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QUATERNIONIC MAPS BETWEEN HYPERKÄHLER MANIFOLDS

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Abstract

Quaternionic maps (Q-maps) between hyperkähler manifolds are quaternionic analogue of Cauchy-Riemann equations between Kähler manifolds. We provide a necessary and sufficient condition on when a quaternionic map becomes holomorphic with respect to some complex structures in the hyperkähler 2-spheres, and give examples of Q-maps which cannot be holomorphic. When the domain is real 4-dimensional, we analyze the structure of the blow-up set of a sequence of Q-maps, and show that the singular set of a stationary Q-map is $\mathcal{H}^1$-rectifiable.

1. Introduction

Quaternionic maps between hyperkähler manifolds arise naturally in higher dimensional gauge theory (cf. [3], [6], [9]). Unlike holomorphic maps, to define quaternionic maps, one needs to use all of the three complex structures, which determine the hyperkähler structures on both the domain and target manifolds. Let M and N be two hyperkähler manifolds and let $\mathbf{I}, \mathbf{J}, \mathbf{K}$ and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be complex structures on them respectively satisfying the quaternionic identities:

$$\begin{aligned} \mathbf{I}^2 &= \mathbf{J}^2 = \mathbf{K}^2 = \mathbf{IJK} = -Id, \\ \mathbf{i}^2 &= \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i}\,\mathbf{j}\,\mathbf{k} = -Id. \end{aligned}$$

A quaternionic map $f : M \to N$ is characterized by

$$\mathbf{i}\, df\, \mathbf{I} + \mathbf{j}\, df\, \mathbf{J} + \mathbf{k}\, df\, \mathbf{K} = df.$$

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In 1935, R. Fueter considered the same equation for $f : \mathbb{H} \to \mathbb{H}$ (cf. [10], [25]) in his effort to generalize complex analysis to the quaternionic setting. Recently, D. Joyce studied $\mathbb{H}$-valued functions defined on hypercomplex manifolds and provided applications to hypercomplex algebraic geometry (cf. [15] and the reference therein). Quaternionic maps automatically minimize the energy functional in their homotopy classes (cf. Proposition 2.2 and [3], [9]), and hence they are harmonic. Since this is a well known property of holomorphic maps between Kähler manifolds, it would be very interesting to know whether quaternionic maps can always be made holomorphic by rotating complex structures or if they constitute a new class of harmonic maps.

In Section 2, we prove Theorem 2.3 which provides a necessary and sufficient condition for a quaternionic map to be holomorphic with respect to some complex structures in the hyperkähler $\mathbb{S}^2$ (here we recall that the complex structures compatible with the hyperkähler metric are parameterized by $\mathbb{S}^2$). We also give examples of quaternionic maps which can not be holomorphic with respect to any complex structures.

In Section 3, we analyze the structure of the blow-up set of a sequence of quaternionic maps. It is shown in [17] that the limit map u of a sequence of quaternionic maps $u_k : M \to N$ with bounded total energy $E(u_k) \leq C$ is a stationary harmonic map, and the blow-up set Σ is stationary. In particular, when M is real 4-dimensional, Σ is 2-rectifiable, so the 2-dimensional Hausdorff measure $\mathcal{H}^2(\Sigma)$ is finite and $\mathcal{H}^2$ almost all points of Σ are contained in a countable union of 2-dimensional C^1-submanifolds of M. When M is a compact hyperkähler surface, we show in Proposition 3.3 that the two dimensional components of Σ are a union of minimal surfaces S_i away from a closed set whose $\mathcal{H}^2$-measure is zero, and moreover if $S_i \cap S_j$ contains a curve $\mathcal{C}$, then $\mathcal{C} \subset \operatorname{sing}(u)$. This is achieved by using the quaternionic map equation. The theory describing the interplay between calibrated submanifolds and higher dimensional instantons has been developed in [6] and [26]. We conjecture that the minimal surfaces S_i are calibrated by a certain closed 2-form. For an energy minimizing map v, the regularity theory of Schoen and Uhlenbeck [20] asserts that the singular set of v always has Hausdorff dimension $\leq m-3$, where m is the real dimension of the domain. For a stationary harmonic map, the $(m-2)$-dimensional Hausdorff measure of its singular set is zero (cf. [2], [7], [12]).

In Section 4, we study the regularity of a stationary quaternionic map u (cf. Definition 4.1) from a hyperkähler surface M. In this section, we assume that the Riemannian metric on N which defines the hyperkähler structure is real analytic. Using a result of Simon [22] on the singular sets of stationary harmonic maps, we prove in Theorem 4.3 that the singular set of u is $\mathcal{H}^1$-rectifiable. For energy minimizing maps, this regularity result was obtained in [22]. If the target N does not admit a holomorphic $\mathbb{S}^2$ with respect to any complex structure in the hyperkähler $\mathbb{S}^2$, a stationary quaternionic map is smooth outside a finite set of points; if N admits a holomorphic $\mathbb{S}^2$ with respect to some complex structure in the hyperkähler $\mathbb{S}^2$, then there exists a stationary quaternionic map whose singular set is a line. Lin's technique in [18] plays an important role in our analysis about stationary quaternionic maps.

In the last section, we take a different approach. Rather than considering the tangent maps, we show that if there are no holomorphic maps from $\mathbb{S}^2$ to N with respect to certain pairs of complex structures on the domain and target spaces, then there is a subsequence, in any sequence of quaternionic maps from a compact hyperkähler surface to N with bounded energy, which converges strongly in $W^{1,2}$ norm to a quaternionic map which is smooth except possibly at finitely many points.

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2. Quaternionic maps

Recall that a hyperkähler manifold is a Riemannian manifold with three covariant constant orthogonal automorphisms $\mathbf{I}$, $\mathbf{J}$ and $\mathbf{K}$ of the tangent bundle which satisfy the quaternionic identities

$$\mathbf{I}^2 = \mathbf{J}^2 = \mathbf{K}^2 = \mathbf{IJK} = -Id.$$

For any real numbers a, b, c with $a^2 + b^2 + c^2 = 1$, we obtain a covariant constant complex structure $a\mathbf{I} + b\mathbf{J} + c\mathbf{K}$. We shall refer this $\mathbb{S}^2$-family of complex structures as *the hyperkähler* $\mathbb{S}^2$. Therefore, $SO(3)$ acts naturally on the covariant constant complex structures. Furthermore, every $SO(3)$ matrix preserves the quaternionic identities. A hyperkähler manifold is of dimension $4k$. $\mathbb{R}^{4k}$ and the standard $4k$ dimensional tori naturally carry hyperkähler structures. It is well known that all $K3$ surfaces are hyperkähler. The moduli space of the irreducible anti-self-dual connections on a $K3$ surface is hyperkähler as well (cf. [14], [19] and other people's work). There is also construction via moment maps by Hitchin and others (cf. [13] and the reference therein).

Definition 2.1. Let M and N be two hyperkähler manifolds with complex structures J^α and $\mathcal{J}^\beta$ respectively for $\alpha, \beta = 1, 2, 3$ which satisfy the quaternionic identities. A smooth map $u : M \to N$ is called a quaternionic map if

$$A_{\alpha\beta}\mathcal{J}^\beta \circ du \circ J^\alpha = du, \tag{1}$$

where $A_{\alpha\beta}$ denote the entries of a matrix A in $SO(3)$.

It can be verified (cf. [3]) that holomorphic and anti-holomorphic maps with respect to some complex structures in the hyperkähler $\mathbb{S}^2$ on M and N are quaternionic maps. In particular, the identity map from M to itself satisfies

$$\mathbf{I}\, du\, \mathbf{I} - \mathbf{J}\, du\, \mathbf{J} - \mathbf{K}\, du\, \mathbf{K} = du.$$

Note that the coefficients in this equation form a $SO(3)$-matrix, i.e., the diagonal matrix with diagonal elements $1, -1, -1$, hence the identity map is quaternionic.

In local coordinates, the equation (1) reads as

$$A_{\alpha\beta}(J^\alpha)_i{}^j(\mathcal{J}^\beta)_m{}^n\partial_j u^m = \partial_i u^n.$$

Here and in sequel, we sum up all repeated upper and lower indices. This equation is a quaternionic analog of the Cauchy-Riemann equation defining holomorphic maps. Since $SO(3)$ preserves the quaternionic identities, we can always choose complex structures J^α for M and $\mathcal{J}^\beta$ for N such that $A_{\alpha\beta} = \delta_{\alpha\beta}$ in (1). In the sequel, we shall assume that $A_{\alpha\beta} = \delta_{\alpha\beta}$.

