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CRITICAL SETS OF SOLUTIONS TO ELLIPTIC EQUATIONS

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Abstract

Let $u \not\equiv$ const. satisfy an elliptic equation $\mathcal{L}_0 u \equiv \sum a_{ij} D_{ij} u + \sum b_j D_j u = 0$ with smooth coefficients in a domain in $\mathbf{R}^n$. It is shown that the critical set $|\nabla u|^{-1}\{0\}$ has locally finite $(n-2)$-dimensional Hausdorff measure. This implies in particular that for a solution $u \not\equiv 0$ of $(\mathcal{L}_0 + c)u = 0$, with $c \in C^\infty$, the singular set $u^{-1}\{0\} \cap |\nabla u|^{-1}\{0\}$ has locally finite $(n-2)$-dimensional Hausdorff measure.

1. Introduction and main results

Let Ω be a domain in $\mathbf{R}^n$, $n \geq 3$, and let $u \not\equiv 0$ be a real-valued classical solution of the elliptic partial differential equation

$$\mathcal{L}u \equiv \sum_{i,j=1}^{n} a_{ij} D_{ij} u + \sum_{i=1}^{n} b_j D_j u + cu = 0 \quad \text{in } \Omega, \tag{1.1}$$

where the real-valued coefficients a_{ij}, b_j, c are C^∞ functions in Ω. We call

$$\Sigma(u) = |\nabla u|^{-1}\{0\} \text{ and } \Sigma_0(u) = \Sigma(u) \cap u^{-1}\{0\}$$

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the critical and singular sets of u, respectively. In the following we shall show that locally the singular set Σ_0 of u has finite $(n-2)$ - dimensional Hausdorff measure, i.e., $\mathcal{H}^{n-2}(\Sigma_0(u) \cap K) < \infty$ for all compact $K \subset \Omega$. The first author and the remaining three authors independently wrote preprints proving this result, and the present paper is a combination of these two works.

For $n = 2$, $\Sigma(u)$ is well-known to consist of isolated points. For $n \geq 3$ an elementary argument (see [20, Section 1.9]), first given by L. Caffarelli and A. Friedman [7] for $\Delta u + f(x,u) = 0$, shows that $\Sigma(u)$ is contained in a countable union of smooth $(n-2)$-dimensional submanifolds. Q. Han [16] obtains similar structural results with much weaker assumptions on the smoothness of the coefficients. In particular, he proved that $\Sigma(u)$ is essentially contained in a countable union of $\mathcal{C}^{1,\alpha}$ graphs if the coefficients are Lipschitz. But, even for smooth coefficients, the question remained concerning the size of $\Sigma(u)$. Last year it was shown in [19] that for $n = 3$, $\Sigma_0(u)$ has locally finite 1-dimensional Hausdorff measure.

Here we generalize this to $n \geq 3$ dimensions. Our result is obtained by showing that the critical set Σ of a solution of (1.1) with $c \equiv 0$ has locally finite $(n-2)$-dimensional Hausdorff measure.

Recently there is a rather rich literature describing the 'size' of the zero set, and in particular the singular set Σ_0 of solutions to elliptic equations in terms of the appropriate Hausdorff measure and Hausdorff dimension respectively. See the list of references in the introduction of [19]. The size of the nodal set was considered in the conjecture of S.T. Yau [27] that $\mathcal{H}^{n-1}(u_\lambda^{-1}\{0\}) \sim c\sqrt{\lambda}$ for the λ-eigenfunction u_λ on a compact Riemannian manifold. This was established for real analytic metrics by H. Donnelly and C. Fefferman [9]. Note that, for real analytic coefficients, the local finiteness, without estimates, of $\mathcal{H}^{n-1}(u^{-1}\{0\})$ (or $\mathcal{H}^{n-2}(\Sigma(u))$) follows just from the real analyticity of u [11, 3.4.8]. For the nonanalytic case, R. Hardt and L. Simon [20] proved the local finiteness of $\mathcal{H}^{n-1}(u^{-1}\{0\})$ with the coefficients being only Lipschitz smooth. However, for the Riemannian manifold application, their upper estimate $C\lambda^{c\sqrt{\lambda}}$ is weaker than Yau's conjecture. F. H. Lin and Q. Han [22], [17], [18] proved a parabolic nodal set estimate (with time-independent coefficients), simplified several arguments in [9] and [20], and made estimates involving the frequency (or order) $N_R \equiv \left[R \int_{\mathbf{B}_R} |\nabla u|^2\, dx\right] / \left[\int_{\partial \mathbf{B}_R} u^2\, d\mathcal{H}^{n-1}\right]$. Lin [22] also conjectured that

$$\mathcal{H}^{n-1}(u^{-1}\{0\} \cap \mathbf{B}_{R/2}) \leq C N_R \quad \text{and } \mathcal{H}^{n-2}(\Sigma(u) \cap \mathbf{B}_{R/2}) \leq C N_R^2 \,.$$

