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HOLOMORPHIC PRINCIPAL BUNDLES OVER ELLIPTIC CURVES II: THE PARABOLIC CONSTRUCTION

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Abstract

This paper continues the study of holomorphic semistable principal G-bundles over an elliptic curve. In this paper, the moduli space of all such bundles is constructed by considering deformations of a minimally unstable G-bundle. The set of all such deformations can be described as the $\mathbb{C}^*$-quotient of the cohomology group of a sheaf of unipotent groups, and we show that this quotient has the structure of a weighted projective space. We identify this weighted projective space with the moduli space of semistable G-bundles, giving a new proof of a theorem of Looijenga.

Introduction

Let E be a smooth elliptic curve and let G be a simple complex algebraic group of rank r. We shall always assume that $\pi_1(G)$ is cyclic and that c is a generator. We shall freely identify c with the corresponding element in the center of the universal cover $\widetilde{G}$ of G. A C^∞ G-bundle ξ_0 over E has a characteristic class $c_1(\xi_0) \in H^2(E; \pi_1(G)) \cong \pi_1(G)$ which determines ξ_0 up to C^∞ isomorphism. The goal of this paper is to continue the study, begun in [10], of the moduli space $\mathcal{M}(G, c)$ of semistable holomorphic G-bundles ξ with $c_1(\xi) = c$. In [10], this space was studied from the transcendental viewpoint of $(0,1)$-connections using the results of Narasimhan-Seshadri and Ramanathan that in every S-equivalence class there is a unique representative whose holomorphic structure is given by a flat connection. This viewpoint, however, is not suitable for

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many questions, such as finding universal bundles, studying singular elliptic curves, or generalizing to families of elliptic curves. In this paper, which is largely independent of [10], we describe $\mathcal{M}(G, c)$ from an algebraic point of view. As we shall show in later papers, this construction is much better adapted for dealing with the questions described above.

Our motivation comes from the theory of deformations of singularities with a $\mathbb{C}^*$-action. In this theory, the deformations corresponding to nonnegative weights are topologically equisingular. Thus, from the point of view of smoothings, it is the negative weight deformation space which is interesting. This space can be studied infinitesimally, by looking at the $\mathbb{C}^*$-action on the Zariski tangent space to the deformation functor. There is also a globalization of this local description. For example, in the case of hypersurface simple elliptic singularities which are given as weighted cones over an elliptic curve E, the global moduli space, modulo the action of $\mathbb{C}^*$, can (essentially) be identified with pairs (X, D), where X is a smooth del Pezzo surface or a del Pezzo surface with rational double points, and D is a hyperplane section of X isomorphic to E.

By analogy, our method is to describe $\mathcal{M}(G, c)$ as a certain set of deformations of a "singular" bundle ξ_{u}, where in this context singular means unstable. Of course, it is natural to require that ξ_{u} be minimally unstable, in the sense that all small deformations which are not roughly speaking topologically equivalent to ξ_{u} should be semistable. As we showed in [11], such minimally unstable bundles always exist, and, in most cases, are unique once we fix the determinant. There is a distinguished subgroup $\mathbb{C}^*$ in the automorphism group of ξ_{u}, and (with our conventions) it acts with nonnegative weights. The positive weight deformations of ξ_{u} correspond to semistable bundles.

More precisely, to every unstable bundle ξ_{u} there is associated a conjugacy class of parabolic subgroups of G. The parabolic subgroup corresponding to a minimally unstable ξ_{u} is a maximal parabolic subgroup. The conjugacy classes of maximal parabolic subgroups are indexed by the simple roots of G. Let P be such a maximal parabolic subgroup, with unipotent radical U and Levi factor L. Then L has a one-dimensional center, and hence there is a natural inclusion of $\mathbb{C}^*$ in L. Correspondingly there is a unique primitive dominant character χ_0 of L. We consider semistable L-bundles η whose degree is -1 with respect to χ_0. In particular, this implies that P_-, the parabolic subgroup opposite to P, is a Harder-Narasimhan parabolic subgroup for the unstable G-bundle $\xi_{\mathrm{u}} = \eta \times_L G$. The central subgroup $\mathbb{C}^*$ of L

acts on the deformation space of $\eta \times_L G$, and we shall be interested in the positive weight deformations. As in the singularities case, there is a global interpretation of this infinitesimal picture: it is the space of isomorphisms classes of pairs (ξ, φ) consisting of a P-bundle ξ and an isomorphism $\varphi\colon \xi/U \to \eta$. The isomorphism classes of such pairs (ξ, φ) are classified by the cohomology group $H^1(E; U(\eta))$, where $U(\eta)$ is the sheaf of unipotent groups $\eta \times_L U$. In general, $U(\eta)$ is not a sheaf of abelian groups, and so $H^1(E; U(\eta))$ is *a priori* simply a set. However, the fact that $U(\eta)$ has a filtration whose successive quotients are vector groups implies, in our situation, that $H^1(E; U(\eta))$ carries in a functorial sense the structure of an affine space. The group $\mathbb{C}^*$ acts on this space, fixing the origin and such that the differential of the action has positive weights at the origin. We show that this action can be linearized. From this point of view, it is the nonabelian nature of U which allows there to be different weights for the $\mathbb{C}^*$-action, so that the quotient $(H^1(E; U(\eta)) - \{0\})/\mathbb{C}^*$ is a weighted projective space $\mathbb{WP}(\eta)$, typically with distinct weights. In fact, choosing η to be minimally unstable, the weights are given as follows in case G is simply connected. Let $\widetilde{\Delta}$ be the extended set of simple roots for G, and for each $\alpha \in \widetilde{\Delta}$, let $\alpha^\vee$ be the corresponding dual coroot. Then there is a unique linear relation

$$\sum_{\alpha \in \widetilde{\Delta}} g_\alpha \alpha^\vee = 0,$$

provided that we require that the coefficient of the coroot dual to the negative of the highest root is 1. Up to multiplying by a common factor, the weights of $\mathbb{WP}(\eta)$ are then the integers g_α, counted with multiplicity. In particular, the dimension of $\mathbb{WP}(\eta)$ is equal to the rank of G, which we know to be the dimension of $\mathcal{M}(G, 1)$. When G is non-simply connected and we consider G-bundles whose first Chern class is a generator $c \in \pi_1(G)$, there is a similar result where, again up to a common factor, the weights are given by the coroot integers $g_{\overline{\beta}}$ on the quotient of the extended Dynkin diagram of G by the action of the central element c in the universal covering group $\widetilde{G}$ of G as described in [7].

Assuming that the bundle $\eta \times_L G$ described above is minimally unstable, all points of $\mathbb{WP}(\eta)$ correspond to semistable bundles. Thus, there is an induced morphism $\Psi\colon \mathbb{WP}(\eta) \to \mathcal{M}(G, c)$. Now every morphism whose domain is a weighted projective space is either constant or finite, and in our case it is easy to see that the morphism is nonconstant. This already yields some information about L and η: assuming that the action of $\mathrm{Aut}_L\,\eta$ on $\mathbb{WP}(\eta)$ is essentially effective, since Ψ is constant on the $\mathrm{Aut}_L\,\eta/\mathbb{C}^*$-orbits, it follows that $\mathbb{C}^*$ is the identity component of $\mathrm{Aut}_L\,\eta$. To go further, we need to use the fact that E is an elliptic curve, which implies that the map $\Psi\colon \mathbb{WP}(\eta) \to \mathcal{M}(G,c)$ is dominant, and more generally that the rational map from $\mathbb{WP}(\eta)$ to $\mathcal{M}(G,c)$ is dominant for every maximal parabolic. This implies that $\dim \mathbb{WP}(\eta) \geq \dim \mathcal{M}(G,c)$, with equality for the minimally unstable case. By contrast, for curves of genus at least 2, even for the case of $SL_2(\mathbb{C})$, the map $\mathbb{WP}(\eta) \to \mathcal{M}(G,c)$ is typically not dominant, but rather maps $\mathbb{WP}(\eta)$ to a proper subvariety of $\mathcal{M}(G,c)$.

To sum up, then, in the minimally unstable case, we have a weighted projective space $\mathbb{WP}(\eta)$ and a finite morphism $\Psi\colon \mathbb{WP}(\eta) \to \mathcal{M}(G,c)$. In fact, we show the following theorem:

Theorem. *Suppose that the L-bundle η is minimally unstable. Then the map $\Psi\colon \mathbb{WP}(\eta) \to \mathcal{M}(G,c)$ is an isomorphism.*

Corollary. *The moduli space $\mathcal{M}(G,c)$ is a weighted projective space, with weights $g_{\overline{\beta}}/n_0$, where n_0 is the* gcd *of the $g_{\overline{\beta}}$. In particular, if G is simply connected, $\mathcal{M}(G,c)$ is a weighted projective space, with weights $g_\alpha, \alpha \in \widetilde{\Delta}$.*

In the simply connected case, the corollary is due to Looijenga [17] (see also [5]). Note however that the theorem goes beyond an abstract description of $\mathcal{M}(G,c)$ as a weighted projective space: it identifies an algebraically defined moduli space, $\mathbb{WP}(\eta)$, with a transcendentally defined moduli space which in some sense is obtained by taking the periods of a flat connection. We view this as a theorem of Torelli type in a nonlinear context.

The bundles produced by the parabolic construction, in addition to being semistable, are regular in the sense that their automorphism groups have minimal possible dimension. This is reminiscent of the Steinberg cross-section of regular elements for the map of G to its adjoint quotient and of the Kostant section of regular elements for the adjoint quotient of the Lie algebra $\mathfrak{g}$ of G. In fact, as we shall show in a future paper, the parabolic construction extends to the case of nodal curves of genus one and to cuspidal curves of genus one when $G \neq E_8$. For nodal curves, the parabolic construction produces a weighted projective space and an open subset which is identified with a Steinberg-like cross-section of regular elements in each conjugacy class. For cuspidal curves (and $G \neq E_8$) the weighted projective space contains an open subset producing a Kostant-like section of the adjoint quotient of $\mathfrak{g}$. Thus, in both cases, the parabolic construction yields a new approach to the proof of the existence of a section of regular elements for the adjoint quotient, and produces a natural compactification of the adjoint quotient which is a weighted projective space.

In case $G = E_8, E_7, E_6$ (as well as D_5 and A_4), the relationship between deformations of minimally unstable G-bundles and deformations of simple elliptic singularities goes far beyond a formal analogy. Indeed, this observation, which is connected to what is called in the physics literature F-theory, was a major motivation for us to study G-bundles over elliptic curves. This connection will be described elsewhere.

This paper is related to [11], where we enumerate the minimally unstable strata in the space of $(0, 1)$-connections on a G-bundle. While we make no use of these results, at least in the simply connected case, that paper helps explain the characterizing properties that a minimally unstable bundle ξ_{u} has to satisfy: it predicts, for example, that its Harder-Narasimhan parabolic is the maximal parabolic associated with what we call a special root and that the L-bundle has degree -1 with respect to the dominant character. But even without knowing that these bundles lie in minimally unstable strata in the Atiyah-Bott formalism, one can establish the isomorphism given in the main theorem above.

The contents of this paper are as follows. In Section 1, we collect together preliminary technical results. Many of these results concern numerical facts related to irreducible representations of the Levi factor of a maximal parabolic on the unipotent radical of that parabolic. These are used in Section 2 to compute the dimensions of various cohomology groups related to bundles over maximal parabolic subgroups of G, as well as to understand the weights of the $\mathbb{C}^*$-action. These dimensions and weights could be computed by case-by-case checking of the root tables. We have tried instead to find classification-free arguments wherever possible. In calculating the $\mathbb{C}^*$-weights, we make use of a property we call circular symmetry. This property was introduced by Witten and established for the coroot integers in [7]. Its name derives from the relation of this property to a symmetry statement for points placed on a circle according to these numbers, as described in [7, §3.8]. Here, we do not need this geometric interpretation. Rather we need to know only that (as was proved in [7]) the coroot integers and the coroot integers on the quotient diagram by the action of the center satisfy circular symmetry, and that numbers satisfying circular symmetry are completely determined by three pieces of information, which in our context are the dimension of the weighted projective space, the highest weight appearing in the $\mathbb{C}^*$-action and the dual Coxeter number of the group. In the minimally unstable case, we are able to show that these invariants agree with the corresponding ones for the coroot integers and this agreement is what allows us to identify the $\mathbb{C}^*$-weights with the coroot integers. We emphasize, however, that circular symmetry holds for all maximal parabolic subgroups, not just those which correspond to minimally unstable bundles. Unfortunately, our proof of this resorts in the end to case-by-case checking. It would be extremely illuminating to have a more conceptual understanding of the meaning of circular symmetry. In §1.5, we discuss the volume of the moduli space of flat connections on E and again relate it to the coroot integers. These two pieces of numerical information, the $\mathbb{C}^*$-weights for $\mathbb{WP}(\eta)$ and the volume of the space of flat connections, will turn out to be crucial in Section 5 for the proof that $\deg \Psi = 1$.

In Section 2, we give a general description, for every maximal parabolic subgroup P, of the bundles η over the Levi factor L. We then compute the dimensions of the nonabelian cohomology space (or rather its tangent space at the origin) and the $\mathbb{C}^*$-weights in terms of the numbers introduced in Section 1. In Section 3, we study the minimally unstable case in detail. As we have mentioned above, we expect from general principles that $\dim \mathrm{Aut}_L \eta = 1$ and that the cohomology dimension must be $\dim \mathcal{M}(G, c) + 1$. We identify the simple roots in the minimally unstable case, verifying the above facts in a classification-free way in the simply connected case, and identify the $\mathbb{C}^*$-weights with the coroot integers via circular symmetry. The description of the minimally unstable case is also given in [11], although the discussion here in the simply connected case is independent of that paper. In §3.4, we consider the non-simply connected case. Here we use the results of [11] as well as a case-by-case analysis to identify the minimally unstable bundles and to identify the cohomology dimension, the dimension of the automorphism group of the bundle, and the $\mathbb{C}^*$-weights.

Section 4 is concerned with the nonabelian cohomology space, i.e., the affine space $H^1(E; U(\eta))$. We show that the $\mathbb{C}^*$-action can be linearized and discuss the obstructions to the existence of a universal bundle. In §4.4, we show that, in the minimally unstable case, the points of $H^1(E; U(\eta)) - \{0\}$ correspond to **regular** semistable bundles. This means that the algebraic families provided by the parabolic construction are different from the families provided by the space of flat connections. Every S-equivalence class of bundles has two extreme representatives which are unique up to isomorphism, the flat representative and the regular representative. On an open dense set of $\mathcal{M}(G,c)$ these representatives agree and, when they do, all bundles of the given S-equivalence class are isomorphic. But, along a codimension one subvariety of $\mathcal{M}(G,c)$, the regular representative does not have a flat connection. It turns out that, because the dimension of the automorphism group of regular representatives is constant, the regular representatives behave better in families.

Finally, in Section 5, we prove the main theorem by calculating the degree of Ψ. The crux of the argument is to study the determinant line bundle on $\mathcal{M}(G,c)$, which pulls back via Ψ^* to the determinant line bundle on $\mathbb{WP}(\eta)$. Thus, once we show that both determinant line bundles have the same top self-intersection, then it follows that the degree of Ψ is one. Some of the technical results concerning the nonabelian cohomology space and its interpretation are deferred to the appendix. We prove that the cohomology space naturally has the structure of an affine space and represents an appropriate functor.

The parabolic construction of semistable G-bundles was originally introduced and explained, for the simply connected case, in [12], in a paper written for an audience of physicists, as well as in the announcement [13]. It is a pleasure to thank Ed Witten for originally raising the questions which led to this work and for the insights he shared with us during the course of our joint work on these subjects. We would also like to thank A. Borel, P. Deligne, and W. Schmid for various helpful conversations and correspondence. Finally, during the preparation of this paper, S. Helmke and P. Slodowy sent us their preprint [16], which has a considerable overlap with the first part of this paper and which also analyzes the case where the automorphism group of $\eta \times_L G$ is just slightly larger than in the minimally unstable case.

1. Preliminaries

1.1 Notation

Throughout this paper, R denotes a reduced and irreducible root system of rank r in a real vector space V, with Weyl group $W = W(R)$, and Δ is a set of simple roots for R. Let R^+ be the set of positive roots corresponding to the choice of Δ. There exists a W-invariant positive definite inner product $\langle\cdot,\cdot\rangle$ on V and it is unique up to scalars.

Given a root α, there is an associated coroot $\alpha^\vee \in V^*$. Using the inner product to identify V with V^*, we have $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$. As usual, we denote the Cartan integer $\alpha(\beta^\vee)$ by $n(\alpha,\beta)$. The *coroot lattice* Λ is the lattice inside V^* spanned by the coroots. Given $\alpha \in \Delta$, we have the *fundamental weight* $\varpi_\alpha \in V$, which satisfies $\varpi_\alpha(\beta^\vee) = \delta_{\alpha\beta}$ for all $\alpha, \beta \in \Delta$. The fundamental coweights $\varpi_\alpha^\vee \in V^*$ are defined similarly. As usual, let ρ be the sum of the fundamental weights, so that $\rho = \sum_{\alpha\in\Delta}\varpi_\alpha = \frac{1}{2}\sum_{\beta\in R^+}\beta$.

Let $\widetilde{\alpha}$ be the highest root of R^+. We have $\widetilde{\alpha} = \sum_{\beta\in\Delta} h_\beta\beta$, with $h_\beta > 0$. We set $\alpha_0 = -\widetilde{\alpha}$, $\widetilde{\Delta} = \Delta \cup \{\alpha_0\}$, and $h_{\alpha_0} = 1$ so that $\sum_{\beta\in\widetilde{\Delta}} h_\beta\beta = 0$. The number $h = 1 + \sum_{\beta\in\Delta} h_\beta$ is the *Coxeter number* of R. Similarly, we have $\widetilde{\alpha}^\vee = \sum_{\beta\in\Delta} g_\beta\beta^\vee$ with $g_\beta > 0$. We set $g_{\alpha_0} = 1$ so that $\sum_{\beta\in\widetilde{\Delta}} g_\beta\beta^\vee = 0$. We call $g = 1 + \sum_{\beta\in\Delta} g_\beta = \sum_{\beta\in\widetilde{\Delta}} g_\beta$ the *dual Coxeter number* of R. An easy calculation then shows:

Lemma 1.1.1. *In the above notation, we have*

$$g_\alpha = \frac{h_\alpha\langle\alpha,\alpha\rangle}{\langle\widetilde{\alpha},\widetilde{\alpha}\rangle}.$$

Thus $g_\alpha | h_\alpha$, and $g_\alpha = h_\alpha$ if and only if α is a long root of R. □

Let $Q \in \mathrm{Sym}^2\Lambda^*$ be the quadratic form defined by

$$Q = \sum_{\alpha\in R}\langle\alpha,\cdot\rangle\langle\alpha,\cdot\rangle.$$

This form was first introduced by Looijenga in [17], where he showed:

Lemma 1.1.2. *Let g be the dual Coxeter number. Then*

$$Q = (2g)I_0,$$

where I_0 is the unique W-invariant quadratic form on Λ such that $I_0(\widetilde{\alpha}^\vee) = 2$. For example, if R is simply laced, then I_0 is the usual intersection form on Λ. □

Throughout this paper, we use the inner product on V^* defined by I_0 and the corresponding dual inner product on V. It has the property that all long roots have length 2.

Lemma 1.1.3. *Let $\{\alpha^*\}_{\alpha\in\Delta}$ be the dual basis to $\{\alpha^\vee\}_{\alpha\in\Delta}$ with respect to I_0. Then $\alpha^* = g_\alpha\varpi_\alpha^\vee/h_\alpha$.*

Proof. This is just the statement that

$$\langle\alpha^\vee, \varpi_\beta^\vee\rangle = \frac{2}{\langle\alpha,\alpha\rangle}\alpha(\varpi_\beta^\vee) = \frac{h_\alpha}{g_\alpha}\delta_{\alpha\beta}.$$ □

1.2 Structure of maximal parabolic subgroups

Let $H \subseteq G$ be a Cartan subgroup, and let $R \subseteq \mathfrak{h}^*$ be the set of roots for the pair (G, H). We denote by $\mathfrak{g}$ the Lie algebra of G and by $\mathfrak{h} \subseteq \mathfrak{g}$ the Cartan subalgebra of $\mathfrak{g}$ corresponding to H. Let R^+ be a set of positive roots, and let Δ denote the corresponding set of simple roots. We shall also view the roots $\alpha \in R$ as characters $\alpha\colon H \to \mathbb{C}^*$ on H. There is a unique dual coroot $\alpha^\vee \in \mathfrak{h}$. We can view $\alpha^\vee$ as defining a linear map $\mathbb{C} \to \mathfrak{h}$. Exponentiating this map gives us a cocharacter of H, i.e., a one-parameter subgroup $\ell_\alpha\colon \mathbb{C}^* \to H$.

If P is a parabolic subgroup of G, then the unipotent radical U of P is a normal subgroup, and there is a semidirect product $P = U \cdot L$, where L is a reductive subgroup of P, unique up to conjugation, called the Levi factor of P. If $\widetilde{G}$ is the universal cover of G, then there is a one-to-one correspondence between parabolic subgroups of G and those of $\widetilde{G}$, which associates to the subgroup P of G its preimage $\widetilde{P} \subseteq \widetilde{G}$. Since a unipotent group is torsion-free, the unipotent radicals of P and $\widetilde{P}$ are isomorphic, and the Levi factor $\widetilde{L}$ of $\widetilde{P}$ is the preimage of the Levi factor L of P. **For the remainder of §1.2, unless otherwise stated, we assume that G is simply connected.**

For $\alpha \in \Delta$, let P^α be the connected subgroup of G whose Lie algebra is spanned by $\mathfrak{h}$ and the root spaces $\mathfrak{g}^\beta$, where either $\beta \in R^+$ or β lies in the span of $\Delta - \{\alpha\}$. Then P^α is a maximal parabolic subgroup of G, and every maximal parabolic subgroup P is conjugate to exactly one P^α, $\alpha \in \Delta$. Thus there are exactly r maximal parabolic subgroups of G up to conjugation. We denote the unipotent radical of P^α by U^α and its Levi factor by L^α. The torus H is a maximal torus of L^α. The semisimple part S^α of L^α (or equivalently the derived subgroup of L^α) has Lie algebra spanned by $\mathfrak{h}'$ and by the root spaces corresponding to the set of roots in the linear span of $\Delta - \{\alpha\}$, where $\mathfrak{h}' \subseteq \mathfrak{h}$ is the subspace spanned by the coroots $\beta^\vee \in \mathfrak{h}$ dual to the simple roots $\beta \in \Delta - \{\alpha\}$. A maximal torus H' of S^α is given by the subtorus which is the image under exponentiation of $\mathfrak{h}' \subseteq \mathfrak{h}$, and $H' = H \cap S^\alpha$. The Dynkin diagram of S^α is the subdiagram of the Dynkin diagram of G obtained by deleting the vertex corresponding to α and the edges incident to this vertex. Since $\{\alpha^\vee\}_{\alpha\in\Delta}$ is a basis for the coroot lattice Λ of G, the intersection of the coroot lattice Λ for G with $\mathfrak{h}'$ is exactly the coroot lattice for S^α. Since G is simply connected, S^α is also simply connected. Note that S^α is a semisimple group of rank $r-1$. Clearly we have:

Lemma 1.2.1. *Let $\Delta - \{\alpha\} = \coprod_{i=1}^t \Delta_i$, where each Δ_i defines a connected component of the Dynkin diagram of $\Delta - \{\alpha\}$. Then $S^\alpha = \prod_{i=1}^t S_i$, where S_i is the simple and simply connected group corresponding to Δ_i.* □

Definition 1.2.2. Let $K(\alpha) = \bigcap_{\beta\in\Delta-\{\alpha\}} \operatorname{Ker}\beta \subseteq \Lambda$. Then $K(\alpha)$ is an infinite cyclic group. Let $\zeta_\alpha = \sum_{\beta\in\Delta} m_\beta\beta^\vee$ be the generator of $K(\alpha)$ such that $m_\alpha > 0$. It then follows that $m_\beta > 0$ for all $\beta \in \Delta$. Define $n_\alpha = \alpha(\zeta_\alpha)$, so that $\zeta_\alpha = n_\alpha\varpi_\alpha^\vee$. Note that $m_\alpha = \varpi_\alpha(\zeta_\alpha)$.

Lemma 1.2.3. *Define the map $\varphi_\alpha\colon \mathbb{C}^* \to H$ by*

$$\varphi_\alpha(\lambda) = \prod_{\beta\in\Delta} \ell_\beta\left(\lambda^{m_\beta}\right).$$

Then φ_α is an isomorphism from $\mathbb{C}^*$ to the identity component of the center of L^α. Moreover,

$$L^\alpha = S^\alpha \times_{\mathbb{Z}/m_\alpha\mathbb{Z}} \mathbb{C}^*,$$

where $1 \in \mathbb{Z}/m_\alpha\mathbb{Z}$ maps to $e^{2\pi i/m_\alpha} \in \mathbb{C}^*$ and to the central element $\prod_{i=1}^t \gamma_i^{-n(\beta_i,\alpha)} \in S^\alpha = \prod_{i=1}^t S_i$, where $\beta_i \in \Delta_i$ is the unique element for which $n(\beta_i,\alpha) \neq 0$ and $\gamma_i = \exp(2\pi\sqrt{-1}\varpi_{\beta_i}^\vee)$.

Proof. With φ_α defined as above, $\varphi_\alpha(\mathbb{C}^*)$ is in the kernel of all simple roots β distinct from α, and thus, since ζ_α is primitive, φ_α is an embedding of $\mathbb{C}^*$ into the center of L^α. Also, since $S^\alpha \cap H = H'$, if $\lambda = e^{2\pi it}$, then $\varphi_\alpha(\lambda) \in S^\alpha$ if and only if $\sum_{\beta\in\Delta} tm_\beta\beta^\vee \equiv 0 \bmod \mathfrak{h}' + \Lambda$, if and only if $m_\alpha t \in \mathbb{Z}$. Thus $\varphi_\alpha(\mathbb{C}^*)\cap S^\alpha$ is the cyclic subgroup of order m_α in $\mathbb{C}^*$, and so

$$L^\alpha = S^\alpha \times_{\mathbb{Z}/m_\alpha\mathbb{Z}} \mathbb{C}^*,$$

where the image of $1 \in \mathbb{Z}/m_\alpha\mathbb{Z}$ in the first factor lies in the center of S^α, and corresponds to the element $c = \zeta_\alpha/m_\alpha - \alpha^\vee$. To describe this central element, let $\beta \in \Delta$ be a root of S_i. Then $\beta(c) = 0$ if $\beta \neq \beta_i$ and $\beta_i(c) = -n(\beta_i,\alpha)$. Thus c is the central element of S_i given by $\exp(-2n(\beta_i,\alpha)\pi\sqrt{-1}\varpi_{\beta_i}^\vee)$. □

The next lemma gives a more precise description of the center of L^α.

