq 1, \quad 0 < \delta < 1. \tag{3.9}$$

For $0 < \sigma \leq 2$ and $|t| < 1$, we use Stieltjes integration to estimate $\beta(s)$. We have

$$\beta(s) = \int_{1^-}^{\infty} \frac{d(\mathcal{S}(x))}{x^s}, \qquad \mathcal{S}(x) := \sum_{n \leq x} \chi_{-4}(n). \tag{3.10}$$

Note that $\chi_{-4}(\cdot)$ is a (non-principal) primitive character of conductor 4, so that by the Pólya-Vinogradov inequality (see [14, Chapter 9, (9.16)]) we have

$$\mathcal{S}(x) < 8 \log 2. \tag{3.11}$$

Thus (3.10) and (3.11) lead to

$$\beta(s) = \left.\frac{\mathcal{S}(x)}{x^s}\right|_{x=1^-}^{\infty} + s \int_1^\infty \frac{\mathcal{S}(x)}{x^{s+1}}dx \ll \frac{|s|}{\sigma}, \quad \sigma > 0.$$

This completes the proof. $\qquad\square$

Next, we show that a slight modification of $r_2(n)$ is multiplicative.

Lemma 3.4. *The function $r_2(n)/4$ is multiplicative.*

Proof. By Theorem 1.3, we have

$$r_2(n)/4 = \sum_{d|n}(-1)^{(d-1)/2} = (1 * \chi_{-4})(n) \tag{3.12}$$

where $1(n) = 1$ for all $n \in \mathbb{N}$. Since 1 and χ_{-4} are multiplicative, it follows from (3.12) that $r_2(n)/4$ is multiplicative too. $\qquad\square$

Lemma 3.5. *The function $\kappa(n)$ is totally multiplicative.*

Proof. Consider two integers $m, \; n \geq 1$. Then $\kappa(mn)$ is either zero or 1. If $\kappa(mn) = 0$, then there is a prime p such that $p|mn$ and either $p = 2$ or $p \equiv 3 \pmod 4$. Since $p|mn$ and p is a prime, we have either $p|m$ or $p|n$. Thus $\kappa(m)\kappa(n) = 0$ and so, $\kappa(mn) = \kappa(m)\kappa(n)$. Similarly, it can be shown that when $\kappa(mn) = 1$, $\kappa(m)\kappa(n) = 1$ and so, $\kappa(mn) = \kappa(m)\kappa(n)$. This completes the proof. $\qquad\square$

Consider the following Dirichlet series:

$$D(s) := \sum_{n=1}^\infty \frac{\kappa(n)2^{\nu(n)}}{n^s}, \quad \sigma > 1. \tag{3.13}$$

Lemma 3.6. *We have*

$$D(s) = \left(1 + \frac{1}{2^s}\right)^{-1} \frac{\zeta(s)\beta(s)}{\zeta(2s)}.$$

Proof. Since $\kappa(n)2^{\nu(n)}$ is multiplicative, $D(s)$ admits an Euler product representation as follows:

$$
\begin{aligned}
D(s) &= \prod_p \left(1 + \frac{\kappa(p)2^{\nu(p)}}{p^s} + \frac{\kappa(p^2)2^{\nu(p^2)}}{p^{2s}} + \cdots\right) \\
&= \prod_{p\equiv 1 \,(\text{mod } 4)} \left(1 + \frac{2}{p^s} + \frac{2}{p^{2s}} + \cdots\right) \\
&= \prod_{p\equiv 1 \,(\text{mod } 4)} \left(1 + \frac{2}{p^s - 1}\right) = \prod_{p\equiv 1 \,(\text{mod } 4)} \left(1 + \frac{1}{p^s}\right)\left(1 - \frac{1}{p^s}\right)^{-1} \\
&= \left(1 + \frac{1}{2^s}\right)^{-1} \frac{\zeta(s)\beta(s)}{\zeta(2s)}.
\end{aligned}
$$

$\qquad\square$

The main reason for studying the sum $\tilde{R}(x)$ instead of $R(x)$ is described as follows. Let us put $q_2(n) = r_2(n)/4$. Then by Lemma 3.4, we have

$$\sum_{n=1}^{\infty} \frac{q_2(n)}{n^s} = \zeta(s)\beta(s). \tag{3.14}$$

Thus we see from Lemma 3.6 that the Dirichlet series for $\kappa(n)2^{\nu(n)}$ differs from that for $q_2(n)$ in (3.14) by the function $((1 + 2^{-s})\zeta(2s))^{-1}$, which is convergent in a larger half-plane. More importantly, $\tilde{R}(x)$ can be handled more conveniently than $R(x)$ as we shall see soon.

We have the following asymptotic estimate for $\tilde{R}(x)$.

Lemma 3.7. *Let $x > 0$. Then we have*

$$\tilde{R}(x) = \frac{x}{\pi} + O\left(\sqrt{x}\log x\right).$$

Proof. We have from Lemma 3.6 and (3.14) that

$$\left(1 + \frac{1}{2^s}\right)\sum_{n=1}^{\infty} \frac{\kappa(n)2^{\nu(n)}}{n^s} = \left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^{2s}}\right)\left(\sum_{n=1}^{\infty} \frac{q_2(n)}{n^s}\right) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}. \tag{3.15}$$

where $f(n) := \sum_{d^2|n} \mu(d)q_2(n/d^2)$. Since both sides of (3.15) are Dirichlet series, which converge for $\sigma > 1$, it follows from the uniqueness of Dirichlet series [2, Chapter 11, Theorem 11.3] that

$$\sum_{n\leq x} \kappa(n)2^{\nu(n)} + \sum_{n\leq x/2} \kappa(n)2^{\nu(n)} = \sum_{n\leq x} f(n) = \sum_{1\leq m\leq\sqrt{x}} \mu(m) \sum_{1\leq \ell\leq x/m^2} q_2(\ell). \tag{3.16}$$

Using Theorem 1.4 in the right-hand side of (3.16), we get

$$P(x) := \tilde{R}(x) + \tilde{R}(x/2) = \frac{1}{4} \sum_{1\leq m\leq\sqrt{x}} \mu(m)\left(\frac{\pi x}{m^2} + O\left(\frac{\sqrt{x}}{m}\right)\right) \tag{3.17}$$

$$= \frac{\pi x}{4\zeta(2)} + O(\sqrt{x}\log x). \tag{3.18}$$

Let $N = \lfloor\frac{\log x}{\log 2}\rfloor$. Substituting x by $x/2, x/2^2, x/2^3, \cdots, x/2^N$ in (3.17) and then considering the telescoping sum, we see that

$$\sum_{k=0}^{N}(-1)^k P(x/2^k) = \begin{cases} \tilde{R}(x) + \tilde{R}(x/2^{N+1}), & N \text{ even} \\ \tilde{R}(x) - \tilde{R}(x/2^{N+1}), & N \text{ odd}. \end{cases} \tag{3.19}$$

Since $\tilde{R}(x/2^{N+1}) = O(1)$, we conclude from (3.17) and (3.19) that

$$\tilde{R}(x) = \frac{\pi x}{4\zeta(2)} \sum_{k=0}^{N} \frac{(-1)^k}{2^k} + O\left(\sqrt{x}\log x \sum_{k=0}^{N} 2^{-k/2}\right). \tag{3.20}$$

Noting that $\sum_{k=N+1}^{\infty} 2^{-k} = O(2^{-N}) = O\left(\frac{1}{x}\right)$, we find from (3.20) that

$$\tilde{R}(x) = \frac{\pi x}{4\zeta(2)} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k} + O\left(\sqrt{x}\log x\right) = \frac{x}{\pi} + O(\sqrt{x}\log x), \qquad (3.21)$$

which completes the proof of the result. $\qquad\square$

Remark 3.8. We note here that the error term in Lemma 3.7 can be improved by using Sierpiński's result [15] or other results with better error terms for $R(x)$ (see [3]). For our purposes, $O(\sqrt{x}\log x)$ suffices since an improved O-term will not really improve the subsequent O-terms in our analysis.