While more precise results are known in 2 dimensions [2], [8], [10], [24], the general Yau and Lin conjectures remain open. Two very recent preprints give some nonexplicit bounds. [14] treats coefficients with finite differentiability, and [15] handles higher order equations. In another preprint [3], C. Bär discusses nodal sets for first order semilinear elliptic systems.

Basic for all these investigations is the asymptotic behaviour of $u(x)$ as $x \to x_0$, where $u(x_0) = 0$. Let $\mathcal{O} \in \Omega$, and let u be a solution of (1.1). Then it is well known (see e.g. [4]) that

$$u(x) = p_M + O(|x|^{M+1}) \text{ as } |x| \to 0, \tag{1.2}$$

where $p_M \not\equiv 0$ is a homogeneous polynomial of degree M satisfying the osculating equation

$$\sum_{i,j} a_{ij}(\mathcal{O}) D_{ij} p_M = 0.$$

Assume without loss of generality that $a_{ij}(\mathcal{O}) = \delta_{ij}$ so that p_M is harmonic. Therefore the investigations of the zero set, respectively singular set, of a solution of (1.1) are motivated by the desire to understand to which extent these sets can be described locally by the zero sets, respectively critical sets, of harmonic homogeneous polynomials. For a harmonic polynomial P_M of degree M in n variables it is known (see e.g. [20]) that for some $C(n) < \infty$

$$\mathcal{H}^{n-2}(\Sigma(P_M) \cap B_1) \leq C(n) M^2, \tag{1.3}$$

B_1 denoting a ball with radius 1.

On the other hand there are examples showing that the singular set of a solution of an elliptic equation can be rather wild. See [19, Section 1]. Conversations with L. Simon also led to the following simple example: For any closed subset K of $\mathbf{R}$, let f be a nonnegative smooth function vanishing exactly on K with $|ff''| + |f'^2| < 1/4$. Then $u(x, y, z) = xy + f^2(z)$ satisfies the elliptic equation $u_{xx} + u_{yy} + u_{zz} - (f^2)''(z) u_{xy} = 0$, and has singular set equaling $\{(0, 0)\} \times K$.

To state now our main results, we define the elliptic operator $\mathcal{L}_0$ by

$$\mathcal{L}_0 = \mathcal{L} - c$$

with $\mathcal{L}$ and c given according to (1.1).

Theorem 1.1. *Let* $u \not\equiv \text{const}$. *satisfy*

$$\mathcal{L}_0 u = 0 \quad \text{in } \Omega, \quad \Omega \subset \mathbf{R}^n. \tag{1.1'}$$

Then for every compact subset K *of* Ω

$$\mathcal{H}^{n-2}(\Sigma(u) \cap K) < \infty.$$

Corollary 1.1. *Let* $u \not\equiv 0$ *satisfy equation (1.1). Then for every compact subset* K *of* Ω

$$\mathcal{H}^{n-2}(\Sigma_0(u) \cap K) < \infty.$$

The Corollary is a rather immediate consequence of Theorem 1.1:

Proof of the Corollary. Given $x_0 \in \Omega$ there is a neighbourhood $U(x_0)$ and a $u_0 \in C^\infty(U(x_0))$ with $u_0 > 0$ and $\mathcal{L}u_0 = 0$ in $U(x_0)$. See e.g. [5, p.228]. It is easily seen that $\mu \equiv u u_0^{-1}$ satisfies in $U(x_0)$ an equation of type (1.1), so that by Theorem 1.1, $H^{n-2}(\Sigma(\mu) \cap U') < \infty$ for every compact subset U' of $U(x_0)$. Furthermore the singular set of u is a subset of the critical set of μ. q.e.d.

Remark. That the assertion of Theorem 1.1 is false if $\mathcal{L}_0$ is replaced by $\mathcal{L}$ can be seen from the following example: Let $v \in C^\infty(B)$, $B \subset \mathbf{R}^n$, with $|v| < 1$. Then with $u = v^2 + 1$ and $c = (\Delta v^2)(v^2+1)^{-1}$, $\Delta u + cu = 0$ and $\Sigma(u) = v^{-1}\{0\}$. But every closed subset of $\mathbf{R}^n$ can be the zero set of a C^∞-function (see e.g. [26])!