Lemma 1.2.4. *The center of L^α is generated by $\varphi_\alpha(\mathbb{C}^*)$ and the center of G. The intersection of $\varphi_\alpha(\mathbb{C}^*)$ with the center of G is a cyclic group of order n_α.*

Proof. We have

$$Z(L^\alpha) = Z(S^\alpha) \times_{\mathbb{Z}/m_\alpha\mathbb{Z}} \mathbb{C}^*.$$

Then $Z(G)$ is the subgroup of $Z(L^\alpha)$ which is in the kernel of the character α. The restriction of α to $\mathbb{C}^* \subseteq Z(L^\alpha)$ is nontrivial, and hence surjective, and for an element of $Z(S^\alpha) \times_{\mathbb{Z}/m_\alpha\mathbb{Z}} \mathbb{C}^*$, written as $[z, \zeta]$, we clearly have $\alpha([z, \zeta]) = \alpha(z)\alpha(\zeta)$. Thus, for each $z \in Z(S^\alpha)$ there is an element $u \in \mathrm{Ker}(\alpha)$ of the form $u = [z, \zeta] \in Z(S^\alpha) \times_{\mathbb{Z}/m_\alpha\mathbb{Z}} \mathbb{C}^*$. This element u is in the center of G, since it is in the kernel of β for all $\beta \neq \alpha$, as well as in $\mathrm{Ker}\,\alpha$. It follows that an arbitrary $[z, \lambda] \in Z(L^\alpha)$ is of the form $u \cdot \mu$, where $\mu \in \mathbb{C}^*$, as claimed.

To see the second statement, note that $\alpha \circ \varphi_\alpha(\lambda) = \lambda^{n_\alpha}$. Since $\varphi_\alpha(\lambda)$ is in the kernel of all of the remaining roots, $\varphi_\alpha(\lambda)$ lies in the center of G if and only if $\lambda^{n_\alpha} = 1$. □

Remark 1.2.5. Along similar lines, one can show that there is an exact sequence

$$\{1\} \to \mathbb{Z}/n_\alpha\mathbb{Z} \to Z(G) \to Z(S^\alpha)/(\mathbb{Z}/m_\alpha\mathbb{Z}) \to \{1\}.$$

In particular, $m_\alpha/n_\alpha = \#Z(S^\alpha)/\#Z(G)$.

To state the next result, recall that a character $\chi\colon H \to \mathbb{C}^*$ is *dominant* if the corresponding linear function $\mathfrak{h} \to \mathbb{C}$ is nonnegative on the positive coroots. A character of any subgroup containing H will be called dominant if its restriction to H is dominant.

Lemma 1.2.6. *The group of characters of L^α is isomorphic to $\mathbb{Z}$. There is a unique primitive dominant character $\chi_0\colon L^\alpha \to \mathbb{C}^*$, and $\chi_0 \circ \varphi_\alpha(\lambda) = \lambda^{m_\alpha}$.*

Proof. Let ϖ_α be the fundamental weight corresponding to the simple root α. The unique primitive dominant character of L^α is ϖ_α, viewed as a character on H, and $\varpi_\alpha \circ \varphi_\alpha(\lambda) = \lambda^{m_\alpha}$. □

If G is not simply connected, we continue to denote the parabolic subgroup associated to the simple root α by P^α, the unipotent radical of P^α by U^α, and the Levi factor of P^α by L^α. The map $\varphi_\alpha\colon \mathbb{C}^* \to L^\alpha$ is defined to be embedding of $\mathbb{C}^*$ into to the center of L^α so that the composition of φ_α followed by the primitive dominant character of L^α is a positive character on $\mathbb{C}^*$. Let $\widetilde{\varphi}_\alpha\colon \mathbb{C}^* \to \widetilde{L}^\alpha$ be as given in Definition 1.2.2 for the simply connected group $\widetilde{G}$ and $\alpha \in \Delta$. Clearly,

$\varphi_\alpha(\mathbb{C}^*)$ is the quotient of $\widetilde{\varphi}_\alpha(\mathbb{C}^*)$ by the finite subgroup $\widetilde{\varphi}_\alpha(\mathbb{C}^*)\cap\langle c\rangle$. It is still the case, of course, that the center of L^α is generated by the center of G and $\varphi_\alpha(\mathbb{C}^*)$. The following lemma is then clear:

Lemma 1.2.7. *Let $n_{c,\alpha}$ be the order of $\varphi_\alpha(\mathbb{C}^*)\cap Z(G)$. Then the order of $\widetilde{\varphi}_\alpha(\mathbb{C}^*)\cap\langle c\rangle$ is $n_\alpha/n_{c,\alpha}$, and the induced map from $\widetilde{\varphi}_\alpha(\mathbb{C}^*)\cong\mathbb{C}^*$ to $\varphi_\alpha(\mathbb{C}^*)\cong\mathbb{C}^*$ is given by raising to the power $n_\alpha/n_{c,\alpha}$.* □

1.3 The unipotent radical of a maximal parabolic subalgebra

In §1.3, we assume that G is simply connected. Fix the simple root α and consider the maximal parabolic subgroup P^α. Here we will describe the Lie algebra $\mathfrak{u}=\mathfrak{u}(\alpha)$ of U^α. It is spanned by the root spaces $\mathfrak{g}^\delta$ such that, if $\delta=\sum_{\beta\in\Delta}x_\beta\beta$, then the coefficient of α in the sum is positive. The action of $\varphi_\alpha(\mathbb{C}^*)$ on $\mathfrak{u}$ is given as follows. Let $\mathfrak{g}^\delta$ be a root space and let $X\in\mathfrak{g}^\delta$. Then $\varphi_\alpha(\lambda)(X)=\lambda^{\delta(\zeta_\alpha)}\cdot X$. Of course, since ζ_α is in the kernel of all the simple roots except α, $\delta(\zeta_\alpha)=\delta(\varpi^\vee_\alpha)\alpha(\zeta_\alpha)$ where $\delta(\varpi^\vee_\alpha)$ is the coefficient of α in the expression for δ as a sum of simple roots. In particular, using Lemma 1.4.5 below, the weights of the action of $\mathbb{C}^*$ on $\mathfrak{u}$ are $n_\alpha, 2n_\alpha,\dots,h_\alpha\cdot n_\alpha$ where $n_\alpha=\alpha(\zeta_\alpha)$.

Let $\mathfrak{u}^k$ be the sum of all such root spaces $\mathfrak{g}^\beta$ where the coefficient of α in β is exactly k. The Lie algebra $\mathfrak{u}$ is a direct sum of the spaces $\mathfrak{u}^k$ for $k>0$. Each space $\mathfrak{u}^k$ is an L^α-module, and as we shall see below it is in fact irreducible.

The top exterior power $\bigwedge^{\text{top}}\mathfrak{u}$ is a one-dimensional L^α-module, and as such it is given by a character χ_+ of L. The next lemma identifies this character:

Lemma 1.3.1. *Let $d_1(\alpha)=2\rho(\zeta_\alpha)/m_\alpha=2\rho(\varpi^\vee_\alpha)/\varpi_\alpha(\varpi^\vee_\alpha)$. In the above notation, $\chi_+=\chi_0^{d_1(\alpha)}$ where χ_0 is the primitive dominant character.*

Proof. Since χ_0 is primitive $\chi_+=\chi_0^{N_+}$ for some integer N_+. Since the character $\chi_0\circ\varphi_\alpha$ of $\mathbb{C}^*$ is given by $\lambda\mapsto\lambda^{m_\alpha}$, the character $\chi_+\circ\varphi_\alpha$ of $\mathbb{C}^*$ is given by $\lambda\mapsto\lambda^{m_\alpha\cdot N_+}$. The action of $\varphi_\alpha(\mathbb{C}^*)$ is diagonal with respect to the decomposition of $\mathfrak{u}$ as a sum of root spaces, and the character on the one-dimensional subspace spanned by a root δ is simply the restriction of δ to $\varphi_\alpha(\mathbb{C}^*)$. The space $\mathfrak{u}$ is the subspace of $\mathfrak{g}$ spanned by the set of all roots $\delta=\sum_{\beta\in\Delta}x_\beta\beta$ with the property that $x_\alpha>0$. Recall that $\varphi_\alpha(\mathbb{C}^*)$ is in the kernel of all the simple roots except α. Thus, to compute the character of $\mathbb{C}^*$ given by the product of φ_α^δ over all δ such that the coefficient of α in δ is positive, we may as well take the product over all of the positive roots δ. In other words, the character of $\mathbb{C}^*$ which gives the degree of the top exterior power is simply the character $\varphi_\alpha^{\sum_{\delta\in R^+}\delta}$. We can rewrite this expression as $\varphi_\alpha^{2\rho}$. Thus, the character that we are computing is $\varphi_\alpha^{2\sum_{\beta\in\Delta}\varpi_\beta}$. Recalling that the embedding of $\varphi_\alpha\colon \mathbb{C}^*\to H$ is given by $\prod_{\beta\in\Delta}\ell_\beta^{m_\beta}$, we see that the character $\chi_+\circ\varphi_\alpha$ is given by raising to the power $2\rho(\zeta_\alpha)$. Thus $N_+ = 2\rho(\zeta_\alpha)/m_\alpha = d_1(\alpha)$. $\square$

Remark 1.3.2. The proof above also shows that the integer $d_1(\alpha)$ is the degree of divisibility of the canonical bundle of the homogeneous space G/P^α.

Now we give a purely root theoretic formula for $d_1(\alpha)$.

Lemma 1.3.3. *$d_1(\alpha) = \sum_{\beta(\varpi_\alpha^\vee)>0} n(\beta,\alpha)$, where the β in the sum range over R.*

Proof. By definition, $\chi_+ = \sum_{\beta(\varpi_\alpha^\vee)>0}\beta$ as additive characters on $\mathfrak{h}$. By the previous lemma, $\chi_+(\zeta_\alpha/m_\alpha) = d_1(\alpha)\chi_0(\zeta_\alpha/m_\alpha) = d_1(\alpha)$. On the other hand, $\zeta_\alpha/m_\alpha = \alpha^\vee + \nu$, where ν lies in the $\mathbb{Q}$-span of the simple coroots $\beta^\vee$, $\beta\neq\alpha$, and hence in the Lie algebra of the derived group S^α. Thus

$$\chi_+(\zeta_\alpha/m_\alpha) = \chi_+(\alpha^\vee) = \sum_{\substack{\beta\in R\\ \beta(\varpi_\alpha^\vee)>0}} \beta(\alpha^\vee),$$

and so $d_1(\alpha) = \sum_{\beta(\varpi_\alpha^\vee)>0} n(\beta,\alpha)$. $\square$

Similarly, we compute the character χ_k of L^α corresponding to $\bigwedge^{\mathrm{top}}\mathfrak{u}^k$:

Lemma 1.3.4. *For $k>0$, let $c(\alpha,k)$ be the dimension of $\mathfrak{u}^k$, in other words the number of roots β such that the coefficient of α in β is k. Let $i(\alpha,k) = kc(\alpha,k)/\varpi_\alpha(\varpi_\alpha^\vee) = kn_\alpha c(\alpha,k)/m_\alpha$. Then the character χ_k of L^α corresponding to $\bigwedge^{\mathrm{top}}\mathfrak{u}^k$ is given by*

$$\chi_k = \chi_0^{i(\alpha,k)}.$$

Proof. As before, $\chi_k = \chi_0^{N_k}$ for some integer N_k. To compute N_k, note that, if $\mathfrak{g}^\beta\subseteq\mathfrak{u}^k$, then φ_α acts on $\mathfrak{g}^\beta$ via raising to the power $\beta(\zeta_\alpha) = kn_\alpha$. Since $\chi_0\circ\varphi_\alpha(\lambda) = \lambda^{m_\alpha}$, we must have

$$m_\alpha N_k = kn_\alpha \dim\mathfrak{u}^k.$$

Hence, $N_k = kn_\alpha c(\alpha, k)/m_\alpha = i(\alpha, k)$. □

Remark 1.3.5. An argument very similar to the proof of Lemma 1.3.3 shows that

$$i(\alpha, k) = \sum_{\beta(\varpi_\alpha^\vee)=k} n(\beta, \alpha).$$

1.4 Some lemmas on root systems

Fix $\alpha \in \Delta$. Our goal now is to analyze the L^α-representations $\mathfrak{u}^k$, and in particular the numbers $d_1(\alpha)$ and $i(\alpha, k)$ introduced above.

Definition 1.4.1. Fix a positive integer k, and consider the set

$$S(\alpha, k) = \{\beta \in R : \beta(\varpi_\alpha^\vee) = k\}.$$

For $k = 0$, we define similarly

$$S^+(\alpha, 0) = \{\beta \in R^+ : \beta(\varpi_\alpha^\vee) = 0\}.$$

The latter is a set of positive roots for the root system $R' = \Delta - \{\alpha\}$, i.e., for S^α. We define $S^-(\alpha, 0)$ similarly. A *lowest root* $\sigma_k(\alpha)$ for $S(\alpha, k)$ is a root $\sigma_k(\alpha) \in S(\alpha, k)$ such that, for every $\beta \in S(\alpha, k)$, $\beta - \sigma_k(\alpha)$ is a sum (possibly empty) of simple roots. For example, $\sigma_1(\alpha) = \alpha$. A lowest root in $S(\alpha, k)$ is clearly unique, if it exists. A highest root $\lambda_k(\alpha) \in S(\alpha, k)$ is defined similarly, and is also clearly unique if it exists.

The following is related to results of Borel-Tits (unpublished) as well as Azad-Barry-Seitz [3].

Proposition 1.4.2. *In the above notation, if $S(\alpha, k) \neq \emptyset$, then lowest roots and highest roots always exist and are unique.*

Proof. Let $R(\alpha, k)$ be the subset of R consisting of roots β such that k divides $\beta(\varpi_\alpha^\vee)$. Clearly $R(\alpha, k)$ is again a root system. Let V be the real span of R and let V' be the subspace of V spanned by $\Delta - \{\alpha\}$. Then clearly $V' \cap R = V' \cap R(\alpha, k) = R'$ is the set of all roots which are linear combinations of elements of $\Delta - \{\alpha\}$. Thus R' is a root system with simple roots $\Delta - \{\alpha\}$. By [8, VI §1, Proposition 24], since $R' \subseteq R(\alpha, k)$, there exists a set of simple roots for $R(\alpha, k)$ containing $\Delta - \{\alpha\}$. In fact, the proof of this proposition shows that there are at least two different sets of simple roots, each of the form $(\Delta - \{\alpha\}) \cup \{\beta\}$. Then $k|\beta(\varpi_\alpha^\vee)$, and since $S(\alpha, k) \neq \emptyset$, in fact $\beta(\varpi_\alpha^\vee) = \pm k$. Suppose for example that $\beta(\varpi_\alpha^\vee) = k$. If $\gamma \in S(\alpha, k)$, then $\gamma = \sum_{\delta \neq \alpha} m_\delta \delta + m\beta$ and $\gamma(\varpi_\alpha^\vee) = \beta(\varpi_\alpha^\vee) = k$. Thus $m = 1$ and $m_\delta \geq 0$ for all $\delta \in \Delta - \{\alpha\}$, so that β is a lowest root for $S(\alpha, k)$. Suppose now that $(\Delta - \{\alpha\}) \cup \{\beta'\}$ is also a set of simple roots for $R(\alpha, k)$, where $\beta' \neq \beta$. It follows that $\varpi_\alpha^\vee(\beta') = -k$. In this case, it is easy to check that $-\beta'$ is a highest root for $S(\alpha, k)$. □

Corollary 1.4.3. *The L^α-modules $\mathfrak{u}^k$ are irreducible.* □

Remark 1.4.4. Using the Borel-de Siebenthal theorem [8, p. 229, ex. 4], the Dynkin diagram of $R(\alpha, k)$ is given abstractly as follows. Begin with the extended Dynkin diagram of R. There exists a root β such that $h_\beta = k$, and such that the Dynkin diagram for $\widetilde{\Delta} - \{\beta\}$ contains a root γ such that the diagram for $\widetilde{\Delta} - \{\beta, \gamma\}$ is the same as the diagram for $\Delta - \{\alpha\}$. In practice, these properties uniquely determine β and γ. The Dynkin diagram for $R(\alpha, k)$ is then the Dynkin diagram of $\widetilde{\Delta} - \{\beta\}$, and γ corresponds to $\lambda_k(\alpha)$.

Lemma 1.4.5. *Let k be a positive integer. Then $S(\alpha, k) \neq \emptyset$ if and only if $1 \leq k \leq h_\alpha$.*

Proof. Since $\alpha \in S(\alpha, 1)$, $S(\alpha, 1) \neq \emptyset$. Suppose inductively we have shown that $S(\alpha, k) \neq \emptyset$ for all $k, 1 \leq k \leq \ell$. If $\beta \in \Delta - \alpha$, then $\lambda_\ell(\alpha) + \beta$ is not a root. If $\lambda_\ell(\alpha) + \alpha$ is not a root, then $\lambda_\ell(\alpha)$ is the highest root and $\ell = h_\alpha$. Otherwise, $\lambda_\ell(\alpha) + \alpha \in S(\alpha, \ell + 1)$. □

For future reference, we record the following properties of $\sigma_k(\alpha)$ and $\lambda_k(\alpha)$:

Lemma 1.4.6. *In the above notation, let $R' \subseteq R$ be the subroot system with simple roots $\Delta - \{\alpha\}$. Suppose that $S(\alpha, k) \neq \emptyset$, and let $w_0' \in W(R') \subseteq W(R)$ be the unique element such that $w_0'(\Delta - \{\alpha\}) = -(\Delta - \{\alpha\})$. Suppose that τ is the permutation of $\Delta - \{\alpha\}$ induced by $-w_0'$, which we can also view as a permutation of Δ fixing α. Let τ act on V and V^* in the natural way. Then:*

(i) $w_0' \sigma_k(\alpha) = \lambda_k(\alpha)$.

(ii) $\lambda_1(\alpha^\vee) = (\lambda_1(\alpha))^\vee$.

(iii) $\sigma_k(\alpha) = k\lambda_1(\alpha) - \tau(\lambda_k(\alpha)) + k\alpha$. *Likewise* $\sigma_k(\alpha)^\vee = k'\lambda_1(\alpha^\vee) - \tau(\lambda_k(\alpha)^\vee) + k'\alpha^\vee$, *where k' is the coefficient of $\alpha^\vee$ in $\lambda_k(\alpha)^\vee$.*

Proof. Clearly $w_0'\sigma_k(\alpha) \in S(\alpha,k)$ has the property that

$$(\Delta - \{\alpha\}) \cup \{-w_0'\sigma_k(\alpha)\}$$

is a set of simple roots for $R(\alpha,k)$. By the proof of Proposition 1.4.2, $w_0'\sigma_k(\alpha) = \lambda_k(\alpha)$, proving (i). To see (ii), note that $\alpha^\vee = \sigma_1(\alpha^\vee)$, and thus $\lambda_1(\alpha^\vee) = w_0'\sigma_1(\alpha^\vee) = w_0'(\alpha^\vee) = (w_0'\alpha)^\vee = (\lambda_1(\alpha))^\vee$. To see (iii), write $\lambda_k(\alpha) = k\alpha + \lambda'$, where $\lambda' \in V'$. Then

$$\sigma_k(\alpha) = w_0'\lambda_k(\alpha) = kw_0'\alpha - \tau(\lambda') = k\lambda_1(\alpha) - \tau(\lambda').$$

On the other hand, $-\tau(\lambda') = k\alpha - \tau(\lambda_k(\alpha))$, and plugging this back in gives the first part of (iii). The second part is proved in a very similar way. □

Recall that, for $k > 0$, $c(\alpha,k)$ is the cardinality of $S(\alpha,k)$, in other words the number of $\beta \in R$ such that $\beta(\varpi_\alpha^\vee) = k$, and that $i(\alpha,k) = kc(\alpha,k)/\varpi_\alpha(\varpi_\alpha^\vee)$.

Definition 1.4.7. Define

$$d_k(\alpha) = \sum_{k|x} i(\alpha,x) = \sum_{\ell>0} i(\alpha,\ell k).$$

In particular $d_1(\alpha)$ agrees with the previous definition. Of course, the $d_k(\alpha)$ determine $i(\alpha,k)$ via Moebius inversion. Since the $i(\alpha,k)$ are all positive integers, the $d_k(\alpha)$ are positive integers as well.

Lemma 1.4.8. *With notation as above,*

$$i(\alpha,k) = \frac{h_\alpha g k c(\alpha,k)}{g_\alpha \sum_{\beta\in R^+}(\beta(\varpi_\alpha^\vee))^2};$$

$$d_1(\alpha) = 2\rho(\varpi_\alpha^\vee)/\varpi_\alpha(\varpi_\alpha^\vee) = \frac{h_\alpha g \sum_{\beta\in R^+}\beta(\varpi_\alpha^\vee)}{g_\alpha \sum_{\beta\in R^+}(\beta(\varpi_\alpha^\vee))^2}.$$

Moreover $\sum_{k>0}\varphi(k)d_k(\alpha) = h_\alpha g/g_\alpha$.

Proof. The first equality follows since, by Lemma 1.1.2 and Lemma 1.1.3,

$$2g\left(\frac{h_\alpha}{g_\alpha}\right)\varpi_\alpha(\varpi_\alpha^\vee) = 2g\langle\varpi_\alpha^\vee,\varpi_\alpha^\vee\rangle = 2gI_0(\varpi_\alpha^\vee) = 2\sum_{\beta\in R^+}(\beta(\varpi_\alpha^\vee))^2.$$

The second follows since

$$\sum_{k>0} kc(\alpha,k) = \sum_{\beta\in R^+} \beta(\varpi_\alpha^\vee).$$

The third is an easy consequence of the fact that $\sum_{k|x}\varphi(k) = x$, using

$$\sum_{k>0} k^2c(\alpha,k) = \sum_{\beta\in R^+} (\beta(\varpi_\alpha^\vee))^2. \qquad \square$$

Proposition 1.4.9. *We have* $d_1(\alpha) = \rho(\lambda_1(\alpha^\vee))+1$, *where* $\lambda_1(\alpha^\vee)$ *is the highest coroot such that the coefficient of* $\alpha^\vee$ *in* $\lambda_1(\alpha^\vee)$ *is* 1.

Proof. By Lemma 1.3.3, it suffices to prove that

$$\rho(\lambda_1(\alpha^\vee)) + 1 = \sum_{\beta(\varpi_\alpha^\vee)>0} n(\beta,\alpha).$$

By Lemma 1.4.6, since $\sigma_1(\alpha^\vee) = \alpha^\vee$, $w_0'(\alpha^\vee) = \lambda_1(\alpha^\vee)$. Since w_0' exchanges positive and negative roots in R', it follows that $-w_0'$ is a permutation of $S^+(\alpha,0)$. Clearly, given $\beta\in S^+(\alpha,0)$,

$$\beta(\lambda_1(\alpha^\vee)) = -(-w_0'(\beta)(\alpha^\vee)).$$

Thus, since $-w_0'$ permutes $S^+(\alpha,0)$, we have

$$\sum_{\substack{\beta\in S^+(\alpha,0)\\ \beta(\lambda_1(\alpha^\vee))>0}} \beta(\lambda_1(\alpha^\vee)) + \sum_{\substack{\beta\in S^+(\alpha,0)\\ \beta(\alpha^\vee)<0}} \beta(\alpha^\vee) = 0.$$

Next we claim:

Lemma 1.4.10.

$$-\sum_{\substack{\beta\in S^+(\alpha,0)\\ \beta(\alpha^\vee)<0}} \beta(\alpha^\vee) \quad + \quad 2 = \sum_{\beta(\varpi_\alpha^\vee)>0} \beta(\alpha^\vee) = \sum_{\beta(\varpi_\alpha^\vee)>0} n(\beta,\alpha).$$

Proof. Suppose that $\beta\neq\pm\alpha\in R$, and consider the α-string defined by β, say $\beta - q\alpha,\dots,\beta+p\alpha$. Since $p-q+1 = -n(\beta,\alpha)+1$, it is easy to see that $\sum_{i=-q}^{p}(\beta+i\alpha)(\alpha^\vee) = 0$. If β is a negative root, then every root in the α-string is negative and thus none of them appears in the right hand side of the above equality. After reindexing, we can assume that β is the origin of the α-string. If $\beta(\alpha^\vee) > 0$, then $(\beta + i\alpha)(\alpha^\vee) > 0$ for all $i > 0$, and so the total contribution to the right hand side from the sum over the α-string is zero. If $\beta \in S^+(\alpha, 0)$, then the contribution to the sum on the right hand side is $\sum_{i\geq 1}(\beta + i\alpha)(\alpha^\vee) = -\beta(\alpha^\vee)$. The remaining possibility for the right hand side is $\beta = \alpha$, and in this case $\beta(\alpha^\vee) = 2$. Thus we see that the right hand side in Lemma 1.4.10 is equal to the left hand side. □

Lemma 1.4.11.

$$\sum_{\substack{\beta\in S^+(\alpha,0)\\ \beta(\lambda_1(\alpha^\vee))>0}} \beta(\lambda_1(\alpha^\vee)) = \rho(\lambda_1(\alpha^\vee)) - 1.$$

Proof. First note that

$$2\rho(\lambda_1(\alpha^\vee)) = \sum_{\beta\in R^+} \beta(\lambda_1(\alpha^\vee)).$$

We consider as before the $\lambda_1(\alpha)$-strings defined by a root $\beta \neq \lambda_1(\alpha)$ which lie in R^+. If the origin of such a string lies in R^+, then so does every γ lying in the string, and the sum over all such γ of $n(\gamma, \lambda_1(\alpha))$ is zero. Next we claim:

Claim 1.4.12. If a $\lambda_1(\alpha)$-string meets R^+ but is not contained in R^+, then either:

(i) The origin of the string lies in $S^-(\alpha, 0)$ and all other elements of the string lie in R^+.