For the purpose of establishing asymptotic estimates for $\tilde{R}(x, y)$ for various ranges of y, we require the following recurrence result:

Lemma 3.9. *Let $2 \leq y \leq y^h \leq x$ and $h \geq 1$. Then there exists a $c_1 > 0$ such that*

$$\tilde{R}(x, y) = \tilde{R}(x, y^h) - \int_y^{y^h} \tilde{R}\left(\frac{x}{t}, t\right) \frac{dt}{\log t} + O\left(\alpha x \log y \exp(-c_1\sqrt{\log y})\right).$$

Proof. We have

$$\tilde{R}(x, y^h) - \tilde{R}(x, y) = \sum_{\substack{n \leq x \\ y < P(n) \leq y^h}} \kappa(n) 2^{\nu(n)}. \qquad (3.22)$$

Let $P(n) = p$. Then we have

$$\sum_{\substack{n \leq x \\ y < P(n) \leq y^h}} \kappa(n) 2^{\nu(n)} = \sum_{y < p \leq y^h} \sum_{\substack{pn \leq x \\ P(n) \leq p}} \kappa(pn) 2^{\nu(np)} = \sum_{\substack{y < p \leq y^h \\ p \equiv 1 \,(\mathrm{mod}\ 4)}} \sum_{\substack{n \leq x/p \\ P(n) \leq p}} \kappa(n) 2^{\nu(np)}. \qquad (3.23)$$

Observe that the inner sum in the right-hand side of (3.23) can be rewritten as follows:

$$\sum_{\substack{n \leq x/p \\ P(n) \leq p}} \kappa(n) 2^{\nu(np)} = \sum_{\substack{n \leq x/p \\ P(n) \leq p \\ (n,p)=1}} \kappa(n) 2^{\nu(np)} + \sum_{\substack{n \leq x/p \\ P(n) \leq p \\ p|n}} \kappa(n) 2^{\nu(np)}$$

$$= 2 \sum_{\substack{n \leq x/p \\ P(n) \leq p \\ (n,p)=1}} \kappa(n) 2^{\nu(n)} + \kappa(p) \sum_{\substack{n \leq x/p^2 \\ P(n) \leq p}} \kappa(n) 2^{\nu(np)}$$

$$= 2\left\{ \sum_{\substack{n \leq x/p \\ P(n) \leq p}} \kappa(n) 2^{\nu(n)} - \kappa(p) \sum_{\substack{n \leq x/p^2 \\ P(n) \leq p}} \kappa(n) 2^{\nu(np)} \right\}$$

$$+ \kappa(p) \sum_{\substack{n \le x/p^2 \\ P(n) \le p}} \kappa(n) 2^{\nu(np)}$$

$$= 2 \sum_{\substack{n \le x/p \\ P(n) \le p}} \kappa(n) 2^{\nu(n)} - \kappa(p) \sum_{\substack{n \le x/p^2 \\ P(n) \le p}} \kappa(n) 2^{\nu(np)}$$

$$= 2 \sum_{\substack{n \le x/p \\ P(n) \le p}} \kappa(n) 2^{\nu(n)} + O\left(\frac{x}{p^2}\right) = 2\tilde{R}\left(\frac{x}{p}, p\right) + O\left(\frac{x}{p^2}\right) \quad (3.24)$$

where the O-term in the penultimate step above follows from Lemma 3.7. Combining (3.23) and (3.24), we get

$$\sum_{\substack{n \le x \\ y < P(n) \le y^h}} \kappa(n) 2^{\nu(n)} = 2 \sum_{\substack{y < p \le y^h \\ p \equiv 1 \,(\mathrm{mod}\ 4)}} \tilde{R}\left(\frac{x}{p}, p\right) + O\left(x \sum_{y < p \le y^h} \frac{1}{p^2}\right)$$

$$= 2 \sum_{\substack{y < p \le y^h \\ p \equiv 1 \,(\mathrm{mod}\ 4)}} \tilde{R}\left(\frac{x}{p}, p\right) + O(xy^{-1}). \quad (3.25)$$

Next, we have

$$\sum_{\substack{y < p \le y^h \\ p \equiv 1 \,(\mathrm{mod}\ 4)}} \tilde{R}\left(\frac{x}{p}, p\right) - \frac{1}{2} \int_y^{y^h} \tilde{R}\left(\frac{x}{t}, t\right) \frac{dt}{\log t} = \sum_{\substack{y < p \le y^h \\ p \equiv 1 \,(\mathrm{mod}\ 4)}} \sum_{\substack{n \le x/p \\ P(n) \le p}} \kappa(n) 2^{\nu(n)}$$

$$- \frac{1}{2} \int_y^{y^h} \left(\sum_{\substack{n \le x/t \\ P(n) \le t}} \kappa(n) 2^{\nu(n)} \right) \frac{dt}{\log t}$$

$$= \sum_{\substack{y < p \le y^h \\ p \equiv 1 \,(\mathrm{mod}\ 4)}} \sum_{\substack{n \le x/p \\ P(n) < p}} \kappa(n) 2^{\nu(n)} - \frac{1}{2} \int_y^{y^h} \left(\sum_{\substack{n \le x/t \\ P(n) \le t}} \kappa(n) 2^{\nu(n)} \right) \frac{dt}{\log t}$$

$$+ \sum_{\substack{y < p \le y^h \\ p \equiv 1 \,(\mathrm{mod}\ 4)}} \sum_{\substack{n \le x/p \\ P(n) = p}} \kappa(n) 2^{\nu(n)}$$

$$= \sum_{\substack{n \le x/y \\ P(n) \le y^h}} \kappa(n) 2^{\nu(n)} \left(\sum_{\substack{\max(P(n), y) < p \le \min(x/n, y^h) \\ p \equiv 1 \,(\mathrm{mod}\ 4)}} 1 - \frac{1}{2} \int_{\max(P(n), y)}^{\min(x/n, y^h)} \frac{dt}{\log t} \right)$$

$$+ \sum_{\substack{y < p \le y^h \\ p \equiv 1 \,(\mathrm{mod}\ 4)}} \sum_{\substack{n \le x/p^2 \\ P(n) \le p}} \kappa(n) 2^{\nu(n)}$$

$$
= \sum_{\substack{n \leq x/y \\ P(n) \leq y^h}} \kappa(n) 2^{\nu(n)} \left(\sum_{\substack{\max(P(n),y) < p \leq \min(x/n,y^h) \\ p \equiv 1 \ (\mathrm{mod} \ 4)}} 1 - \frac{1}{2} \int_{\max(P(n),y)}^{\min(x/n,y^h)} \frac{dt}{\log t} \right)
$$
$$
+ \ O(xy^{-1})
$$
(3.26)

where the O-term in the last step above follows from Lemma 3.7. We now invoke Theorem 3.2 to find that

$$
\sum_{\substack{\max(P(n),y) < p \leq \min(x/n,y^h) \\ p \equiv 1 \ (\mathrm{mod} \ 4)}} 1 - \frac{1}{2} \int_{\max(P(n),y)}^{\min(x/n,y^h)} \frac{dt}{\log t} = O\left(\frac{x}{n} \exp(-c_1 \sqrt{\log y}) \right).
$$