The structure of the proof of Theorem 1.1 is similar to the 3-dimensional case in [19]. For this it was crucial to show [19, Theorem 3.1] that *in 3-dimensions, the complex dimension of the complex critical set of a homogeneous real harmonic polynomial is at most one.* Here it is shown that the complex critical set of a homogeneous harmonic polynomial P with real coefficients has at most complex dimension $n-2$ (Theorem 2.1). With this result it can be proven that for suitable complex 2-planes ϵ_{ij}, $1 \le i < j \le n$, $P|_{\epsilon_{ij}}$ has an isolated critical point in the origin of $\mathbf{C}^2$ for all i, j. Using results from singularity theory, [1], this implies that the algebraic multiplicity of the gradient map of $P|_{\epsilon_{ij}}$ at the origin is finite. Further looking at the restriction of the solution u to affine 2-planes it follows via a C^∞-perturbation argument that the number of critical points of u restricted to these affine 2-plane slices is

uniformly bounded in a small enough neighborhood of the origin. This estimate together with the countable rectifiability of $\Sigma(u)$ (which follows immediately using the arguments from the proof of Lemma 1.9 in [20], or see also [7]) allows us to apply a geometric measure inequality of Federer [11] which yields the desired result.

2. The critical set of a harmonic homogeneous polynomial

A *homogeneous polynomial* of degree $k \geq 1$ on $\mathbf{R}^n$ is a nonzero function in the form

$$u(x) \;=\; \sum_{\|\alpha\|=k} a_\alpha x^\alpha,$$

where $a_\alpha \in \mathbf{R}$, $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ for $x = (x_1, \dots, x_n) \in \mathbf{R}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \{0, 1, \dots\}^n$, and $\|\alpha\| \equiv \alpha_1 + \cdots + \alpha_n$.

The critical set $\Sigma(v)$ of a polynomial $v(x) = \sum_{\|\alpha\| \leq k} b_\alpha x^\alpha$ is a real algebraic variety that is a cone in case v is homogeneous. Extending v to a complex-valued polynomial, also denoted v, on $\mathbf{C}^n$ by replacing each x_i by z_i, we also have the *complex critical (zero) set*

$$\begin{aligned} \Sigma_{\mathbf{C}}(v) \;\equiv\; \{z = (z_1, \dots, z_n) \in \mathbf{C}^n \;:\; \\ v(0) = \frac{\partial v}{\partial z_1}(z) = \cdots = \frac{\partial v}{\partial z_n}(z) = 0\} \;, \end{aligned}$$

which is a complex algebraic variety that satisfies

$$\Sigma_{\mathbf{C}}(v) \cap \mathbf{R}^n = \; \Sigma(v) \;.$$

Analogously we denote $\Sigma_{0\mathbf{C}}(v) = \Sigma_{\mathbf{C}}(v) \cap v^{-1}\{0\}$.

For a nonconstant polynomial v on $\mathbf{R}^n$, one thus always has the rough estimates

$$\dim_{\mathbf{R}} \Sigma(v) \;\leq\; n-1 \quad \text{and} \quad \dim_{\mathbf{C}} \Sigma_{\mathbf{C}}(v) \;\leq\; n-1 \;.$$

Suppose now that v is a nonconstant *harmonic* polynomial on $\mathbf{R}^n$. From [7], [20], we know that

$$\dim_{\mathbf{R}} \Sigma(v) \;\leq\; n-2 \;.$$

[19, Theorem 3.1] also shows that

$$\dim_{\mathbf{C}} \Sigma_{\mathbf{C}}(v) \;\leq\; 1 \quad \text{in case} \quad n = 3 \;,$$

and v is homogeneous.

For the proof of Theorem 1.1 we need the generalisation of this last result to n dimensions, which is given below. Thereby we thank H. Knörrer for crucial remarks.

Theorem 2.1. *Let P be a harmonic homogeneous polynomial in $\mathbf{C}^n$ with real coefficients, $P \not\equiv \text{const}$. Then $\dim \Sigma_{\mathbf{C}}(P) \leq n-2$.*

Proof of Theorem 2.1. For any nonconstant irreducible polynomial in $\mathbf{C}^n$ the conclusion is true. See e.g. [25, Chapter II, 1.4].

So now we assume P be reducible, and, for contradiction, that $\dim \Sigma_{\mathbf{C}}(P) = n-1$. Then P can be represented as

(2.1) $P = p^2 q$, where p and q are homogenous and p is irreducible.