(ii) The extremity of the string lies in $S^+(\alpha, 0)$, and all other elements lie in R^-.

Moreover, there is a length-preserving bijection between strings of types (i) and (ii) above.

Proof. First note that, if $\beta \in S(\alpha, 1)$, then $\beta - 2\lambda_1(\alpha)$ cannot be a root. For then $\beta - \lambda_1(\alpha)$ would also be a root, necessarily negative, and then $2\lambda_1(\alpha) - \beta$ would be an element of $S(\alpha, 1)$ higher than $\lambda_1(\alpha)$. Thus, every $\lambda_1(\alpha)$-string meeting R^+ but not contained in it must either begin or terminate in $S^+(\alpha, 0) \cup S^-(\alpha, 0)$. Clearly, the only possibilities are (i) and (ii) above, and the bijection is given by sending the origin of a string of type (i) to its negative, which is the extremity of a string of type (ii). □

Returning to the proof of Lemma 1.4.11, the only nonzero contributions to the sum $\sum_{\beta\in R^+}\beta(\lambda_1(\alpha^\vee))$ come from

(i) $\lambda_1(\alpha)$-strings whose origin is $\gamma = -\beta \in S^-(\alpha,0)$, and these contribute $-\gamma(\lambda_1(\alpha^\vee)) = \beta(\lambda_1(\alpha^\vee))$;

(ii) $\lambda_1(\alpha)$-strings whose extremity is $\beta \in S^+(\alpha,0)$, and these contribute $\beta(\lambda_1(\alpha^\vee))$;

(iii) the root $\lambda_1(\alpha)$ and this contributes $\lambda_1(\alpha)(\lambda_1(\alpha^\vee)) = 2$.

Summing these up, we see that

$$2\rho(\lambda_1(\alpha^\vee)) = \sum_{\beta\in R^+}\beta(\lambda_1(\alpha^\vee)) = 2\sum_{\substack{\beta\in S^+(\alpha,0)\\ \beta(\lambda_1(\alpha^\vee))>0}}\beta(\lambda_1(\alpha^\vee)) \quad + \quad 2.$$

Dividing by 2 gives the final formula of Lemma 1.4.11. □

To complete the proof of Proposition 1.4.9, we have

$$\begin{aligned}\sum_{\beta(\varpi_\alpha^\vee)>0} n(\beta,\alpha) &= -\sum_{\substack{\beta\in S^+(\alpha,0)\\ \beta(\alpha^\vee)<0}}\beta(\alpha^\vee) \;+\; 2\\ &= \sum_{\substack{\beta\in S^+(\alpha,0)\\ \beta(\lambda_1(\alpha^\vee))>0}}\beta(\lambda_1(\alpha^\vee)) \;+\; 2\\ &= \rho(\lambda_1(\alpha^\vee)) + 1,\end{aligned}$$

as claimed. □

There is a generalization of the previous proposition to the computation of $d_k(\alpha)$ for every $k > 0$:

Proposition 1.4.13. *Let*

$$k' = k\langle\alpha,\alpha\rangle/\langle\lambda_k(\alpha),\lambda_k(\alpha)\rangle$$

be the coefficient of $\alpha^\vee$ *in* $\lambda_k(\alpha)^\vee$. *Then*

$$d_1(\alpha) + d_k(\alpha) = \frac{2}{k'}(\rho(\lambda_k(\alpha)^\vee) + 1).$$

Proof. In the notation of Proposition 1.4.2, set $\alpha_k = -\lambda_k(\alpha)$ and let $R_k = R(\alpha, k)$, with simple roots $(\Delta - \{\alpha\}) \cup \{\alpha_k\}$. Although R_k need not be irreducible, we can still define the integer $d_1^{R_k}(\alpha_k)$ with respect to the root system R_k as in Lemma 1.3.1. By Proposition 1.4.9, which holds even if R_k is reducible, $d_1^{R_k}(\alpha_k) = \rho_k(\lambda_1((\alpha_k)^\vee)) + 1$, where ρ_k is the sum of the fundamental weights of the root system R_k. Applying Lemma 1.4.6, and using the notation introduced in its statement, we have

$$\begin{aligned}\lambda_1((\alpha_k)^\vee) = \lambda_1(\alpha_k)^\vee = w_0'(\alpha_k)^\vee &= (w_0'\alpha_k)^\vee = -(\sigma_k(\alpha))^\vee \\ &= -k'\lambda_1(\alpha^\vee) + \tau(\lambda_k(\alpha))^\vee) - k'\alpha^\vee,\end{aligned}$$

where $k' = k\langle\alpha,\alpha\rangle/\langle\lambda_k(\alpha),\lambda_k(\alpha)\rangle$ is the coefficient of $\alpha^\vee$ in $\lambda_k(\alpha)^\vee$. Next we compute ρ_k. Write

$$(\alpha_k)^\vee = \sum_{\beta\in\Delta-\{\alpha\}} c_\beta\beta^\vee - k'\alpha^\vee = \sum_{\beta\in\Delta} c_\beta\beta^\vee,$$

where $c_\alpha = -k'$. Denote the fundamental weights of R_k by ϖ_β^k, $\beta \neq \alpha$, and ϖ_{α_k}. Then it is easy to check that

$$\begin{aligned}\varpi_{\alpha_k} &= -\frac{1}{k'}\varpi_\alpha; \\ \varpi_\beta^k &= \varpi_\beta + \frac{c_\beta}{k'}\varpi_\alpha \qquad \text{for } \beta \neq \alpha.\end{aligned}$$

Thus

$$\begin{aligned}\rho_k &= \sum_{\beta\neq\alpha}\varpi_\beta^k + \varpi_{\alpha_k} = \sum_{\beta\neq\alpha}\varpi_\beta + \frac{1}{k'}\left(\sum_{\beta\neq\alpha}c_\beta - 1\right)\varpi_\alpha \\ &= \sum_\beta \varpi_\beta + \frac{1}{k'}\left(\sum_\beta c_\beta - 1\right)\varpi_\alpha \\ &= \rho - \frac{1}{k'}\left(\rho(\lambda_k(\alpha)^\vee) + 1\right)\varpi_\alpha.\end{aligned}$$

Since

$$\varpi_\alpha(-k'\lambda_1(\alpha^\vee) + \tau(\lambda_k(\alpha))^\vee) - k'\alpha^\vee) = -k'$$

and $\rho(\tau(\gamma)) = \rho(\gamma)$ for all γ, we see that

$$\begin{aligned}d_1^{R_k}(\alpha_k) &= -k'(\rho(\lambda_1(\alpha^\vee) + 1) + \rho(\lambda_k(\alpha)^\vee)) + 1 + \rho(\lambda_k(\alpha)^\vee)) + 1 \\ &= -k'd_1(\alpha) + 2(\rho(\lambda_k(\alpha)^\vee)) + 1).\end{aligned}$$

Now an argument similar to the calculation of ρ_k above shows that $\varpi^\vee_{\alpha_k}$, the fundamental coweight for R_k dual to α_k, is given by $-(1/k)\varpi^\vee_\alpha$. Also, $c(\alpha, nk) = c(\alpha_k, n)$ for all n. Thus $i(\alpha_k, n) = k' i(\alpha, nk)$ for all positive integers n. It follows that $d_1^{R_k}(\alpha_k) = k' d_k(\alpha)$. Putting this together with the above gives $k' d_1(\alpha) + k' d_k(\alpha) = 2(\rho(\lambda_k(\alpha)^\vee)) + 1)$, which is the statement of the proposition. $\square$

Corollary 1.4.14. $d_1(\alpha) + d_{h_\alpha}(\alpha) = \dfrac{2g}{g_\alpha}$.

Proof. For $k = h_\alpha$, we have $\lambda_k(\alpha) = \widetilde{\alpha}$ and $k' = kg_\alpha/h_\alpha = g_\alpha$, and the corollary is clear. $\square$

To put the corollary in a more general context, we have the following definition which is taken from [7]:

Definition 1.4.15. For a sequence $d_1, \dots, d_N$ of positive integers, let $M = \sum_{k>0} \varphi(k) d_k$. Let the Farey sequence

$$\mathcal{F}_N = \{0/1, 1/N, 1/(N-1), \dots\}$$

be the sequence of rational numbers between 0 and 1, written in lowest terms, whose denominator is at most N, written in increasing order. We say that the sequence $\{d_k\}$ has the *circular symmetry property* with respect to N and M if, for all consecutive terms r/x and s/y in $\mathcal{F}_N$,

$$d_x + d_y = \frac{2M}{xy}.$$

The geometric meaning of this property is explained in [7].

Since every integer x, $1 \le k \le N$, appears as a denominator of some element of $\mathcal{F}_N$, the following is clear:

Lemma 1.4.16. *Suppose that $d_1, \dots, d_N$ and $d'_1, \dots, d'_N$ are two sequences of positive integers which both satisfy the circular symmetry property with respect to N and M. If $d_1 = d'_1$, then $d_x = d'_x$ for all x, $1 \le k \le N$.* $\square$

For the sequence $\{d_k(\alpha)\}$, the largest integer N such that $d_N(\alpha) \neq 0$ is h_α, and $\sum_{k>0} \varphi(k) d_k(\alpha) = gh_\alpha/g_\alpha$, by Lemma 1.4.8. Moreover $d_1(\alpha)$ is given by Proposition 1.4.9. Then Corollary 1.4.14 is the first case $x = 1, y = h_\alpha$ of the following:

Proposition 1.4.17. *With notation as above, the sequence $\{d_k(\alpha)\}$ has the circular symmetry property with respect to h_α and gh_α/g_α.*

Proof. If $h_\alpha \le 3$, it is easy to check that the above conditions follow from Corollary 1.4.14 and the fact that $\sum_{k>0}\varphi(k)d_k(\alpha) = gh_\alpha/g_\alpha$. The remaining cases are: F_4 with α the root such that $h_\alpha = 4$, E_7 with α the root such that $h_\alpha = 4$, or E_8 with α a root such that $h_\alpha = 4, 5, 6$. These cases may be checked by hand. □

It would be very interesting to find a more conceptual proof of Proposition 1.4.17.

1.5 The moduli space of semistable G-bundles

If ξ_0 is a C^∞ principal G-bundle over E, then there is a characteristic class $c_1(\xi_0) \in H^2(E;\pi_1(G)) \cong \pi_1(G) = \langle c\rangle$. Let $\mathcal{M}(G,c)$ be the set of S-equivalence classes of holomorphic semistable G-bundles ξ over E with $c_1(\xi) = c$. Suppose that K is the compact form of G. Then $\langle c\rangle = \pi_1(K)$. Let $\widetilde{K}$ be the universal covering group of K. By the theorem of Narasimhan-Seshadri and Ramanathan and [10, 5.8(i)], $\mathcal{M}(G,c)$ is homeomorphic to the the space

$$\{(x,y) \in \widetilde{K} \times \widetilde{K} : xyx^{-1}y^{-1} = c\}/\widetilde{K},$$

where the action of $\widetilde{K}$ is by simultaneous conjugation. By [7, 3.2], corresponding to $c \in Z(\widetilde{G})$ there is an element $w_c \in W$ and an affine isomorphism φ_c of V^* which permutes the set $\widetilde{\Delta}$ and thus acts on the fundamental alcove A corresponding to the Weyl chamber defined by Δ.

Definition 1.5.1. The Weyl element w_c induces a permutation of the set $\widetilde{\Delta}$ which induces an automorphism of the extended Dynkin diagram of G. Thus $g_{w_c(\alpha)} = g_\alpha$ for all $\alpha \in \widetilde{\Delta}$. Let $\overline{\alpha}$ denote the w_c-orbit of α and let $n_{\overline{\alpha}}$ be the cardinality of $\overline{\alpha}$. Set $g_{\overline{\alpha}} = n_{\overline{\alpha}}g_\alpha$, for any choice of $\alpha \in \overline{\alpha}$. Let n_0 be the gcd of the integers $g_{\overline{\alpha}}$. Let $\widetilde{\Lambda}$ be the free abelian group with basis $\widetilde{\Delta}$. Then w_c acts on $\widetilde{\Lambda}$, preserving the relation $\sum_{\beta\in\widetilde{\Delta}} g_\beta\beta^\vee = 0$, and so w_c acts on the quotient which is Λ. Define r_c by: $r_c + 1$ is the cardinality of $\widetilde{\Delta}/w_c$.

Let A^c be the fixed subspace of φ_c, acting on A. With $V^* = \Lambda \otimes \mathbb{R}$, let $T = V^*/\Lambda$, so that T is a real torus of dimension r. Then W and w_c act on T. Let $T_0 = (T^{w_c})^0$ be the identity component of the group T^{w_c}. Thus $r_c = \dim T_0$.

For an abelian group $\mathcal{A}$ and an automorphism σ of $\mathcal{A}$, we denote as usual by $\mathcal{A}^\sigma$ the subgroup of invariants of σ and by $\mathcal{A}_\sigma$ the group

$\mathcal{A}/\operatorname{Im}(\mathrm{Id}-\sigma)$ of coinvariants of σ. The automorphism w_c acts on $T = V^*/\Lambda$, and we can define T^{w_c} and T_{w_c} as before. Let $T_0 = (T^{w_c})^0$ be the identity component of the group T. There is an induced map from T_0 to T_{w_c} and it is finite. We then have:

Theorem 1.5.2. *Fix $x_0 \in A^c$ and y_0 an element in the normalizer $N_T(\widetilde{K})$ of T in $\widetilde{K}$ which projects to w_c in $W = N_T(\widetilde{K})/T$. Then for $s, t \in T_0$ the pair $x = sx_0, y = ty_0$ satisfies $[x, y] = c$. We define a map $T_0 \times T_0 \to \mathcal{M}(G, c)$ by $(s, t) \mapsto [(sx_0, ty_0)]$. This map is finite and surjective. Its degree is*

$$(r_c)!\frac{\det(I_0|\Lambda^{w_c})}{n_0}\prod_{\overline{\alpha}} g_{\overline{\alpha}},$$

where the product is over the w_c-orbits of $\widetilde{\Delta}$.

Proof. By [7, Lemma 6.1.7], with $\widetilde{K}$ and c as above, every pair (x, y) such that $xyx^{-1}y^{-1} = c$ is conjugate to such a pair with $x \in A^c$ and $y \in T_0 \cdot w_c$. This proves that the map is surjective. Clearly, it is finite-to-one. If x is in the interior of A^c, then it is regular, and the only further possible conjugation is via an element $t \in T$, which acts on y via $t - w_c(t)$. Thus, a fundamental domain for the map $T_0 \times T_0 \to \mathcal{M}(G, c)$ is given by $A^c \times S$, where S is a fundamental domain for the quotient map $T_0 \to T_{w_c}$. It follows that the degree of the map $T_0 \times T_0 \to \mathcal{M}(G, c)$ is the product of the degree of the map from T_0 to T_{w_c} with the ratio $\operatorname{vol}(T_0)/\operatorname{vol}(A^c)$, where volume is computed with respect to any Weyl invariant metric. We consider these two integers separately.

Lemma 1.5.3. *Let $\overline{\alpha}^\vee = \sum_{\alpha \in \overline{\alpha}} \alpha^\vee$. Then*

$$\Lambda^{w_c} \cong \bigoplus_{\overline{\alpha}} \mathbb{Z} \cdot \overline{\alpha}^\vee \Big/ \sum_{\overline{\alpha}} g_\alpha \overline{\alpha}^\vee.$$

Moreover, the set $\{\overline{\alpha}^\vee : \overline{\alpha} \neq \overline{\alpha}_0\}$ is an integral basis for Λ^{w_c}. Finally, for each orbit $\overline{\alpha}$, choose $\alpha \in \overline{\alpha}$ and let $e_{\overline{\alpha}}$ be the image of α in Λ_{w_c}. Then

$$\Lambda_{w_c} \cong \bigoplus_{\overline{\alpha}} \mathbb{Z} \cdot e_{\overline{\alpha}} \Big/ \sum_{\overline{\alpha}} g_{\overline{\alpha}} e_{\overline{\alpha}}.$$

Proof. There is an exact sequence

$$0 \to \mathbb{Z} \to \widetilde{\Lambda} \to \Lambda \to 0,$$

where $1 \in \mathbb{Z} \mapsto \sum_{\beta \in \widetilde{\Delta}} g_\beta \beta^\vee$. The homomorphisms in this sequence are equivariant with respect to the action of w_c. Moreover, w_c acts on $\widetilde{\Lambda}$ by a permutation of the basis. The proof of the lemma follows easily by considering the associated long exact sequence

$$0 \to \mathbb{Z}^{w_c} \to (\widetilde{\Lambda})^{w_c} \to \Lambda^{w_c} \to \mathbb{Z}_{w_c} \to (\widetilde{\Lambda})_{w_c} \to \Lambda_{w_c} \to 0. \qquad \square$$

Corollary 1.5.4. *The torsion subgroup* $\operatorname{Tor} \Lambda_{w_c} \cong \mathbb{Z}/n_0\mathbb{Z}$, *and*

$$\Lambda_{w_c}/\operatorname{Tor}\Lambda_{w_c} \cong \bigoplus_{\overline{\alpha}} \mathbb{Z} \cdot e_{\overline{\alpha}} \Big/ \sum_{\overline{\alpha}} \frac{g_{\overline{\alpha}}}{n_0} e_{\overline{\alpha}}. \qquad \square$$

Lemma 1.5.5. *The order of* T^{w_c}/T_0 *is* n_0. *The natural map* $T_0 \to T_{w_c}$ *is finite and surjective of degree* $\prod_{\overline{\alpha}} n_{\overline{\alpha}} \Big/ n_0$.

Proof. Beginning with the short exact sequence of w_c-modules

$$0 \to \Lambda \to V \to T \to 0,$$

we get a long exact sequence

$$0 \to \Lambda^{w_c} \to V^{w_c} \to T^{w_c} \to \Lambda_{w_c} \to V_{w_c} \to T_{w_c} \to 0.$$

The quotient $V^{w_c}/\Lambda^{w_c} = T_0$. Since T^{w_c}/T_0 is a finite group and V_{w_c} is torsion free, the induced map $T^{w_c}/T_0 \to \Lambda_{w_c}$ is an isomorphism from T^{w_c}/T_0 to $\operatorname{Tor}\Lambda_{w_c}$, and hence $T^{w_c}/T_0 \cong \mathbb{Z}/n_0\mathbb{Z}$. Moreover, it is clear that the degree of the map from T_0 to T_{w_c} is the index of the image of Λ^{w_c} in $\Lambda_{w_c}/\operatorname{Tor}\Lambda_{w_c}$. By Lemma 1.5.3 and Corollary 1.5.4, it suffices to compute the order of the quotient of $\bigoplus_{\overline{\alpha}} \mathbb{Z} \cdot e_{\overline{\alpha}}$ by the relations $n_{\overline{\alpha}} e_{\overline{\alpha}}$ and $\sum_{\overline{\alpha}} g_{\overline{\alpha}} e_{\overline{\alpha}}$. Since $g_{\overline{\alpha}_0} = 1$, it is clear that the quotient has order $\prod_{\overline{\alpha}} n_{\overline{\alpha}} \Big/ n_0$. $\square$

Now we compute $\operatorname{vol}(T_0)/\operatorname{vol}(A^c)$ using the volume determined by the inner product I_0. The alcove A has vertices equal to 0 and $\varpi_\alpha^\vee/h_\alpha$, $\alpha \in \Delta$. By Lemma 1.1.3, the vertices are 0 and α^*/g_α, where the α^* are the dual basis with respect to I_0 to the basis $\alpha^\vee$ of Λ. Now we have the following elementary lemma, whose proof is left to the reader:

Lemma 1.5.6. *Let* A *be a simplex in* $\mathbb{R}^n$ *with vertices* $0 = e_0, e_1, \dots, e_n$. *Let* φ *be an affine linear transformation of* $\mathbb{R}^n$ *which acts via a permutation of the vertices of* A. *Suppose that the orbits of*

w on the vertices are $\mathbf{o}_0, \dots, \mathbf{o}_s$, with $0 \in \mathbf{o}_0$. If $n_{\mathbf{o}}$ is the order of the orbit $\mathbf{o}$, set

$$v_{\mathbf{o}} = \frac{1}{n_{\mathbf{o}}} \sum_{e_i \in \mathbf{o}} e_i.$$

Then the fixed set of A^{φ} for the action of φ on A is a simplex with vertices

$$v_{\mathbf{o}_0}, v_{\mathbf{o}_1} + v_{\mathbf{o}_0}, \dots, v_{\mathbf{o}_s} + v_{\mathbf{o}_0}. \qquad \square$$

Applying the lemma to A^c, we see that A^c is a translate of the simplex in $(V^*)^{w_c}$ spanned by 0 and $1/g_\alpha v_{\overline{\alpha}}$, where $v_{\overline{\alpha}} = 1/n_{\overline{\alpha}} \sum_{\alpha \in \overline{\alpha}} \alpha^*$ and α is any representative for $\overline{\alpha}$. It follows by Lemma 1.5.3 that $\{\overline{\alpha}^\vee; \overline{\alpha} \neq \overline{\alpha}_0\}$ is an integral basis for Λ^{w_c}, where $\overline{\alpha}^\vee = \sum_{\alpha \in \overline{\alpha}} \alpha^\vee$. Since

$$I_0(v_{\overline{\alpha}}, \overline{\beta}^\vee) = \begin{cases} 0, & \text{if } \overline{\alpha} \neq \overline{\beta}; \\ 1, & \text{if } \overline{\alpha} = \overline{\beta}, \end{cases}$$

we see that $\{\overline{\alpha}^\vee\}$ and $\{v_{\overline{\alpha}}\}$ are dual bases for the restriction of I_0 to $(V^*)^{w_c}$. Now

$$\mathrm{vol}(A^c) = \frac{\mathrm{vol}(C_1)}{(r_c)! \prod_{\overline{\alpha}} g_\alpha},$$

where C_1 is the parallelepiped spanned by the basis $\{v_{\overline{\alpha}}\}$ and as usual α is any representative for $\overline{\alpha}$. On the other hand, $\mathrm{vol}(T_0) = \mathrm{vol}(C_2)$, where C_2 is the parallelepiped spanned by the dual basis $\{\overline{\alpha}^\vee\}$. Thus

$$\frac{\mathrm{vol}(T_0)}{\mathrm{vol}(A^c)} = (r_c)! \left(\prod_{\overline{\alpha}} g_\alpha \right) \frac{\mathrm{vol}(C_2)}{\mathrm{vol}(C_1)} = (r_c)! \left(\prod_{\overline{\alpha}} g_\alpha \right) \det(I_0 | \Lambda^{w_c}).$$

To complete the proof of Theorem 1.5.2, the degree in question is the product

$$(r_c)! \left(\prod_{\overline{\alpha}} g_\alpha \right) \det(I_0 | \Lambda^{w_c}) \cdot \left(\frac{\prod_{\overline{\alpha}} n_{\overline{\alpha}}}{n_0} \right) = (r_c)! \frac{\det(I_0 | \Lambda^{w_c})}{n_0} \prod_{\overline{\alpha}} g_{\overline{\alpha}},$$

as claimed. $\square$

2. Bundles over maximal parabolic subgroups

2.1 Description of bundles and their automorphisms

Fix $\alpha \in \Delta$. We consider L^α-bundles η over E such that $c_1(\eta \times_{L^\alpha} G) = c$, and will refer to such a bundle as an *unliftable bundle of type* c. The primitive dominant character χ_0 of P^α lifts to a character on $\widetilde{P}^\alpha$ which is a positive power of the primitive dominant character ϖ_α of $\widetilde{P}^\alpha$. We denote this power by $o_{c,\alpha}$. Note that $\varpi_\alpha(c)$ is well-defined as an element of $\mathbb{Q}/\mathbb{Z}$ and $o_{c,\alpha}$ is its order. In fact, we have:

Lemma 2.1.1. *Let β be a root such that $c \equiv \varpi_\beta^\vee \pmod \Lambda$. Then $o_{c,\alpha}$, the order of $\varpi_\alpha(c)$, is the order of $\varpi_\alpha(\varpi_\beta^\vee) \mod \mathbb{Z}$.* □

In the notation of [7, §3.4], $o_{c,\alpha} = 1$ if and only if $\alpha \notin \Delta(c)$. The Dynkin diagram for $\Delta(c)$ is a union of diagrams of A-type and is described in the tables at the end of [7].

If η is an L^α-bundle, then the character χ_0 defines an associated line bundle $\eta \times_{L^\alpha} \mathbb{C}$. This line bundle is the *determinant* of η, which we write as $\det \eta$. Its degree is called the *degree* of η and is denoted $\deg \eta$.