(3.27)

Using the estimate (3.27) in (3.26), we obtain

$$
\sum_{\substack{y < p \leq y^h \\ p \equiv 1 \ (\mathrm{mod} \ 4)}} \tilde{R}\left(\frac{x}{p}, p \right) - \frac{1}{2} \int_y^{y^h} \tilde{R}\left(\frac{x}{t}, t \right) \frac{dt}{\log t}
$$

$$
= O\left(x \exp(-c_1 \sqrt{\log y}) \sum_{n \leq x/y} \frac{\kappa(n) 2^{\nu(n)}}{n} \right) + O\left(xy^{-1} \right). \qquad (3.28)
$$

Using Theorem 1.4 or Lemma 3.7, we obtain by Stieltjes integration that

$$
\sum_{\substack{y < p \leq y^h \\ p \equiv 1 \ (\mathrm{mod} \ 4)}} \tilde{R}\left(\frac{x}{p}, p \right) - \frac{1}{2} \int_y^{y^h} \tilde{R}\left(\frac{x}{t}, t \right) \frac{dt}{\log t}
$$

$$
= O\left(\alpha x \exp(-c_1 \sqrt{\log y}) \log y \right) + O(xy^{-1}). \qquad (3.29)
$$

The lemma follows from (3.25), (3.29) and the fact that $O(xy^{-1})$ is suppressed by the first O-term in (3.29). $\qquad \square$

Lemma 3.10. *Let* $2 \leq y \leq y^h \leq x$, $h \geq 1$. *Then*

$$
\tilde{\mathcal{R}}(x, y) = \tilde{\mathcal{R}}(x, y^h) - \int_y^{y^h} \tilde{\mathcal{R}}\left(\frac{x}{u}, u \right) \frac{du}{\log u} \qquad (3.30)
$$

where $\tilde{\mathcal{R}}(x, y)$ *is defined in* (3.3).

Proof. Using the identity (see [7, pp. 7, Eq. 3.4])

$$
x\rho\left(\frac{\log x - \log t}{\log y} \right) = x\rho\left(\frac{\log x - \log t}{\log y^h} \right) - x \int_y^{y^h} \frac{\rho\left(\dfrac{\log(x/u) - \log t}{\log u} \right)}{u} \frac{du}{\log u}
$$

we get the required result. $\qquad \square$

We also need the following useful result in the sequel.

Lemma 3.11. *For $x, y \geq 1$ we have*

$$\sum_{n \in S(x,y)} \frac{1}{n^2} = \zeta(2) + O\left(\frac{1}{x}\right) + O\left(\frac{1}{y}\right). \tag{3.31}$$

Proof. First, let $y \geq x$. Then

$$\sum_{n \in S(x,y)} \frac{1}{n^2} = \sum_{n \leq x} \frac{1}{n^2} = \zeta(2) + O\left(\frac{1}{x}\right). \tag{3.32}$$

Next, let $y < x$. Then

$$\sum_{n \in S(x,y)} \frac{1}{n^2} = \sum_{n \leq x} \frac{1}{n^2} - \sum_{y < p \leq x} \frac{1}{p^2} \sum_{n \in S(x/p,p)} \frac{1}{n^2}$$

$$= \zeta(2) + O\left(\frac{1}{x}\right) + \sum_{y < p \leq x} \frac{1}{p^2} \cdot O(1)$$

$$= \zeta(2) + O\left(\frac{1}{x}\right) + O\left(\frac{1}{y}\right). \tag{3.33}$$

The lemma now follows from (3.32) and (3.33). $\qquad\qquad\qquad\square$

We conclude this section with the following result required in the proof of Theorem 2.3.

Lemma 3.12. *Let $B \geq 2$ be an integer and $\varepsilon > 0$. Then for $x > 0$ and some $0 < \delta_B < 1$, we have*

$$R_B(x) = x P_B(\log x) + O\left(x^{\delta_B + \varepsilon}\right)$$

where $P_B(t)$ is a polynomial in t of degree $B - 2$, and for $z \in \mathbb{C}$, $R_z(x)$ is defined as in (1.8).

Proof. We use induction to prove this result. For $B = 2$, this is clearly true from Theorem 1.4. By induction hypothesis, let us assume that

$$R_\ell(x) = x P_\ell(\log x) + O(x^{\delta_\ell}) \tag{3.34}$$

for $3 \leq \ell \leq B-1$ with $0 < \delta_\ell < 1$ and $P_\ell(x)$ is a polynomial in x of degree $\ell - 2$. Noting that $(\zeta(s)\beta(s))^{B-1} = (\zeta(s)\beta(s)) \cdot (\zeta(s)\beta(s))^{B-2}$, the Dirichlet hyperbola method (see [2, Chapter 3, Theorem 3.17]) yields

$$R_B(x) = \sum_{n \leq a} q_2(n) R_{B-1}(x/n) + \sum_{n \leq b} q_{B-1}(n) R_2(x/n) - R_2(a)R_{B-1}(b) \tag{3.35}$$

where a, b are positive reals with $ab = x$. Next, we estimate the sums in the right-hand side of (3.35). Using induction hypothesis and (3.34), we obtain

$$\sum_{n \leq a} q_2(n) R_{B-1}(x/n) = \sum_{n \leq a} q_2(n) \left(\frac{x P_{B-1}(\log x/n)}{n} + O\left(\left(\frac{x}{n}\right)^{\delta_{B-1} + \varepsilon} \right) \right). \tag{3.36}$$

The first sum in the right-hand side of (3.36) can be estimated using Theorem 1.4 and Abel's summation formula to obtain

$$\sum_{n \leq a} q_2(n) \frac{P_{B-1}(\log x/n)}{n}$$

$$= \frac{P_{B-1}(\log x/a)R_2(a)}{a} + \int_1^a R_2(t)\, d\left(\frac{P_{B-1}(\log x/t)}{t}\right)$$

$$= \pi P_{B-1}(\log x/a) + O(x^\varepsilon a^{-\frac{1}{2}}) + \int_1^a R_2(t) \frac{Q_{B-1}(\log x/t)}{t^2} dt$$

$$= \pi P_{B-1}(\log x/a) + O(x^\varepsilon a^{-\frac{1}{2}}) + \pi \int_1^a \frac{Q_{B-1}(\log x/t)}{t} dt$$

$$+ \int_1^a \frac{(R_2(t) - \pi t) \cdot Q_{B-1}(\log x/t)}{t^2} dt$$

$$= \pi P_{B-1}(\log x/a) + O(x^\varepsilon a^{-\frac{1}{2}}) + \pi \int_1^a \frac{Q_{B-1}(\log x/t)}{t} dt$$

$$+ \int_1^\infty \frac{(R_2(t) - \pi t) \cdot Q_{B-1}(\log x/t)}{t^2} dt - \int_a^\infty \frac{(R_2(t) - \pi t) \cdot Q_{B-1}(\log x/t)}{t^2} dt$$

$$= T_B(\log x, \log a) + O\left(x^\varepsilon a^{-\frac{1}{2}}\right) \tag{3.37}$$

where $Q_{B-1}(t) := P'_{B-1}(t) + P_{B-1}(t)$ and $T_B(X, Y)$ is a polynomial in X and Y of degree $B - 2$. The second sum in the right-hand side of (3.36) is

$$\ll x^{\delta_{B-1}+\varepsilon} \sum_{n \leq a} \frac{q_2(n)}{n^{\delta_{B-1}+\varepsilon}} \ll x^{\delta_{B-1}+\varepsilon}(1 + a^{1-\delta_{B-1}+\varepsilon}). \tag{3.38}$$