This can be seen as follows: Let $P = \prod_{j=1}^{k} q_j$, q_j irreducible $\forall j$, with $k \geq 2$, and denote $N_j = q_j^{-1}\{0\}$. If for $i \neq j$, $\dim N_i \cap N_j < n-1$, then clearly $\dim \Sigma(P) < n-1$. Without loss of generality we assume $\dim N_1 \cap N_2 = n-1$. Since q_1, q_2 are irreducible, this implies (see e.g. [23, Lemma 2.5]) $q_1 = \text{const}\, q_2$, proving (2.1).

(2.1) now implies a nontrivial factorization

$$P = \tilde{p}^2 \tilde{q}, \tag{2.2}$$

where $\tilde{p}, \tilde{q}$ are homogeneous polynomials with real coefficients.

This can be seen as follows: Let $\underline{f}$ denote the polynomial which is obtained from the polynomial f by complex conjugations of its coefficients. Since P has real coefficients, we conclude from (2.1) that $P = \underline{p}^2 \underline{q} = p^2 q$. Since p is irreducible, $q = \underline{p}^2 \tilde{q}$ follows for some homogeneous polynomial $\tilde{q}$. Hence $P = (p\underline{p})^2 \tilde{q}$ and $p\underline{p}$ has real coefficients.

Finally we use:

Proposition 2.2. *If P is a harmonic polynomial in $\mathbf{R}^n$ given by $P = p^2 q$, where p and q are homogeneous polynomials with real coefficients and $p \not\equiv \text{const}$, then $P \equiv 0$.*

Proof of Proposition 2.2. Let M denote the degree of P. Then for some spherical harmonic $Y_M(x|x|^{-1})$, $P(x) = |x|^M Y_M$ in polar coordinates. Further we have

$$q|_{S^{n-1}} = \sum_{j=1}^{M-1} a_j Y_j,$$

where each $a_j \in \mathbf{R}$, and the Y_j's are spherical harmonics of degree $\leq M-1$, which can be taken to be orthonormal on S^{n-1}. Hence $\int_{S^{n-1}} Y_j Y_M d\omega = 0$ for $j \neq M$, and

$$\int_{S^{n-1}} p^2 q^2 d\omega = \int_{S^{n-1}} Pq d\omega = 0$$

implying $pq \equiv 0$. q.e.d.

Now we may combine (2.2) and Proposition 2.2 to conclude that $P \equiv 0$. This contradicts that $P \not\equiv$ const and finishes the proof of Theorem 2.1. q.e.d.

3. Restriction to 2-plane slices

Suppose now $p : \mathbf{C}^n \to \mathbf{C}$ is a complex *homogeneous* polynomial; hence,

$$\Sigma_{0\mathbf{C}}(p) = \Sigma_{\mathbf{C}}(p).$$

For any complex 2-dimensional subspace $\epsilon \subset \mathbf{C}^n$, the restriction $p|_\epsilon$ is essentially a complex homogeneous polynomial of two variables. Moreover, *for* $z \in \epsilon \setminus \{0\}$,

$$\nabla(p|_\epsilon)(z) = 0$$

if and only if either

$$z \in \Sigma_{\mathbf{C}}(p)$$

or

(3.1) $\quad z \in p^{-1}\{0\} \setminus \Sigma_{\mathbf{C}}(p)$ and ϵ is tangent to $p^{-1}\{0\}$ at z .

For each pair i, j of integers with $1 \leq i < j \leq n$ and point $(z_1, \ldots, z_n) \in \mathbf{C}^n$, let

$$\pi_{ij}(z_1, \ldots, z_n) = (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \in \mathbf{C}^{n-2} .$$

For each real rotation $\gamma \in O(n)$, let $\gamma : \mathbf{C}^n \to \mathbf{C}^n$ also denote the complex linear extension of γ. Thus each set $(\pi_{ij} \circ \gamma)^{-1}\{y\}$, for $y \in \mathbf{C}^{n-2}$, is a complex affine 2 plane in $\mathbf{C}^n$.