Let g and h be the Riemannian metrics on M and N respectively. Consider the energy functional

$$E(u) = \frac{1}{2}\|du\|^2 = \frac{1}{2}\int_M g^{ij}h_{mn}\partial_i u^m\partial_j u^n dV,$$

where dV is the volume element, and the functional

$$\begin{aligned} E_T(u) &= \int_M \sum_\alpha \langle J^\alpha, u^*\mathcal{J}^\alpha\rangle dV \\ &= \frac{1}{2}\int_M \sum_\alpha (J^\alpha)^{pq}\mathcal{J}^\alpha_{mk}\partial_p u^m \partial_q u^k dV, \end{aligned}$$

and set

$$I(u) = \frac{1}{2}\int_M |du - \sum_\alpha \mathcal{J}^\alpha \circ du \circ J^\alpha|^2 dV.$$

It is clear that $I(u) = 0$ if and only if u is a quaternionic map. Let J be a complex structure on M and let $\mathcal{J}$ be a complex structure on N. That holomorphic maps between Kähler manifolds are energy minimizers in their homotopy classes follows easily from

$$E(u) + \int_M \langle J, u^*\mathcal{J}\rangle dV = \frac{1}{4}\int_M |du - \mathcal{J}\circ du \circ J|^2 dV. \tag{2}$$

Note that $\int_M \langle J, u^*\mathcal{J}\rangle dV$ is just $\int_M \langle \omega_J, u^*\omega_{\mathcal{J}}\rangle dV$ where ω_J is the Kähler form on M defined by $\omega_J(X,Y) = g(JX,Y)$ and $\omega_{\mathcal{J}}$ is the Kähler form on N with respect to $\mathcal{J}$. In order to investigate similar properties for quaternionic maps, we have (cf. [3], [17], [9]):

Proposition 2.2. *For any smooth map $u : M \to N$, we have*

$$E(u) + E_T(u) = \frac{1}{4} I(u). \tag{3}$$

If u is a quaternionic map, then it minimizes energy in its homotopy class.

Proof. It is clear that

$$\begin{aligned} I(u) &= \frac{1}{2}\int_M g^{ij}h_{mn}(\partial_i u^m - \sum_\alpha (J^\alpha)_i{}^p(\mathcal{J}^\alpha)_k{}^m \partial_p u^k) \\ &\quad (\partial_j u^n - \sum_\alpha (J^\alpha)_j{}^q (\mathcal{J}^\alpha)_k{}^n \partial_q u^k) dV \\ &= E(u) - \int_M \sum_\alpha (J^\alpha)^{pq}\mathcal{J}^\alpha_{km}\partial_p u^m \partial_q u^k dV \\ &\quad + \frac{1}{2}\int_M \sum_{\alpha,\gamma} g^{ij}h_{mn}(J^\alpha)_i{}^p(\mathcal{J}^\alpha)_k{}^m (J^\gamma)_j{}^q(\mathcal{J}^\gamma)_l{}^n \partial_p u^k \partial_q u^l dV. \end{aligned}$$

We claim that

$$\sum_{\alpha,\gamma} g^{ij}h_{mn}(J^\alpha)_i{}^p(\mathcal{J}^\alpha)_k{}^m(J^\gamma)_j{}^q(\mathcal{J}^\gamma)_l{}^n = 3\delta_{pq}\delta_{kl} + 2\sum_\alpha (J^\alpha)^{pq}(\mathcal{J}^\alpha)_{kl}.$$

It suffices to prove the above identity in the normal coordinates. Take a system of normal coordinates around any point $p \in M$ and a system of normal coordinates around $u(p) \in N$. Then we have

$$\begin{aligned}
&\sum_{\alpha,\gamma} g^{ij}h_{mn}(J^\alpha)_i{}^p(\mathcal{J}^\alpha)_k{}^m(J^\gamma)_j{}^q(\mathcal{J}^\gamma)_l{}^n \\
&\quad = \sum_{\alpha,\gamma} J^\alpha_{ip}\mathcal{J}^\alpha_{km}J^\gamma_{iq}\mathcal{J}^\gamma_{lm} \\
&\quad = \sum_{\alpha,\gamma} J^\alpha_{pi}J^\gamma_{iq}\mathcal{J}^\alpha_{km}\mathcal{J}^\gamma_{ml} \\
&\quad = 3\delta_{pq}\delta_{kl} + 2\left(J^1_{pq}\mathcal{J}^1_{kl} + J^2_{pq}\mathcal{J}^2_{kl} + J^3_{pq}\mathcal{J}^3_{kl}\right),
\end{aligned}$$

and the claim follows. Then it is clear that

$$\begin{aligned}
I(u) &= E(u) + 2E_T(u) + 3E(u) + 2E_T(u) \\
&= 4(E(u) + E_T(u)).
\end{aligned}$$

This proves (3). For fixed complex structures J^α and $\mathcal{J}^\beta$, E_T only depends on the homotopy class of u (cf. [16]). Therefore, if u is a quaternionic map, the right side of (3) vanishes and in turn u is an energy minimizer in its homotopy class. q.e.d.

We now establish a criterion which detects when a quaternionic map becomes holomorphic with respect to some complex structures in the hyperkähler $\mathbb{S}^2$.

Theorem 2.3. *Suppose that u is a quaternionic map. Let A be a 3×3-matrix whose (α,β)-entries are $-\int_M \langle J^\alpha, u^*\mathcal{J}^\beta\rangle dV$ for $\alpha,\beta = 1,2,3$. Then*

$$(trA)^2 \geq \max\{\textit{eigenvalues of } AA^t\}$$

and the equality holds if and only if u is a holomorphic map with respect to some complex structures in the hyperkähler $\mathbb{S}^2$ on M and on N.

Proof. Setting $J = X_\alpha J^\alpha$ with $|X| = 1$ and $\mathcal{J} = Y_\beta\mathcal{J}^\beta$ with $|Y| = 1$, then from (2) we have

$$E(u) = XAY^t + \frac{1}{4}\int_M |du - \mathcal{J}\circ du\circ J|^2 dV.$$

Since u is a quaternionic map, by (3) we have

$$E(u) = trA.$$

It follows that

$$trA \geq XAY^t \tag{4}$$

for any unit vectors X and Y, and in fact the equality holds if and only if u is holomorphic with respect to J and $\mathcal{J}$. The eigenvalues of AA^t are all nonnegative. If $4\lambda^2$ is an eigenvalue of the real symmetric 3×3 matrix AA^t where $\lambda \geq 0$, there is a unit vector Y_λ in $\mathbb{R}^3$ such that

$$AA^tY_\lambda^t = 4\lambda^2 Y_\lambda^t,$$

and hence

$$Y_\lambda(A^tAY_\lambda^t) = Y_\lambda(AA^tY_\lambda^t) = 4\lambda^2 Y_\lambda Y_\lambda^t.$$

In turn, we have $|AY_\lambda^t| = 2\lambda$. If $\lambda \neq 0$, we choose $X_\lambda^t = \frac{1}{2\lambda}AY_\lambda^t$ and get

$$X_\lambda AY_\lambda^t = 2\lambda.$$

We therefore have $trA \geq 2\lambda$ and consequently

$$(trA)^2 \geq \max\{\text{eigenvalues of } AA^t\}.$$

This is trivially true if all of the eigenvalues of AA^t are 0 since A is the zero matrix in this case. This proves the first part of the theorem.

As for the second part, we introduce the Lagrange multiplier:

$$F(X, Y) = XAY^t - \lambda(|X|^2 - 1) - \mu(|Y|^2 - 1).$$

If XAY^t attains its maximum at two unit vectors $V, W \in \mathbb{R}^3$, we have $F_X = F_Y = 0$ at $X = V, \ \ Y = W$. This leads to

$$AW^t = 2\lambda V^t \ \text{ and } \ VA = 2\mu W.$$

It follows that

$$2\lambda = 2\lambda|V|^2 = VAW^t = 2\mu|W|^2 = 2\mu.$$

This implies

$$A^tAW^t = 2\lambda A^tV^t = 4\lambda\mu W^t = 4\lambda^2 W^t.$$

Therefore $4\lambda^2$ is an eigenvalue of A^tA.

If u is holomorphic with respect to some complex structures $X_\alpha J^\alpha$ on M and $Y_\alpha \mathcal{J}^\alpha$ on N in the hyperkähler $\mathbb{S}^2$, then XAY^t attains the maximum trA by (4). The discussion above asserts that $trA = XAY^t = 2\lambda$ and $4\lambda^2$ is an eigenvalue of A^tA, so $(trA)^2 = \max\{\text{eigenvalues of } AA^t\}$ according to the first part of the theorem.

Conversely, if $(trA)^2 = \max\{\text{eigenvalues of } AA^t\} > 0$, we set $2\lambda = trA$, then $4\lambda^2$ is an eigenvalue of A^tA. Suppose that $|Y| = 1$ and

$$A^tAY^t = 4\lambda^2 Y^t.$$

Then we have $YA^tAY^t = 4\lambda^2$, and hence $|AY^t|^2 = 4\lambda^2$. We choose $X^t = \frac{1}{2\lambda}AY^t$ and we get $XAY^t = 2\lambda = trA$. So by (4) u is a holomorphic map with respect to $J = X_\alpha J^\alpha$ and $\mathcal{J} = Y_\alpha \mathcal{J}^\alpha$. If A is the zero matrix, the quaternionic map u is constant. q.e.d.