Let $\eta \to E$ be a principal L^α-bundle whose degree d is negative. We shall study the corresponding bundles $\eta \times_{L^\alpha} P^\alpha$ and $\xi = \eta \times_{L^\alpha} G$. Associated to η and the action of L^α on the Lie algebra $\mathfrak{g}$ there is the vector bundle $\eta \times_{L^\alpha} \mathfrak{g} = \operatorname{ad} \xi$. The Lie algebra $\mathfrak{g}$ decomposes under L^α as $\mathfrak{u} \oplus \mathfrak{l} \oplus \mathfrak{u}_-$ where $\mathfrak{l}$ is the Lie algebra of L^α, $\mathfrak{u}$ is the subspace of $\mathfrak{g}$ on which $\varphi_\alpha(\mathbb{C}^*) \subseteq L^\alpha$ acts with positive weights, and $\mathfrak{u}_-$ is the subspace of $\mathfrak{g}$ on which $\varphi_\alpha(\mathbb{C}^*) \subseteq L^\alpha$ acts with negative weights. Since the coefficients of ζ_α are nonnegative, $\mathfrak{u}$ is the Lie algebra of U^α, $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ is the Lie algebra of P^α, and $\mathfrak{u}_-$ is the orthogonal space to $\mathfrak{u}$ under the Killing form. Clearly:

Lemma 2.1.2. *There is a direct sum decomposition*

$$\operatorname{ad} \xi = \eta \times_{L^\alpha} \mathfrak{g} = \operatorname{ad}_{L^\alpha} \eta \oplus \mathfrak{u}(\eta) \oplus \mathfrak{u}_-(\eta).$$

The action of the $\mathbb{C}^* \subseteq L^\alpha$ is trivial on $\operatorname{ad}_{L^\alpha} \eta$ and has positive (resp. negative) weights on $\mathfrak{u}(\eta)$ (resp. $\mathfrak{u}_-(\eta)$). □

We now have:

Lemma 2.1.3. *For every negative integer d, there exists a semistable L^α-bundle η over E of degree d. There is an unliftable semistable L^α-bundle η of type c if and only if $\deg \eta / o_{c,\alpha} \equiv \varpi_\alpha(c) \mod \mathbb{Z}$. In* *particular, if η is unliftable of type c and degree -1, then $-1/o_{c,\alpha} \equiv \varpi_\alpha(c) \mod \mathbb{Z}$. For a semistable L^α-bundle η of degree $d < 0$, we have:*

(i) *The bundle $\eta \times_{L^\alpha} G$ is unstable.*

(ii) *The parabolic P^α_- opposite to P^α is a Harder-Narasimhan parabolic of ξ.*

(iii) *$\mathfrak{u}(\eta)$ is a direct sum of semistable vector bundles of strictly negative degrees.*

(iv) *The Atiyah-Bott point of η as defined in [2, 10, 11] is given by*

$$\mu(\eta) = \frac{d\zeta_\alpha}{o_{c,\alpha}m_\alpha} = \frac{dn_\alpha}{o_{c,\alpha}m_\alpha}\varpi^\vee_\alpha.$$

Proof. The dominant character χ_0 lifts to the character $\varpi_\alpha^{o_{c,\alpha}}$ on $\widetilde{L}^\alpha$. By [11, Definition 2.1.1], $\mu(\eta)$ is the unique point μ in the center of $\mathfrak{l}$ such that $o_{c,\alpha}\varpi_\alpha(\mu) = d$. Thus $\mu = d\zeta_\alpha/o_{c,\alpha}m_\alpha$, showing (iv). The congruence condition $\deg \eta/o_{c,\alpha} \equiv \varpi_\alpha(c) \mod \mathbb{Z}$ follows from [11, Lemma 2.1.2 (ii)]. Statement (iii) follows from the fact that the $\mathfrak{u}^k$ are irreducible L-modules, by Corollary 1.4.3, and from [11, Lemma 2.2.1]. The remaining statements are clear. $\square$

Our goal now is to study the spaces $H^1(E; \operatorname{ad}\xi)$ and $H^0(E; \operatorname{ad}\xi)$. We shall primarily be interested in the case where η is semistable. It is enough to study the spaces $H^i(E; \operatorname{ad}_{L^\alpha}\eta)$, $H^i(E; \mathfrak{u}(\eta))$, and $H^i(E; \mathfrak{u}_-(\eta))$. Since L^α has a one-dimensional center, regardless of the choice of η we must always have $\dim H^0(\operatorname{ad}_{L^\alpha}\eta) \geq 1$, and thus, since $\deg \operatorname{ad}_{L^\alpha}\eta = 0$, $\dim H^1(\operatorname{ad}_{L^\alpha}\eta) \geq 1$ as well by Riemann-Roch on E. More precisely, we have:

Lemma 2.1.4. *Let $\widehat{S}^\alpha$ be the quotient of L^α by its center. Let η be a semistable L^α-bundle and let $\widehat{\eta}$ be the induced $\widehat{S}^\alpha$-bundle. Let $r(\widehat{\eta})$ be the dimension of $\operatorname{Aut}_{\widehat{S}^\alpha}(\widehat{\eta})$. Then*

$$\dim H^0(E; \operatorname{ad}_{L^\alpha}\eta) = \dim H^1(E; \operatorname{ad}_{L^\alpha}\eta) = 1 + r(\widehat{\eta}).$$

Proof. On the level of Lie algebras, there is a direct sum decomposition $\mathfrak{l} = \mathbb{C} \oplus \operatorname{Lie}(\widehat{S}^\alpha)$, where $\mathbb{C} = \operatorname{Lie}(Z(L^\alpha))$, and the proof follows. $\square$

We turn next to the groups $H^i(E; \mathfrak{u}(\eta))$ and $H^i(E; \mathfrak{u}_-(\eta))$:

Lemma 2.1.5. *Let η be a principal L^α-bundle of negative degree. Then*

$$\dim H^1(E;\mathfrak{u}(\eta)) \geq -\deg(\mathfrak{u}(\eta)),$$

with equality holding if and only if $H^0(E;\mathfrak{u}(\eta)) = 0$. Likewise,

$$\dim H^0(E;\mathfrak{u}_-(\eta)) \geq \deg(\mathfrak{u}_-(\eta)),$$

with equality holding if and only if $H^1(E;\mathfrak{u}_-(\eta)) = 0$. Finally, if η is semistable, then $H^0(E;\mathfrak{u}(\eta)) = H^1(E;\mathfrak{u}_-(\eta)) = 0$.

Proof. The first two statements are immediate from Riemann-Roch on E. The final one follows from Statement (iii) of Lemma 2.1.3. □

From the decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$, it follows that $H^0(E;\mathfrak{u}(\eta)) = 0$ if and only if

$$\dim \operatorname{Aut}_{L^\alpha} \eta = \dim \operatorname{Aut}_{P^\alpha}(\eta \times_{L^\alpha} P^\alpha).$$

The vanishing of $H^1(E;\mathfrak{u}_-(\eta))$ says on the other hand that the map

$$H^1(E;\operatorname{ad}_{P^\alpha}(\eta \times_{L^\alpha} P^\alpha)) \to H^1(E;\operatorname{ad}_G(\eta \times_{L^\alpha} G))$$

is an isomorphism. In particular, every small deformation of the G-bundle $\eta \times_{L^\alpha} G$ arises from a small P^α-deformation of $\eta \times_{L^\alpha} P^\alpha$.

To complete the determination of $H^1(E;\mathfrak{u}(\eta))$, we must compute the degree of $\mathfrak{u}(\eta)$.

Proposition 2.1.6. *We have* $\deg \mathfrak{u}(\eta) = (\deg \eta) d_1(\alpha)/o_{c,\alpha}$, *where* $d_1(\alpha)$ *is the integer defined in Lemma 1.3.1 for* $\widetilde{G}$. *Likewise,* $\deg \mathfrak{u}_-(\eta) = -(\deg \eta) d_1(\alpha)/o_{c,\alpha}$.

Proof. We compute the degree of the line bundle $\bigwedge^{\mathrm{top}} \mathfrak{u}(\eta)$. Let $\widetilde{\chi}_0$ be the dominant character for $\widetilde{P}^\alpha$. The line bundle $\bigwedge^{\mathrm{top}} \mathfrak{u}(\eta)$ is associated to η by the character $\chi_+ \colon L^\alpha \to \mathbb{C}^*$. By Lemma 1.3.1, χ_+ lifts to the character $\widetilde{\chi}_+ = \widetilde{\chi}_0^{d_1(\alpha)}$ of $\widetilde{L}^\alpha$. Since χ_0 lifts to $\widetilde{\chi}_0^{o_{c,\alpha}}$ on $\widetilde{L}^\alpha$ and since the line bundle associated to η by the character χ_0 has degree $\deg \eta$, the degree of $\mathfrak{u}(\eta)$ is $(\deg \eta) \cdot d_1(\alpha)/o_{c,\alpha}$. A similar argument (or duality) handles the case of $\mathfrak{u}_-(\eta)$. □

Corollary 2.1.7. *If η is semistable of negative degree, then*

$$\dim H^1(E;\mathfrak{u}(\eta)) = \dim H^0(E;\mathfrak{u}_-(\eta)) = -(\deg \eta) d_1(\alpha)/o_{c,\alpha}.$$

Thus

$$\begin{aligned}\dim H^0(E;\mathrm{ad}(\eta\times_{L^\alpha}G)) &= \dim H^1(E;\mathrm{ad}(\eta\times_{L^\alpha}G))\\ &= 1+r(\widehat{\eta})-(\deg\eta)d_1(\alpha)/o_{c,\alpha}.\end{aligned}$$

Proof. By (iii) of Lemma 2.1.3, $\mathfrak{u}(\eta)$ is a direct sum of semistable vector bundles of negative degrees, so that $H^0(E;\mathfrak{u}(\eta))=0$. The proof now follows by combining Lemmas 2.1.4 and 2.1.5 with Proposition 2.1.6. ☐

2.2 The $\mathbb{C}^*$-action in cohomology

The Lie algebra $\mathfrak{u}$ is a direct sum of the subspaces $\mathfrak{u}^k$, $k>0$, where $\mathfrak{u}^k$ is the sum of all the root spaces $\mathfrak{g}^\beta$ where the coefficient of β is exactly k. By Proposition 1.4.2, $\mathfrak{u}^k$ is an irreducible L^α-module. Thus $\mathfrak{u}(\eta)$ is the direct sum of vector bundles $\mathfrak{u}^k(\eta)$ associated to irreducible representations of L^α. By [21], $\mathfrak{u}^k(\eta)$ is semistable. Our goal now is to study the action of $\mathbb{C}^*=\varphi_\alpha(\mathbb{C}^*)$ on $\mathfrak{u}(\eta)$ and on $H^1(E;\mathfrak{u}(\eta))$. As we saw in Section 1.3, $\widetilde{\varphi}_\alpha(\mathbb{C}^*)\subseteq\widetilde{L}^\alpha$ acts on $\mathfrak{u}^k$ with weight kn_α. Thus, by Lemma 1.2.7, $\varphi_\alpha(\mathbb{C}^*)$ acts on $\mathfrak{u}^k$ with weight $kn_{c,\alpha}$.

Lemma 2.2.1. *The degree* $\deg\mathfrak{u}^k(\eta)$ *is equal to* $(\deg\eta)\cdot i(\alpha,k)/o_{c,\alpha}$, *where*

$$i(\alpha,k)=kc(\alpha,k)/\varpi_\alpha(\varpi_\alpha^\vee)=kn_\alpha c(\alpha,k)/m_\alpha.$$

The slope of $\mathfrak{u}^k(\eta)$ *is* $k\varpi_\alpha(\mu(\eta))/\varpi_\alpha(\varpi_\alpha^\vee)$, *where* $\mu(\eta)$ *is the Atiyah-Bott point of* η. *Thus, if* η *is semistable of negative degree, then*

$$\dim H^1(E;\mathfrak{u}^k(\eta))=-(\deg\eta)\cdot i(\alpha,k)/o_{c,\alpha}.$$

Proof. By Lemma 1.3.4 applied to $\widetilde{G}$, the character of L^α defined by the determinant on $\mathfrak{u}^k$ lifts to the character $\chi_0^{i(\alpha,k)}$ on $\widetilde{L}^\alpha$. The degree of $\mathfrak{u}^k(\eta)$ is thus $(\deg\eta)\cdot i(\alpha,k)/o_{c,\alpha}$. It follows that the slope of $\mathfrak{u}^k(\eta)$ is $kn_\alpha\deg\eta/o_{c,\alpha}m_\alpha=k\varpi_\alpha(\mu)/\varpi_\alpha(\varpi_\alpha^\vee)$. ☐

3. Special roots and the associated bundles

In §3.1–3.3, we assume that G is simply connected. We will defer the discussion of the non-simply connected case until §3.4.

3.1 Definition of special roots

Definition 3.1.1. A simple root α is *special* if

(i) The Dynkin diagram associated to $\Delta - \{\alpha\}$ is a union of diagrams of A-type.

(ii) The simple root α meets each component of the Dynkin diagram associated to $\Delta - \{\alpha\}$ at an end of the component.

(iii) The root α is a long root.

If R is of type A_n, then every simple root is special. All other irreducible root systems have a unique special simple root. It corresponds to the unique trivalent vertex if the Dynkin diagram is of type $D_n, n \geq 4$ or $E_n, n = 6, 7, 8$. For $R = C_n, n \geq 2$ and G_2, it is the long simple root. For $R = B_n, n \geq 2$, and F_4 it is the unique long simple root which is not orthogonal to a short simple root.

We shall investigate the structure of the group L^α and the space $H^1(E; \mathfrak{u}(\eta))$ more closely in case α is special.

Let α be special and let η be a semistable bundle over P^α of degree -1. By the results of [11], the unstable bundles $\eta \times_{L^\alpha} G$ are minimally unstable bundles, in the sense that every small deformation of such a bundle is either of the same type or semistable. Moreover, if G is not of A-type, then for every unstable G-bundle ξ, there is a small deformation of ξ to a bundle of the form $\eta \times_{L^\alpha} G$. Now for every unstable G-bundle ξ, there is the Harder-Narasimhan reduction to a parabolic subgroup, not necessarily maximal, and in fact ξ reduces to a bundle η the Levi factor L. Let P be the opposite parabolic to the Harder-Narasimhan parabolic and let $\mathfrak{u}$ be the Lie algebra of the unipotent radical of P. It is easy to see that the function $\xi \mapsto \dim H^1(E; \mathfrak{u}(\eta))$ is strictly decreasing for the Atiyah-Bott ordering, and hence attains its minimum in the case where P is a maximal parabolic corresponding to a special root and $-\deg \eta$ is minimal. We shall see this directly below.

3.2 Bundles associated to special roots

Our first lemma determines the structure of S^α and L^α in case α is special:

Lemma 3.2.1. *Suppose that α is special, and let t be the number of components in the Dynkin diagram of S^α. Then there exist integers*

$n_i \geq 2$ such that $S^\alpha \cong \prod_{i=1}^t SL_{n_i}(\mathbb{C})$ and $m_\alpha = \operatorname{lcm}(n_i)$. Moreover,

$$L^\alpha \cong \left\{ (A_1, \dots, A_t) \in \prod_{i=1}^t GL_{n_i}(\mathbb{C}) : \det A_1 = \cdots = \det A_t \right\},$$

in such a way that the primitive dominant character of L^α corresponds to the common value of the determinant.

Proof. It follows from the definition of a special root that $S^\alpha \cong \prod_{i=1}^t SL_{n_i}(\mathbb{C})$. By Lemma 1.2.3, there is an isomorphism

$$L^\alpha \cong S^\alpha \times_{\mathbb{Z}/m_\alpha\mathbb{Z}} \mathbb{C}^*,$$

where the image of $1 \in \mathbb{Z}/m_\alpha\mathbb{Z}$ is mapped to $e^{2\pi i/m_\alpha} \in \mathbb{C}^*$ and to $e^{-2\pi i/n_i}\operatorname{Id} \in SL_{n_i}(\mathbb{C})$. From this, we must have $m_\alpha = \operatorname{lcm}(n_i)$. The map from $S^\alpha \times \mathbb{C}^*$ to $\prod_{i=1}^t GL_{n_i}(\mathbb{C})$ which is the natural inclusion

$$\prod_{i=1}^t SL_{n_i}(\mathbb{C}) \subseteq \prod_{i=1}^t GL_{n_i}(\mathbb{C})$$

and which maps $\lambda \in \mathbb{C}^*$ to

$$(\lambda^{m_\alpha/n_1}\operatorname{Id}, \dots, \lambda^{m_\alpha/n_t}\operatorname{Id})$$

then factors to give an induced homomorphism

$$S^\alpha \times_{\mathbb{Z}/m_\alpha\mathbb{Z}} \mathbb{C}^* \to \prod_{i=1}^t GL_{n_i}(\mathbb{C}).$$

It is clear from the construction that this induced homomorphism is injective and that its image is the subgroup of matrices of equal determinant. Let $\det L^\alpha \to \mathbb{C}^*$ denote the value of any of these determinants under the inverse isomorphism. For $\lambda \in \mathbb{C}^*$, we see that $\det \circ \varphi_\alpha = \lambda^{m_\alpha}$, and hence $\det = \chi_0$. □

Lemma 3.2.2. *If α is special, the positive integers n_α are equal to* 1 *except in the following cases:*

(a) *If $G = SL_n(\mathbb{C})$ and α corresponds to the k^{th} vertex in the usual ordering of the simple roots, then $n_\alpha = n/\gcd(k, n)$.*

(b) *If G is of type B_n and n is even, then $n_\alpha = 2$.*

(c) *If G is of type C_n, then $n_\alpha = 2$.*

(d) *If G is of type D_n and n is odd, then $n_\alpha = 2$.*

Proof. If the center of G is trivial, then $\varpi_\alpha^\vee$ is a primitive element of Λ, and hence $n_\alpha = 1$. This handles the cases E_8, F_4, G_2. Next suppose that R is simply laced and not of type A_n, so that the Dynkin diagram of R is a $T_{p,q,r}$ diagram, with $(p,q,r) = (2,2,n)$ or $(2,3,s)$ with $s = 3,4,5$. Let $N = (1/p+1/q+1/r-1)^{-1}$, so that $N = n, 6, 12, 30$ in the respective cases above. In particular $N \in \mathbb{Z}$. There exists a labeling of the roots as $\{\alpha, \beta_1, \dots, \beta_{p-1}, \gamma_1, \dots, \gamma_{q-1}, \delta_1, \dots, \delta_{r-1}\}$, where $\beta_1, \gamma_1, \delta_1$ are ends of the diagram, $\langle \beta_i, \beta_{i+1}\rangle = -1$, $1 \le i \le p-2$ and similarly for the γ_j and δ_k, and $\beta_{p-1}, \gamma_{q-1}, \delta_{r-1}$ meet α, such that

$$\varpi_\alpha^\vee = \frac{N}{p}\sum_{i=1}^{p-1} i\beta_i^\vee + \frac{N}{q}\sum_{j=1}^{q-1} j\gamma_j^\vee + \frac{N}{r}\sum_{k=1}^{r-1} k\delta_k^\vee + N\alpha.$$

It follows that $\varpi_\alpha^\vee$ is integral, and hence $n_\alpha = 1$, unless $(p,q,r) = (2,2,n)$ and n is odd, in which case $n_\alpha = 2$. A similar argument handles the case of A_n. In case B_n, if we number the roots as in [8] beginning at the long end of the Dynkin diagram, then $\alpha = \alpha_{n-1}$ and

$$\varpi_{\alpha_{n-1}}^\vee = \sum_{i=1}^{n-1} i\alpha_i^\vee + \frac{n-1}{2}\alpha_n^\vee.$$

Thus $n_\alpha = 1$ if n is odd and 2 if n is even. Finally, for the case of C_n, again numbering the roots in order as in [8] beginning at a short root, so that $\alpha = \alpha_n$, we have

$$\varpi_{\alpha_n}^\vee = \frac{1}{2}\sum_{i=1}^{n} i\alpha_i^\vee.$$

Thus $n_\alpha = 2$. □

We turn now to the existence of special bundles over L^α.

Proposition 3.2.3. *Suppose that α is special. Then there is a unique principal L^α-bundle η_0 over E with the following properties:*

(i) $\det \eta_0 = \mathcal{O}_E(-p_0)$.

(ii) *For $1 \leq i \leq t$, if V_i is the vector bundle associated to the principal $GL_{n_i}(\mathbb{C})$-bundle obtained from the composition of the inclusion $L^\alpha \subseteq \prod_{i=1}^t GL_{n_i}(\mathbb{C})$ followed by projection onto the i^{th} factor, then each V_i is a stable vector bundle.*

The automorphism group of η_0, as an L^α-bundle, is identified with the center of L^α which is of the form $Z(S^\alpha) \times_{\mathbb{Z}/m_\alpha\mathbb{Z}} \varphi_\alpha(\mathbb{C}^*)$, acting by multiplication.

Proof. Recall that, for every $d \geq 1$, there is a unique stable vector bundle W_d of rank d over E such that $\det W_d = \mathcal{O}_E(p_0)$. Given the structure of L^α as in Lemma 3.2.1, it is clear that there is a unique principal L^α-bundle, up to isomorphism, satisfying (i) and (ii) above, with $V_i = W_{n_i}^*$ for every i. Since the vector bundles V_i in (i) are simple, the automorphism group of each of these is isomorphic to the center of $GL_{n_i}(\mathbb{C})$ acting by multiplication. It then follows that the L^α-automorphisms of η_0 are given by the action of the center of L^α acting by multiplication. □

Definition 3.2.4. If η is an L^α-bundle which satisfies (ii) of the Proposition, together with the condition that $\deg \eta = -1$, then η is the pullback of η_0 under a translation map $E \to E$. We call η a *translate* of η_0.

Let us describe the unstable bundle $\eta_0 \times_{L^\alpha} G$ for the classical groups. First, we need the following notation. As above, let W_d be the unique stable vector bundle over E of rank d and such that $\det W_d = \mathcal{O}_E(p_0)$. Let θ_i, $i = 1, 2, 3$ be the three nontrivial line bundles of order two on E. Let

$$Q_3 = \theta_1 \oplus \theta_2 \oplus \theta_3$$

be the corresponding rank three vector bundle. Fix isomorphisms $\theta_i \otimes \theta_i \to \mathcal{O}_E$ and give Q_3 the corresponding diagonal symmetric bilinear form. Define similarly

$$Q_4 = \mathcal{O}_E \oplus \theta_1 \oplus \theta_2 \oplus \theta_3,$$

together with a similar choice of a diagonal symmetric bilinear form. We then have:

Proposition 3.2.5. *With notation as above, and supposing that α is special, let η_0 be the principal L^α-bundle constructed in Proposition 3.2.3. Then the vector bundle associated to $\eta_0 \times_{L^\alpha} G$ under the standard representation of G is:*

(i) $W_k^* \oplus W_{n-k}$, *if* $G = SL_n(\mathbb{C})$ *and* α *is the root corresponding to the* k^{th} *vertex in the Dynkin diagram, ordered in the usual way.*

(ii) $W_n^* \oplus W_n$, *if* $G = Sp(2n)$, *where each factor is isotropic and we choose an isomorphism* $W_n^* \to W_n^*$, *unique up to a scalar, to define the alternating form on the direct sum.*

(iii) $W_{n-2}^* \oplus Q_4 \oplus W_{n-2}$, *if* $G = Spin(2n)$, *where* Q_4 *is given the form described above,* W_{n-2}^* *and* W_{n-2} *are isotropic, we choose an isomorphism* $W_{n-2}^* \to W_{n-2}^*$, *unique up to a scalar, to define the symmetric form on the direct sum* $W_{n-2}^* \oplus W_{n-2}$, *and* Q_4 *is orthogonal to this direct sum.*

(iv) $W_{n-1}^* \oplus Q_3 \oplus W_{n-1}$, *if* $G = Spin(2n+1)$, *where* Q_3 *is given the form described above,* W_{n-1}^* *and* W_{n-1} *are isotropic, we choose an isomorphism* $W_{n-1}^* \to W_{n-1}^*$, *unique up to a scalar, to define the symmetric form on the direct sum* $W_{n-1}^* \oplus W_{n-1}$, *and* Q_3 *is orthogonal to this direct sum.*

Proof. The cases $G = SL_n(\mathbb{C})$ and $G = Sp(2n)$ follow easily from the explicit descriptions of the maximal parabolic subgroups and are left to the reader. In case $G = Spin(2n)$, the corresponding maximal parabolic of $SO(2n)$ is the set of $g \in SO(2n)$ preserving an isotropic subspace of $\mathbb{C}^{2n}$ of dimension $n-2$. The corresponding Levi factor L is the subgroup of matrices in $GL_{n-2}(\mathbb{C}) \times GL_2(\mathbb{C}) \times GL_2(\mathbb{C})$ with equal determinant. If ρ_1 is the representation of L induced by the standard representation of $GL_{n-2}(\mathbb{C})$ on $\mathbb{C}^{n-2}$ and ρ_2, ρ_3 are the two representations of L induced by the standard representations of the second and third factors of L on $\mathbb{C}^2$, then it is easy to check that the representation of L on $\mathbb{C}^{2n}$ which is the restriction of the standard representation of $Spin(2n)$ is just

$$\rho_1 \oplus \rho_1^* \oplus (\rho_2 \otimes \rho_3^*).$$

The vector bundle associated to η_0 is thus

$$W_{n-2}^* \oplus W_{n-2} \oplus (W_2 \otimes W_2^*).$$

Moreover this is an orthogonal direct sum with respect to the induced form and W_{n-2}^* and W_{n-2} are isotropic subspaces. Furthermore, by a result of Atiyah [1], $W_2 \otimes W_2^* \cong Q_4$, and since the each line bundle summand of Q_4 is not isomorphic to the dual of any other summand, the direct sum decomposition of Q_4 must be orthogonal with respect to the quadratic form and thus as described above.

The case of $Spin(2n+1)$ is similar. □

3.3 Cohomology dimensions and weight spaces

We have seen that the bundles η_0 are minimally unstable in the sense of deformation theory. Here we begin by showing that their deformation spaces have minimal dimension among all unstable bundles.

Theorem 3.3.1. *Let η_0 be the bundle described in Proposition 3.2.3. Then*

$$\dim H^0(E; \mathrm{ad}(\eta_0 \times_{L^\alpha} G)) = \dim H^1(E; \mathrm{ad}(\eta_0 \times_{L^\alpha} G)) = r + 2.$$

If ξ is any unstable G-bundle, then $\dim H^1(E; \mathrm{ad}\,\xi) \geq r+2$, with equality if and only if ξ is isomorphic to $\eta \times_{L^\alpha} G$, where η is a translate of η_0. Finally, if U is the unipotent radical of a parabolic subgroup P of G with Levi factor L and η is a semistable L-bundle of negative degree, then

$$\dim H^1(E; \mathfrak{u}(\eta)) \geq r + 1,$$

with equality if and only if P is conjugate to P^α, where α is special, and η is a translate of η_0.

Proof. First, by Corollary 2.1.7,

$$\begin{aligned}\dim H^0(E; \mathrm{ad}(\eta_0 \times_{L^\alpha} G)) = &\dim H^1(E; \mathrm{ad}(\eta_0 \times_{L^\alpha} G)) \\ &= 1 + r(\widehat{\eta}_0) + d_1(\alpha) = 1 + d_1(\alpha),\end{aligned}$$

since $\dim \mathrm{Aut}_{\widehat{S}^\alpha}\,\widehat{\eta}_0 = 0$.