3.1 Lemma. *For any nonconstant homogeneous polynomial* $p : \mathbf{C}^n \to \mathbf{C}$ *having* $\dim_{\mathbf{C}} \Sigma_{\mathbf{C}}(p) \leq n-2$, *there exists a rotation* $\gamma \in O(n)$ *so that, for all integers* $1 \leq i < j \leq n$,

$$\Sigma_{\mathbf{C}}(p) \cap (\pi_{ij} \circ \gamma)^{-1}\{0\} = \{0\}$$

and each complex 2*-plane* $(\pi_{ij}\circ\gamma)^{-1}\{0\}$ *is transverse to* $p^{-1}\{0\}\setminus\Sigma_{\mathbf{C}}(p)$. *Hence*

$$\left[\nabla p|_{(\pi_{ij}\circ\gamma)^{-1}\{0\}}\right]^{-1}\{0\} \;=\; \{0\}\ .$$

Proof. For $a\in\mathbf{S}^{n-1}$, let π_a be the complex rank $n-1$ projection of $\mathbf{C}^n$ corresponding to the orthogonal projection of $\mathbf{R}^n$ onto $a^\perp$. Thus, $\pi_a(z)=z-(z\cdot a)a$, and π_a has kernel $\pi_a^{-1}\{0\}$ equaling the complex span of a and image $\pi_a(\mathbf{C}^n)$ equaling the complex span of $a^\perp$ in $\mathbf{C}^n$.

We also define, for $y\in\pi_a(\mathbf{C}^n)$,

$$D_a(y) \;=\; \prod_{z\in\pi_a^{-1}(y)\cap p^{-1}\{0\}} \left[a_1\frac{\partial p}{\partial z_1}(z)+\cdots+a_n\frac{\partial p}{\partial z_n}(z)\right]\ ,$$

which is the discriminant of p, with respect to the direction a. Then D_a is a polynomial that is homogeneous because, for $0\neq\lambda\in\mathbf{C}$,

$$z\in\pi_a^{-1}(\lambda y)\cap p^{-1}\{0\} \;\text{ if and only if }\; \lambda^{-1}z\in\pi_a^{-1}(y)\cap p^{-1}\{0\},$$

and $\frac{\partial p}{\partial z_i}(z)=\lambda^{k-1}\frac{\partial p}{\partial z_i}(\lambda^{-1}z)$. Moreover, $D_a\not\equiv 0$, because, otherwise, ∇p would vanish on a complex $(n-1)$-dimensional stratum of $p^{-1}\{0\}$, contradicting that $\dim_{\mathbf{C}}\Sigma_{\mathbf{C}}(p)\leq n-2$.

For $b\in\mathbf{S}^{n-1}$ with $a\cdot b=0$, $\pi_b^{-1}\{0\}\subset\pi_a(\mathbf{C}^n)$ and

$$\epsilon_{a,b} \;=\; \pi_a^{-1}\left[\pi_b^{-1}\{0\}\right]$$

is the complex span of $\{a,b\}$ in $\mathbf{C}^n$. Note that this complex 2-plane is transverse to $p^{-1}\{0\}\setminus\{0\}$ if $D_a(b)\neq 0$. In fact, if $z\in\epsilon_{a,b}\cap p^{-1}\{0\}\setminus\{0\}$, then $\pi_a(z)=\lambda b$ for some $0\neq\lambda\in\mathbf{C}$. Since $D_a(\lambda b)\neq 0$, $a\cdot(\nabla p)(z)\neq 0$, and, by the complex implicit function theorem, $p^{-1}\{0\}$ is locally near z, a holomorphic graph over a domain in $\pi_a(\mathbf{C}^n)$. In particular, $p^{-1}\{0\}$ is transverse at z to $\pi_a^{-1}\left[\pi_b^{-1}\{0\}\right]=\epsilon_{a,b}$.

In the set of all pairs

$$\mathcal{A}=\{(a,b)\in\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}\ :a\cdot b=0\}\ ,$$

we are now interested in the "bad" set

$$\mathcal{B} \;=\{(a,b)\in\mathcal{A}:\ \text{ either }\Sigma_{\mathbf{C}}(p)\cap\epsilon_{a,b}\setminus\{0\}\neq\emptyset\ \text{ or }D_a(b)=0\}\ .$$

Since $\mathcal{B}$ is a semi-algebraic set [6], we may show that $\dim_{\mathbf{R}}\mathcal{B}<\dim_{\mathbf{R}}\mathcal{A}=(n-1)(n-2)$ by simply verifying that $\mathcal{B}$ contains no nonempty open subset U of $\mathcal{A}$.