Corollary 2.4. *Let Ω be an open domain in M with smooth boundary $\partial\Omega$. Suppose that u is a quaternionic map from Ω to N which extends smoothly to $\overline{\Omega} \to N$. Let A be a 3×3-matrix whose (α,β)-entries are $-\int_\Omega \langle J^\alpha, u^*\mathcal{J}^\beta\rangle dV$ for $\alpha,\beta = 1,2,3$. Then*

$$(trA)^2 \geq \max\{\textit{eigenvalues of } AA^t\}$$

and the equality holds if and only if u is a holomorphic map with respect to some complex structures in the hyperkähler $\mathbb{S}^2$ on M and on N.

Proof. If Ω is a domain in M with smooth boundary and $u : \Omega \to N$ is a smooth map, we still have

$$E(u) + E_T(u) = \frac{1}{4}I(u).$$

Now $E_T(u)$ is a homotopy invariant among maps $v : \Omega \to N$ which are homotopic to u relative to $\partial\Omega$ with $v|_{\partial\Omega} = u|_{\partial\Omega}$ (i.e., there exists a continuous family of maps $u_t, 0 \leq t \leq 1$, with $u_0 = u, u_1 = v$ and $u_t \equiv u$ on $\partial\Omega$). To see this, we observe that the pull-back 2-forms of the Kähler form $\omega_{\mathcal{J}^\beta}$ by u and v stay in the same cohomology class $H^2(\Omega, \mathrm{rel}\,\partial\Omega)$:

$$v^*\omega_{\mathcal{J}^\beta} - v^*\omega_{\mathcal{J}^\beta} = d\eta$$

for some 1-form η on Ω and $\eta(X) = 0$ for any vector X tangent to $\partial\Omega$.

It follows that

$$\begin{aligned}\int_\Omega \langle \omega_{J^\alpha}, v^*\omega_{\mathcal{J}^\beta}\rangle\, dV - \int_\Omega \langle \omega_{J^\alpha}, u^*\omega_{\mathcal{J}^\beta}\rangle\, dV &= \int_\Omega \langle \omega_{J^\alpha}, d\eta\rangle\, dV \\ &= \int_\Omega d\eta \wedge \frac{\omega_{J^\alpha}^{m-1}}{(m-1)!} \\ &= \int_\Omega d\left(\eta \wedge \frac{\omega_{J^\alpha}^{m-1}}{(m-1)!}\right) \\ &= \int_{\partial\Omega} \eta \wedge \frac{\omega_{J^\alpha}^{m-1}}{(m-1)!} \\ &= 0\end{aligned}$$

by Stokes' theorem and η vanishes along $\partial\Omega$. The rest of the argument is the same as that in the proof of Theorem 2.3. q.e.d.

Assume the real dimensions of M and N are four. To write (1) in local coordinates, we choose a coordinate system at a point x in M and a coordinate system at $f(x)$ in N, so that the matrix expressions of the complex structures take the following form (cf. [14]): let id be the 2×2 identity matrix and let I be the matrix for the standard complex structure on $\mathbb{C}$, then we take

$$\begin{aligned} &J^1 = \mathcal{J}^1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad J^2 = \mathcal{J}^2 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \\ &J^3 = \mathcal{J}^3 = \begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix}. \end{aligned} \tag{5}$$

Denote the differential du of u by the matrix $\left(\frac{\partial u^\alpha}{\partial x^i}\right)$ for $\alpha, i = 1, 2, 3, 4$. Set

$$u_i = \left(\frac{\partial u^1}{\partial x^i}, \frac{\partial u^2}{\partial x^i}, \frac{\partial u^3}{\partial x^i}, \frac{\partial u^4}{\partial x^i}\right)^t.$$

Simple computation then shows that the quaternionic equation (1) in dimension four is equivalent to

$$u_1 - \mathcal{J}^2 u_2 + \mathcal{J}^3 u_3 - \mathcal{J}^1 u_4 = 0, \tag{6}$$

which leads to

$$\begin{cases} u_1^1 + u_2^2 + u_3^3 + u_4^4 &= 0 \\ u_1^2 - u_2^1 + u_3^4 - u_4^3 &= 0 \\ u_1^3 - u_3^1 - u_2^4 + u_4^2 &= 0 \\ u_1^4 - u_4^1 - u_3^2 + u_2^3 &= 0. \end{cases} \tag{7}$$

If $\dim_{\mathbb{R}} M = 4$ and $\dim_{\mathbb{R}} N = 4n$, to write (1) in local coordinates, we choose a coordinate system $\{x_1, x_2, x_3, x_4\}$ around x and a coordinate system around $u(x)$ in N, so that the matrix expressions of the complex structures take the following form: for each $\alpha = 1, 2, 3$, J^α is as before and $\mathcal{J}^\alpha$ has n copies of J^α along the diagonal and 0 elsewhere in block form. Then the quaternionic equation is given by

$$\left\{\begin{array}{lcl} u_1^{4i-3} + u_2^{4i-2} + u_3^{4i-1} + u_4^{4i} & = & 0 \\ u_1^{4i-2} - u_2^{4i-3} + u_3^{4i} - u_4^{4i-1} & = & 0 \\ u_1^{4i-1} - u_3^{4i-3} - u_2^{4i} + u_4^{4i-2} & = & 0 \\ u_1^{4i} - u_4^{4i-3} - u_3^{4i-2} + u_2^{4i-1} & = & 0 \end{array}\right. \tag{8}$$

for $1 \leq i \leq n$.

Now we provide examples of quaternionic maps.

Example 2.5. Let M and N be $\mathbb{R}^4$ with complex structures $J^\alpha = \mathcal{J}^\alpha$ which are defined by (5). Let $u: \mathbb{R}^4 \to \mathbb{R}^4$ defined by

$$u(x_1, x_2, x_3, x_4) = (0, x_1x_2, -x_1x_3, -x_2x_3).$$

Using (7), one can check that u is a quaternionic map. Direct computation leads to

$$Du = \begin{pmatrix} 0 & 0 & 0 & 0 \\ x_2 & x_1 & 0 & 0 \\ -x_3 & 0 & -x_1 & 0 \\ 0 & -x_3 & -x_2 & 0 \end{pmatrix}.$$

Since

$$\det \begin{pmatrix} x_2 & x_1 & 0 \\ -x_3 & 0 & -x_1 \\ 0 & -x_3 & -x_2 \end{pmatrix} = -x_1x_2x_3,$$

we conclude that the rank of Du is three at (x_1, x_2, x_3, x_4) as long as $x_1x_2x_3 \neq 0$. However, the rank of a holomorphic map should be even. It follows that u can not be a holomorphic map with respect to any complex structures on $\mathbb{R}^4$.

Example 2.6. Consider a map $u: (0,1)\times(0,1)\times(0,1)\times(0,1) \to \mathbb{R}^4$ given by

$$u(x_1, x_2, x_3, x_4) = (a_1x_1, a_2x_2, a_3x_3, a_4x_4),$$

where $a_i \in \mathbb{R}$ for $i = 1, 2, 3, 4$. By (7), if

$$a_1 + a_2 + a_3 + a_4 = 0,$$

then u is a quaternionic map. Suppose that the elements of the matrix A is $A_{\alpha\beta}$ $(\alpha, \beta = 1, 2, 3)$. We have

$$A_{\alpha\beta} = -\frac{1}{2}\sum_{p,q=1}^{4} a_p a_q (J^\alpha)^{pq}(\mathcal{J}^\beta)_{pq}.$$

A simple computation shows that

$$A = -\begin{pmatrix} a_1a_4 + a_2a_3 & 0 & 0 \\ 0 & a_1a_2 + a_3a_4 & 0 \\ 0 & 0 & a_1a_3 + a_2a_4 \end{pmatrix}.$$

Choosing $a_2 = a_3 = a_4 = \mu > 0$ and $a_1 = -3\mu$. Then u is a quaternionic map. But

$$A = 2\begin{pmatrix} \mu^2 & 0 & 0 \\ 0 & \mu^2 & 0 \\ 0 & 0 & \mu^2 \end{pmatrix}.$$

So $trA = 6\mu^2$ and

$$AA^t = 4\begin{pmatrix} \mu^4 & 0 & 0 \\ 0 & \mu^4 & 0 \\ 0 & 0 & \mu^4 \end{pmatrix}.$$

Applying Corollary 2.4, we see that u is not a holomorphic map with respect to any complex structures on $(0,1)\times(0,1)\times(0,1)\times(0,1)$ and $\mathbb{R}^4$.

Example 2.7. Let $\mathbb{T}^4 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ be the standard 4-dimensional torus. Consider a map $u: \mathbb{T}^4 \to \mathbb{T}^4$ determined by

$$u(e^{2\pi i x_1}, e^{2\pi i x_2}, e^{2\pi i x_3}, e^{2\pi i x_4}) = (e^{2\pi i x_1}, e^{2\pi i x_2}, e^{2\pi i x_3}, e^{-6\pi i x_4}).$$

In local coordinates,

$$u(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, -3x_4).$$

It is clear from (7) that u is a quaternionic map. By an argument similar to the one used in the second example, we conclude that u is not holomorphic with respect to any complex structures on $\mathbb{T}^4$ by Theorem 2.3.