Next, we have

$$\sum_{n \leq b} q_{B-1}(n) R_2(x/n) = \sum_{n \leq b} q_{B-1}(n)\left(\frac{\pi x}{n} + O\left(\left(\frac{x}{n}\right)^{1/2}\right)\right)$$

$$= \pi x \sum_{n \leq b} \frac{q_{B-1}(n)}{n} + O\left(\sqrt{x} \sum_{n \leq b} \frac{q_{B-1}(n)}{\sqrt{n}}\right). \tag{3.39}$$

To estimate the first sum in the right-hand side of (3.39), we use (3.34) and Abel's summation formula to obtain

$$\sum_{n \leq b} \frac{q_{B-1}(n)}{n} = \frac{R_{B-1}(b)}{b} + \int_1^b \frac{R_{B-1}(t)}{t^2} dt$$

$$= P_{B-1}(\log b) + O\left(b^{\delta_{B-1}-1+\varepsilon}\right) + \int_1^b \frac{t P_{B-1}(\log t)}{t^2} dt$$

$$+ \int_1^b \frac{R_{B-1}(t) - tP_{B-1}(\log t)}{t^2} dt$$

$$= P_{B-1}(\log b) + O\left(b^{\delta_{B-1}-1+\varepsilon}\right) + \int_1^b \frac{tP_{B-1}(\log t)}{t^2} dt$$

$$+ \int_1^\infty \frac{R_{B-1}(t) - tP_{B-1}(\log t)}{t^2} dt - \int_b^\infty \frac{R_{B-1}(t) - tP_{B-1}(\log t)}{t^2} dt$$

$$= V_B(\log b) + O\left(b^{\delta_{B-1}-1+\varepsilon}\right) \tag{3.40}$$

where $V_B(x)$ is a polynomial in x of degree $B - 2$. In a similar way, the second sum in the right-hand side of (3.39) can be estimated as follows:

$$\sum_{n \le b} \frac{q_{B-1}(n)}{\sqrt{n}} \ll b^{\frac{1}{2}+\varepsilon}. \tag{3.41}$$

Finally, we have

$$R_2(a)R_{B-1}(b) = (\pi a + O(a^{\frac{1}{2}}))(bP_{B-1}(\log b) + O(b^{\delta_{B-1}+\varepsilon}))$$

$$= \pi ab P_{B-1}(\log b) + O\left(ab^{\delta_{B-1}+\varepsilon} + a^{\frac{1}{2}}b^{1+\varepsilon}\right). \tag{3.42}$$

From (3.35)–(3.42), we obtain

$$R_B(x) = xT_B(\log x, \log a) + \pi x V_B(\log b) - \pi x P_{B-1}(\log b) \tag{3.43}$$

$$+ O\left(x^{1+\varepsilon}a^{-\frac{1}{2}} + x^{\delta_{B-1}+\varepsilon} + x^{\delta_{B-1}+\varepsilon}a^{1-\delta_{B-1}+\varepsilon} + xb^{\delta_{B-1}-1+\varepsilon} + x^{\frac{1}{2}}b^{\frac{1}{2}+\varepsilon}\right.$$

$$\left. + ab^{\delta_{B-1}+\varepsilon} + a^{\frac{1}{2}}b^{1+\varepsilon}\right).$$

At this point, we substitute $b = \frac{x}{a}$ to find that the O-term is

$$\ll x^{1+\varepsilon}a^{-\frac{1}{2}} + x^{\delta_{B-1}+\varepsilon}a^{1-\delta_{B-1}+\varepsilon} \tag{3.44}$$

and to minimize this error, we choose $a = x^{\frac{\frac{1-\delta_{B-1}}{3}}{\frac{1}{2}-\delta_{B-1}}}$ to find that the error in (3.44) is $\ll x^{\delta_B+\varepsilon}$ for ε sufficiently small where $\delta_B = \frac{2-\delta_{B-1}}{3-2\delta_{B-1}} < 1$. From preceding discussions, the first two terms in the right-hand side of (3.43) are polynomials of degree $B - 2$ and under the above substitutions, their sum still yields a polynomial in $\log x$ of degree $B - 2$. The result now follows from noting that the first three terms in the right-hand side of (3.43) together yields $xP_B(\log x)$ where $P_B(x)$ is a polynomial in x of degree $B - 2$. □

4. Proof of main results

4.1. Proof of Theorem 2.1. We first show that the continuous function $\tilde{\mathcal{R}}(x, y)$ is a good approximation to $\tilde{R}(x, y)$ for a certain range of y. To this end, let $h \ge 1$.

For $2 \leq y \leq t \leq t^h \leq x$, let us put

$$\mathcal{E}(x,t) = \tilde{R}(x,t) - \hat{\mathcal{R}}(x,t).$$

Then it follows from Lemma 3.9 and Lemma 3.10 that

$$\mathcal{E}(x,t) = \mathcal{E}(x,t^h) - \int_t^{t^h} \mathcal{E}\left(\frac{x}{u},u\right)\frac{du}{\log u} + O\left(x\log x \exp(-c_1\sqrt{\log t})\right). \quad (4.1)$$

For $y \geq 2$ and $k \in \mathbb{N}$, we further define

$$\phi_k(y) := \sup_{\substack{0<x\leq t^k \\ t\geq y}} x^{-1}|\mathcal{E}(x,t)|. \quad (4.2)$$

We shall prove an inequality for $\phi_k(y)$ by induction on k. It is clear that $\phi_1(y) = 0$. For $k \geq 2$, choose $h = k/(k-1)$ and take $x \leq y^k \leq t^k$. Then $x \leq (t^h)^{k-1}$ and for $t \leq u \leq t^h$, we have $x/u \leq u^{k-1}$. Thus, (4.1) yields

$$|\mathcal{E}(x,t)| \leq x \cdot \phi_{k-1}(y)\left(1 + \int_t^{t^h} \frac{du}{u\log u}\right) + O\left(x\log x \exp(-c_1\sqrt{\log t})\right)$$

$$\leq x \cdot \phi_{k-1}(y)\left(1 + \log\left(\frac{k}{k-1}\right)\right) + O\left(\alpha x \log y \exp(-c_1\sqrt{\log y})\right). \quad (4.3)$$

Thus, (4.2), (4.3) and the facts that $\log(k/(k-1)) < (k-1)^{-1}$ and $\phi_1(y) = 0$ yield

$$\frac{1}{k}\phi_k(y) < \frac{1}{k-1}\phi_{k-1}(y) + O\left(k^{-1}\alpha \log y \exp(-c_1\sqrt{\log y})\right), \; k = 2,\, 3,\cdots. \quad (4.4)$$