Next we show that $d_1(\alpha)$ has the following minimality property:

Lemma 3.3.2. *If α is special, then $d_1(\alpha) = r + 1$. If β is not special, then $d_1(\beta) > r + 1$.*

Proof. To see that $d_1(\alpha) = r + 1$, it suffices by Proposition 1.4.9 to show that $\lambda_1(\alpha^\vee) = \sum_{\beta\in\Delta} \beta^\vee$. First, by [8, cor. 3, p. 160], $\lambda = \sum_{\beta\in\Delta} \beta^\vee$ is always a coroot. Clearly, $n(\lambda^\vee, \beta^\vee) \geq 0$ for all $\beta \in \Delta - \{\alpha\}$, $n(\lambda^\vee, \beta^\vee) > 0$ if $\beta \neq \alpha$ is an end of the Dynkin diagram and $\lambda - \beta^\vee$ can only be a coroot if $\beta \neq \alpha$ is an end of the Dynkin diagram. These properties say that, if $\beta \neq \alpha$, then $\lambda + \beta^\vee$ is not a coroot. Thus $\lambda = \lambda_1(\alpha^\vee)$.

Now suppose that R is not of A-type, and hence that $R^\vee$ is not of A-type. Then $\lambda_1(\alpha^\vee)$ is not the highest coroot of $R^\vee$. Thus there exists a simple root β such that $\lambda_1(\alpha^\vee)+\beta^\vee$ is again a coroot. By what we have just seen, we must have $\beta=\alpha$. It follows that, for $\alpha\neq\beta$, $\lambda_1(\beta^\vee)$ is equal to $\lambda_1(\alpha^\vee)+\alpha^\vee$ plus a sum of simple coroots. Hence $d_1(\beta)=\rho(\lambda_1(\beta^\vee))+1\geq\rho(\lambda_1(\alpha^\vee))+2=r+2$. □

Thus, we have proved the first statement in the theorem. To see the second, first assume that the Harder-Narasimhan parabolic subgroup for ξ is maximal. In this case, we can assume that the Harder-Narasimhan parabolic for ξ is P^β_- for some β. Thus ξ is isomorphic to $\eta\times_{L^\beta}G$, where η is a semistable bundle of negative degree $-n$ on L^β. Now

$$\dim H^1(E;\mathrm{ad}(\eta\times_{L^\beta}G))=1+r(\widehat{\eta})+nd_1(\beta)\geq r+2,$$

with equality holding if and only if $r(\widehat{\eta})=0$, $n=1$, and β is special. In this last case, it follows from Definition 3.2.4 that η is a translate of η_0.

Now suppose that the Harder-Narasimhan parabolic for ξ is not maximal. There exists a maximal parabolic subgroup P^β such that ξ has a reduction to an L^β-bundle η, where η has degree $-n<0$. By Lemma 2.1.3, η is unstable, for otherwise the Harder-Narasimhan parabolic for ξ would be P^β_-, which is maximal. Hence, the associated $\widehat{S}^\beta$-bundle $\widehat{\eta}$ is also unstable. Thus the vector bundle $\mathrm{ad}_{\widehat{S}^\beta}\widehat{\eta}$ is unstable of degree zero, and hence contains a semistable summand of negative degree. It follows that $\dim H^1(E;\mathrm{ad}_{\widehat{S}^\beta}\widehat{\eta})\geq 1$. Applying Lemmas 2.1.4 and 2.1.5 and Proposition 2.1.6, we see that

$$\begin{aligned}h^1(E;\mathrm{ad}\,\xi)&=1+h^1(E;\mathrm{ad}_{\widehat{S}^\beta}\widehat{\eta})+h^1(E;\mathfrak{u}(\eta))+h^1(E;\mathfrak{u}_-(\eta))\\&\geq 2+nd_1(\beta)\geq r+3.\end{aligned}$$

This completes the proof of the second statement in case the Harder-Narasimhan parabolic for ξ is not maximal. The proof of the final statement of the theorem is implicit in the above discussion. □

We turn now to the $\mathbb{C}^*$-weights for the action of $\mathbb{C}^*$ on $\mathfrak{u}(\eta_0)$.

Proposition 3.3.3. *Suppose that α is special. Then the $\mathbb{C}^*$-weights for the action of the center of L^α on $H^1(E;\mathfrak{u}(\eta_0))$, with multiplicity, are the integers $n_\alpha g_\beta,\beta\in\widetilde{\Delta}$.*

Proof. The group $\mathbb{C}^*$ acts on $H^1(E;\mathfrak{u}^k(\eta))$ with weight kn_α. By Lemma 2.2.1,

$$\dim H^1(E;\mathfrak{u}^k(\eta))=-(\deg\eta_0)\cdot i(\alpha,k)=i(\alpha,k).$$

Thus, it suffices to show that $i(\alpha,k) = \#\{\beta \in \widetilde{\Delta} : g_\beta = k\}$. If we define

$$i(k) = \#\{\beta \in \widetilde{\Delta} : g_\beta = k\};$$
$$d(k) = \sum_{k|x} i(x) = \sum_{\ell \geq 1} i(\ell k),$$

then it clearly suffices to show that, for all k, $d(k) = d_k(\alpha)$, in the notation of Definition 1.4.7. By Proposition 1.4.17, the integers $d_k(\alpha)$ have the circular symmetry property with respect to g_α and g, since α is a long root, and by Lemma 3.3.2, $d_1(\alpha) = r + 1$. By the proof of Theorem 3.8.7 in [7], the integers $d(k)$ have the circular symmetry property with respect to N and g, where $N = \max\{g_\beta : \beta \in \Delta\}$. By Corollary 6.2.5 of [11], $N = g_\alpha$. (We will give another proof of this fact in Part III.) Clearly $d(1) = \#\widetilde{\Delta} = r+1$. By Lemma 1.4.16, $d(k) = d_k(\alpha)$ for all k. □

3.4 The non-simply connected case

We now establish the analogues in the non-simply connected case of the previous results. While we believe there should be classification-free arguments for these results, we argue here in a case-by-case analysis.

Definition 3.4.1. Let $o(c)$ denote the order of $c \in \pi_1(G)$. A root $\alpha \in \Delta$ is *c-special* if there exists an integer $d < 0$ such that:

(i) $d/o_{c,\alpha} \equiv \varpi_\alpha(c) \mod \mathbb{Z}$.

(ii) The Weyl orbit of the point $\mu_{c,\alpha} = d\zeta_\alpha/o_{c,\alpha}m_\alpha$ is minimal in the Atiyah-Bott ordering [11] among all Weyl orbits of nonzero points of Atiyah-Bott type for c.

Remark 3.4.2. a) The first condition means that that there is a holomorphic semistable L^α-bundle η with $c_1(\eta \times_{L^\alpha} G) = c$ and Atiyah-Bott point equal to $\mu_{c,\alpha}$, and the topological type of the L^α-bundle η is uniquely determined by $\mu_{c,\alpha}$. This topological type is specified by an element $c_1(\eta) = \gamma \in \pi_1(L^\alpha)$. The second condition means that the point $\mu_{c,\alpha}$, or the corresponding stratum of $(0,1)$-connections, is minimally unstable in the sense of [11, Definition 6.1.1].

b) It is not always true that there is a unique c-special root. Uniqueness fails exactly when $G = SL_n(\mathbb{C})/\langle c\rangle$ and c does not generate the center, where there are $n/o(c)$ special roots, and for $G = SO(2n)$, where there are two special roots.

c) As defined here, c-special roots are certain simple roots. In [11] we used the simple roots to index strata of the space of $(0,1)$-connections, by associating to α the stratum lying in the Lie algebra of the center of L^α and having the smallest possible **positive** value under the dominant character. The convention here differs by a sign from the one of [11]. This means that the image under the automorphism of the Dynkin diagram induced by $-w_0$ of the roots which are c-special as defined here correspond to the roots indexing the minimally unstable strata in [11]. This automorphism sends c to c^{-1} and thus fixes the set of c-special roots if and only if c is of order 1 or 2. Otherwise, the c-special roots correspond to the roots indexing the minimally unstable strata for c^{-1} in [11].

Theorem 3.4.3. *Let α be a c-special root for G, and let $\gamma \in \pi_1(L^\alpha)$ be the first Chern class of the L^α-bundle η corresponding to $\mu_{c,\alpha}$. Then:*

(i) *The integer $d = -1$ and $o_{c,\alpha} = o(c)$.*

(ii) *The adjoint quotient $\mathrm{ad}(L^\alpha) = L^\alpha/Z(L^\alpha)$ is a product $\prod_{i=1}^k \widehat{S}_i$, where the $\widehat{S}_i$ are simple groups of A-type.*

(iii) *Let $\widehat{\gamma}$ be the image of $\gamma \in \pi_1(L^\alpha)$ under the projection $\pi_1(L^\alpha) \to \pi_1(\prod_{i=1}^k \widehat{S}_i) = \prod_{i=1}^k \pi_1(\widehat{S}_i)$. For $i = 1, \dots, k$, the image of $\widehat{\gamma}$ in $\pi_1(\widehat{S}_i)$ generates the cyclic group $\pi_1(\widehat{S}_i)$.*

(iv) *$d_1(\alpha)/o_{c,\alpha} = r_c + 1$.*

Proof. The minimally unstable points $\mu_{c,\alpha}$ are listed in §6.3 of [11]. From this list, it is easy to check that $d = -1$ and $o_{c,\alpha} = o(c)$. To prove the remaining statements, we make a case-by-case analysis.

$\widetilde{G} = SL_n(\mathbb{C})$:

We choose an identification of Λ with

$$\Big\{(x_1, \dots, x_n) \in \mathbb{Z}^n \,\Big|\, \sum_{i=1}^n x_i = 0\Big\}$$

in such a way that $\Delta = \{\alpha_1, \dots, \alpha_{n-1}\}$ with $\alpha_i = e_i - e_{i+1}$ with e_i being the standard unit vector in the i^{th}-coordinate direction. We write c as the image of an element of the form

$$(m/n)(\alpha_1^\vee + 2\alpha_2^\vee + \cdots + (n-1)\alpha_{n-1}^\vee)$$

for some $1 \leq m < n$. We factor $m = m_0 \cdot \ell$ where $(m_0, n) = 1$ and $\ell | n$. Then c is an element of order $f = n/\ell$. By [11, §6.3], the c-special roots are those α for which $\varpi_\alpha(c) \equiv -1/f \pmod{\mathbb{Z}}$. There are exactly n/f such roots. Suppose $\alpha = \alpha_k$. Then

$$-k \cdot m/n \equiv -1/f \pmod{\mathbb{Z}}.$$

In particular, $(k, f) = 1$. Since α is a special root for $SL_n(\mathbb{C})$, $d_1(\alpha) = n$, and hence $d_1(\alpha)/f = n/f = \ell$. On the other hand, since $\langle c \rangle$ acts freely on the Dynkin diagram for G we see that $r_c + 1 = \ell$. Let us consider the group L^α. By Lemma 3.2.1, $\widetilde{L}^\alpha$ is isomorphic to the subgroup of $GL_k(\mathbb{C}) \times GL_{n-k}(\mathbb{C})$ matrices of equal determinant. Hence

$$\mathrm{ad}(L^\alpha) = \mathrm{ad}(\widetilde{L}^\alpha) = PGL_k(\mathbb{C}) \times PGL_{n-k}(\mathbb{C}).$$

The map $\det\colon \widetilde{L}^\alpha \to \mathbb{C}^*$ induces an identification $\pi_1(\widetilde{L}^\alpha) = \mathbb{Z}$ and the projection $\widetilde{L}^\alpha \to \mathrm{ad}(L^\alpha)$ sends $1 \in \mathbb{Z}$ to the element

$$(a_k, a_{n-k}) \in \pi_1(PGL_k(\mathbb{C})) \times \pi_1(PGL_{n-k}(\mathbb{C}))$$

where a_k, resp. a_{n-k} generates $\pi_1(PGL_k(\mathbb{C}))$, resp. $\pi_1(PGL_{n-k}(\mathbb{C}))$. Direct computation shows that $\pi_1(L^\alpha) = \mathbb{Z}$ and that the natural map $\pi_1(\widetilde{L}^\alpha) \to \pi_1(L^\alpha)$ is multiplication by f. Thus, we have an identification $\pi_1(L^\alpha) = \mathbb{Z}[1/f]$. Under this identification the element $\gamma \in \pi_1(L^\alpha)$ is $-1/f$. $\pi_1(L^\alpha)$ and projects to

$$\widehat{\gamma} = (f^{-1}a_k, f^{-1}a_{n-k}) \in \pi_1(PGL_k(\mathbb{C})) \times \pi_1(PGL_{n-k}(\mathbb{C})).$$

Since $(k, f) = 1$, the projection of this element to either factor generates that factor.

$G = SO(2n+1)$, $n \geq 3$.

The c-special root is the unique short root α in the Dynkin diagram. Direct inspection shows that $L^\alpha = GL_n(\mathbb{C})$, and that $\gamma \in \pi_1(L^\alpha)$ is a generator of the fundamental group. Thus, $\mathrm{ad}(L^\alpha) = PGL_n(\mathbb{C})$ and $\widehat{\gamma}$ generates $\pi_1(PGL_n(\mathbb{C}))$. Furthermore, $d_1(\alpha) = 2n$ so that $d_1(\alpha)/o(c) = n$. Since c acts on the extended Dynkin diagram for G with one free orbit and $n-1$ fixed points, we see that $r_c + 1 = n = d_1(\alpha)/o(c)$.

$G = Sp(2n)/\langle c \rangle$, $n \geq 2$:

Suppose first that n is odd. Then there is a unique c-special root, the unique long root α. In this case, $L^\alpha = GL_n(\mathbb{C})/(\mathbb{Z}/2\mathbb{Z})$ and γ generates the fundamental group of L^α. Hence, $\mathrm{ad}(L^\alpha) = PGL_n(\mathbb{C})$ and $\widehat{\gamma}$ is the square of a generator for this group. Since n is odd, $\widehat{\gamma}$ is a generator of $\pi_1(PGL_n(\mathbb{C}))$. Since α is special for the simply connected form of the group, $d_1(\alpha) = n+1$. In this case the element c acts freely on the nodes of the extended Dynkin diagram so that $r_c+1 = (n+1)/2 = d_1(\alpha)/o(c)$.

Now suppose that n is even. Then there is a unique c-special root, the unique short simple root α which is not orthogonal to the unique long simple root. Direct computation shows that $\widetilde{L}^\alpha$ is isomorphic to $GL_{n-1}(\mathbb{C}) \times SL_2(\mathbb{C})$ and that c is the diagonal element $(-1,-1)$. Thus, $\pi_1(L^\alpha) = \mathbb{Z}$ and γ is a generator of this group. Furthermore, $\mathrm{ad}(L^\alpha) = PGL_{n-1}(\mathbb{C}) \times PGL_2(\mathbb{C})$ and the map $\pi_1(L^\alpha) \to \pi_1(\mathrm{ad}(L^\alpha))$ is onto. Thus, the image $\widehat{\gamma}$ of γ generates $\pi_1(\mathrm{ad}(L^\alpha))$, and hence its projection to each factor generates the fundamental group of that factor. Lastly, direct computation shows that $d_1(\alpha) = n+2$. Since c acts on the extended Dynkin diagram for G with one fixed point and $n/2$ free orbits, we see that $r_c + 1 = (n+2)/2 = d_1(\alpha)/o(c)$.

$G = SO(2n)$, $n \geq 4$:

For $\widetilde{G} = Spin(2n)$ we identify Λ with the even integral lattice inside $\mathbb{R}^n$. Let e_i be the standard unit vector in the i^{th}-coordinate direction. Then $\Delta = \{\alpha_1, \cdots, \alpha_{n-1}, \alpha_n\}$ where $\alpha_i = e_i - e_{i+1}$ for $1 \leq i < n$ and $\alpha_n = e_{n-1} + e_n$.

There are two c-special roots $\alpha_{n-1} = e_{n-1} - e_n$ and $\alpha_n = e_{n-1} + e_n$. (Of course, these elements are interchanged by an outer automorphism of $SO(2n)$.) Let α be one of the c-special roots. Then $L^\alpha = GL_n(\mathbb{C})$ and γ is a generator of $\pi_1(L^\alpha) \cong \mathbb{Z}$. Thus, $\mathrm{ad}(L^\alpha) = PGL_n(\mathbb{C})$ and $\widehat{\gamma}$ is a generator of this group. Direct computation shows that $d_1(\alpha) = 2(n-1)$. Since c acts on the Dynkin diagram for G with two free orbits and $n-1$ fixed points, we see that $r_c + 1 = n-1 = d_1(\alpha)/o(c)$.

$\widetilde{G} = Spin(4n+2)$, $n \geq 2$ and c is an element of order 4:

There is one c-special root. It is the simple root α corresponding to the "ear" of the Dynkin diagram (i.e., either α_{n-1} or α_n) with the property that $\varpi_\alpha(c) = -1/4 \pmod{\mathbb{Z}}$. In this case

$$L^\alpha = GL_{2n+1}(\mathbb{C})/(\mathbb{Z}/2\mathbb{Z}).$$

Hence $\pi_1(L^\alpha) = \mathbb{Z}$ and γ is a generator. Under the projection to $\mathrm{ad}(L^\alpha) = PGL_{2n+1}(\mathbb{C})$ the image $\widehat{\gamma}$ of γ is the square of the usual generator. This is clearly still a generator. Lastly, as above $d_1(\alpha) = 2((2n+1)-1) = 4n$ whereas $r_c + 1 = n$. Thus, $d_1(\alpha)/o(c) = r_c + 1$.

$\widetilde{G} = Spin(4n)$, $n \geq 2$ and c is an element of order two not contained in $\pi_1(SO(4n))$:

There is one c-special root. It is the simple root α_{n-3} corresponding to the node of the "long" arm of the Dynkin diagram next to the trivalent node. Thus, $\widetilde{L}^\alpha$ is isomorphic to $(SL_{2n-3}(\mathbb{C})\times SL_4(\mathbb{C}))\times_{(\mathbb{Z}/(4n-6)\mathbb{Z})}$ $\mathbb{C}^*$ where the cyclic group is embedded in the standard way in $\mathbb{C}^*$ and the usual generator maps to the standard generator of $Z(SL_{2n-3}(\mathbb{C}))$ and to the element of order 2 in $Z(SL_4(\mathbb{C}))$. Thus, we can identify $\widetilde{L}^\alpha$ with $GL_{2n-3}(\mathbb{C})\times_{(\mathbb{Z}/2\mathbb{Z})} SL_4(\mathbb{C})$. The element c is the image of the element (a,b) where a and b are central elements of order 4 in $GL_{2n-3}(\mathbb{C})$ and $SL_4(\mathbb{C})$ under the inclusion of $\widetilde{L}^\alpha \subset \widetilde{G}$. Thus, L^α is isomorphic $GL_{2n-3}(\mathbb{C})\times_{(\mathbb{Z}/4\mathbb{Z})} SL_4(\mathbb{C})$. Hence γ is a generator of $\pi_1(L^\alpha)=\mathbb{Z}$, the image $\widehat{\gamma}$ of γ is a generator for $\pi_1(PGL_{2n-3}(\mathbb{C}))\times\pi_1(PGL_4(\mathbb{C}))$, and hence the projection of $\widehat{\gamma}$ into either factor generates the fundamental group of that factor. Direct computation shows that $d_1(\alpha)=2(n+1)$. Since the action of c on the extended Dynkin diagram for G has one fixed point and n free orbits, we see that $r_c+1=n+1=d_1(\alpha)/o(c)$.

$G=\mathrm{ad}(E_6)$:

There is one c-special root. It is a simple root α corresponding to the node next to the trivalent node on one of the arms of length 3 with the property $\varpi_\alpha(c)=-1/3$. In this case $\widetilde{L}^\alpha$ is isomorphic to $(SL_5(\mathbb{C})\times SL_2(\mathbb{C}))\times_{(\mathbb{Z}/10\mathbb{Z})}\mathbb{C}^*$ where the element in $\mathbb{Z}/10\mathbb{Z}$ that maps to $\exp(2\pi i/10)$ maps to the generator in $Z(SL_2(\mathbb{C}))$ and to the square of the usual generator in $Z(SL_5(\mathbb{C}))$. Hence, L^α is isomorphic to $(SL_5(\mathbb{C})\times SL_2(\mathbb{C}))\times_{(\mathbb{Z}/10\mathbb{Z})}\mathbb{C}^*$ where the element in $\mathbb{Z}/10\mathbb{Z}$ that maps to $\exp(2\pi i/10)$ maps to the generator in $Z(SL_2(\mathbb{C}))$ and to the usual generator in $Z(SL_5(\mathbb{C}))$. Thus, $\mathrm{ad}(L^\alpha)=PGL_2(\mathbb{C})\times PGL_5(\mathbb{C})$, $\pi_1(L^\alpha)$ is cyclic and γ is a generator of this group. It follows that $\widehat{\gamma}\in\pi_1(PGL_5(\mathbb{C})\times PGL_2(\mathbb{C}))$ generates and hence the image of $\widehat{\gamma}$ under projection to either factor is a generator of the fundamental group of that factor. Direct computation shows that $d_1(\alpha)=9$. Since the action of $\langle c\rangle$ on the extended Dynkin diagram of E_6 has two free orbits and one fixed point, we see that $r_c+1=3=d_1(\alpha)/o(c)$.

$G=\mathrm{ad}(E_7)$:

There is one c-special root. It corresponds to the node of the Dynkin diagram next to the trivalent node on the arm of length 4. In this case $\widetilde{L}^\alpha$ is isomorphic to $(SL_3(\mathbb{C})\times SL_5(\mathbb{C}))\times_{(\mathbb{Z}/15\mathbb{Z})}\mathbb{C}^*$, where the element in $\mathbb{Z}/15\mathbb{Z}$ that maps to $\exp(2\pi i/15)\in\mathbb{C}^*$ maps to the usual generator of $Z(SL_3(\mathbb{C}))$ and the square of the usual generator of $Z(SL_5(\mathbb{C}))$. Thus, $L^\alpha=(SL_3(\mathbb{C})\times SL_5(\mathbb{C}))\times_{(\mathbb{Z}/15\mathbb{Z})}\mathbb{C}^*$ where the element in $\mathbb{Z}/15\mathbb{Z}$ that maps to $\exp(2\pi i/15)\in\mathbb{C}^*$ maps to the inverse of the usual generator of

$Z(SL_3(\mathbb{C}))$ and the inverse of the usual generator of $Z(SL_5(\mathbb{C}))$. Thus, $\mathrm{ad}(L^\alpha) = PGL_3(\mathbb{C}) \times PGL_5(\mathbb{C})$, $\pi_1(L^\alpha)$ is isomorphic to $\mathbb{Z}$ and γ is a generator. Consequently, $\widehat{\gamma}$ is a generator of $\pi_1(\mathrm{ad}(L^\alpha))$.

Direct computation shows that $d_1(\alpha) = 10$. The action of c on the extended Dynkin diagram of E_7 has two fixed points and 3 free orbits so that $r_c + 1 = 5 = d_1(\alpha)/o(c)$. □

Next we compute the integer $n_{c,\alpha}$ defined in Lemma 1.2.7:

Lemma 3.4.4. *If α is c-special and c is nontrivial, then $n_{c,\alpha} = 1$ except in the following cases:*

(i) *If $\widetilde{G} = SL_n(\mathbb{C})$, α corresponds to the k^{th} vertex in the usual ordering, and $o(c) = d$, then $n_{c,\alpha} = n/d \cdot \gcd(k,n)$.*

(ii) *If $\widetilde{G} = Spin(2n)$ and c is of order 2, then $n_{c,\alpha} = 2$.*

Proof. If $Z(\widetilde{G})$ is cyclic and c is a generator, then $n_{c,\alpha} = 1$. The remaining cases are $\widetilde{G} = SL_n(\mathbb{C})$ and $\widetilde{G} = Spin(2n)$, and these can be checked directly. □

As in the simply connected case, we have:

Lemma 3.4.5. *There is a unique semistable L^α-bundle η_0 with the following properties:*

(i) $c_1(\eta_0 \times_{L^\alpha} G) = c$.

(ii) *The Atiyah-Bott point of η_0 is $\mu_{c,\alpha}$.*

(iii) $\det \eta_0 = \mathcal{O}_E(-p_0)$. □

As before, a bundle η satisfying (i) and (ii) above is the pullback of η_0 via a translation of E, and we will call such an η a *translate* of η_0. The following is then proved via arguments similar to those used in the proof of Theorem 3.3.1.

Theorem 3.4.6. *Let η_0 be the bundle described in Lemma 3.4.5. Then*

$$\dim H^0(E; \mathrm{ad}(\eta_0 \times_{L^\alpha} G)) = \dim H^1(E; \mathrm{ad}(\eta_0 \times_{L^\alpha} G)) = r_c + 2.$$

If ξ is any unstable G-bundle which is C^∞ isomorphic to $\eta_0 \times_{L^\alpha} G$, then $\dim H^1(E; \mathrm{ad}\,\xi) \geq r_c + 2$, with equality if and only if ξ is isomorphic to $\eta \times_{L^\alpha} G$ for some translate η of η_0. □

Finally, we must determine the weights for the action of $\mathbb{C}^*$ on $H^1(E; \mathfrak{u}(\eta_0))$:

Proposition 3.4.7. *Suppose that α is c-special. Then the $\mathbb{C}^*$ weights for the action of $\overline{\varphi}_\alpha(\mathbb{C}^*)$ on $H^1(E; \mathfrak{u}(\eta_0))$, with multiplicity, are the integers $n_{c,\alpha} g_{\overline{\beta}}/n_0, \overline{\beta} \in \widetilde{\Delta}/w_c$.*

Proof. The weight for the action of $\mathbb{C}^*$ on $H^1(E; \mathfrak{u}^k(\eta_0))$ is $kn_{c,\alpha}$. By Lemma 2.2.1, the dimension of $H^1(E; \mathfrak{u}^k(\eta_0))$ is

$$i(\alpha, k)/o_{c,\alpha} = i(\alpha, k)/o(c).$$

By Proposition 1.4.17, the integers $d_k(\alpha)/o(c)$ have the circular symmetry property with respect to h_α and $gh_\alpha/g_\alpha o(c)$. By Theorem 3.4.3, $d_1(\alpha)/o(c) = r_c + 1$. If we define

$$\begin{aligned} i_c(k) &= \#\{\beta \in \widetilde{\Delta} : g_{\overline{\beta}} = kn_0\} \\ d_c(k) &= \sum_{k|x} i_c(x) = \sum_{\ell \geq 1} i_c(\ell k), \end{aligned}$$

then the integers $d_c(k)$ satisfy: $d_c(1) = r_c + 1$, and the $d_c(k)$ have the circular symmetry property with respect to N/n_0 and g/n_0, where N is the maximum value of the $g_{\overline{\beta}}$. It follows by inspection or from [7, Proposition 10.1.8] that $n_0 = o(c)g_\alpha/h_\alpha$, i.e., $h_\alpha/g_\alpha o(c) = 1/n_0$. By inspection, $h_\alpha = N/n_0$. Thus $i(\alpha, k)/o(c) = i_c(k)$, and the proof follows. □

4. The nonabelian cohomology space

4.1 The affine space and the universal bundle

Let α be an arbitrary simple root. We abbreviate $P^\alpha = P$, $L^\alpha = L$, and $U^\alpha = U$. Let η be an unliftable semistable principal L-bundle of type c. There is the associated sheaf of (not necessarily abelian) groups $U(\eta)$. The cohomology set $H^1(E; U(\eta))$ classifies pairs (ξ, φ), where ξ is a P-bundle and φ is an isomorphism from the induced bundle ξ/U on L to η. There is a marked point $0 \in H^1(E; U(\eta))$, corresponding to the pair $(\eta \times_L P, I)$, where I is the canonical identification of the bundle $(\eta \times_L P)/U$ with η. There is a corresponding functor $\mathbf{F}$ from schemes to sets defined as follows: for a scheme of finite type over $\mathbb{C}$, $\mathbf{F}(S)$ is the set of isomorphism classes of pairs (Ξ, Φ), where Ξ is a P-bundle over $E \times S$ and Φ is an isomorphism from Ξ/U to $\pi_1^*\eta$.