3. The blow-up set

Let M and N be two compact hyperkähler manifolds. Let $m = \dim_{\mathbb{R}} M$. Suppose that u_k is a sequence of smooth quaternionic maps with $E(u_k) \leq C$. We recall that as a sequence of harmonic maps with bounded energy the *blow-up set of* u_k can be defined as

$$\Sigma = \bigcap_{r>0} \{x \in M | \liminf_{k\to\infty} r^{2-m} \int_{B_r(x)} |\bigtriangledown u_k|^2 dy \geq \epsilon_0\}.$$

We can always assume that $u_k \rightharpoonup u$ weakly in $W^{1,2}(M,N)$ and that

$$|\bigtriangledown u_k|^2 dx \rightharpoonup |\bigtriangledown u|^2 dx + \nu,$$

in the sense of measure as $k \to \infty$. Here ν is a nonnegative Radon measure on M with support in Σ. It is known that Σ is a closed $\mathcal{H}^{m-2}$-rectifiable set, so we can define the gradient operator $\bigtriangledown^\Sigma$ (cf. Section 12.1 in [23]) and the divergence along Σ

$$\mathrm{div}_\Sigma X = \sum_{j=1}^{m-2} (\bigtriangledown^\Sigma_{e_j} X) \cdot e_j$$

for any smooth vector field X on M, where $e_1, \cdots, e_{m-2}$ is any orthonormal basis for $T_x\Sigma$, $x \in \Sigma$ (cf. Section 16 in [23]). We say that Σ is *stationary* if for any smooth vector field X on M, we have

$$\int_\Sigma \mathrm{div}_\Sigma X \nu = 0.$$

A map f from M to N is said to be a *stationary harmonic map*, if it is a weakly harmonic map and for any smooth vector field X on M, we have

$$\int_M \left(|\bigtriangledown f|^2 \mathrm{div}(X) - 2\langle df(\bigtriangledown_\alpha X), df(\frac{\partial}{\partial x^\alpha})\rangle \right) dV = 0.$$

The following result is proved in [17] (Theorem 4.4 and Theorem 4.5 in [17]).

Theorem 3.1 ([17]). *Let M and N be two compact hyperkähler manifolds. Suppose that u_k is a sequence of quaternionic maps with bounded energy $E(u_k) \leq C$. Then by taking a subsequence if necessary, $u_k \rightharpoonup u$ weakly in $W^{1,2}(M,N)$ and the limit map u is a stationary harmonic map which is also a stationary quaternionic map. Moreover, the blow-up set Σ is stationary and has no boundary.*

From now on, we assume that M is a hyperkähler surface, that is $m = 4$.

Since Σ is stationary, it is a union of smooth real 2-dimensional surfaces outside a closed set Σ_0 with $\mathcal{H}^2(\Sigma_0) = 0$.

Definition 3.2. Let $B_r(x)$ be a geodesic ball and let

$$\Sigma_1 = \{\ x \in \Sigma \mid \mathcal{H}^2(\Sigma \cap B_r(x)) > 0 \text{ for any } r > 0\}.$$

We call Σ_1 the two dimensional component of Σ.

It is clear that if $x \in \Sigma \setminus \Sigma_1$, there is $r_0 > 0$ such that $\mathcal{H}^2(\Sigma \cap B_{r_0}(x)) = 0$. Consequently, $u_k \to u$ strongly in $W^{1,2}(B_{r_0}(x), N)$.

Proposition 3.3. *Let M be a compact hyperkähler surface and let N be a compact hyperkähler manifold. Suppose that u_k is a sequence of smooth quaternionic maps with bounded energy $E(u_k) \leq C$. Let Σ_1 be the two dimensional component of the blow-up set Σ for the sequence. Then $\Sigma_1 \setminus \Sigma_0 = \cup_i S_i$ where S_i are minimal surfaces. Let u be the weakly limiting map of the sequence u_k. If S_i and S_j meets transversely along a curve $\mathcal{C}$, then $\mathcal{C} \subset \operatorname{sing}(u)$.*

Proof. By the constancy theorem for stationary varifolds (cf. [1], [23]), the density function of Σ is constant on each connected component. Then the first part of the proposition is just a re-statement of Theorem 3.1. It suffices to show that, if S_i and S_j meets transversely along a curve $\mathcal{C}$, then $\mathcal{C} \subset \operatorname{sing}(u)$. Since each S_i is smooth, we can choose three tangent vectors $e'_1,\ e'_2,\ e'_3 \in T_xS_i \cup T_xS_j$ for any $x \in \mathcal{C}$. If $\mathcal{C} \not\subset \operatorname{sing}(u)$, we may choose x to be the point where u is smooth. So u is smooth in a neighborhood of x. By Lemma 2.2 in [17], the nonnegative Radon measure $\nu_{e'_i} = 0$ and in the measure convergence $|\nabla_{e'_i} u_k|^2 dV \rightharpoonup |\nabla_{e'_i} u|^2 dV + \nu_{e'_i}$ for all tangential direction $e'_i, i = 1, 2, 3$, therefore we have

$$\begin{aligned} \lim_{r\to 0}\lim_{k\to\infty} r^{-2}\int_{B_r(x)} |\nabla_{e'_i} u_k|^2 dV &= \lim_{r\to 0} r^{-2}\int_{B_r(x)} |\nabla_{e'_i} u|^2 dV \\ &= 0, \end{aligned}$$

for $i = 1, 2, 3$, here the last equality follows from the smoothness of u at x. We choose an orthonormal frame $\{e_1, e_2, e_3, e_4\}$ on the tangent bundle T_UM over a neighborhood U of x in M such that $e_i = \sum_{j=1}^{3} A_{ij} e'_j$ at x, for $i = 1, 2, 3$, where $A = (A_{ij})$ is an invertible 3×3-matrix. So we have

$$\lim_{r\to 0}\lim_{k\to\infty} r^{-2}\int_{B_r(x)}\sum_{i=1}^{3}|\bigtriangledown_{e_i} u_k|^2 dV = 0.$$

By the quaternionic map equation, we get

$$\lim_{r\to 0}\lim_{k\to\infty} r^{-2}\int_{B_r(x)}|\bigtriangledown_{e_4} u_k|^2 dV = 0.$$

This implies

$$\lim_{r\to 0}\lim_{k\to\infty} r^{-2}\int_{B_r(x)}|\bigtriangledown u_k|^2 dV = 0,$$

which contradicts that $x \in \Sigma$ is a blow-up point. q.e.d.

4. Removable singularity

In this section, we prove that the singular set of a stationary quaternionic map defined in [17] is a one dimensional rectifiable set.

Definition 4.1. Let M and N be two hyperkähler manifolds. A map u from M to N is a stationary quaternionic map if (1) $u \in W^{1,2}(M,N)$ and (2) u is a smooth quaternionic map outside a singular set Σ which is of Hausdorff codimension at least two.

It is proved in [17] that a stationary quaternionic map is a stationary harmonic map. Let us recall some basic terminologies and facts about stationary harmonic maps. Let $u(x)$ be a stationary harmonic map from M to N. The *regular set* reg (u) of u is defined as the set of points $x \in M$ such that u is smooth in some neighborhood of x. It is clear that $\mathrm{reg}(u)$ is open in M. The *singular set* $\mathrm{sing}(u)$ of u is the complement of $\mathrm{reg}(u)$. The *density function* Θ_u of u is defined by

$$\Theta_u(x) = \lim_{r\to 0} r^{2-m}\int_{B_r(x)}|\bigtriangledown u|^2 dV,$$

where m is the real dimension of M. The monotonicity inequality for u guarantees that the limit exists, and the density function Θ_u of u is upper semi-continuous. It is proved (cf. Theorem I.4 in [2], and [7]) that $x \in \mathrm{reg}(u)$ if and only if $\Theta_u(x) = 0$. To study the local behavior of the map u around a point $x \in M$, we take a small convex geodesic ball $B(x)$ centered at x in M. For any $y \in B(x)$, there is a unique geodesic $\gamma(x,y)$ which connects x and y with unit speed. We shall denote the point on $\gamma(x, y)$ with distance r from x by $x + ry$ for any $r \in [0, 1)$. At each singular point, the tangent maps exist:

Theorem 4.2 ([17]). *Let u be a stationary harmonic map from M to N. Assume that $x \in \operatorname{sing}(u)$ and set $u_{x,r}(y) = u(x + ry)$. Then for any sequence $r_{i'} \to 0$ there is a subsequence $r_i \to 0$ such that $u_{x,r_i} \rightharpoonup \phi$ weakly in $W^{1,2}(\mathbb{R}^m, N)$ and*

$$|\bigtriangledown u_{x,r_i}|^2 dy \rightharpoonup |\bigtriangledown \phi|^2 dy + \theta(y)\mathcal{H}^{m-2} \lfloor \Sigma_x$$

in the sense of measure. Moreover,

(1) *$\phi(\lambda y) = \phi(y)$ for all $\lambda > 0$ and $y \in \mathbb{R}^m$, i.e., $\phi(y)$ is of homogeneous degree zero.*