Suppose, for contradiction, that there were such a U. Since $p \not\equiv 0$, $p|_{\mathbf{S}^{n-1}} \not\equiv 0$, because the coefficients of p are determined by $p|_{\mathbf{R}^n}$. Thus,

$$\dim_{\mathbf{R}} \left(\mathbf{S}^{n-1} \cap p^{-1}\{0\}\right) \ \leq \ n-2 \ ,$$

and we may choose a pair $(a,b) \in U$ with $p(a) \neq 0$. By homogeneity, $p(\lambda a) \neq 0$ for all $0 \neq \lambda \in \mathbf{C}$, hence,

$$p^{-1}\{0\} \cap \pi_a^{-1}\{0\} \ = \ \{0\} \ .$$

By the Proper Mapping Theorem and Chow's Theorem [13, pp.162,170], the projection $\pi_a(\Sigma_{\mathbf{C}}(p))$ is a complex homogeneous algebraic subvariety of $\pi_a(\mathbf{C}^n)$ of complex dimension $\leq n-2$. Moreover, the discriminant locus $D_a^{-1}\{0\}$ is also a complex homogeneous algebraic subvariety of $\pi_a(\mathbf{C}^n)$ of complex dimension $\leq n-2$. Thus,

$$\pi_a(\Sigma_{\mathbf{C}}(p)) \cup D_a^{-1}\{0\} \ \subset \ q^{-1}\{0\}$$

for some non-identically-zero complex homogeneous polynomial q on $\pi_a(\mathbf{C}^n)$, and we may similarly find a point $c \in \mathbf{S}^{n-1} \cap a^{\perp}$ near b so that $(a,c) \in U \subset \mathcal{B}$ and $q(c) \neq 0$; hence, $q^{-1}\{0\} \cap \pi_c^{-1}\{0\} = \{0\}$. But then

$$\begin{aligned} \Sigma_{\mathbf{C}}(p) \cap \epsilon_{a,c} \ &\subset \ p^{-1}\{0\} \cap \pi_a^{-1}\left[\pi_a\left(\Sigma_{\mathbf{C}}(p)\right)\right] \cap \pi_a^{-1}\left[\pi_c^{-1}\{0\}\right] \\ &\subset \ p^{-1}\{0\} \cap \pi_a^{-1}\left[q^{-1}\{0\} \cap \pi_c^{-1}\{0\}\right] \ \subset \ p^{-1}\{0\} \cap \pi_a^{-1}\{0\} \ = \{0\} \ , \end{aligned}$$

and $D_a(c) \neq 0$, contradicting that $(a,c) \in \mathcal{B}$.

Thus, $\dim_{\mathbf{R}} \mathcal{B} < (n-1)(n-2)$. For each pair of integers $1 \leq i < j \leq n$, we deduce that, in the space $\mathcal{C}$ of ordered orthonormal bases of $\mathbf{R}^n$, the set of ordered bases $(a_1, \ldots, a_n)$ with $(a_i, a_j) \in \mathcal{B}$ has dimension $< \dim_{\mathbf{R}} \mathcal{C} = (n-1)!$. In particular we are able to choose a "good" basis $(a_1, \ldots, a_n)$ so that $(a_i, a_j) \notin \mathcal{B}$ for every $1 \leq i < j \leq n$. Such a basis readily determines the desired rotation $\gamma \in O(n)$. q.e.d.

Corollary 3.2. *Let $P : \mathbf{C}^n \to \mathbf{C}$ be a nonconstant homogeneous harmonic polynomial with real coefficients. Then for some real rotation $\gamma \in O(n)$, $P|_{(\pi_{ij} \circ \gamma)^{-1}\{0\}}$ has an isolated critical zero in the origin of $\mathbf{C}^2$, $\forall i,j$ with $1 \leq i < j \leq n$.*

Proof of Corollary 3.2. Because of Theorem 2.1,

$$\dim_{\mathbf{C}} \Sigma_{\mathbf{C}}(P) \leq n-2.$$

Therefore Lemma 3.1 is applicable and yields the desired result. q.e.d.

4. Stability under smooth perturbations

Let $u \not\equiv \text{const.}$ satisfy (1.1'), i.e.,

$$\mathcal{L}_0 u = 0 \quad \text{in } \Omega, \quad \Omega \subset \mathbf{R}^n.$$

Without loss of generality, we assume that $\mathcal{O} \in \Omega$, that

$$a_{ij}(\mathcal{O}) = \delta_{ij}, \quad 1 \leq i < j \leq n$$

and that u has a critical zero in $\mathcal{O}$. Then due to (1.2)

$$u(x) = P_M(x) + O(|x|^{M+1}) \quad \text{for } |x| \to 0 \tag{4.1}$$

for some harmonic homogeneous polynomial $P_M \not\equiv 0$ of degree $M \geq 2$.