By summing both sides over $k = 2, 3, \cdots, N$ for some positive integer N and noting that the inequality in (4.4) telescopes, we end up getting

$$\phi_N(y) \ll \alpha N(\log N)(\log y)\exp(-c_1\sqrt{\log y}). \quad (4.5)$$

By choosing $N = \lfloor\alpha\rfloor$ in (4.5), we obtain

$$|\mathcal{E}(x,y)| = |\tilde{R}(x,y) - \hat{\mathcal{R}}(x,y)| \leq x\phi_{\lfloor\alpha\rfloor}(y) \ll \alpha^2 \log \alpha \; x \log y \exp(-c_1\sqrt{\log y}). \quad (4.6)$$

We estimate $\hat{\mathcal{R}}(x,y)$. First, we note that $\rho(\alpha) = 0$, $\alpha < 0$ and $\rho(\alpha) = 1$, $0 \leq \alpha \leq 1$. Also, $\tilde{R}(t) = 0$, $0 < t \leq 1$. Thus, the limits of the integral in (3.3) reduce to 1 and x and we have

$$\hat{\mathcal{R}}(x,y) \;=\; x\int_1^x \rho\left(\frac{\log x - \log t}{\log y}\right)d\left(\frac{\tilde{R}(t) - t/\pi}{t}\right). \quad (4.7)$$

Using integration by parts in (4.7) and Lemma 3.7 we get

$$x\int_1^x \rho\left(\frac{\log x - \log t}{\log y}\right)d\left(\frac{\tilde{R}(t) - t/\pi}{t}\right) = x\,\rho\left(\frac{\log x - \log t}{\log y}\right)\frac{\tilde{R}(t) - t/\pi}{t}\bigg|_{1^-}^x$$

$$+ \frac{x}{\log y} \int_1^x \rho' \left(\frac{\log x - \log t}{\log y} \right) \frac{\tilde{R}(t) - t/\pi}{t^2} dt$$

$$= \frac{x \cdot \rho(\alpha)}{\pi} + O(\sqrt{x} \log x) + \frac{x}{\log y} \int_1^x \rho' \left(\frac{\log x - \log t}{\log y} \right) \frac{\tilde{R}(t) - t/\pi}{t^2} dt.$$

$$(4.8)$$

We will now estimate the integral in the right-hand side of (4.8). First, note the well-known inequalities satisfied by $\rho(\alpha)$, (see [7, Equations (4.1)-(4.2)]), namely

$$0 > \frac{\rho'(\alpha)}{\rho(\alpha)} > -C_1 \log(\alpha + 1) \ (\alpha > 1), \quad 0 > \frac{\rho''(\alpha)}{\rho'(\alpha)} > -C_2 \log(\alpha) \ (\alpha > 2)$$

$$(4.9)$$

for some absolute constants $C_1, C_2 > 0$. From this, we infer (see the offset equation between (4.4) and (4.5) in [7, Page 8]) that

$$\frac{\rho'(\alpha - s)}{\rho'(\alpha)} < C_3 \cdot \alpha^{C_4 s}, \quad \alpha \geq 2, \ 0 \leq s \leq \alpha, \ C_3, C_4 > 0. \tag{4.10}$$

Using Lemma 3.7 and (4.10), we have

$$\frac{1}{\log y} \int_1^x \rho' \left(\frac{\log x - \log t}{\log y} \right) \frac{\tilde{R}(t) - t/\pi}{t^2} dt \ll \frac{1}{\log y} \int_1^x \rho' \left(\frac{\log x - \log t}{\log y} \right) \frac{\log t \ dt}{t^{3/2}}$$

$$\ll \frac{|\rho'(\alpha)|}{\log y} \int_1^x \frac{t^{C_4 \log \alpha / \log y} \log t}{t^{3/2}} dt.$$

$$(4.11)$$

If $C_4 \log \alpha < \frac{1}{4} \log y$, then in view of (4.9) and (4.11), we get

$$\frac{1}{\log y} \int_1^x \rho' \left(\frac{\log x - \log t}{\log y} \right) \frac{\tilde{R}(t) - t/\pi}{t^2} \ll \frac{\rho(\alpha) \log(\alpha + 1)}{\log y}. \tag{4.12}$$

Hence, (4.7), (4.8) and (4.12) give

$$\tilde{R}(x, y) = \quad \frac{x \cdot \rho(\alpha)}{\pi} + O \left(\frac{x \rho(\alpha) \log(\alpha + 1)}{\log y} \right) + O(\sqrt{x} \log x). \tag{4.13}$$

Combining (4.6) and (4.13), we find that

$$\tilde{R}(x, y) = \frac{x \cdot \rho(\alpha)}{\pi} + O \left(\frac{x \rho(\alpha) \log(\alpha + 1)}{\log y} \right)$$

$$+ O \left(\alpha^2 \log \alpha \ x \log y \exp(-c_1 \sqrt{\log y}) \right). \tag{4.14}$$

In particular, (4.14) shows that for any fixed $\alpha > 1$, we have

$$\lim_{x \to \infty} \frac{\tilde{R}(x, x^{1/\alpha})}{x} = \frac{\rho(\alpha)}{\pi}. \tag{4.15}$$

We now find the range where the asymptotic estimate in (4.14) is valid. First, note that (4.14) is valid when $C_4 \log \alpha < \frac{1}{4} \log y$ for some $C_4 > 0$ in which case the first O-term in (4.14) is smaller than the main term. If α is fixed (does not tend to infinity with x), then we can obtain a very large range of y where (4.14) holds; however, for α not necessarily fixed, we find that the range of y where (4.14) holds is much smaller. Taking into account these facts, we choose y in the range

$$\exp((\log x)^{1-\delta}) \le y \le x \tag{4.16}$$

where $\delta \in (0,1)$. Clearly, the inequality $C_4 \log \alpha < \frac{1}{4} \log y$ is satisfied for the range of y in (4.16) and $x \gg_\delta 1$. We want to choose an optimal $\delta > 0$ in (4.16) such that both O-terms in (4.14) are smaller than the main term. Since $\rho(\alpha) = \exp(-\alpha \log \alpha - \alpha \log \log \alpha + O(\alpha))$ as $\alpha \to \infty$ (see [7]), it suffices to consider the following inequality

$$\alpha^2 \log \alpha \log y \exp(-c_1 \sqrt{\log y}) < \exp(-\alpha \log \alpha),$$

which is satisfied if

$$(2\delta + 1) \log \log x + \log \delta + \log \log \log x - c_1 (\log x)^{\frac{1-\delta}{2}} < -\delta (\log x)^\delta \log \log x. \tag{4.17}$$

Thus, for sufficiently large x, (4.17) is true provided $\frac{1-\delta}{2} > \delta$, which yields $\delta < 1/3$, so we can choose $\delta = 1/3 - \epsilon$ for any $\epsilon > 0$. Thus the range in which the asymptotic estimate in (4.14) is valid is

$$\exp((\log x)^{2/3+\epsilon}) \le y \le x. \tag{4.18}$$

Thus, we get the required result.