(2) *$\theta(\lambda y) = \theta(y)$ for all $\lambda > 0$ and $y \in \Sigma_x$, i.e., $\theta(y)$ is of homogeneous degree zero.*

(3) *Σ_x is a tangent cone, which means $\lambda \Sigma_x = \Sigma_x$ for all $\lambda > 0$.*

The map ϕ obtained in this theorem is called *a tangent map of u at x*. Let $x \in M$ and take a sequence of points $x_i \to x$ with $\Theta_u(x_i) \geq \Theta_u(x)$. We define

$$u_{x_i,r}(y) = u(x_i + ry).$$

By a similar argument as in the proof of Theorem 4.2, we can show that there is a subsequence $r_i \to 0$ such that $u_{x_i,r_i} \rightharpoonup \phi$ weakly in $W^{1,2}(\mathbb{R}^m, N)$ and

$$|\bigtriangledown u_{x_i,r_i}|^2 dy \rightharpoonup |\bigtriangledown \phi|^2 dy + \theta(y)\mathcal{H}^{m-2} \lfloor \Sigma_x$$

in the sense of measure. $\phi(y)$ and $\theta(y)$ are of homogeneous degree zero and Σ_x is a tangent cone. We call such a map ϕ *the pseudo-tangent map of u at x* as in [22]. If u is a quaternionic map, by Theorem 3.1 and Theorem 4.2 we know that $\phi(y)$ is a stationary harmonic map and $\theta(y)$ is constant in each connected domain in Σ_x by the constancy theorem (cf. [23] Theorem 41.1).

Theorem 4.3. *Let u be a stationary quaternionic map from a hyperkähler surface M to a hyperkähler manifold N with a real analytic metric which defines the hyperkähler structure. Then the Hausdorff dimension of the singular set of u is at most one, and if it is one, the singular set is rectifiable.*

Proof. Since u is a stationary harmonic map by Theorem 4.4 in [17], it is smooth outside a closed subset in M with $\mathcal{H}^2$ measure zero.

There is a very useful result in Section 9 of [22] about the singular set of stationary harmonic maps. Simon shows the following: Let u be a stationary harmonic map from a m-dimensional domain Ω with a smooth metric to a Riemannian manifold with a real analytic metric. If all tangent map and all pseudo-tangent maps ϕ of u are stationary for the energy functional, and if the set of all such ϕ with density $\Theta_\phi(0) \leq \beta$ (for any given β) lies in a compact set (relative to the local L^2 metric) of stationary maps all with singular set of dimension $\leq m_1$, then $\text{sing}(u)$ is locally a finite union of locally m_1-rectifiable locally compact subsets. In general, unlike the minimizing maps, neither the stationary assumption on tangent maps and pseudo-tangent maps nor the compactness is known to hold automatically if u is stationary.

We only prove the case that N is also a hyperkähler surface, the proof for the general case is the same.

Step 1. We first show the compactness of the pseudo-tangent maps of u.

Suppose that u_k is a sequence of pseudo-tangent maps of u. We can always assume that $u_k \rightharpoonup u$ weakly in $W^{1,2}_{loc}(\mathbb{R}^4, N)$ and that $|\triangledown u_k|^2 dx \rightharpoonup |\triangledown u|^2 dx + \nu$ in the sense of measure as $k \to \infty$. Here ν is a nonnegative Radon measure on $\mathbb{R}^4$ with support in Σ, and Σ is the blow-up set of the sequence u_k. It is clear that Σ is a stationary tangent cone. In particular, we know that the origin 0 of $\mathbb{R}^4$ belongs to Σ and θ is constant. We will prove $\mathcal{H}^2(\Sigma) = 0$ by deriving a contradiction if $\mathcal{H}^2(\Sigma) > 0$.

Because Σ is $\mathcal{H}^2$-rectifiable, it decomposes into

$$\Sigma = \bigcup_{i=1}^{\infty} \Sigma_i, \quad \Sigma_i \cap \Sigma_j = \emptyset$$

if $i \neq j$, where $\mathcal{H}^2(\Sigma_0) = 0$, Σ_i is $\mathcal{H}^2$-rectifiable and $\Sigma_i \subset M_i$ with M_i an embedded 2-dimensional C^1 submanifold of $\mathbb{R}^4$ for all $i \geq 1$; and $T_x\Sigma = T_x M_i$ for $\mathcal{H}^2$-a.e. $x \in \Sigma_i$ (cf. p.61 in [23]). Suppose that (x^1, x^2, x^3, x^4) is a coordinate system around 0 such that

$$\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2} \in T(\Sigma)$$

for $\mathcal{H}^2$-a.e. $x \in \Sigma$. We recall the following result in [17]: if $T =$

$\sum_{\alpha=1}^{2} \xi^\alpha \frac{\partial}{\partial x^\alpha}$ then

$$\lim_{\epsilon\to 0}\lim_{k\to\infty}\int_{B_\epsilon(\Sigma)} |\bigtriangledown_T u_k|^2 dV = 0. \tag{9}$$

Here and in the sequel we denote by $B_\epsilon(\Sigma) = \{x \in M |\ dist\,(x,\Sigma) < \epsilon\}$. Set

$$F_{k\epsilon}(x) = \int_{B^2_\epsilon(0)} |\bigtriangledown_T u_k|^2(x,x')dx'$$

for $x \in \Sigma$. Here and in the sequel we denote by $B^2_r(x)$ the metric ball centered at x with radius r in $\mathbb{R}^2$. We consider the Hardy-Littlewood maximal function $M_{F_{k\epsilon}}(x)$ of $F_{k\epsilon}(x)$ (cf. [24]), which is defined by

$$M_{F_{k\epsilon}}(x) = \sup_{0<r<1} r^{-2}\int_{B^2_r(x)} F_{k\epsilon}(x)dx.$$

By the weak type $(1,1)$ inequality (cf. [24]) for $M_{F_{k\epsilon}}(x)$, we have

$$\mathcal{H}^2\{x\in\Sigma | M_{F_{k\epsilon}}(x)\geq\lambda\} \leq \frac{C}{\lambda}\int_{B_\epsilon(\Sigma)}\sum_{\alpha=1}^{2} |\bigtriangledown_\alpha u_k|^2 dV$$

for any $\lambda > 0$. By (9), we have

$$\lim_{\epsilon\to 0}\lim_{k\to\infty}\mathcal{H}^2\{x\in\Sigma | M_{F_{k\epsilon}}(x)\geq\lambda\} = 0.$$

Thus, for any integer $l > 0$,

$$\mathcal{H}^2\left(\bigcup_{n_0=1}^{\infty}\bigcap_{n=n_0}^{\infty}\bigcup_{k_0=1}^{\infty}\bigcap_{k=k_0}^{\infty}\left\{x\in\Sigma\ \middle|\ M_{F_{k(\frac{1}{n})}}(x)\geq\frac{1}{l}\right\}\right) = 0.$$

By the partial regularity result for stationary harmonic maps in [2] and [7], we can find $x_l \in \Sigma \subset B^2_1(0)\times\{0\}$, such that for any $n_0 > 0$ and any $k_0 > 0$ there are $n_l > n_0$ and $k_l > k_0$ such that

$$x_l \to 0 \quad\text{and}\quad M_{F_{k_l(\frac{1}{n_l})}}(x_l) < \frac{1}{l} \tag{10}$$

and u_k is smooth near (x_l, x') for all $x' \in B^2_1(0)$. We claim that for all k sufficiently large there exist $\delta_k \to 0$ such that

$$\max_{x'\in B^2_1(0)} \delta_k^{-2}\int_{B^2_{\delta_k}(x_k)\times B^2_{\delta_k}(x')} |\bigtriangledown u_k|^2(x,x')dxdx' = \frac{\epsilon_0}{8\cdot 2^8}, \tag{11}$$

and

$$\frac{|x_k|}{\delta_k} \le C, \tag{12}$$

where ϵ_0 is the small constant in the small energy regularity theorem (cf. [2], [7]). In fact, since $u_k(x)$ is smooth at (x_k, x'), for any given k and for $\delta < \delta(k)$, we have

$$\delta^{-2}\int_{B^2_\delta(x_k)\times B^2_\delta(x')} |\bigtriangledown u_k|^2(x,x')dxdx' \le \frac{\epsilon_0}{16\cdot 2^8}.$$

On the other hand, since $x \in \Sigma$, for fixed $\delta > 0$ and sufficiently large k,

$$\delta^{-2}\int_{B^2_\delta(x_k)\times B^2_\delta(0)} |\bigtriangledown u_k|^2(x,x')dxdx' \ge \frac{\epsilon_0}{4\cdot 2^8}.$$

Therefore we can choose $\delta_k > 0$ so that (11) and (12) hold.