Denoting the complexification of P_M for simplicity again by P_M it follows from Corollary 3.2 that

$$P_M|_{(\pi_{ij}\circ\gamma)^{-1}\{0\}} \text{ has an isolated critical zero in the origin of } \mathbf{C}^2, \forall i, j. \tag{4.2}$$

This will be essential to show

Lemma 4.1. *There exists $R > 0$ such that*

$$\operatorname{card} \Sigma(u) \cap (\pi_{ij} \circ \gamma)^{-1}\{y\} \cap B_R \leq (M-1)^2 \tag{4.3}$$

for all $y \in B_R^{(n-2)}$ and for all i, j such that $1 \leq i < j \leq n$.

Proof of Lemma 4.1. The proof is similar to that of Lemma 2.3 in [19]. From there we use Proposition 2.2, namely:

Proposition 4.2. *Let $p(z_1, z_2)$ be a homogeneous polynomial in $\mathbf{C}^2$ of degree k with real coefficients, and assume that p has an isolated critical point at the origin in $\mathbf{C}^2$. Let further $\phi \in C^\infty(D_r(\mathcal{O}))$, $D_r(\mathcal{O}) = \{y \in \mathbf{R}^2 : |y| < r\}$, and $r > 0$, with*

$$\phi(y) = p(y) + o(|y|^k) \quad \text{for } |y| \to 0$$

and let $\phi_t(y) \in C^\infty(D(\mathcal{O}) \times I)$ for $t \in I$ where $I = [-t_0, t_0]$, with $\phi_0 = \phi$. Then there exists $\tilde{r}$, $0 < \tilde{r} < r$ such that for $|t| \leq t_0$, t_0 small enough, the number of critical points of $\phi_t(.)$ in $D_{\tilde{r}}(\mathcal{O})$ is uniformly bounded by $(k-1)^2$.

For the proof we use results in [1], namely: for a homogeneous polynomial $p(z)$, $z \in \mathbf{C}^2$ of degree k, with an isolated critical point at the

origin $\mathcal{O}$ the algebraic multiplicity of the gradient map of p in $\mathcal{O}$ is $(k-1)^2$. This together with the subadditivity of the algebraic multiplicity yields the result, which can be stated as

$$\operatorname{card}(\Sigma(\phi_t) \cap D_r(\mathcal{O})) \leq (k-1)^2 \quad \forall t, \quad |t| \leq t_0.$$

We apply this to our case and identify $\forall i, j \quad 1 \leq i < j \leq n$

$$p = P_M|_{(\pi_{ij} \circ \gamma)^{-1}\{0\}}$$

and

$$\phi_0 = u|_{(\pi_{ij} \circ \gamma)^{-1}\{0\}}.$$

Let $\sigma \in C^\infty$ denote a curve in $\mathbf{R}^{n-2}$ passing through the origin, parametrized such that $\sigma(0) = \mathcal{O}$. We define for $t \in I$, I being an interval about 0 in $\mathbf{R}$,

$$\phi_t = u|_{(\pi_{ij} \circ \gamma)^{-1}\{\sigma(t)\}}.$$

Due to Lemma 3.2 and (4.1) we can apply Proposition 4.2 and obtain for some $\tilde{r} > 0$

$$\operatorname{card}(\Sigma(u|_{(\pi_{ij} \circ \gamma)^{-1}\{\sigma(t)\}}) \cap D_{\tilde{r}}(\mathcal{O})) \leq (M-1)^2$$

for t, $|t| \leq t_0$, t_0 small enough. This implies further that for some $\overline{R} > 0$ and $\overline{t} > 0$

$$\operatorname{card}(\Sigma(u) \cap (\pi_{ij} \circ \gamma)^{-1}\{\sigma(t)\} \cap B_{\overline{R}}) \leq (M-1)^2 \quad \forall t, |t| \leq \overline{t}. \tag{4.4}$$

Suppose now for contradiction that Lemma 4.1 is false. Then for some i, j there are sequences $\{R_k\}$ and $\{y^{(k)}\}$ with $y^{(k)} \in \mathbf{R}^{n-2}$, $R_k \to 0$, $|y^{(k)}| \to 0$ for $k \to \infty$ such that

$$\operatorname{card}(\Sigma(u) \cap (\pi_{ij} \circ \gamma)^{-1}\{y^{(k)}\} \cap B_{R_k}) > (M-1)^2.$$

Proposition 4.3. *Let $\{y^{(k)}\}$ denote a sequence in $\mathbf{R}^n$ convergent to some $\overline{y}$. Then there is a subsequence which is a subset of a C^∞-curve in $\mathbf{R}^n$.*