4.2. Proof of Theorem 2.2. Let $\psi(\cdot)$ denote the characteristic function on y-smooth numbers $S(x,y) := \{n \le x : P(n) \le y\}$. Then clearly ψ is multiplicative (in fact, totally multiplicative). Let $q_2(n) = r_2(n)/4$, which is multiplicative in view of Lemma 3.4. It is easy to verify that

$$\sum_{n=1}^{\infty} \frac{q_2(n)\psi(n)}{n^s} = \prod_{p<y}(1-p^{-s})^{-1} \prod_{\substack{p \equiv 1 \,(\mathrm{mod}\, 4) \\ p<y}} (1-p^{-s})^{-1} \prod_{\substack{p \equiv 3 \,(\mathrm{mod}\, 4) \\ p<y}} (1+p^{-s})^{-1}. \tag{4.19}$$

It can also be verified easily that

$$\sum_{n=1}^{\infty} \frac{\kappa(n)2^{\nu(n)}\psi(n)}{n^s} \tag{4.20}$$

$$\prod_{p<y}(1-p^{-s})^{-1}\prod_{\substack{p\equiv 1\ (\text{mod }4)\\p<y}}(1-p^{-s})^{-1}\prod_{\substack{p\equiv 3\ (\text{mod }4)\\p<y}}(1+p^{-s})^{-1}$$

$$= (1+2^{-s})^{-1}\frac{}{\prod_{p<y}(1-p^{-2s})^{-1}}.$$

Then it follows from (4.19) and (4.20) by Dirichlet convolution that

$$\sum_{n\in S(x,y)}q_2(n) = \sum_{n\in S(\sqrt{x},y)}\tilde{R}(x/n,y) + \sum_{n\in S(\sqrt{x/2},y)}\tilde{R}(x/n,y). \qquad (4.21)$$

Using Theorem 2.1 in (4.21) we have

$$\sum_{n\in S(\sqrt{x},y)}\tilde{R}(x/n,y) + \sum_{n\in S(\sqrt{x/2},y)}\tilde{R}(x/n,y) = \frac{x}{\pi}\sum_{n\in S(\sqrt{x},y)}\frac{\rho\left(\dfrac{\log(x/n^2)}{\log y}\right)}{n^2}$$

$$+ \frac{x}{2\pi}\sum_{n\in S(\sqrt{x/2},y)}\frac{\rho\left(\dfrac{\log(x/(2n^2))}{\log y}\right)}{n^2} + D_1 + D_2 \quad (4.22)$$

where

$$D_1 := O\left(\frac{x}{\log y}\sum_{n\in S(\sqrt{x},y)}\frac{\rho(\alpha_n)\log(\alpha_n+1)}{n^2}\right)$$

$$+ O\left(x\log y\exp(-c_1\sqrt{\log y})\sum_{n\in S(\sqrt{x},y)}\frac{\alpha_n^2\log\alpha_n}{n^2}\right)$$

$$D_2 := O\left(\frac{x}{\log y}\sum_{n\in S\left(\sqrt{x/2},y\right)}\frac{\rho(\tilde{\alpha}_n)\log(\tilde{\alpha}_n+1)}{n^2}\right)$$

$$+ O\left(x\log y\exp(-c_1\sqrt{\log y})\sum_{n\in S\left(\sqrt{x/2},y\right)}\frac{\tilde{\alpha}_n^2\log\tilde{\alpha}_n}{n^2}\right)$$

where for convenience we put

$$\alpha_n = \log(x/n^2)/\log y, \quad \tilde{\alpha}_n = \log(x/(2n^2))/\log y.$$

We estimate the first sum in the right-hand side of (4.22) by Stieltjes integration and integration by parts to get

$$
\sum_{n \in S(\sqrt{x}, y)} \frac{\rho\left(\dfrac{\log(x/n^2)}{\log y}\right)}{n^2} = \int_{1^-}^{\sqrt{x}} \rho\left(\frac{\log(x/t^2)}{\log y}\right) d\left(\sum_{n \in S(t,y)} n^{-2}\right)
$$

$$
= \rho\left(\frac{\log(x/t^2)}{\log y}\right)\left(\sum_{n \in S(t,y)} n^{-2}\right)\Bigg|_{1^-}^{\sqrt{x}}
$$

$$
+ \frac{2}{\log y}\int_1^{\sqrt{x}} t^{-1}\rho'\left(\frac{\log(x/t^2)}{\log y}\right)\left(\sum_{n \in S(t,y)} n^{-2}\right) dt
$$

$$
= \sum_{n \in S(\sqrt{x}, y)} \frac{1}{n^2} + \frac{2}{\log y}\int_1^{\sqrt{x}} t^{-1}\rho'\left(\frac{\log(x/t^2)}{\log y}\right)\left(\sum_{n \in S(t,y)} n^{-2}\right) dt. \tag{4.23}
$$

We now use Lemma 3.11 in the right-hand side of (4.23) to get

$$
\sum_{n \in S(\sqrt{x}, y)} \frac{\rho\left(\dfrac{\log(x/n^2)}{\log y}\right)}{n^2} = \zeta(2) + O(x^{-1/2}) + O(y^{-1})
$$

$$
- \zeta(2)\int_1^{\sqrt{x}} d\left(\rho\left(\frac{\log(x/t^2)}{\log y}\right)\right) + O\left(\frac{1}{\log y}\int_1^{\sqrt{x}} \rho'\left(\frac{\log(x/t^2)}{\log y}\right)\frac{dt}{t^2}\right)
$$

$$
+ O\left(\frac{1}{y\log y}\int_1^{\sqrt{x}} \rho'\left(\frac{\log(x/t^2)}{\log y}\right)\frac{dt}{t}\right)
$$

$$
= \zeta(2)\rho(\alpha) + O(x^{-1/2}) + O(y^{-1}) + E_1 + E_2 \tag{4.24}
$$

where E_1 and E_2 are respectively the last two O-terms in the penultimate step above. We estimate E_1 first. To do so, we use (4.9) and (4.10) to find that for $C_4 \log \alpha < \frac{1}{4}\log y$ (satisfied if y is in the range specified in the theorem and $x \gg 1$), we have

$$
E_1 \ll \frac{\rho(\alpha)\log(\alpha + 1)}{\log y}. \tag{4.25}
$$

To estimate E_2 we notice that

$$
E_2 \ll \frac{1}{y}\int_1^{\sqrt{x}} d\left(\rho\left(\frac{\log(x/t^2)}{\log y}\right)\right) = \frac{1 - \rho(\alpha)}{y} \ll \frac{1}{y}. \tag{4.26}
$$

Thus, (4.24), (4.25), (4.26) and the fact that $\rho(\alpha) = O\left(e^{-\alpha \log \alpha}\right)$ as $\alpha \to \infty$ yield the following in the range specified in the theorem:

$$\frac{x}{\pi} \sum_{n \in S(\sqrt{x}, y)} \frac{\rho\left(\dfrac{\log(x/n^2)}{\log y}\right)}{n^2} = \frac{\zeta(2)\rho(\alpha)x}{\pi} + O\left(\frac{x\rho(\alpha)\log(\alpha + 1)}{\log y}\right) + O(xy^{-1}).$$

(4.27)

Replacing x by $x/2$, it follows that

$$\frac{x}{2\pi} \sum_{n \in S\left(\sqrt{x/2}, y\right)} \frac{\rho\left(\dfrac{\log(x/(2n^2))}{\log y}\right)}{n^2} = \frac{\zeta(2)\rho(\alpha)x}{2\pi} + O\left(\frac{x\rho(\alpha)\log(\alpha + 1)}{\log y}\right)$$

$$+ O(xy^{-1}).$$

(4.28)

Let us now estimate D_1. In a similar way, D_2 can be estimated. First, we note that the second O-term in D_1 can be trivially estimated and we have that

$$x \log y \exp(-c_1\sqrt{\log y}) \sum_{n \in S(\sqrt{x}, y)} \frac{\alpha_n^2 \log \alpha_n}{n^2} \ll \alpha^2 \log \alpha \, x \log y \exp(-c_1\sqrt{\log y}).$$

(4.29)