By (10), (11) and (12), we can find $\epsilon_k \to 0$, $r_k \to 0$, $(x_k, x'_k) \in \Sigma$ with

$$\lim_{k\to\infty} \frac{(x_k, x'_k)}{r_k} = p = (p^1, p^2, p^3, p^4), \tag{13}$$

where $|p^1| < \infty$, $|p^2| < \infty$, $|p^3| \le \infty$, $|p^4| \le \infty$, and a subsequence of u_k, which we also denote by u_k for simplicity, such that

$$r_k^{-2}\int_{B^2_{r_k}(x_k)\times B^2_{r_k}(x'_k)} |\bigtriangledown u_k|^2(x,x')dxdx' = \frac{\epsilon_0}{8\cdot 2^8},$$

$$M_{F_{k\epsilon_k}}(x_k, x'_k) < \frac{1}{k}$$

and

$$\lim_{k\to\infty}\frac{\epsilon_k}{r_k} = \infty.$$

We define

$$v_k(y) = u_k((x_k, x'_k) + r_k y).$$

It is obvious that

$$\int_{B^2_1(0)\times B^2_1(0)} |\bigtriangledown v_k|^2(x,x')dxdx' = \frac{\epsilon_0}{8\cdot 2^8} \tag{14}$$

and

$$\sup_{0<r<1}\left(\frac{r}{r_k}\right)^{-2}\int_{B^2_{\frac{r}{r_k}}(0)\times B^2_{\frac{\epsilon_k}{r_k}}(0)}\sum_{i=1}^{2}|\bigtriangledown_i v_k|^2dV<\frac{1}{k}. \tag{15}$$

By the monotonicity inequality for stationary harmonic maps, for any $R>0$, we have

$$R^{-2}\int_{B^2_R(0)\times B^2_R(0)}|\bigtriangledown v_k|^2dV\le C.$$

By the diagonal subsequence argument, we obtain a subsequence of v_k, still denoted by v_k, such that $v_k\rightharpoonup v$ weakly in $W^{1,2}(\mathbb{R}^4,N)$, and v is a weak harmonic map. It follows that

$$R^{-2}\int_{B^2_R(0)\times B^2_R(0)}|\bigtriangledown v|^2dV\le C. \tag{16}$$

Since $r_k\to 0$ and $\epsilon_k/r_k\to\infty$ as $k\to\infty$, we get by (15) that

$$\sum_{i=1}^{2}\int_{B^2_R(0)\times B^2_R(0)}|\bigtriangledown_i v|^2dV=0$$

for any $R>0$, hence $\bigtriangledown_i v\equiv 0$ for $i=1,2$. By (16), we have

$$\sum_{i=3}^{4}\int_{B^2_R(0)}|\bigtriangledown_i v|^2dV\le C,$$

where C does not depend on R. Therefore, we have

$$\sum_{i=3}^{4}\int_{\mathbb{R}^2}|\bigtriangledown_i v|^2dV\le C.$$

By the removable singularity theorem in [21], we can extend v to a smooth harmonic map from $\mathbb{S}^2$ to N.

Let $\phi(y)\in C_0^\infty(B_1^2(0))$ be a cut-off function with $\phi(y)=1$ in $B^2_{1/2}(0)$. Let $\psi(y')\in C_0^\infty(B_1^2(0))$ be a cut-off function with $\psi(y')=1$ in $B^2_{1/2}(0)$. We consider

$$f_k(a)=\int_{B^2_1(0)\times B^2_1(0)}|\bigtriangledown v_k|^2((x,x')+a)\phi(x)\psi(x')dxdx',$$

for $a \in B_2^2(0) \times B_4^2(0)$. Since v_k are harmonic maps, integration by parts leads to

$$\frac{\partial f_k(a)}{\partial a_i} = 2\sum_{l=1}^{4} \int_{B_1^2(0)\times B_1^2(0)} \frac{\partial v_k}{\partial x_l} \frac{\partial v_k}{\partial x_i}((x, x') + a) \frac{\partial(\phi\psi)}{\partial x_l} dx dx'$$

for $i = 1, 2$. By (15), we obtain

$$\frac{\partial f_k(a)}{\partial a_i} \to 0 \quad \text{as} \quad k \to \infty,$$

uniformly for $a \in B_2^2(0) \times B_4^2(0)$. It follows

$$\int_{B_2^2(0)\times B_1^2(0)} | \bigtriangledown v_k|^2(x, x' + b) dx dx' \leq \frac{\epsilon_0}{8 \cdot 2^4}$$

for each $b \in B_4^2(0)$, and consequently we get

$$\int_{B_2^2(0)\times B_2^2(0)} | \bigtriangledown v_k|^2(x, x') dx dx' \leq \epsilon_0.$$

Noting that v_k are harmonic maps, we see that $v_k \to v$ in

$$C^2(B_1^2(0) \times B_1^2(0))$$

as $k \to \infty$. By (14), v is a nonconstant quaternionic map. We choose $x_0 = (x_0^1, x_0^2, x_0^3, x_0^4)$ so that $x_0^3 + p^3 \neq 0, x_0^4 + p^4 \neq 0$, where $p = (p^1, p^2, p^3, p^4)$ is the point in (13). We may assume that at x_0 in $\mathbb{R}^4$ and at $v(x_0)$ in N the matrix expressions of the complex structures are given by (5). By (6), v satisfies

$$\mathcal{J}^3 v_3 - \mathcal{J}^1 v_4 = 0. \tag{17}$$

Since u_k is a pseudo-tangent map of u, we have

$$\sum_{i=1}^{4} x^i \frac{\partial u_k}{\partial x^i} = 0.$$

It follows that

$$\begin{aligned}
&\sum_{i=1}^{4} \left(\frac{(x_k, x_k') + r_k x_0}{r_k}\right)^i \frac{\partial v_k}{\partial x^i}(x_0) \\
&\qquad = \sum_{i=1}^{4} \left((x_k, x_k') + r_k x_0\right)^i \frac{\partial u_k}{\partial x^i}((x_k, x_k') + r_k x_0) \\
&\qquad = 0.
\end{aligned}$$

Since $x_0^3 + p^3 \neq 0$, we have $x_0^3 + r_k^{-1} x_k'^3 \neq 0$ for large k by (13) and then

$$\begin{aligned} \frac{\partial v_k}{\partial x^3}(x_0) = & -\left(\frac{x_0^4 + r_k^{-1} x_k'^4}{x_0^3 + r_k^{-1} x_k'^3}\right) \frac{\partial v_k}{\partial x^4}(x_0) \\ & - \sum_{i=1}^{2} \left(\frac{x_0^i + r_k^{-1} x_k^i}{x_0^3 + r_k^{-1} x_k'^3}\right) \frac{\partial v_k}{\partial x^i}(x_0). \end{aligned} \tag{18}$$

Since $0 \in \Sigma \backslash \operatorname{sing}(v)$ and $|\nabla_i v| \equiv 0$ for $i = 1, 2$,

$$\lim_{k \to \infty} \sum_{i=1}^{2} \left(\frac{x_0^i + r_k^{-1} x_k^i}{x_0^3 + r_k^{-1} x_k'^3}\right) \frac{\partial v_k}{\partial x^i}(x_0) = 0.$$

Notice that

$$\frac{\partial v}{\partial x^j}(x_0) = \lim_{k \to \infty} \frac{\partial v_k}{\partial x^j}(x_0)$$

exist for $j = 3, 4$. Letting $k \to \infty$ in (18), we have

$$\frac{\partial v}{\partial x^3}(x_0) = -\frac{x_0^4 + p^4}{x_0^3 + p^3} \frac{\partial v}{\partial x^4}(x_0). \tag{19}$$

If the coefficient in (19) is infinite, then $\frac{\partial v}{\partial x^4}(x_0)$ is zero and in turn $\frac{\partial v}{\partial x^3}(x_0)$ is zero as well because v is quaternionic; if the coefficient is finite, then (19) means $\frac{\partial v}{\partial x^3}(x_0)$ and $\frac{\partial v}{\partial x^4}(x_0)$ are colinear, but by (17)

$$\frac{\partial v}{\partial x^3}(x_0) = -\mathcal{J}^3 \mathcal{J}^1 \frac{\partial v}{\partial x^4}(x_0) = -\mathcal{J}^2 \frac{\partial v}{\partial x^4}(x_0),$$

which implies $\frac{\partial v}{\partial x^3}(x_0)$, $\frac{\partial v}{\partial x^4}(x_0)$ are orthogonal, therefore we still conclude that

$$\frac{\partial v}{\partial x^3}(x_0) = \frac{\partial v}{\partial x^4}(x_0) = 0.$$

In sum, $v \equiv$ *constant* since x_0 is arbitrary as long as $x_0^3 + p^3 \neq 0, x_0^4 + p^4 \neq 0$. This contradicts to v is a nonconstant bubble, and consequently implies the compactness.

Step 2. We show that every pseudo-tangent map of u is smooth outside some rays.