Lemma 4.1.1. *The functor* $\mathbf{F}$ *is represented by an affine space.*

Proof. Let U_i be the closed subgroup of U whose Lie algebra is $\bigoplus_{k\geq i}\mathfrak{u}^k$. Then the filtration $\{U_i\}$ is a decreasing filtration of U by normal, L-invariant subgroups such that U_i/U_{i+1} is in the center of U/U_{i+1} for every i, and $U_i/U_{i+1} \cong \mathfrak{u}^i$. By Theorem A.2.2 of the appendix, it suffices to check that $H^0(E; (U_i/U_{i+1})(\eta)) = H^2(E; (U_i/U_{i+1})(\eta)) = 0$. The second statement is clear since $\dim E = 1$, and the first follows from Lemma 2.1.5, which implies that $H^0(E; \mathfrak{u}^k(\eta)) = 0$ for every $k > 0$. □

Thus, there is a structure of an affine space on $H^1(E; U(\eta))$ and a universal pair (Ξ_0, Φ_0) over the scheme $E \times H^1(E; U(\eta))$ which represents the functor $\mathbf{F}$. We will somewhat carelessly identify Ξ_0 with the associated G-bundle $\Xi_0 \times_P G$.

We now identify $\varphi_\alpha(\mathbb{C}^*)$ with $\mathbb{C}^*$. Thus we have fixed the embedding of $\mathbb{C}^*$ in L. Since L acts on U, there are induced actions of $\mathbb{C}^*$ on $U(\eta)$ and $\mathfrak{u}(\eta)$, and hence on $H^1(E; U(\eta))$ and on $H^1(E; \mathfrak{u}(\eta))$. Viewing $H^1(E; U(\eta))$ as the set of pairs (ξ, φ) as above, the action of $\mathbb{C}^*$ is via the action of $\operatorname{Aut} L$ on the isomorphism φ, and this action fixes the origin in $H^1(E; U(\eta))$, i.e., the bundle $\eta \times_L P$. By Theorem A.2.2, the action of $\mathbb{C}^*$ lifts to an action on the universal principal bundle Ξ_0 over $E \times H^1(E; U(\eta))$. The first goal of this section is to prove that the action of $\varphi_\alpha(\mathbb{C}^*)$ on $H^1(E; U(\eta))$ is linearizable, and in fact there is a $\mathbb{C}^*$-equivariant isomorphism from $H^1(E; U(\eta))$ to $H^1(E; \mathfrak{u}(\eta))$. Thus the quotient $(H^1(E; U(\eta)) - \{0\})/\mathbb{C}^*$ is a weighted projective space $\mathbb{WP}(\eta)$. In §4.3, we give a sufficient condition for the existence of universal bundles over $E \times \mathbb{WP}(\eta)$. Next we show that, in the case where α is c-special and η_0 is the bundle of Proposition 3.2.3 or Lemma 3.4.5, the points of $\mathbb{WP}(\eta_0)$ correspond to semistable bundles whose automorphism groups have minimal possible dimensions. In §4.5, we analyze the Kodaira-Spencer homomorphism. The results of §4.5 will not however be used in this paper. Finally, we discuss the singular locus of the weighted projective space and relate it to moduli spaces of bundles with a non-simply connected structure group.

4.2 Linearization of the action

We analyze the $\mathbb{C}^*$-action on the affine space $H^1(E; U(\eta))$ more closely. Our goal will be to show that this action can be linearized and to calculate the $\mathbb{C}^*$-weights.

Lemma 4.2.1. *Every $\mathbb{C}^*$-orbit in $H^1(E; U(\eta))$ contains the origin in its closure.*

Proof. Let $x \in H^1(E; U(\eta))$ be represented by the 1-cocycle $\{u_{ij}\}$, where $\{\Omega_i\}$ is an open cover of E and $u_{ij} \colon \Omega_i \cap \Omega_j \to U$ is a morphism. Then $\lambda \in \mathbb{C}^* \subseteq L$ acts on the cocycle $\{u_{ij}\}$. Define morphisms

$$\widetilde{u}_{ij}(e, \lambda) \colon (\Omega_i \cap \Omega_j) \times \mathbb{C} \to U$$

as follows:

$$\widetilde{u}_{ij}(e, \lambda) = \begin{cases} \lambda \cdot u_{ij}(e), & \text{if } \lambda \neq 0; \\ 1, & \text{if } \lambda = 0. \end{cases}$$

There is a $\mathbb{C}^*$-equivariant morphism from the unipotent subgroup U to the affine space $\mathfrak{u}$ (see for example [6, Remark, p. 183]). Using this $\mathbb{C}^*$-equivariant isomorphism, and the fact that all of the $\mathbb{C}^*$-weights on $\mathfrak{u}$ are positive, it is easy to check that the $\widetilde{u}_{ij}$ are morphisms and so define a 1-cocycle for the sheaf $U(\pi_1^*\eta)$ over $E \times \mathbb{C}$. Thus, they define a bundle Ξ over $E \times \mathbb{C}$, reducing to $\pi_1^*\eta$ mod U and such that $\Xi | E \times \{0\} = \eta \times_L P$. By the functorial property of $H^1(E; U(\eta))$, there is a morphism from $\mathbb{C}$ to $H^1(E; U(\eta))$ corresponding to Ξ. Clearly, the image of $0 \in \mathbb{C}$ is the origin of $H^1(E; U(\eta))$, and the image of $\mathbb{C}^*$ is exactly the $\mathbb{C}^*$-orbit of the cocycle $\{u_{ij}\}$. This proves Lemma 4.2.1. □

Lemma 4.2.2. *Let T be the tangent space of $H^1(E; U(\eta))$ at the fixed point 0, with the natural $\mathbb{C}^*$-action. Then the Kodaira-Spencer homomorphism from T to $H^1(E; \mathrm{ad}(\eta \times_L G))$ induced by the bundle Ξ_0 is given by a $\mathbb{C}^*$-equivariant isomorphism $T \to H^1(E; \mathfrak{u}(\eta))$ followed by the inclusion of $H^1(E; \mathfrak{u}(\eta))$ as a direct summand in*

$$H^1(E; \mathrm{ad}(\eta \times_L G)) \cong H^1(E; \mathrm{ad}_L \eta) \oplus H^1(E; \mathfrak{u}_-(\eta)) \oplus H^1(E; \mathfrak{u}(\eta)).$$

Proof. Let $\mathbb{C}[\varepsilon]$ denote the dual numbers. The space T is the set of maps from $\operatorname{Spec} \mathbb{C}[\varepsilon]$ to $H^1(E; U(\eta))$ such that the closed point is mapped to the origin. By the functorial interpretation of $H^1(E; U(\eta))$, such a morphism corresponds to a P-bundle Ξ over $E \times \operatorname{Spec} \mathbb{C}[\varepsilon]$, which is the pullback of Ξ_0, together with an isomorphism from Ξ/U to $\pi_1^*\eta$, and such that Ξ restricts to $\eta \times_L P$ over the closed point. The second condition says that Ξ is a first order deformation of the P-bundle $\eta \times_L P$. Such deformations are classified by

$$H^1(E; \mathrm{ad}_P(\eta \times_L P)) = H^1(E; \mathrm{ad}_L \eta) \oplus H^1(E; \mathfrak{u}(\eta)).$$

The first condition says that the corresponding first order deformation of the L-bundle η is trivial, or equivalently that the projection of the Kodaira-Spencer class of Ξ to $H^1(E; \operatorname{ad}_L \eta)$ is zero. Thus we have defined a canonical map from T to $H^1(E; \mathfrak{u}(\eta))$. Conversely, by reversing this construction, every element of $H^1(E; \mathfrak{u}(\eta))$ defines a first order deformation of $\eta \times_L P$ which reduces to $\pi_1^*\eta$ mod U, so that in fact the map from T to $H^1(E; \mathfrak{u}(\eta))$ is an isomorphism. Since this isomorphism is canonical, it is easily seen to be $\mathbb{C}^*$-equivariant. The last statement is clear by construction. □

Using the previous two lemmas, we show that the $\mathbb{C}^*$-action on $H^1(E; U(\eta))$ can be linearized. There is the following general result about $\mathbb{C}^*$-actions on an affine space.

Lemma 4.2.3. *Let $\mathbb{A}^n$ be an affine space with a $\mathbb{C}^*$-action, and suppose that $0 \in \mathbb{A}^n$ is a fixed point for the action. Let T be the tangent space of $\mathbb{A}^n$ at the origin, together with the induced linear $\mathbb{C}^*$-action on T. Further suppose that:*

(i) *Every $\mathbb{C}^*$-orbit in $\mathbb{A}^n$ contains 0 in its closure.*

(ii) *All of the weights in the $\mathbb{C}^*$-action on T are strictly positive.*

Then there is a $\mathbb{C}^*$-equivariant isomorphism from $\mathbb{A}^n$ to T. Hence, the $\mathbb{C}^*$-action on $\mathbb{A}^n$ is linearizable, and the $\mathbb{C}^*$-weights for this action are those for the action on T.

Proof. Let $A = \mathbb{C}[z_1, \dots, z_n]$ be the affine coordinate ring of $\mathbb{A}^n$, where 0 is defined by $z_1 = \cdots = z_n = 0$, and let $x_1, \dots, x_n$ be a basis for the linear functions on T. The finite-dimensional subspace of A spanned by the z_i is contained in a finite-dimensional $\mathbb{C}^*$-invariant subspace V of A, by the Cartier lemma [20, p. 25] (or by using the grading on A induced by the $\mathbb{C}^*$-action). The map $p \in A \mapsto (dp)_0$ is a $\mathbb{C}^*$-equivariant map from A to T^*, and hence restricts to a $\mathbb{C}^*$-equivariant map from V to T^*. Choosing a $\mathbb{C}^*$-equivariant splitting of the map $V \to T^*$ defines a $\mathbb{C}^*$-equivariant map $T^* \to A$ and thus a $\mathbb{C}^*$-equivariant homomorphism $\mathbb{C}[x_1, \dots, x_n] \to A$. Let $f : \mathbb{A}^n \to T$ be the corresponding morphism. By construction, f has an invertible differential at the origin and is $\mathbb{C}^*$-equivariant. Thus, f is injective in a neighborhood Ω of the origin, and the image of f contains an open set Ω' about the origin. Since the weights on T are positive, every point of T lies in the $\mathbb{C}^*$-orbit of some point of Ω'. Thus f is surjective. Likewise, f is injective:

if $f(x_1) = f(x_2)$, then since the closures of the $\mathbb{C}^*$-orbits of x_1 and x_2 contain the origin, and the weights of the action on the tangent space at 0 are all positive, it follows that there is a $\lambda \in \mathbb{C}^*$ such that $\lambda \cdot x_1$ and $\lambda \cdot x_2$ both lie in Ω. By assumption $f(\lambda \cdot x_1) = \lambda \cdot f(x_1) = \lambda \cdot f(x_2) = f(\lambda \cdot x_2)$. But since f is injective on Ω, $\lambda \cdot x_1 = \lambda \cdot x_2$, and hence $x_1 = x_2$. It follows that f is a $\mathbb{C}^*$-equivariant bijection from $\mathbb{A}^n$ to T and thus it is an isomorphism. $\square$

Corollary 4.2.4. *The $\mathbb{C}^*$-equivariant morphism from $H^1(E; U(\eta))$ to $H^1(E; \mathfrak{u}(\eta))$ defined in Lemma 4.2.2 is an isomorphism. Hence, the $\mathbb{C}^*$-action on $H^1(E; U(\eta))$ can be linearized, and $(H^1(E; U(\eta)) - \{0\})/\mathbb{C}^*$ is a weighted projective space $\mathbb{WP}(\eta)$.* $\square$

In case α is c-special and $\eta = \eta_0$, we have calculated the corresponding weights of the weighted projective space in Proposition 3.3.3 and Proposition 3.4.7.

4.3 Existence of universal bundles on the weighted projective space

Next we discuss the existence of universal bundles over the $\mathbb{C}^*$-quotient. It is easy to see that such bundles cannot exist at the orbifold singular points of the weighted projective space, essentially because there are no local sections from the weighted projective space back to the affine space at such points. We shall show that, away from such points, we can almost find a universal bundle. In particular, there is a universal adjoint bundle away from the orbifold singular points of the weighted projective space.

Recall that $Z(G) \cap \varphi_\alpha(\mathbb{C}^*)$ is a finite cyclic group which we have denoted $\mathbb{Z}/n_{c,\alpha}\mathbb{Z}$.

Lemma 4.3.1. *The subgroup $\mathbb{Z}/n_{c,\alpha}\mathbb{Z}$ of $\mathbb{C}^*$ acts trivially on $H^1(E; U(\eta))$, and the quotient group $\mathbb{C}^*/(\mathbb{Z}/n_{c,\alpha}\mathbb{Z})$ acts faithfully on $H^1(E; U(\eta))$.*

Proof. The $\mathbb{C}^*$-weights are of the form $kn_{c,\alpha}$ for $1 \leq k \leq h_\alpha$, and so the lemma is clear. $\square$

The fact that $\mathbb{Z}/n_{c,\alpha}\mathbb{Z}$ acts trivially on $H^1(E; U(\eta))$ also follows from the fact that it is contained in $Z(G)$. Note however that, if $n_{c,\alpha} > 1$, then the associated action of $\mathbb{Z}/n_{c,\alpha}\mathbb{Z}$ on Ξ_0 is not in general trivial, and in fact is just multiplication by the corresponding subgroup of the center of G.

Proposition 4.3.2. *Let $\mathbb{WP}(\eta)$ be the weighted projective space $(H^1(E;U(\eta)) - \{0\})/\mathbb{C}^*$, and let $\mathbb{WP}_{\text{reg}}(\eta)$ denote the open subset of $\mathbb{WP}(\eta)$ which is the $\mathbb{C}^*$-quotient of the set of points of $H^1(E;U(\eta))$ where $\mathbb{C}^*/(\mathbb{Z}/n_{c,\alpha}\mathbb{Z})$ acts freely. Let $\widehat{G}$ be the quotient of G by the subgroup $\mathbb{Z}/n_{c,\alpha}\mathbb{Z}$. Then the universal bundle Ξ_0 over $E \times H^1(E;U(\eta))$ induces a principal $\widehat{G}$-bundle over $E \times \mathbb{WP}_{\text{reg}}(\eta)$.*

Proof. Let $H^1(E;U(\eta))_{\text{reg}}$ be the set of points of $H^1(E;U(\eta))$ where $\mathbb{C}^*/(\mathbb{Z}/n_{c,\alpha}\mathbb{Z})$ acts freely and effectively. We have seen that there is a lifted action of $\mathbb{C}^*$ on $\Xi_0|E \times H^1(E;U(\eta))_{\text{reg}}$, which in fact is free. The action of the isotropy group $\mathbb{Z}/n_{c,\alpha}\mathbb{Z}$ of a point in the base on the fiber is via multiplication by elements of the center of G, and thus there is an induced $\widehat{G}$-bundle on the $\mathbb{C}^*$-quotient of $H^1(E;U(\eta))_{\text{reg}}$ with the desired properties. □

It is easy to see that we could replace $\widehat{G}$-bundles with bundles over an appropriate conformal form $G \times_{\mathbb{Z}/n_{c,\alpha}\mathbb{Z}} \mathbb{C}^*$ of the group.

If $n_{c,\alpha} = 1$, then there is an induced G-bundle over $E \times \mathbb{WP}_{\text{reg}}(\eta)$. If $n_{c,\alpha} > 1$, then it is easy to see that the corresponding $\widehat{G}$-bundle does not lift to a G-bundle. For example, in case $G = SL_n(\mathbb{C})$, the vector bundle over $E \times \mathbb{P}^{n-1}$ constructed by taking the k^{th} vertex of the Dynkin diagram is given as an extension

$$0 \to \pi_1^* W_k^* \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^{n-1}}(1) \to \mathbf{U} \to \pi_1^* W_{n-k} \to 0,$$

and no twist of this bundle by a line bundle will have trivial determinant. In a future paper, we shall discuss methods for constructing universal bundles in case $n_{c,\alpha} > 1$ via spectral covers.

4.4 The case of a c-special root

Definition 4.4.1. Let ζ be a semistable G-bundle with $c_1(\zeta) = c$. Then ζ is *regular* if $\dim \operatorname{Aut} \zeta = r_c$. By [10, Corollary 6.3], every ζ is S-equivalent to a regular semistable G-bundle, which is unique up to isomorphism. Moreover, by [10, Corollary 6.3], if ζ is not regular, then $\dim \operatorname{Aut} \zeta \geq r_c + 2$.

Let η_0 be the distinguished bundle defined in Proposition 3.2.3 and Lemma 3.4.5. We next show that the nonzero elements of $H^1(E;U(\eta_0))$ correspond to regular semistable G-bundles.

Proposition 4.4.2. *With η_0 as above, for every $x \in H^1(E;U(\eta_0)) - \{0\}$, let ξ_x denote the principal P-bundle $\Xi|E \times \{x\}$ induced by the re-*

striction of Ξ to the slice over x. Then $\xi_x \times_G P$ is a regular semistable G-bundle.

Proof. By Theorem 3.3.1 and Theorem 3.4.6, if ζ is unstable and $c_1(\zeta) = c$, then $\dim \operatorname{Aut}\zeta \geq r_c + 2$. As we have noted above, the same holds for a semistable G-bundle ζ which is not regular. Thus, a G-bundle ζ is semistable and regular if and only if $\dim \operatorname{Aut}\zeta \leq r_c + 1$. To prove Proposition 4.4.2, we shall show that, for all $x \neq 0$, $\dim \operatorname{Aut}_G(\xi_x \times_P G) \leq r_c + 1$.

Let $\xi = \xi_x$. We have the inclusion of the Lie algebra $\mathfrak{p}$ in $\mathfrak{g}$. Clearly, viewing $\mathfrak{g}$ as a representation of P, the vector bundle $\mathfrak{g}(\xi)$ is the same as $\mathfrak{g}(\xi \times_P G) = \operatorname{ad}_G(\xi \times_P G)$. Moreover $\mathfrak{p}(\xi) = \operatorname{ad}_P \xi$. Thus there is an exact sequence of vector bundles

$$0 \to \operatorname{ad}_P \xi \to \operatorname{ad}_G(\xi \times_P G) \to (\mathfrak{g}/\mathfrak{p})(\xi) \to 0.$$

Now replacing ξ by $\eta_0 \times_L P$ gives the corresponding exact sequence

$$0 \to \operatorname{ad}_P(\eta_0 \times_L P) \to \operatorname{ad}_G(\eta_0 \times_L G) \to \mathfrak{u}_-(\eta_0) \to 0,$$

since $(\mathfrak{g}/\mathfrak{p})(\eta_0 \times_L P) = \mathfrak{u}_-(\eta_0)$. Furthermore, by Corollary 2.1.7 and Lemma 3.3.2 in the simply connected case and Theorem 3.4.3 in the non-simply connected case, $H^0(E; \mathfrak{u}_-(\eta_0))$ has dimension $r_c + 1$. By semicontinuity, there is a neighborhood Ω of the origin in $H^1(E; U(\eta_0))$ such that, if $\xi = \xi_x$ corresponds to an $x \in \Omega$, then $\dim H^0(E; (\mathfrak{g}/\mathfrak{p})(\xi)) \leq r_c + 1$ as well. As every point of $H^1(E; U(\eta_0))$ is $\mathbb{C}^*$-equivalent to such an x, we must have $\dim H^0(E; (\mathfrak{g}/\mathfrak{p})(\xi)) \leq r_c + 1$ for all possible ξ.

Next consider the exact sequence of Lie algebras

$$0 \to \mathfrak{u} \to \mathfrak{p} \to \mathfrak{l} \to 0.$$

There is the associated bundle sequence

$$0 \to \mathfrak{u}(\xi) \to \operatorname{ad}_P \xi \to \operatorname{ad}_L \eta_0 \to 0.$$

Since $\mathfrak{u}(\eta_0)$ is a direct sum of semistable bundles of negative degrees, $H^0(E; \mathfrak{u}(\eta_0)) = 0$. It follows as before from semicontinuity and $\mathbb{C}^*$-equivariance that $H^0(E; \mathfrak{u}(\xi)) = 0$ for all ξ. So

$$H^0(E; \operatorname{ad}_P \xi) \subseteq H^0(E; \operatorname{ad}_L \eta_0) \cong \mathbb{C}.$$

Thus

$$h^0(E; \operatorname{ad}_G(\xi \times_P G)) \leq h^0(E; (\mathfrak{g}/\mathfrak{p})(\xi)) + h^0(E; \operatorname{ad}_P \xi) \leq r_c + 2,$$

with equality holding if and only if the map

$$H^0(E;\mathrm{ad}_P\,\xi) \to H^0(E;\mathrm{ad}_L\,\eta_0)$$

is surjective. This can only happen if the natural homomorphism $\mathrm{Aut}_P\,\xi \to \mathrm{Aut}_L\,\eta_0$ is surjective on the connected component of the identity, which would say that every $\lambda \in \mathbb{C}^*$ in $\mathrm{Aut}_L\,\eta_0$ lifts to an element of $\mathrm{Aut}_P\,\xi$. But then x must be a fixed point for the $\mathbb{C}^*$-action, and hence x is the origin. Conversely, if $x \neq 0$, then $H^0(E;\mathrm{ad}_P\,\xi) = 0$ and $\dim \mathrm{Aut}_G(\xi \times_P G) \leq r_c + 1$, and as we have seen above, this statement implies Proposition 4.4.2. □

4.5 The Kodaira-Spencer homomorphism

For the moment, we return to the case of an arbitrary root α. Let $x \in H^1(E;U(\eta))$, and let ξ be the corresponding P-bundle. The bundle Ξ_0 induces a Kodaira-Spencer homomorphism from the tangent space T_x of $H^1(E;U(\eta))$ at x to $H^1(E;\mathrm{ad}(\xi \times_P G))$, and we wish to find some general circumstances where this map is surjective.

Theorem 4.5.1. *Suppose that the differential of the action of* $\mathrm{Aut}_L\,\eta$ *on* $H^1(E;U(\eta))$ *at* $1 \in \mathrm{Aut}_L\,\eta$ *and* $x \in H^1(E;U(\eta))$ *is an injective homomorphism* $H^0(E;\mathrm{ad}_L\,\eta) \to T_x$*. Then the Kodaira-Spencer homomorphism* $T_x \to H^1(E;\mathrm{ad}(\xi \times_P G))$ *is surjective.*

Proof. As in the proof of Lemma 4.2.2, the tangent space T_x can be identified with bundles ξ_ε over $E \times \mathrm{Spec}\,\mathbb{C}[\varepsilon]$ which restrict to ξ over the closed fiber and reduce mod U to $\pi_1^*\eta$. If η is given by the 1-cocycle $\{\ell_{ij}\}$, where the ℓ_{ij} take values in L, and ξ by the 1-cocycle $\{\ell_{ij}u_{ij}\}$, where the u_{ij} take values in U, then it is easy to see that ξ_ε is given by a 1-cocycle $\{\ell_{ij}(u_{ij} + \varepsilon v_{ij})\}$, where the v_{ij} are also U-valued. Moreover $w_{ij} = u_{ij}^{-1}v_{ij}$ defines an element of $H^1(E;\mathfrak{u}(\xi))$. In this way, we identify T_x with $H^1(E;\mathfrak{u}(\xi))$.

There is a long exact sequence

$$0 \to \mathfrak{u}(\xi) \to \mathrm{ad}_P\,\xi \to \mathrm{ad}_L\,\eta \to 0.$$

The natural map $H^1(E;\mathfrak{u}(\xi)) \to H^1(E;\mathrm{ad}_P\,\xi)$ is the Kodaira-Spencer map for deformations of the P-bundle ξ, and its kernel is the image of the coboundary map $\delta\colon H^0(E;\mathrm{ad}_L\,\eta) \to H^1(E;\mathfrak{u}(\xi))$. This kernel also contains the image of the differential of the action of $\mathrm{Aut}_L\,\eta$ on

$H^1(E; U(\eta))$ at x, which has dimension equal to $\dim H^0(E; \mathrm{ad}_L \eta)$. Thus it follows by hypothesis that δ is injective.