The first O-term in D_1 is

$$\frac{x}{\log y} \sum_{n \in S(\sqrt{x}, y)} \frac{\rho(\alpha_n)\log(\alpha_n + 1)}{n^2} \ll \frac{x \log(\alpha + 1)}{\log y} \sum_{n \in S(\sqrt{x}, y)} \frac{\rho(\alpha_n)}{n^2}$$

$$\ll \frac{x \log^2(\alpha + 1)\rho(\alpha)}{\log y} + \frac{x \log(\alpha + 1)}{y \log y} \quad (4.30)$$

where we have used (4.27) in the last line above. We have similar estimates for the two O-terms in D_2. Thus we have from (4.21), (4.22), (4.27), (4.28), (4.29) and (4.30) that

$$\sum_{n \in S(x, y)} q_2(n) = \frac{\pi\rho(\alpha)x}{4} + O\left(\frac{x\rho(\alpha)\log^2(\alpha + 1)}{\log y}\right)$$

$$+ O\left(\alpha^2 \log \alpha \, x \log y \exp(-c_1\sqrt{\log y})\right) + O\left(\frac{x \log(\alpha + 1)}{y \log y}\right) + O\left(xy^{-1}\right).$$

(4.31)

Finally, if y is in the range specified in the theorem, then it follows that the four O-terms in (4.31) are smaller than the main term and the last two O-terms can be absorbed in the first two, and we are done.

4.3. Proof of Theorem 2.3. The proof of this result is in the same vein as the proof of the asymptotic formula for the partial sum of the *generalized divisor function* in [14, Theorem 7.17]. Thus, we freely skip certain details if the arguments are exactly as laid out in [14, Theorem 7.17]. Using Euler product formula for $\zeta(s)\beta(s)$, we obtain

$$\sum_{n\geq 1} \frac{q_z(n)}{n^s} = (\zeta(s)\beta(s))^{z-1} = \prod_p \left(1 - \frac{1}{p^s}\right)^{1-z} \left(1 - \frac{\chi_{-4}(p)}{p^s}\right)^{1-z}$$

$$= \left(1 - \frac{1}{2^s}\right)^{1-z} \prod_{p\equiv 1 \,(\mathrm{mod}\ 4)} \left(1 - \frac{1}{p^s}\right)^{2-2z} \prod_{p\equiv 3 \,(\mathrm{mod}\ 4)} \left(1 - \frac{1}{p^{2s}}\right)^{1-z}.$$

(4.32)

Next, for $|\xi| < 1$ and $w \in \mathbb{C}$, we have

$$\frac{1}{(1-\xi)^w} = \sum_{\nu\geq 0} \binom{w+\nu-1}{\nu}\xi^\nu, \quad \text{where} \quad \binom{\tau}{\nu} := \frac{1}{\nu!}\prod_{0\leq j<\nu}(\tau - j), \quad \tau \in \mathbb{C}$$

and using this in the right-hand side of (4.32) we get

$$\sum_{n\geq 1} \frac{q_z(n)}{n^s} = \left(1 + \sum_{\nu_1\geq 1} \binom{z+\nu_1-2}{\nu_1}2^{-s\nu_1}\right)$$

$$\times \prod_{p\equiv 1 \,(\mathrm{mod}\ 4)} \left(1 + \sum_{\nu_2\geq 1} \binom{2z+\nu_2-3}{\nu_2}p^{-s\nu_2}\right)$$

$$\times \prod_{p\equiv 3 \,(\mathrm{mod}\ 4)} \left(1 + \sum_{\nu_3\geq 1} \binom{z+\nu_3-2}{\nu_3}p^{-2s\nu_3}\right)$$

from which it follows that for a prime p and $\nu \in \mathbb{N}$, we have

$$\begin{cases} q_z(p^\nu) = \binom{2z+\nu-3}{\nu}, & p \equiv 1 \,(\mathrm{mod}\ 4), \\ q_z(p^{2\nu}) = \binom{z+\nu-2}{\nu}, & p \equiv 3 \,(\mathrm{mod}\ 4), \\ q_z(p^{2\nu+1}) = 0, & p \equiv 3 \,(\mathrm{mod}\ 4), \\ q_z(p^\nu) = \binom{z+\nu-2}{\nu}, & p = 2. \end{cases}$$

We note that for $\nu \in \mathbb{N}$, and $|z| < B$, $\left|\binom{z+\nu-2}{\nu}\right| \leq \binom{|z-1|+\nu-1}{\nu}$ and $\left|\binom{2z+\nu-3}{\nu}\right| \leq \binom{2|z-1|+\nu-1}{\nu}$ which imply that $|q_z(p^\nu)| \leq q_{|z-1|+1}(p^\nu)$, and by multiplicativity of $q_z(n)$, it follows that

$$|q_z(n)| \leq q_{|z-1|+1}(n) \leq q_{B+2}(n). \tag{4.33}$$

Next, let $c = 1 + (\log x)^{-1}$. Then by effective Perron integral formula [14, Chapter 5, Corollary 5.3] and (1.7), we have that

$$S_z(x) := \sum_{n \leq x} q_z(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} (\zeta(s)\beta(s))^{z-1} \frac{x^s}{s} ds + R \qquad (4.34)$$

where $1 \leq T \leq x$ (to be chosen later) and

$$R \ll \sum_{\substack{x/2 < n < 2x \\ n \neq x}} |q_z(n)| \min\left(1, \frac{x}{T|x-n|}\right) + \frac{4^c + x^c}{T} \sum_{n=1}^{\infty} \frac{|q_z(n)|}{n^c}. \qquad (4.35)$$

For $U > 0$ and large, consider the set $\mathcal{B} := \{n \leq x : |x-n| \leq x/U\}$.

Using Lemma 3.12 and (4.33), it now follows that the contribution of the first sum in the error term in (4.35) for $n \in \mathcal{B}$ is

$$\ll \sum_{n \in \mathcal{B}} |q_z(n)| \ll \sum_{n \in \mathcal{B}} q_{B+2}(n) \ll \frac{x(\log x)^B}{U} + x^{\delta_{B+2}+\varepsilon} \qquad (4.36)$$

and that for $n \notin \mathcal{B}$ is

$$\ll \frac{U}{T} \sum_{n \leq 2x} |q_z(n)| \ll \frac{Ux(\log x)^B}{T}. \qquad (4.37)$$

In (4.37) we choose $T = \exp(\sqrt{\log x})$ and to minimize the error in (4.36) and (4.37), we choose $U = \sqrt{T} = \exp\left(\frac{1}{2}\sqrt{\log x}\right)$. Thus, the contribution of error from the first sum in (4.35) is $\ll x(\log x)^{-B-3}$. The second sum in the error term in (4.35) is $\ll (\zeta(c)\beta(c))^{B-1} \ll (\log x)^{2B-2}$ in view of Lemma 3.3, thus the total error from the second expression in (4.35) is $\ll x(\log x)^{2B-2}/T \ll x(\log x)^{-B-3}$. This combined with (4.34), (4.36) and (4.37) yield

$$S_z(x) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} (\zeta(s)\beta(s))^{z-1} \frac{x^s}{s} ds + O\left(x(\log x)^{-B-3}\right). \qquad (4.38)$$