Let ϕ be a pseudo-tangent map of u. It is a stationary quaternionic map of homogeneous degree zero in $\mathbb{R}^4$. We claim that $\phi|_{\mathbb{S}^3}$ is also stationary. Let X be a smooth vector field with compact support in $\mathbb{R}^4$, and let $e_1 = \frac{\partial}{\partial r}$ and e_2, e_3, e_4 be an orthonormal frame on $\mathbb{S}^3$. Assume that $X = X^i e_i$ and $\widetilde{X} = X|_{\mathbb{S}^3} = \widetilde{X}^2 e_2 + \widetilde{X}^3 e_3 + \widetilde{X}^4 e_4$. We have

$$\operatorname{div}(X) = \frac{\partial X^1}{\partial r} + \frac{3X^1}{r} + \frac{1}{r^2}\operatorname{div}_{\mathbb{S}^3}\widetilde{X}$$

and

$$X^i_j = \frac{X^1}{r}\delta_{ij} + \frac{1}{r^2}\widetilde{X}^i_j \qquad \textit{for } i,\ j = 2,\ 3,\ 4.$$

Then

$$\begin{aligned}
&\int_{\mathbb{R}^4}\left(|\bigtriangledown\phi|^2\operatorname{div}(X) - 2\langle d\phi(\bigtriangledown_i X), d\phi(\frac{\partial}{\partial x^i})\rangle\right)dV \\
= &\int_{\mathbb{R}^4}|\bigtriangledown\phi|^2\left(\frac{\partial X^1}{\partial r} + \frac{X^1}{r}\right)dV \\
&+ \int_{\mathbb{R}^4}\frac{1}{r^4}\left(|\bigtriangledown_{\mathbb{S}^3}\phi|^2\operatorname{div}_{\mathbb{S}^3}\widetilde{X} - 2\bigtriangledown_i^{\mathbb{S}^3}\phi\bigtriangledown_j^{\mathbb{S}^3}\phi\widetilde{X}^i_j\right)dV \\
= &\ 0.
\end{aligned}$$

For any smooth vector field $\widetilde{Y}$ on $\mathbb{S}^3$, we choose $X = \eta(r)\widetilde{Y}$, where $\eta \in C_0^\infty(\mathbb{R}^4)$ is a cut-off function with $\eta = 0$ in $(0, \epsilon)$. Using the above identity, we obtain

$$\int_{\mathbb{S}^3}\left(|\bigtriangledown_{\mathbb{S}^3}\phi|^2\operatorname{div}_{\mathbb{S}^3}\widetilde{Y} - 2\bigtriangledown_i^{\mathbb{S}^3}\phi\bigtriangledown_j^{\mathbb{S}^3}\phi\widetilde{Y}^i_j\right)dV = 0.$$

It follows that $\phi|_{\mathbb{S}^3}$ is stationary.

To prove ϕ is smooth outside some rays, it suffices to show that $\psi = \phi|_{\mathbb{S}^3}$ is smooth outside a finite set of points. The pseudo-tangent map of ψ is stationary, hence the 1-dimensional Hausdorff measure of its singular set is zero by Bethuel's theorem in [2]. We then conclude that the pseudo-tangent maps of ψ are smooth except at 0, because otherwise the whole ray passing through a singular point different from the origin would be in the singular set.

We now derive the equation for the pseudo-tangent maps of ψ. For simplicity, we consider the pseudo-tangent map f of ψ at $(1, 0, 0, 0)$. We choose a spherical coordinates in $\mathbb{R}^4$ as follows:

$$\left\{\begin{array}{l}
x^1 = r\sin\alpha\sin\beta\cos\gamma \\
x^2 = r\sin\alpha\sin\beta\sin\gamma \\
x^3 = r\sin\alpha\cos\beta \\
x^4 = r\cos\alpha.
\end{array}\right. \tag{20}$$

In this coordinate system, $(1,0,0,0) = (1,\pi/2,\pi/2,0)$. Suppose that J^α is expressed by J_s^α in the spherical coordinates. We can write $J_s^\alpha = A^{-1}J^\alpha A$ for $\alpha = 1,2,3$, where A is the Jacobi matrix of the change of coordinates (20). Then at $(1,\pi/2,\pi/2,0)$, we have

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

A simple computation shows that

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad J_s^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$J_s^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad J_s^3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

We will use lower indices in f_j^i to denote the derivatives in $\mathbb{R}^4$ at the blow-up point. Set

$$df = \begin{pmatrix} f_1^1 & f_2^1 & f_3^1 & f_4^1 \\ f_1^2 & f_2^2 & f_3^2 & f_4^2 \\ f_1^3 & f_2^3 & f_3^3 & f_4^3 \\ f_1^4 & f_2^4 & f_3^4 & f_4^4 \end{pmatrix}.$$

Then we have

$$\mathcal{J}^1\, df\, J_s^1 = \begin{pmatrix} f_2^4 & -f_1^4 & f_4^4 & -f_3^4 \\ -f_2^3 & f_1^3 & -f_4^3 & f_3^3 \\ f_2^2 & -f_1^2 & f_4^2 & -f_3^2 \\ -f_2^1 & f_1^1 & -f_4^1 & f_3^1 \end{pmatrix}$$

$$\mathcal{J}^2\, df\, J_s^2 = \begin{pmatrix} -f_4^2 & -f_3^2 & f_2^2 & f_1^2 \\ f_4^1 & f_3^1 & -f_2^1 & -f_1^1 \\ f_4^4 & f_3^4 & -f_2^4 & -f_1^4 \\ -f_4^3 & -f_3^3 & f_2^3 & f_1^3 \end{pmatrix}$$

$$\mathcal{J}^3\, df\, J_s^3 = \begin{pmatrix} f_3^3 & -f_4^3 & -f_1^3 & f_2^3 \\ f_3^4 & -f_4^4 & -f_1^4 & f_2^4 \\ -f_3^1 & f_4^1 & f_1^1 & -f_2^1 \\ -f_3^2 & f_4^2 & f_1^2 & -f_2^2 \end{pmatrix}.$$

Noting that $f_1 = 0$ and f is a quaternionic map, we can see that f satisfies the equation

$$\begin{cases} f_2^1 = -f_3^2 - f_4^3 \\ f_2^2 = f_3^1 - f_4^4 \\ f_2^3 = f_4^1 + f_3^4 \\ f_2^4 = f_4^2 - f_3^3. \end{cases} \tag{21}$$

When f is viewed as a map on $\mathbb{R}^3$, it is of homogeneous degree zero in $\mathbb{R}^3$. By an argument similar to the one used in the first step, we can show that, if a weakly convergence sequence of the pseudo-tangent maps of ψ did not converge strongly, we would get a nonconstant map bubble w satisfying the equations

$$\begin{aligned} 0 &= w_3^2 + w_4^3 \\ 0 &= w_3^1 - w_4^4 \\ 0 &= w_4^1 + w_3^4 \\ 0 &= w_4^2 - w_3^3 \end{aligned}$$

and

$$x^3 w_3^i + x^4 w_4^i = 0 \quad \text{for} \quad i = 1, 2, 3, 4.$$

However, the above equations imply that $w \equiv \mathit{constant}$. This contradiction yields the compactness of the pseudo-tangent maps of ψ.

Step 3. Now the two requirements in Simon's theorem are satisfied by ψ. Therefore ψ is smooth outside a finite set of points, and consequently ϕ is smooth outside the rays which pass through the singular points of ψ. Then the two requirements in Simon's theorem are satisfied by u, so the singular set of u is a one dimensional rectifiable set. q.e.d.

Proposition 4.4. *Let M be a compact hyperkähler surface and let N be a compact hyperkähler manifold with a real analytic metric which defines the hyperkähler structure. If N does not admit holomorphic $\mathbb{S}^2$ with respect to any complex structure in the hyperkähler $\mathbb{S}^2$ and $u : M \to N$ is a stationary quaternionic map, then it is smooth outside a finite set of points. Conversely, if N admits a holomorphic $\mathbb{S}^2$ with respect to some complex structure in the hyperkähler $\mathbb{S}^2$, then there is a stationary quaternionic map $u : \mathbb{R}^4 \to N$ whose singular set is of one dimension.*

Proof. We continue the argument in the proof of Theorem 4.3, and we will provide in next section an alternative proof for the first part of the theorem.

We will show now that a nonconstant pseudo-tangent map f of ψ induces a holomorphic $\mathbb{S}^2$ in N. Note that f is of homogeneous degree zero in $\mathbb{R}^3$. Set $h = f|_{\mathbb{S}^2}$, we will show that h is a holomorphic $\mathbb{S}^2$ in N.