The Killing form identifies the vector bundle $(\mathfrak{g}/\mathfrak{p})(\xi)$ with the dual of $\mathfrak{u}(\xi)$. In particular, $(\mathfrak{g}/\mathfrak{p})(\xi)$ has a filtration whose successive quotients are stable bundles of positive degrees. Hence $H^1(E; (\mathfrak{g}/\mathfrak{p})(\xi)) = 0$ and the natural map $H^1(E; \mathrm{ad}_P \xi) \to H^1(E; \mathrm{ad}_G(\xi \times_P G))$ is surjective. Now consider the commutative diagram

$$\begin{array}{ccccccc}
 & & H^1(E;\mathfrak{u}(\xi)) & \longrightarrow & H^1(E;\mathrm{ad}_G(\xi\times_P G)) & & \\
 & & \downarrow & & \| & & \\
H^0(E;(\mathfrak{g}/\mathfrak{p})(\xi)) & \longrightarrow & H^1(E;\mathrm{ad}_P\,\xi) & \longrightarrow & H^1(E;\mathrm{ad}_G(\xi\times_P G)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
H^1(E;\mathrm{ad}_L\,\eta) & = & H^1(E;\mathrm{ad}_L\,\eta) & & & & \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}$$

where the middle row and column are exact. A diagram chase shows that, if the map $H^0(E; (\mathfrak{g}/\mathfrak{p})(\xi)) \to H^1(E; \mathrm{ad}_L \eta)$ is surjective, then so is the map

$$H^1(E; \mathfrak{u}(\xi)) \to H^1(E; \mathrm{ad}_G(\xi \times_P G)),$$

which is the statement of the theorem. The map $H^0(E; (\mathfrak{g}/\mathfrak{p})(\xi)) \to H^1(E; \mathrm{ad}_L \eta)$ is given by the composition

$$H^0(E; (\mathfrak{g}/\mathfrak{p})(\xi)) \to H^1(E; \mathrm{ad}_P \xi) \to H^1(E; \mathrm{ad}_L \eta),$$

and the above sequence is Serre dual to the sequence

$$H^1(E; \mathfrak{u}(\xi)) \leftarrow H^0(E; (\mathfrak{g}/\mathfrak{u})(\xi)) \leftarrow H^0(E; \mathrm{ad}_L \eta).$$

By the naturality of the connecting homomorphisms associated to the following commutative diagram of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{u}(\xi) & \longrightarrow & \mathrm{ad}_P\,\xi & \longrightarrow & \mathrm{ad}_L\,\xi & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{u}(\xi) & \longrightarrow & \mathrm{ad}_G(\xi\times_P G) & \longrightarrow & (\mathfrak{g}/\mathfrak{u})(\xi) & \longrightarrow & 0,
\end{array}$$

the composition $H^0(E; \mathrm{ad}_L \eta) \to H^1(E; \mathfrak{u}(\xi))$ is just the homomorphism δ, which is injective by hypothesis. Hence by duality $H^0(E; (\mathfrak{g}/\mathfrak{p})(\xi)) \to H^1(E; \mathrm{ad}_L \eta)$ is surjective, which completes the proof. $\square$

Corollary 4.5.2. *If* $\dim \mathrm{Aut}_L \eta = 1$*, and* x *is not the origin of* $H^1(E; U(\eta))$*, then the Kodaira-Spencer homomorphism*

$$T_x \to H^1(E; \mathrm{ad}(\xi \times_P G))$$

is surjective. If α *is c-special,* $\eta = \eta_0$*, and* x *is not the origin, then the Kodaira-Spencer homomorphism induces an isomorphism from* T_x *modulo the tangent space to* $\mathbb{C}^* \cdot x$ *to* $H^1(E; \mathrm{ad}(\xi \times_P G))$*.*

Proof. Since the $\mathbb{C}^*$-action is via strictly positive weights, the differential of the action is injective at every nonzero point $H^1(E; U(\eta))$, so the hypothesis of the previous theorem is satisfied. This proves the first statement. To see the second statement, the induced map from T_x modulo the tangent space to $\mathbb{C}^* \cdot x$ to $H^1(E; \mathrm{ad}(\xi \times_P G))$ is surjective. The dimension of T_x modulo the tangent space to $\mathbb{C}^* \cdot x$ is r_c. Since, for every ξ, $\dim H^1(E; \mathrm{ad}(\xi \times_P G)) \geq r_c$, equality must hold, giving a new proof of Proposition 4.4.2, and the induced map from T_x modulo the tangent space to $\mathbb{C}^* \cdot x$ to $H^1(E; \mathrm{ad}(\xi \times_P G))$ is an isomorphism. □

This shows that the map $\Psi\colon \mathbb{WP}(\eta_0) \to \mathcal{M}(G, c)$ is a local diffeomorphism over the smooth points of $\mathcal{M}(G, c)$. Moreover, giving $\mathcal{M}(G, c)$ and $\mathbb{WP}(\eta_0)$ their natural orbifold structures, it follows from Corollary 4.5.2 that Ψ is an orbifold covering. Let $d_k(\mathbb{WP}(\eta_0))$ and $d_k(\mathcal{M}(G, c))$ be the dimensions of the subspaces of $\mathbb{WP}(\eta_0)$ and $\mathcal{M}(G, c)$ where k divides the order of the orbifold isotropy. The orbifold covering property implies that

$$d_k(\mathbb{WP}(\eta_0)) \leq d_k(\mathcal{M}(G, c)) \quad \text{for all} \quad k \geq 1.$$

The results in [7] and [10] imply that $d_k(\mathcal{M}(G, c)) = d(k)$ in the notation of Proposition 3.3.3 in the simply connected case.

In Section 5 we shall prove that Ψ is an isomorphism. In proving this result we do not appeal to Corollary 4.5.2, but rather use the fact that both $d_k(\mathbb{WP}(\eta_0))$ and $d_k(\mathcal{M}(G, c))$ satisfy circular symmetry. One can in fact turn this argument around. Using the above inequality and the fact that the sums of the weights for $\mathbb{WP}(\eta_0)$ and $\mathcal{M}(G, c)$ add up to g, one can prove directly that $d_k(\mathbb{WP}(\eta_0)) = d_k(\mathcal{M}(G, c))$ for all $k \geq 1$, and hence apply the results of [7] to show that the $d_k(\mathbb{WP}(\eta_0))$ satisfy circular symmetry. In the simply connected case, this gives a classification-free proof of circular symmetry for the numbers $d_k(\alpha)$, where α is a special root.

4.6 The singular locus of the weighted projective space

The weighted projective space $\mathbb{WP}(\eta)$ is naturally an orbifold. Its singular locus (as an orbifold) corresponds to the set of points in the affine space $H^1(E; U(\eta))$ whose isotropy group is larger than that of the generic point, i.e., is larger than $\mathbb{Z}/n_{c,\alpha}\mathbb{Z}$. This will be the singular locus of $\mathbb{WP}(\eta)$ as a variety provided that its codimension is at least two. If we choose a linear structure and a diagonal basis of $H^1(E; U(\eta))$, then the orbifold singular locus of $\mathbb{WP}(\eta)$ is a union of weighted projective subspaces $\mathbb{WP}(\eta)_k$ for $k > 1$, where $\mathbb{WP}(\eta)_k$ is the subvariety of $\mathbb{WP}(\eta)$ where all of the coordinates are equal to zero except for those for which the weights are divisible by $kn_{c,\alpha}$. Our goal is to show that, when G is simply connected and α is special, each such subspace can be naturally identified the moduli space for a non-simply connected subgroup of G.

Given $\alpha \in \Delta$, fix an integer $k > 1$ such that $k = \beta(\varpi_\alpha^\vee)$ for some root β. Recall from the proof of Proposition 1.4.2 that the set of all $\beta \in R$ such that $k|\beta(\varpi_\alpha^\vee)$ is a root system $R(\alpha, k)$. Moreover, $\Delta(\alpha, k) = (\Delta - \{\alpha\}) \cup \{-\lambda_k(\alpha)\}$ is a set of simple roots for $R(\alpha, k)$. There is a semisimple subalgebra

$$\mathfrak{g}(\alpha, k) = \mathfrak{h} \oplus \bigoplus_{\beta \in R(\alpha,k)} \mathfrak{g}^\beta \subseteq \mathfrak{g}.$$

Let $G(\alpha, k)$ be the corresponding closed connected subgroup of G. Of course, $\mathfrak{g}(\alpha, k)$ will not be simple in general. Let

$$\Delta(\alpha, k) = \coprod_{i \geq 1} \Delta(\alpha, k)_i,$$

where each subset $\Delta(\alpha, k)_i$ corresponds to a connected component of the Dynkin diagram of $R(\alpha, k)$, and where $-\lambda_k(\alpha) \in \Delta(\alpha, k)_1$. Since $G(\alpha, k)$ is semisimple, we can write

$$G(\alpha, k) = \left(\prod_i G_i\right) \Big/ F,$$

where each G_i is simple and simply connected and corresponds to the subset $\Delta(\alpha, k)_i$ of $\Delta(\alpha, k)$, and where F is finite. If G is simply connected, then F is cyclic of order k, generated by an element c_k.

Viewing $-\lambda_k(\alpha)$ as an element of $\Delta(\alpha, k)$, i.e., a simple root for $G(\alpha, k)$, $-\lambda_k(\alpha)$ defines a maximal parabolic subgroup $P(\alpha, k)$ of $G(\alpha, k)$ contained in the maximal parabolic subgroup

$P^\alpha = P$ of G determined by α. The parabolic subgroup $P(\alpha, k)$ is of the form $(P_1 \times \prod_{i\geq 2} G_i)/F$, where P_1 is the maximal parabolic subgroup in G_1 corresponding to $-\lambda_k(\alpha)$. Clearly, the Levi factor $L(\alpha, k)$ of $P(\alpha, k)$ is just $L^\alpha = L = (L_1 \times \prod_{i\geq 2} G_i)/F$, where L_1 is the Levi factor of P_1. The unipotent radical $U(\alpha, k)$ of $P(\alpha, k)$ has Lie algebra $\bigoplus_{k|j} \mathfrak{u}^j$. Let η be a semistable L-bundle of negative degree. The degree of η is of course independent of whether we view L as the Levi factor of P or of $P(\alpha, k)$, and we can define the cohomology set $H^1(E; U(\alpha, k)(\eta))$. The inclusion of $U(\alpha, k)$ in U defines an $\mathrm{Aut}(L)$-equivariant function $H^1(E; U(\alpha, k)(\eta)) \to H^1(E; U(\eta))$ on the level of cohomology sets, as well as a morphism between the corresponding functors. Since the two associated functors are both represented by affine spaces, the function $H^1(E; U(\alpha, k)(\eta)) \to H^1(E; U(\eta))$ corresponds to an $\mathrm{Aut}(L)$-equivariant morphism of affine spaces. The geometric meaning of this morphism is as follows: let F_i be the projection of F to the factor G_i, so that there is a homomorphism from $(L_1 \times \prod_{i\geq 2} G_i)/F$ to $(L_1/F_1) \times \prod_{i\geq 2}(G_i/F_i)$ for $i \geq 2$. The bundle η thus induces an $(L_1/F_1) \times \prod_{i\geq 2}(G_i/F_i)$-bundle $\prod_i \eta_i$, where η_1 is an (L_1/F_1)-bundle and the η_i are (G_i/F_i)-bundles for $i \geq 2$, Moreover η defines a canonical lifting of the $(L_1/F_1) \times \prod_{i\geq 2}(G_i/F_i)$-bundle $\prod_i \eta_i$ to an $(L_1 \times \prod_{i\geq 2} G_i)/F$-bundle. A class x in $H^1(E; U(\alpha, k)(\eta))$ defines a lifting of η_1 to a (P_1/F_1)-bundle ξ_1. The lift η then defines a lift of the $(P_1/F_1) \times \prod_{i\geq 2}(G_i/F_i)$-bundle defined by ξ_1 and the η_i, $i \geq 2$, to a $(P_1 \times \prod_{i\geq 2} G_i)/F$-bundle ξ'. Since $P(\alpha, k) = (P_1 \times \prod_{i\geq 2} G_i)/F$ is a subgroup of P, we can form the associated bundle $\xi = \xi' \times_{P(\alpha,k)} P$, and this bundle is clearly the lift of η corresponding to the image of x in $H^1(E; U(\eta))$.

Proposition 4.6.1. *Let η be a semistable L-bundle of negative degree $-d$. There are compatible linear structures on $H^1(E; U(\alpha, k)(\eta))$ and on $H^1(E; U(\eta))$ so that the morphism*

$$H^1(E; U(\alpha, k)(\eta)) \to H^1(E; U(\eta))$$

is a $\mathbb{C}^*$-equivariant embedding of $H^1(E; U(\alpha, k)(\eta))$ onto the linear subspace of $H^1(E; U(\eta))$ defined by the span of all of the eigenvectors of the $\mathbb{C}^*$-action on $H^1(E; U(\eta))$ whose weights are divisible by $kn_{c,\alpha}d$.

Proof. It is an elementary exercise to check that, if $\mathbb{C}^*$ acts linearly and with positive weights on two affine spaces $\mathbb{A}_1$ and $\mathbb{A}_2$ and if $f\colon \mathbb{A}_1 \to \mathbb{A}_2$ is a $\mathbb{C}^*$-equivariant morphism whose differential at the origin is injective, then there exist coordinates on $\mathbb{A}_2$ for which $\mathbb{C}^*$ acts linearly and such that f is a linear embedding, and the $\mathbb{C}^*$-weights for the image of f can be determined from the differential of f at the origin. The differential of the morphism $H^1(E; U(\alpha, k)(\eta)) \to H^1(E; U(\eta))$ is given by the inclusion

$$\bigoplus_{k|j} H^1(E; \mathfrak{u}^j(\eta)) \to \bigoplus_{j>0} H^1(E; \mathfrak{u}^j(\eta)).$$

Hence the image of the differential of f at the origin is the span of the eigenvectors of the $\mathbb{C}^*$-action on $H^1(E; \mathfrak{u}(\eta))$ whose weights are divisible by $kn_{c,\alpha}d$. Thus, the same is true for the morphism f. □

We turn now to the case of a special root α. For simplicity, we assume that G is simply connected, so that the finite group $F = \pi_1(G(\alpha, k))$ is generated by an element c_k of order k.

Proposition 4.6.2. *Suppose that G is simply connected and that α is special. Let η_0 be the L-bundle of Proposition 3.2.3. Then:*

(i) *For $i > 1$, $\Delta(\alpha, k)_i$ is of type A and c_k projects to a generator of the corresponding fundamental group.*

(ii) *The (L_1/F_1)-bundle η_1 induced by η_0 has the property that $\eta_1 \times_{L_1/F_1} (G_1/F_1)$ is a minimally unstable (G_1/F_1)-bundle, and the root $-\lambda_k(\alpha)$ is a c_k-special simple root in $\Delta(\alpha, k)_1$.*

Proof. Part (i) follows easily from the explicit description of the special root. To see Part (ii), it follows from Proposition 4.6.1 that, if x is a nonzero element of $H^1(E; U(\alpha, k)(\eta_0))$, then the image of x in $H^1(E; U(\eta_0))$ is also nonzero. In particular, if ξ is the corresponding P-bundle, then $\xi \times_P G$ is semistable. It is easy to check that, in this case, the (P_1/F_1)-bundle corresponding to x is again semistable. Thus, $\eta_1 \times_{(P_1/F_1)} (G_1/F_1)$ is a minimally unstable (G_1/F_1)-bundle, and so $-\lambda_k(\alpha)$ is a c_k-special simple root for G_1/F_1. □

Of course, we could also check the above proposition by a case-by-case analysis. This also shows that the projection $F \to G_1$ is always an embedding of F into the center of G_1. For $i \geq 2$, the factor G_i is of type A and F_i is the full center, and hence the bundle η_i is always rigid. On the other hand, the image F_1 need not be the full center of G_1. For example, if G is of type E_8 and $k = 4$, then G_1 is of type A_7, and thus its center is isomorphic to $\mathbb{Z}/8\mathbb{Z}$, whereas F_1 has order 4.

There is thus an induced morphism of weighted projective spaces. In terms of moduli spaces, if we grant Looijenga's theorem (Theorem 5.1.1) in both the simply connected and the non-simply connected cases, this morphism identifies the sub-projective space $\mathbb{WP}(\eta_0)_k$ of $\mathbb{WP}(\eta_0)$ corresponding to setting all the coordinates in the weight spaces with weights not divisible by kn_α equal to zero with the weighted projective space which is the moduli space of unliftable semistable (G_1/F_1)-bundles. Of course, on the level of G-bundles, this shows that up to S-equivalence a bundle corresponding to a point of the moduli space lying in this sub-projective space has a reduction of structure to a semistable unliftable $(\prod_i G_i)/F$-bundle, and conversely such a bundle defines a semistable G-bundle whose moduli point lies in this sub-projective space. In terms of the $\mathbb{C}^*$-weights, the number of such weights divisible by kn_α can be related to the weights appearing for the appropriate non-simply connected form of a subgroup. For example, if G is of type E_8, the root system $R(\alpha, 2)$ is of type $E_7 \times A_1$ and the $\mathbb{C}^*$-weights divisible by 2 for G, in other words the g_α such that $2|g_\alpha$, are the weights occurring in the weighted projective space for the adjoint form of E_7. These are exactly the weights appearing in the quotient diagram for $\widetilde{E}_7$ modulo the action of the nontrivial element of the center, namely twice the weights for the group of type F_4: $2, 2, 4, 4, 6$. In [7, §9], these quotient root systems appear in a different context, unrelated to the special roots, as certain root systems $\Phi(\mathfrak{t}(k))$ constructed on certain subtori of H. It would be nice to understand this somewhat mysterious connection more directly.

5. A new proof of Looijenga's theorem

5.1 Statement of the theorem

Fix a c-special root α. We denote the corresponding parabolic subgroup simply by P, and similarly for L. We have defined the bundle η_0 in Proposition 3.2.3 and Lemma 3.4.5. As in Section 1, let $\mathcal{M}(G, c)$ denote the coarse moduli space of semistable G-bundles ξ with $c_1(\xi) = c$, modulo S-equivalence. We have seen that there is a universal family of regular semistable G-bundles Ξ_0 over $E \times (H^1(E; U(\eta_0)) - \{0\})$. Thus there is an induced morphism

$$\widetilde{\Psi}\colon H^1(E; U(\eta_0)) - \{0\} \to \mathcal{M}(G, c).$$

The morphism $\widetilde{\Psi}$ is constant on $\mathbb{C}^*$-orbits. Let $\mathbb{WP}(\eta_0)$ be the weighted projective space which is the $\mathbb{C}^*$-quotient of $H^1(E; U(\eta_0)) - \{0\}$ and let

$$\Psi\colon \mathbb{WP}(\eta_0) \to \mathcal{M}(G, c)$$

be the morphism induced by $\widetilde{\Psi}$. We can then state our version of Looijenga's theorem as follows:

Theorem 5.1.1. *Let E be an elliptic curve, let G be a simple group and let $c \in \pi_1(G)$ be a generator. The morphism $\Psi\colon \mathbb{WP}(\eta_0) \to \mathcal{M}(G, c)$ defined above is an isomorphism.*

Corollary 5.1.2. *The moduli space $\mathcal{M}(G, c)$ is isomorphic to a weighted projective space with weights $g_{\overline{\beta}}/n_0, \overline{\beta} \in \widetilde{\Delta}/w_c$ as defined in Definition 1.5.1.*

In particular, if G is simply connected, then we obtain a new proof of Looijenga's theorem [17].

5.2 The classical cases

Let us first sketch the proof of the theorem above for the case of the classical groups, bearing in mind the description of the bundle η_0 given in Proposition 3.2.5. In case $G = SL_n(\mathbb{C})$, the theorem asserts that every regular semistable vector bundle V of rank n and trivial determinant is S-equivalent to a unique extension

$$0 \to W_k^* \to V \to W_{n-k} \to 0.$$

In this form, the theorem is proved in [14], Theorem 3.2(iv).

Let us next consider the case of the symplectic group. We shall show that the morphism Ψ has degree one in this case (as we shall see below, this implies that Ψ is an isomorphism). Let V be a generic regular semistable symplectic vector bundle, in other words a regular semistable vector bundle of rank $2n$ with a nondegenerate symplectic form A. Here generic will mean that V is a direct sum

$$\bigoplus_{i=1}^{n}(\lambda_i \oplus \lambda_i^{-1}),$$

where the λ_i are line bundles of degree zero, not of order 2, and such that, for $i \neq j$, $\lambda_i \neq \lambda_j^{\pm 1}$. In this case, the symplectic form on V is an orthogonal sum of symplectic forms A_i on $\lambda_i \oplus \lambda_i^{-1}$. The space of such forms which are nondegenerate corresponds to the choice of an isomorphism from λ_i^{-1} to itself, in other words to a nonzero multiple of A_i, and the group of symplectic automorphisms of A_i is also isomorphic to $\mathbb{C}^*$. For each i, the space of surjections $\varphi_i\colon W_n^* \to \lambda_i^{\pm 1}$ is a $\mathbb{C}^*$. Thus, the space of morphisms $W_n^* \to \lambda_i \oplus \lambda_i^{-1}$ is a $\mathbb{C}^* \times \mathbb{C}^*$. It is clear that the pullback of A_i to W_n under such a morphism is a nonzero symplectic form, which we denote by B_i. Moreover, by varying the morphism from $W_n^* \to \lambda_i \oplus \lambda_i^{-1}$, we exactly get all symplectic forms on W_n^* of the form t_iB_i, where $t_i \in \mathbb{C}^*$.

Now suppose we have found, for every i, a morphism $f_i\colon W_n^* \to \lambda_i \oplus \lambda_i^{-1}$ such that, if t_iB_i is the pulled back morphism, then $t_i \neq 0$ for every i and such that $\sum_{i=1}^n t_iB_i = 0$. It follows that the morphism $W_n^* \to \lambda_i^{\pm 1}$ is nonzero for every i. By Proposition 3.6 of [14], the morphism $W_n^* \to V$ embeds W_n^* as an isotropic subbundle of V, and the quotient V/W_n^* is isomorphic to W_n. This implies that we have reduced the structure group of V to a maximal parabolic subgroup P of $Sp(2n)$ corresponding to the special root, and the corresponding L-bundle is η_0. Hence (V, A) is in the image of Ψ, and the statement that Ψ has degree 1 is the statement that the t_i above are uniquely determined up to multiplying by a fixed nonzero scalar.

Thus, we must find t_i such that $\sum_{i=1}^n t_iB_i = 0$ and show that the t_i are unique up to a scalar. To see this, note that the space of alternating forms on W_n^* is $H^0(E; \bigwedge^2 W_n)$. One easily computes that $\deg \bigwedge^2 W_n = n-1$. Since W_n is stable, $\bigwedge^2 W_n$ is a semistable vector bundle of positive degree. Thus, $h^1(E; \bigwedge^2 W_n) = 0$, and so by Riemann-Roch, $h^0(E; \bigwedge^2 W_n) = n-1$. It follows that every collection of n forms B_i is linearly dependent, and so some linear combination of the B_i is zero. To prove uniqueness, and also to prove that the quotient is a W_n, it will suffice to show that no smaller linear combination is zero. Suppose, say, that $\sum_{i=1}^k t_iB_i = 0$, with $k < n$ and $t_i \neq 0$ for all $i \leq k$. Consider the induced morphism from W_n^* to $\bigoplus_{i=1}^k (\lambda_i \oplus \lambda_i^{-1})$. If $k > n/2$, then by Proposition 3.6 of [14] there would exist an embedding of W_n^* as an isotropic subbundle of a vector bundle of rank $< 2n$, which is impossible since the symplectic form on $\bigoplus_{i=1}^k (\lambda_i \oplus \lambda_i^{-1})$ is nondegenerate. If $k \leq n/2$, then in fact the argument of Proposition 3.6 of [14] shows that the image of W_n^* would be a subbundle of rank equal to $2k$, and hence that the symplectic form on $\bigoplus_{i=1}^k (\lambda_i \oplus \lambda_i^{-1})$ is identically zero, which is again a contradiction. Thus no smaller linear combination of the B_i is zero, proving that Ψ has degree one. A similar argument shows that, even for a regular symplectic bundle (V, A) which is not generic in the above sense, there still exists an embedding of W_n^* in V as a maximal isotropic subbundle, and this embedding is unique up to the automorphism group of (V, A). Thus, we can also show directly that Ψ is a bijection instead of merely having degree one in this case.

The case of $Spin(n)$ is similar. By a generic $Spin(2n)$-bundle, we mean one whose associated $SO(2n)$-bundle V is of the form $\bigoplus_{i=1}^n(\lambda_i \oplus \lambda_i^{-1})$, where the λ_i are line bundles of degree zero, not of order 2, and such that, for $i \neq j$, $\lambda_i \neq \lambda_j^{\pm 1}$, and the quadratic form is the orthogonal sum of quadratic forms A_i on $\lambda_i \oplus \lambda_i^{-1}$. We consider maps $W_{n-2}^* \to V$ whose image is isotropic. Arguments as in the symplectic case show that there is an embedding ι of W_{n-2}^* as an isotropic subbundle of V, and such that the projection to each summand $\lambda_i^{\pm 1}$ is nonzero, and such an embedding is unique up to orthogonal isomorphisms of V. Dually, we have a map $V^* \cong V \to W_{n-2}$. Thus there is a complex $W_{n-2}^* \xrightarrow{\iota} V \xrightarrow{\iota^*} W_{n-2}$. The symmetric form identifies $V/(W_{n-2}^*)^\perp$ with W_{n-2}. It is easy to check that the bundle $\mathrm{Ker}(\iota^*)/\mathrm{Im}(\iota)$ is a semistable $SO(4)$-bundle which does not lift to $Spin(4)$, and hence is of the form $W_2 \otimes W_2^* \cong Q_4$. The filtration on V then reduces the structure group to the appropriate maximal parabolic subgroup as before. A very similar argument handles the case of $SO(2n+1)$, using instead an isotropic subbundle isomorphic to W_{n-1}^*, and showing that $(W_{n-1}^*)^\perp/W_{n-1}^*$ is isomorphic to $\mathrm{ad}\, W_2$. Similar statements can also handle the case of unliftable bundles.

5.3 Proof of the main theorem: determinant line bundles

We turn to the proof of Theorem 5.1.1.

First, it is an elementary result that a morphism from a weighted projective space to a quasiprojective variety is either constant or finite. Indeed, since every weighted projective space has a finite ramified cover which is an ordinary projective space, this follows from the analogous and well-known result for $\mathbb{P}^n$. Thus, since $\mathbb{WP}(\eta_0)$ and $\mathcal{M}(G, c)$ are normal, if we can show that Ψ has degree one, then it follows from Zariski's main theorem that Ψ is an isomorphism.