Proof of Proposition 4.3. We use a result of Kriegl [21] (see also Lemma 4.2.15 in [12]):

Let $x_m \in \mathbf{R}^n$, $x_m \to \overline{x}$ for $m \to \infty$ and let $t_m \in \mathbf{R}$, $t_m \downarrow 0$ for $m \to \infty$. If $\forall k$, $k \in \mathbf{N}$, $\{(x_m - x_{m+1})(t_m - t_{m+1})^{-k}\}$ is bounded, then for some C^∞-curve γ, $\gamma(t_m) = x_m$, $\forall m$ and $\gamma^{(j)}(t_m) = 0$, $\forall j \in \mathbf{N}$.

From any convergent sequence $\{y^{(k)}\}$ it is easily seen that we can pick a subsequence converging fast enough so that the assumptions above are satisfied. q.e.d.

Returning to the proof of Lemma 4.1, we conclude that we can pick a subsequence of $\{y^{(k)}\}$ (again denoted by $\{y^{(k)}\}$) such that for some $\sigma \in C^\infty$, $\sigma(t_k) = y^{(k)}$, $\forall k$ and

$$\text{(4.5)} \qquad \operatorname{card}(\Sigma(u) \cap (\pi_{ij} \circ \gamma)^{-1}\{\sigma(t_k)\} \cap B_{R_k}) > (M-1)^2 \quad \forall k.$$

On the other hand given σ, there are $\overline{R}, \overline{t} > 0$ such that (4.4) holds. But this contradicts (4.5) and completes the proof of Lemma 4.1. q.e.d.

5. Finiteness of the measure of the critical set

We first need

Lemma 5.1. *Let $u \not\equiv$ const satisfy (1.1') and B be a ball with $\overline{B} \subset \Omega$. Then $\Sigma(u) \cap B$ decomposes into the countable union of subsets of a pairwise disjoint collection of smooth $n-2$ dimensional submanifolds, i.e., $\Sigma(u) \cap B$ is a countably $(n-2)$-rectifiable subset in the sense of Federer [11].*

Proof of Lemma 5.1. The proof in principle is the same as the one of Lemma 1.9 in [20]: Thereby the argument is essentially that used by Cafarelli and Friedman [7]:

Let for $q = 1, 2, 3, \ldots$

$$S_q = \{x | D^\alpha u(x) = 0, \forall \alpha \text{ with } 0 < |\alpha| \le q, \quad D^{q+1} u(x) \neq 0\}.$$

For any ball B_{2R} with $\overline{B}_{2R} \subset \Omega$ and any point $x_0 \in \overline{B}_R$, consider the equation $\mathcal{L}_0(u - u(x_0)) = 0$. Since the coefficients are smooth, it follows via unique continuation that $u - u(x_0)$ vanishes to some finite order $M(x_0)$ at x_0 and

$$\sup_{x_0 \in \overline{B}_{2R}} M(x_0) \equiv \overline{M} < \infty \ .$$

Thus $\forall a \in \Sigma(u) \cap B_R$,

$$B_R(a) \cap \{x | \nabla u(x) = 0\} = B_R(a) \cap \bigcup_{q=1}^{\overline{M}} S_q.$$

The remaining part of the proof is the same as in e.g. [20] or [7].
q.e.d.

Due to Lemma 5.1 we have in particular

$$\Sigma(u) \cap B_R = \cup_{m=1}^{\infty} E_m,$$

where $E_1 \subset E_2 \subset \dots$ are Borel subsets of $\Sigma(u)$ of finite $\mathcal{H}^{n-2}$-measure.

Without loss of generality we change coordinates to make $\gamma = \mathrm{Id}$ in Lemma 4.1. Then we use the integral geometric inequality 3.2.27 in [11] and Lemma 4.1 to obtain the following estimate:

With $R > 0$ given in Lemma 4.1

$$\begin{aligned}\mathcal{H}^{n-1}(\Sigma(u) \cap \overline{B}_R) &= \lim_{m\to\infty} \mathcal{H}^{n-2}(E_m \cap \overline{B}_R) \\ &\le \limsup_{m\to\infty} \sum_{1\le i<j\le n} \int_{B_R^{n-2}} \mathrm{card}[\pi_{ij}^{-1}\{y\} \cap E_m \cap \overline{B}_R] d\mathcal{H}^{n-2} y \\ &\le \binom{n}{2} \mathcal{H}^{n-2}(B_R^{n-2})(M-1)^2.\end{aligned}$$

This finishes the proof of Theorem 1.1. q.e.d.

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