Now, we deform the truncated contour $[c - iT, c + iT]$ to a path consisting of $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ where $\mathcal{C}_1$ is polygonal with vertices $c - iT$, $\sigma_1 - iT$, $\sigma_1 - i/\log x$; $\mathcal{C}_2$ begins with the line segment from $\sigma_1 - i/\log x$ to $1 - i/\log x$, continues with the semicircle $\{1 + e^{i\theta}/\log x : -\pi/2 \leq \theta \leq \pi/2\}$, and concludes with the line segment from $1 + i/\log x$ to $\sigma_1 + i/\log x$; and finally, $\mathcal{C}_3$ is polygonal with vertices $\sigma_1 + i/\log x$, $\sigma_1 + iT$, $c + iT$ where we choose $\sigma_1 = 1 - c_0(\log T)^{-1}$, for some positive constant c_0. Then Theorem 3.1 and Lemma 3.3 yield that $(\zeta(s)\beta(s))^{z-1} \ll (\log x)^{2B-2}$ on $\mathcal{C}_1 \cup \mathcal{C}_3$ whence the contribution of the integral on $\mathcal{C}_1 \cup \mathcal{C}_3$ is $\ll x(\log x)^{-B-3}$. On $\mathcal{C}_2$, we have $(\zeta(s)\beta(s))^{z-1}/s = \beta(1)^{z-1}(s-1)^{1-z}(1 + O(|s-1|))$. Hence,

$$\frac{1}{2\pi i} \int_{\mathcal{C}_2} (\zeta(s)\beta(s))^{z-1} \frac{x^s}{s} ds = \frac{\beta(1)^{z-1}}{2\pi i} \int_{\mathcal{C}_2} (s-1)^{1-z} x^s ds$$

$$+ O\left(\int_{\mathcal{C}_2} |s-1|^{2-\mathrm{Re}(z)} x^{\sigma} |ds|\right) \qquad (4.39)$$

To estimate the integrals in the right-hand side of (4.39), we follow the same strategy as in the proof of [14, Theorem 7.17]. By changing variable $s = 1 + w/\log x$ and relating the integral on $\mathcal{C}_2$ to Hankel's formula

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{\mathcal{H}} e^s s^{-z} ds \qquad (4.40)$$

where $\mathcal{H}$ is the *Hankel contour* (consisting of a path encircling zero in the positive direction beginning at and returning to negative infinity with respect to the branch cut along the negative real axis), it follows from (4.38)–(4.40) that

$$S_z(x) = \frac{x(\log x)^{z-2}\beta(1)^{z-1}}{\Gamma(z-1)} + O\left(x(\log x)^{\mathrm{Re}(z)-3}\right).$$

5. Conclusions

The range of y in Theorems 2.1 or 2.2 cannot be improved by our methods since this requires improving the error term in the Siegel-Walfisz theorem. However, on the Generalized Riemann Hypothesis (GRH), one can show that the range of y where these theorems hold is

$$\exp((\log x)^{\delta}) \le y \le x, \quad \delta > 0. \qquad (5.1)$$

To further improve the range in these theorems, one might need to adopt the methods of Hildebrand [10].

6. Acknowledgements

References

[1] ALLADI, K. Asymptotic Estimates of Sums Involving the Moebius Function. II. *Trans. Amer. Math. Soc.* **272:1** (1982), 87–105. MR0656482, Zbl 0499.10050, doi: 10.2307/1998952. 273

[2] APOSTOL, T. Introduction to Analytic Number Theory. *Springer-Verlag, New York-Heidelberg*, Undergraduate Texts in Mathematics, 1976. MR0434929, Zbl 0335.10001, doi: 10.1007/978-1-4757-5579-4. 276, 278, 282

[3] BERNDT, B.C.; KIM, S.; ZAHARESCU, A. The circle problem of Gauss and the divisor problem of Dirichlet–still unsolved. *The American Mathematical Monthly* **125** (2018), 99–114. MR3756337, Zbl 1396.11001, doi: 10.1080/00029890.2018.1401853. 271, 279

[4] DAVENPORT, H. Multiplicative number theory. *Springer–Verlag, New York, Third edition, Grad. Texts in Math.* **74** (2000). MR1790423, Zbl 1002.11001. 275

[5] DE LA VALLÉE POUSSIN, C.-J. Sur la fonction $\zeta(s)$ de Riemann et le nombres des nombres premiers inférieurs à une limite donnée. *Colloque sur la Théorie des Nombres, Bruxelles* (1956), 9–66. MR0079602, Zbl 0072.03703. 275

[6] DICKMAN, K. On the frequency of numbers containing prime factors of a certain relative magnitude. *Ark. Mat. Astr. Fys.* **22** (1930), 1–14. Zbl JFM 56.0178.04. 275

[7] DE BRUIJN, N.G. On the number of positive integers $\leq$ x and free of prime factors $> y$. *Nederl. Akad. Wetensch. Proc. Ser. A* **54** (1951), 50–60. MR0046375, Zbl 0042.04204, doi: 10.1016/S1385-7258(51)50008-2. 281, 286, 287

[8] GAUSS, C.F. Disquistiones Arithmeticae. *Springer-Verlag, New York* (1986). MR0837656, Zbl 0136.32301, doi: 10.1007/978-1-4939-7560-0. 271

[9] GRANVILLE, A. Smooth numbers: Computational number theory and beyond. *Algorithmic Number Theory: lattices, number fields, curves and cryptography, MSRI Publications* **44** (2008), 267–323. MR2467549, Zbl 1230.11157. 273

[10] HILDEBRAND, A. On the number of positive integers $\leq x$ and free of prime factors $> y$. *J. Number Theory* **22** (1986), 289–307. MR0831874, Zbl 0575.10038, doi: 10.1016/0022-314X(86)90013-2. 293

[11] JACOBI, C.G.J. Fundamenta Nova Theoriae Functionum Ellipticarum. *Regiomonti. Sumptibus fratrum Bornträger* **45** (1829); reprinted in Jacobi's Gesammelte Werke, vol. 1, Reimer, Berlin, 1881–1891, pp. 49–239; reprinted by Chelsea, New York, 1969; Now distributed by The American Mathematical Society, Providence, RI. 271

[12] LANDAU, E. Über einen Satz des Herrn Sierpiński. *Giornale di Matematiche di Battaglini.* **51** (1913), 73–81. 271

[13] LANDAU, E. Über die Gitterpunkte in einem Kreise (Erste Mitteilung). *Nachr. Akad. Wiss. Göttingen.* (1915), 148–160. 271

[14] MONTGOMERY, H.L.;VAUGHAN, R.C. Multiplicative Number Theory I-Classical Theory. *Cambridge University Press* (2006). 275, 276, 291, 292, 293

[15] SIERPIŃSKI, W. O pewnem zagadnieniu z rachunku funkcyj asymptotycznych. *Prace Mat. Fiz.* **17** (1906), 77–118. Zbl JFM 37.0236.02. 279

[16] ROSSER, B. The nth prime is greater than $n \log n$. *Proc. London Math. Soc.* **45** (1939), 21–44. MR1576808, Zbl 0019.39401, doi: 10.1112/plms/s2-45.1.21.

[17] SELBERG, A. Note on a paper by L. G. Sathe. *J. Indian Math. Soc.* **18** (1954), 83–87. MR0067143, Zbl 0057.28502, doi: 10.18311/jims/1954/17018. 272

(Ankush Goswami) SCHOOL OF MATHEMATICAL AND STATISTICAL SCIENCES, THE UNIVERSITY OF TEXAS RIO GRANDE VALLEY, 1201 W. UNIVERSITY DR., EDINBURG, TX 78539, USA
ankush.goswami@utrgv.edu, ankushgoswami3@gmail.com