Using the spherical coordinates

$$\begin{cases} x^1 = r\sin\alpha\cos\theta \\ x^2 = r\sin\alpha\sin\theta \\ x^3 = r\cos\alpha, \end{cases}$$

by (21) and

$$\begin{aligned} \frac{\partial}{\partial x^1} &= \sin\alpha\cos\theta\frac{\partial}{\partial r} + \frac{\cos\alpha\cos\theta}{r}\frac{\partial}{\partial\alpha} - \frac{\sin\theta}{r\sin\alpha}\frac{\partial}{\partial\theta} \\ \frac{\partial}{\partial x^2} &= \sin\alpha\sin\theta\frac{\partial}{\partial r} + \frac{\cos\alpha\sin\theta}{r}\frac{\partial}{\partial\alpha} + \frac{\cos\theta}{r\sin\alpha}\frac{\partial}{\partial\theta} \\ \frac{\partial}{\partial x^2} &= \cos\alpha\frac{\partial}{\partial r} - \frac{\sin\alpha}{r}\frac{\partial}{\partial\alpha}, \end{aligned}$$

we can see that h satisfies the equation

$$\frac{1}{\sin\alpha}\begin{pmatrix} \frac{\partial h^1}{\partial\theta} \\ \frac{\partial h^2}{\partial\theta} \\ \frac{\partial h^3}{\partial\theta} \\ \frac{\partial h^4}{\partial\theta} \end{pmatrix} = \begin{pmatrix} 0 & x_3 & -x_2 & -x_1 \\ -x_3 & 0 & x_1 & -x_2 \\ x_2 & -x_1 & 0 & -x_3 \\ x_1 & x_2 & x_3 & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial h^1}{\partial\alpha} \\ \frac{\partial h^2}{\partial\alpha} \\ \frac{\partial h^3}{\partial\alpha} \\ \frac{\partial h^4}{\partial\alpha} \end{pmatrix}. \tag{22}$$

One can check that h is a conformal harmonic map from $\mathbb{S}^2$ to N. In fact, h is holomorphic with respect to the the complex structures given by the 2×2-matrix with 0 on its diagonal and $\sin\alpha$ and $-\sin^{-1}\alpha$ as its $(1,2),(2,1)$ entries respectively on $\mathbb{S}^2$ and the 4×4-matrix in (22) on N. This proves the first part of the proposition.

Conversely, if we have a holomorphic $\mathbb{S}^2$ in N given by h, then h satisfies the equation (22). We extend h to $f:\mathbb{R}^3\to N$ by letting $f(r,\alpha,\theta) = h(\alpha,\theta)$ in spherical coordinates. Then in the standard coordinates, f satisfies the equation (21). We define $u:\mathbb{R}^4\to N$ by setting $u(x^1,x^2,x^3,x^4) = f(x^2,x^3,x^4)$. One can check that u is a quaternionic map from $\mathbb{R}^4$ to N with respect to the complex structures J_s^α at $(1,\pi/2,\pi/2,0)$ and $\mathcal{J}^\alpha$ $(\alpha = 1,2,3)$. q.e.d.

By an argument similar to the one used in the proof of Theorem 4.3, we obtain the following well-known result for holomorphic maps.

Proposition 4.5. *Let M be a compact complex surface and let N be a compact complex manifold with a real analytic Hermitian metric. Suppose that $u \in W^{1,2}(M,N)$ is holomorphic outside a closed set Σ in M with $\mathcal{H}^2(\Sigma) < \infty$. Then u can be extended to a holomorphic map from M possibly away from finitely many points to N.*

Proof. It suffice to show that the pseudo-tangent maps of u are smooth except at 0. Suppose that ϕ is a pseudo-tangent map of u at a singular point, say $(1,0,0,0)$, let $\psi = \phi|_{\mathbb{S}^3}$, we need only show that the pseudo-tangent maps of ψ are constant. Suppose that u is holomorphic with respect to J^1 and $\mathcal{J}^1$, then a pseudo-tangent map f of ψ satisfies the equation

$$f_4^4 = f_3^1, \quad -f_3^4 = f_4^1, \quad -f_4^3 = f_3^2, \quad f_3^3 = f_4^2.$$

Since f also satisfies that $x^3 f_3 + x^4 f_4 = 0$, we can see that $f \equiv \mathit{constant}$. q.e.d.

5. Strong convergence

Let M compact hyperkähler surface and N be a compact hyperkähler manifold. Suppose that u_k is a sequence of quaternionic maps in a fixed homotopy class. We consider in this section when u_k will converge strongly in $W^{1,2}(M,N)$.

Theorem 5.1. *Let M be a compact hyperkähler surface and N be a compact hyperkähler manifold. Suppose that u_k is a sequence of quaternionic maps in a fixed homotopy class. If N does not admit holomorphic $\mathbb{S}^2$ with respect to the complex structure $a_i J^i$ on $\mathbb{R}^4$ restricts to $\mathbb{S}^2$ and the complex structure $a_i \mathcal{J}^i$ on N for some a_i $(i = 1,2,3)$ with $\sum_i a_i^2 = 1$, then a subsequence of $\{u_k\}$ converges strongly in $W^{1,2}(M,N)$ to a quaternionic map u. The singular set of the limit map u is a finite set of points.*

The above theorem follows from the following theorem and the standard dimension reduction argument (cf. [8], [18], [20]).

Theorem 5.2. *Let M compact hyperkähler surface and N be a compact hyperkähler manifold. Suppose that u_k is a sequence of quaternionic maps with bounded energies. If N does not admit holomorphic $\mathbb{S}^2$ with* *respect to the complex structure a_iJ^i on $\mathbb{R}^4$ restrict to $\mathbb{S}^2$ and the complex structure $a_i\mathcal{J}^i$ on N for some a_i $(i = 1, 2, 3)$ with $\sum_i a_i^2 = 1$, then there is a subsequence of $\{u_k\}$ which converges strongly to a stationary quaternionic map u.*

Proof. We can always assume that $u_k \rightharpoonup u$ weakly in $W^{1,2}(M, N)$ and that $|\bigtriangledown u_k|^2 dx \rightharpoonup |\bigtriangledown u|^2 dx + \nu$ in the sense of measure as $k \to \infty$. Here ν is a nonnegative Radon measure on M with support in Σ, and Σ is the blow-up set of the sequence u_k. We will prove $\mathcal{H}^2(\Sigma) = 0$. Otherwise, by an argument similar to the one used in the first step of the proof of Theorem 4.3, we get a nonconstant quaternionic map v with $v_i = 0$ for $i = 1, 2$.

Assume that e, a_iJ^ie is an orthogonal basis of the normal bundle of $T\Sigma$ at a point where $T\Sigma$ exists, for some real valued functions a_1, a_2, a_3 with $a_1^2+a_2^2+a_3^2 = 1$. a_i's may not be constant. Since v is a nonconstant harmonic map from $\mathbb{S}^2$ into N, viewed as extension of $(T\Sigma)^\perp$, v in fact is conformal. In particular,

$$\begin{aligned} \langle dv(e), dv(a_jJ^je)\rangle &= 0 \\ |dv(a_jJ^je)| &= |dv(e)|. \end{aligned}$$

If we set

$$dv(e) = \xi, \tag{23}$$

then we may write

$$dv(a_iJ^ie) = -b_i\mathcal{J}^i\xi \tag{24}$$

for some b_i, where $b_1^2 + b_2^2 + b_3^2 = 1$. Note that dv restricts to 0 along Σ. We can check that

$$(a_2J^1 - a_1J^2)e \in Ker(dv) \tag{25}$$

$$(a_3J^1 - a_1J^3)e \in Ker(dv). \tag{26}$$

Solving the linear system (24), (25) and (26), we get

$$\left\{ \begin{aligned} dv(J^1e) &= -a_1(b_i\mathcal{J}^i\xi) \\ dv(J^2e) &= -a_2(b_i\mathcal{J}^i\xi) \\ dv(J^3e) &= -a_3(b_i\mathcal{J}^i\xi). \end{aligned} \right. \tag{27}$$

Since v is a quaternionic map, we have

$$\sum_\alpha \mathcal{J}^\alpha dv J^\alpha e = dv \cdot e.$$

It follows from (27) that

$$\begin{aligned}\xi &= \sum_\alpha \mathcal{J}^\alpha dv J^\alpha e \\ &= a_1b_1\xi - a_1b_2\mathcal{J}^3\xi + a_1b_3\mathcal{J}^2\xi + a_2b_1\mathcal{J}^3\xi \\ &\quad +a_2b_2\xi - a_2b_3\mathcal{J}^1\xi - a_3b_1\mathcal{J}^2\xi + a_3b_2\mathcal{J}^1\xi + a_3b_3\xi.\end{aligned}$$

Notice that $\xi, \mathcal{J}^1\xi, \mathcal{J}^2\xi, \mathcal{J}^3\xi$ are linearly independent. Comparing coefficients in the above identity leads to

$$\left\{\begin{array}{ccc} a_1b_1 + a_2b_2 + a_3b_3 & = & 1 \\ a_2b_3 - a_3b_2 & = & 0 \\ a_1b_3 - a_3b_1 & = & 0 \\ a_1b_2 - a_2b_1 & = & 0. \end{array}\right.$$

Solving this linear system, we have

$$a_1 = b_1 \quad a_2 = b_2 \quad a_3 = b_3.$$

It follows that v is a holomorphic map from $\mathbb{S}^2$ to N with respect to the complex structure a_iJ^i restrict to $\mathbb{S}^2$ and the complex structure $a_i\mathcal{J}^i$ on N. But no such holomorphic can exist by assumption. So we must have $\mathcal{H}^2(\Sigma) = 0$ and in turn u_k converge strongly to u in $W^{1,2}$ norm. q.e.d.

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