To calculate the degree of Ψ we shall compare determinant line bundles on the two sides. The idea will be to find line bundles $\mathbb{D}_1$, $\mathbb{D}_2$ on $\mathbb{WP}(\eta_0)$ and $\mathcal{M}(G, c)$ respectively, such that $[\mathbb{D}_1] = \Psi^*[\mathbb{D}_2]$ in $H^2(\mathbb{WP}(\eta_0); \mathbb{Q})$ and such that $\int_{\mathbb{WP}(\eta_0)} c_1(\mathbb{D}_1)^{r_c} = \int_{\mathcal{M}(G,c)} c_1(\mathbb{D}_2)^{r_c} \neq 0$. Since on the other hand we have $c_1(\mathbb{D}_1)^{r_c} = \deg \Psi \cdot c_1(\mathbb{D}_2)^{r_c}$, it will then follow that $\deg \Psi = 1$.

To define the line bundles $\mathbb{D}_i$ (which in fact will be *a priori* only $\mathbb{Q}$-

Cartier divisors), we recall the definition of the determinant line bundle on the moduli functor. First recall the definition of the moduli functor $\mathbf{M}$ itself: for a scheme S over $\mathbb{C}$, $\mathbf{M}(S)$ is the set of isomorphism classes of principal G-bundles Ξ over $E\times S$ such that $\Xi|E\times\{x\}$ is semistable for every $x \in S$. The moduli functor is coarsely represented by $\mathcal{M}(G,c)$. Given an element of $\mathbf{M}(S)$, corresponding to a principal G-bundle Ξ over $E \times S$, we have the associated vector bundle $\operatorname{ad}\Xi$ over $E \times S$, and thus we can form the determinant line bundle $\det R\pi_{2*}\operatorname{ad}\Xi$ over S (see for example Chapter 5, Section 3 of [9]). Since π_2 has relative dimension one, $H^i(\operatorname{ad}\Xi|E \times \{s\}) = 0$ and likewise $R^i\pi_{2*}\operatorname{ad}\Xi = 0$ for $i > 1$. The fiber of $\det R\pi_{2*}\operatorname{ad}\Xi$ over $s \in S$ is then the complex line

$$\bigwedge^{\text{top}} H^0(\operatorname{ad}\Xi|E \times \{s\}) \otimes \left(\bigwedge^{\text{top}} H^1(\operatorname{ad}\Xi|E \times \{s\})\right)^{-1}.$$

Here are some of the basic properties of this line bundle:

1. If $R^0\pi_{2*}\operatorname{ad}\Xi$ and $R^1\pi_{2*}\operatorname{ad}\Xi$ are locally free, for example if $\Xi|E \times \{x\}$ is regular for every $x \in S$, then
$$\det R\pi_{2*}\operatorname{ad}\Xi = \left(\bigwedge^{\text{top}} R^0\pi_{2*}\operatorname{ad}\Xi\right) \otimes \left(\bigwedge^{\text{top}} R^1\pi_{2*}\operatorname{ad}\Xi\right)^{-1}.$$
2. Suppose that S is smooth, that λ is a line bundle of degree zero on E, and that
$$D_\lambda = \{\, x \in S : h^0(E; (\operatorname{ad}\Xi|E \times \{x\}) \otimes \lambda) \neq 0\}$$
is a hypersurface in S. Let Z_i be the irreducible components of D_λ, and let n_i be the length of the torsion sheaf $R^1\pi_{2*}(\operatorname{ad}\Xi \otimes \pi_1^*\lambda)$ at a generic point of Z_i. Then (see e.g., [9], Chapter 5, Corollary 1.2 and Proposition 3.9) there is a canonical section div of $\big(\det R\pi_{2*}(\operatorname{ad}\Xi \otimes \pi_1^*\lambda)\big)^{-1}$ whose divisor of zeroes is $\sum_i n_iZ_i$, and hence
$$c_1(\det R\pi_{2*}\operatorname{ad}\Xi) = c_1(\det R\pi_{2*}(\operatorname{ad}\Xi \otimes \pi_1^*\lambda)) = -\sum_i n_i[Z_i].$$

By general results, there is an associated $\mathbb{Q}$-divisor on $\mathcal{M}(G,c)$ (i.e., an element in $\operatorname{Pic}(\mathcal{M}(G,c)) \otimes \mathbb{Q}$, which we shall denote by $\mathbb{D}_2$. As we shall see, $\mathbb{D}_2$ is in fact Cartier, in other words, a line bundle. We will find an analogous divisor $\mathbb{D}_1$ over $\mathbb{WP}(\eta_0)$ and show that (1) $\int_{\mathbb{WP}(\eta_0)} c_1(\mathbb{D}_1)^{r_c} = \int_{\mathcal{M}(G,c)} c_1(\mathbb{D}_2)^{r_c}$ and (2) $[\mathbb{D}_1] = \Psi^*[\mathbb{D}_2]$. This will prove Theorem 5.1.1.

5.4 The divisor on the weighted projective space

Consider the universal G-bundle Ξ_0 over $E \times H^1(E; U(\eta_0))$. The action of $\mathbb{C}^*$ on Ξ_0 gives a linearization of the action on $\mathbb{C}^*$ on the associated vector bundle $\operatorname{ad}\Xi_0$ and thus on the line bundle $\det R\pi_{2*}\operatorname{ad}\Xi_0$. Since $H^1(E; U(\eta_0))$ is an affine space, $\det R\pi_{2*}\operatorname{ad}\Xi_0$ is the trivial line bundle. Every linearization of the $\mathbb{C}^*$-action on the trivial bundle is given by a character $\chi\colon \mathbb{C}^* \to \mathbb{C}^*$ of the form $\chi(z) = z^a$ for a unique $a \in \mathbb{Z}$. There is the corresponding coherent sheaf $\mathcal{O}_{\mathbb{WP}(\eta_0)}(a)$, which is given by viewing $\mathbb{WP}(\eta_0)$ as $\operatorname{Proj}\mathbb{C}[z_0, \dots, z_{r_c}]$ with the appropriate grading. By a general result on weighted projective spaces [19], the coherent sheaf $\mathcal{O}_{\mathbb{WP}(\eta_0)}(a)$ is a line bundle if and only if the $\mathbb{C}^*$-weights all divide a. Note that, on the open set $\mathbb{WP}_{\mathrm{reg}}(\eta_0)$ of free $\mathbb{C}^*$-orbits, the vector bundle $\operatorname{ad}\Xi_0$ is defined, and in fact $R^0\pi_{2*}\operatorname{ad}\Xi_0$ and $R^1\pi_{2*}\operatorname{ad}\Xi_0$ are both locally free of rank r_c. Thus there is a well-defined line bundle $\left(\bigwedge^{r_c} R^0\pi_{2*}\operatorname{ad}\Xi_0\right) \otimes \left(\bigwedge^{r_c} R^1\pi_{2*}\operatorname{ad}\Xi_0\right)^{-1}$, and this line bundle clearly agrees with the restriction of $\mathcal{O}_{\mathbb{WP}(\eta_0)}(a)$ to $\mathbb{WP}_{\mathrm{reg}}(\eta_0)$. We next identify the integer a:

Lemma 5.4.1. *The natural $\mathbb{C}^*$-linearization on $\det R\pi_{2*}\operatorname{ad}\Xi_0$ corresponds to the line bundle $\mathcal{O}_{\mathbb{WP}(\eta_0)}(-2gn_{c,\alpha}/n_0)$.*

Proof. We must show that the $\mathbb{C}^*$-linearization on $\det R\pi_{2*}\operatorname{ad}\Xi_0$ is given by the character which is raising to the power $-2gn_{c,\alpha}/n_0$. To compute the $\mathbb{C}^*$-linearization, it suffices to compute the action of $\mathbb{C}^*$ on the fiber of $\det R\pi_{2*}\operatorname{ad}\Xi_0$ over the origin, which is a fixed point for the $\mathbb{C}^*$-action on $H^1(E; U(\eta_0))$. The fiber over 0 is canonically

$$\bigwedge^{\mathrm{top}} H^0(E; \operatorname{ad}_G(\eta_0 \times_L G)) \otimes \left(\bigwedge^{\mathrm{top}} H^1(E; \operatorname{ad}_G(\eta_0 \times_L G))\right)^{-1}.$$

Now, by Lemma 2.1.2 and Lemma 2.1.5,

$$H^0(E; \operatorname{ad}_G(\eta_0 \times_L G)) = H^0(E; \mathfrak{u}_-(\eta_0)) \oplus H^0(E; \operatorname{ad}_L \eta_0).$$

Since $\mathbb{C}^*$ is contained in the center of L, the action of $\mathbb{C}^*$ on $H^0(E; \operatorname{ad}_L \eta_0)$ is trivial. By Proposition 3.3.3 and Proposition 3.4.7, $\mathbb{C}^*$ acts on $H^0(E; \mathfrak{u}_-(\eta_0))$ with weights $-n_{c,\alpha}g_{\overline{\beta}}/n_0$. Thus the action on $\bigwedge^{\mathrm{top}} H^0(E; \operatorname{ad}_G(\eta_0 \times_L G))$ is via $-\sum_{\overline{\beta}} n_{c,\alpha}g_{\overline{\beta}}/n_0 = -gn_{c,\alpha}/n_0$. A similar argument (or duality) handles the case of the $\mathbb{C}^*$-action on $\bigwedge^{\mathrm{top}} H^1(E; \operatorname{ad}_G(\eta_0 \times_L G))$. Putting these together, we get the power $-2gn_{c,\alpha}/n_0$. □

Remark 5.4.2. In case $\mathbb{WP}(\eta_0)$ is the weighted projective space arising from the moduli space of G-bundles, where G is simply connected, one can use the Kodaira-Spencer map to check that $\mathbb{D}_1 = K^{\otimes 2}_{\mathbb{WP}(\eta_0)}$. Now, by a standard fact about weighted projective spaces, $K_{\mathbb{WP}(\eta_0)} = \mathcal{O}_{\mathbb{WP}(\eta_0)}(-g)$, and thus $K^{\otimes 2}_{\mathbb{WP}(\eta_0)} = \mathcal{O}_{\mathbb{WP}(\eta_0)}(-2g)$.

Define $\mathbb{D}_1 = \mathcal{O}_{\mathbb{WP}(\eta_0)}(-2gn_{c,\alpha}/n_0)$. We now compute the top self-intersection of $c_1(\mathbb{D}_1)$:

Lemma 5.4.3. *Let $\mathbb{WP}^r$ be the weighted projective space which is the quotient of $\mathbb{C}^{r+1} - \{0\}$ by the action of $\mathbb{C}^*$ acting with positive weights $w_0, \dots, w_r$, and let a be an integer such that $w_i|a$ for every i. Then*

$$\int_{\mathbb{WP}^r} c_1(\mathcal{O}_{\mathbb{WP}^r}(a))^r = a^r d/(w_0 \cdots w_r),$$

where $d = \gcd\{w_0, \dots, w_r\}$.

Proof. The morphism $\mathbb{C}^{r+1} \to \mathbb{C}^{r+1}$ defined by

$$(z_0, \dots, z_r) \mapsto (z_0^{w_0}, \dots, z_r^{w_r})$$

is $\mathbb{C}^*$-equivariant, where $\mathbb{C}^*$ acts with all weights equal to 1 on the domain and with weights w_i on the range. Thus there is an induced cover $f\colon \mathbb{P}^r \to \mathbb{WP}^r$, and it is easy to check that the degree of this cover is $w_0 \cdots w_r/d$. There is always a natural inclusion $f^*\mathcal{O}_{\mathbb{WP}^r}(a) \to \mathcal{O}_{\mathbb{P}^r}(a)$, and one checks that this inclusion is an isomorphism if $w_i|a$ for every a. In this case,

$$\int_{\mathbb{WP}^r} c_1(\mathcal{O}_{\mathbb{WP}^r}(a))^r = \int_{\mathbb{P}^r} c_1(\mathcal{O}_{\mathbb{P}^r}(a))^r/\deg f = a^r d/(w_0 \cdots w_r).$$

This proves the formula of Lemma 5.4.3. □

Corollary 5.4.4. $c_1(\mathbb{D}_1)^{r_c} = (-2g)^{r_c} n_0 / \prod_{\overline{\beta}} g_{\overline{\beta}}$.

Proof. This is immediate from Lemma 5.4.1 and Lemma 5.4.3, with $w_i = n_{c,\alpha} g_{\overline{\beta}}/n_0$, $a = -2gn_{c,\alpha}/n_0$, and $d = n_{c,\alpha}$. □

5.5 The divisor on the moduli space: the simply connected case

We turn now to the calculation of $c_1(\mathbb{D}_2)^r$. In order to make the argument easier to follow, we begin by working out the simply connected case. Recall that we have the finite morphism $E \otimes_{\mathbb{Z}} \Lambda \to \mathcal{M}(G,c)$. Let $\mathbb{D}_3$ be the pullback of $\mathbb{D}_2$ to a divisor on $E \otimes_{\mathbb{Z}} \Lambda$. Clearly, $c_1(\mathbb{D}_2)^r = c_1(\mathbb{D}_3)^r/\#(W)$. Thus, we shall begin by computing $c_1(\mathbb{D}_3)^r$. For a point $\rho \in E \otimes_{\mathbb{Z}} \Lambda$, there is an associated flat G-bundle ξ_0, and as we have seen in Lemma 3.1 of [10],

$$\operatorname{ad}\xi_0 \cong \mathcal{O}_E^r \oplus \bigoplus_{\alpha\in R} \lambda_{\alpha(\rho)},$$

where $\lambda_{\alpha(\rho)}$ is the line bundle of degree zero corresponding to the point $\alpha(\rho) \in E \cong \operatorname{Pic}^0 E$, or equivalently is the line bundle associated to the flat $U(1)$-bundle whose holonomy is given by $\alpha(\rho)$. In particular, we see that for $\lambda = \mathcal{O}_E$, the set D_λ defined in the discussion on determinant line bundles is all of $E \otimes_{\mathbb{Z}} \Lambda$, whereas for a nontrivial line bundle λ of degree zero, D_λ is a hypersurface in $E \otimes_{\mathbb{Z}} \Lambda$. In fact, D_λ is a union of distinct hypersurfaces $D_{\lambda,\alpha}$, where if λ corresponds to the point $e \in E$, then

$$D_{\lambda,\alpha} = \{\, \rho \in E \otimes_{\mathbb{Z}} \Lambda : \alpha(\rho) = -e \,\}.$$

Each $D_{\lambda,\alpha}$ is a union of translates of abelian subvarieties of $E \otimes_{\mathbb{Z}} \Lambda$. In particular, the hypersurface $D_{\lambda,\alpha}$ is smooth. The next lemma says that every component of $D_{\lambda,\alpha}$ counts with multiplicity one in the expression for $-c_1(\mathbb{D}_3)$.

Lemma 5.5.1. *For λ a nontrivial line bundle of degree zero,*

$$c_1(\mathbb{D}_3) = -\sum_{\alpha\in R} [D_{\lambda,\alpha}].$$

Proof. There is a universal G-bundle Ξ_1 over $E \times (E \otimes_{\mathbb{Z}} \Lambda)$, which in fact arises from a universal H-bundle, which we shall also denote by Ξ_1. One can describe Ξ_1 as follows. An H-bundle over $E \times (E \otimes_{\mathbb{Z}} \Lambda)$ is the same thing as an element of $\operatorname{Pic}(E \times (E \otimes_{\mathbb{Z}} \Lambda)) \otimes \Lambda$. The inclusion

$$\operatorname{Pic}(E \times E) \otimes \Lambda^* \to \operatorname{Pic}(E \times (E \otimes_{\mathbb{Z}} \Lambda))$$

induces an inclusion

$$\operatorname{Pic}(E\times E)\otimes\Lambda^*\otimes\Lambda = \operatorname{Pic}(E\times E)\otimes\operatorname{Hom}(\Lambda,\Lambda) \to \operatorname{Pic}(E\times(E\otimes_{\mathbb{Z}}\Lambda))\otimes\Lambda,$$

and we take the image of the element $\mathcal{P} \otimes \mathrm{Id}$. As vector bundles over $E \times (E \otimes_{\mathbb{Z}} \Lambda)$,

$$\operatorname{ad}\Xi_1 = \mathcal{O}^r_{E\times(E\otimes_{\mathbb{Z}}\Lambda)} \oplus \bigoplus_{\alpha\in R} \mathcal{P}_\alpha,$$

where $\mathcal{P}_\alpha$ is the pullback to $E\times(E\otimes_{\mathbb{Z}}\Lambda)$ of the Poincaré bundle $\mathcal{P}$ over $E\times E$, via the morphism induced from α from $E\otimes_{\mathbb{Z}}\Lambda$ to E. Thus, by functorial properties of determinant line bundles (cf. [9, Chapter 5, Proposition 3.8]), it will suffice to show that, over $E\times E$,

$$\det R\pi_{2*}(\mathcal{P}\otimes\pi_1^*\lambda)=\mathcal{O}_E(-e),$$

where as before $\lambda=\mathcal{O}_E(e-p_0)$. It is clear in any case that the inverse of $\det R\pi_{2*}(\mathcal{P}\otimes\pi_1^*\lambda)$ is represented by an effective divisor supported at e, and the only question is the length of $R^1\pi_{2*}(\mathcal{P}\otimes\pi_1^*\lambda)$. A standard calculation using the Grothendieck-Riemann-Roch theorem shows that this length is one (cf. [9, Chapter 7, Lemma 1.6] for the case of the trivial line bundle). □

Next we identify the divisor $\sum_{\alpha\in R}D_{\lambda,\alpha}$. Using the identifications

$$H^2(E\otimes_{\mathbb{Z}}\Lambda;\mathbb{Z})=\bigwedge^2 H^1(E\otimes_{\mathbb{Z}}\Lambda;\mathbb{Z})=\bigwedge^2\left(H^1(E;\mathbb{Z})\otimes\Lambda^*\right)$$

there is an inclusion

$$\bigwedge^2 H^1(E;\mathbb{Z})\otimes\operatorname{Sym}^2\Lambda^*\subseteq\bigwedge^2\left(H^1(E;\mathbb{Z})\otimes\Lambda^*\right),$$

and hence, since there is a canonical identification $\bigwedge^2 H^1(E;\mathbb{Z})\cong\mathbb{Z}$ there is a natural inclusion of $\operatorname{Sym}^2\Lambda^*$ in $H^2(E\otimes_{\mathbb{Z}}\Lambda;\mathbb{Z})$. Let $Q\in\operatorname{Sym}^2\Lambda^*$ be the quadratic form described in Section 1 defined by

$$Q=\sum_{\alpha\in R}\langle\alpha,\cdot\rangle\langle\alpha,\cdot\rangle,$$

and let $\widehat{Q}$ be the corresponding element of $H^2(E\otimes_{\mathbb{Z}}\Lambda;\mathbb{Z})$. By Lemma 1.1.2, $Q=(2g)I_0$, where I_0 is the unique W-invariant quadratic form on Λ such that $I_0(\widetilde{\alpha}^\vee,\widetilde{\alpha}^\vee)=2$.

Lemma 5.5.2. $-c_1(\mathbb{D}_3)=\widehat{Q}$.

Proof. By Lemma 5.5.1, it clearly suffices to show that, for every $\alpha\in R$, we have an equality (under the obvious identifications)

$$[D_{\lambda,\alpha}]=\langle\alpha,\cdot\rangle\langle\alpha,\cdot\rangle.$$

As such, this equality is a general fact about lattices Λ: suppose that Λ is a lattice and $\alpha\colon\Lambda\to\mathbb{Z}$ is a homomorphism. There is an associated morphism $E \otimes_{\mathbb{Z}} \Lambda \to E$ which we shall denote by e_α. We can define the divisor $D_\alpha = e_\alpha^*(p)$ for $p \in E$, as well as the cohomology class $\langle\alpha, \cdot\rangle\langle\alpha, \cdot\rangle \in \mathrm{Sym}^2 \Lambda^* \subset H^2(E; \mathbb{Z})$. To prove Lemma 5.5.2, it is enough to prove:

Claim 5.5.3. The class of the divisor $D_\alpha = e_\alpha^*(p)$ is equal to $\langle\alpha, \cdot\rangle\langle\alpha, \cdot\rangle$.

Proof of Claim 5.5.3. First assume that α is primitive. Then after a suitable choice of a basis of Λ we can assume that $\Lambda \cong \mathbb{Z}^r$ and that α is projection onto the last factor. In this case, $D_\alpha = E^{r-1} \times \{p\}$ and $\langle\alpha, \cdot\rangle\langle\alpha, \cdot\rangle = \pi_r^*(e \wedge f)$, where $e \wedge f$ is a positive generator of $H^2(E; \mathbb{Z}) \cong \mathbb{Z}$. Clearly, equality holds in this case. If α is not primitive, we can write $\alpha = n\alpha_0$, where n is a nonnegative integer and α_0 is primitive. In this case, e_α factors as the morphism α_0 followed by multiplication by n on E, and so D_α is cohomologous to n^2 copies of D_{α_0}. Likewise

$$\langle\alpha, \cdot\rangle\langle\alpha, \cdot\rangle = n^2\langle\alpha_0, \cdot\rangle\langle\alpha_0, \cdot\rangle,$$

and so the claim follows from the case where α is primitive. □

5.6 The divisor on the moduli space: the non-simply connected case

We now redo the above calculations to handle the non-simply connected case. We have the moduli space $\mathcal{M}(G, c)$ and the corresponding determinant line bundle as before, and we use the notation of §1.5. To make the calculation, we can pull the determinant line bundle up to the space $E \otimes \Lambda^{w_c} = T_0 \times T_0$, where there is a universal flat bundle. Let $\mathbb{D}_3$ be the class of the determinant line bundle pulled back to $E \otimes \Lambda^{w_c}$. As before, we have an inclusion $\mathrm{Sym}^2(\Lambda^{w_c})^* \to H^2(E \otimes \Lambda^{w_c}; \mathbb{Z})$. Let Q_0 be the element $(2g)(I_0|\Lambda^{w_c}) \in \mathrm{Sym}^2(\Lambda^{w_c})^*$ and let $\widehat{Q}_0$ be the corresponding element of $H^2(E \otimes \Lambda^{w_c}; \mathbb{Z})$. We have the following analogue of Lemma 5.5.2:

Lemma 5.6.1. $-c_1(\mathbb{D}_3) = \widehat{Q}_0$.

Proof. The proof is similar to that in the simply connected case for Lemmas 5.5.1 and 5.5.2, and we shall be a little sketchy. Suppose that ξ is a flat K-bundle corresponding to the c-pair (x, y). Let λ be a fixed, general line bundle of degree zero on E. We compute when ξ is in the support of $(R\det \pi_{2*}(\mathrm{ad}\, \Xi \otimes \pi_1^*\lambda))^{-1} = -\mathbb{D}_3$. As we have seen in

Lemma 4.5 of [10],

$$\operatorname{ad}\xi \cong (\mathfrak{h}^{w_c} \otimes \mathcal{O}_E) \oplus V_0' \oplus \bigoplus_{\mathbf{o}} (V_{y,\mathbf{o}} \otimes L_{x,\mathbf{o}}),$$

where the $\mathbf{o}$ are the orbits for the action of w_c on R. Here V_0' is a sum of certain torsion line bundles, $L_{x,\mathbf{o}}$ is a line bundle with holonomy $\alpha(x)$ for any fixed choice of $\alpha \in \mathbf{o}$, and $V_{y,\mathbf{o}}$ is the sum of the root spaces $\mathfrak{g}^{\alpha}, \alpha \in \mathbf{o}$, with the action defined by y. It follows that $V_{y,\mathbf{o}} \otimes L_{x,\mathbf{o}}$ is a direct sum of **distinct** line bundles of degree zero. Given a w_c-orbit $\mathbf{o}$, let $\alpha_{\mathbf{o}}$ be a choice of $\alpha \in \mathbf{o}$. Next we construct a universal bundle Ξ_1 as in the simply connected case, along the lines of [10, Lemma 5.21]. The construction outlined there shows that

$$\operatorname{ad}\Xi_1 = (\mathfrak{h}^{w_c} \otimes \mathcal{O}_{E\times(E\otimes\Lambda^{w_c})}) \oplus \pi_1^* V_0' \oplus \bigoplus_{\mathbf{o}} (V_{y_0,\mathbf{o}} \otimes L_{x_0,\mathbf{o}} \otimes \mathcal{P}_{\mathbf{o}}),$$

where $\pi_1 \colon E \times (E \otimes \Lambda^{w_c}) \to E$ is the projection onto the first factor, (x_0, y_0) is a fixed c-pair and, as in the simply connected case, $\mathcal{P}_{\mathbf{o}}$ is the pullback to $E \times (E \otimes \Lambda^{w_c})$ of the Poincaré bundle $\mathcal{P}$ on $E \times E$ via the morphism $E \otimes \Lambda^{w_c} \to E$ induced by $\alpha_{\mathbf{o}}$. The proof of Lemma 5.5.1 then shows that the divisor $\mathbb{D}_3$ is reduced.

For a general choice of λ, there exist c_x and c_y depending only on λ such that ξ is in the support of $-\mathbb{D}_3$ if and only if there exists an $\mathbf{o}$ such that

$$\alpha_{\mathbf{o}}(x) = c_x;$$
$$\sum_{\alpha\in\mathbf{o}} \alpha(y) = c_y.$$

Thus, in cohomology, $c_1(\mathbb{D}_3)$ corresponds to the element

$$\sum_{\mathbf{o}} \left(\alpha_{\mathbf{o}} \otimes \sum_{\alpha\in\mathbf{o}} \alpha \right) \in (\Lambda^{w_c})^* \otimes (\Lambda^{w_c})^*.$$

Now every $\alpha \in \mathbf{o}$ has the same restriction to Λ^{w_c} as $\alpha_{\mathbf{o}}$. Thus the above sum become

$$\sum_{\mathbf{o}} d_{\mathbf{o}}(\alpha_{\mathbf{o}} \otimes \alpha_{\mathbf{o}}),$$

where $d_{\mathbf{o}}$ is the order of $\mathbf{o}$. On the other hand, we have $(2g)I_0 = \sum_{\alpha\in R}(\alpha\otimes\alpha)$. Clearly, the restriction of this form to Λ^{w_c} is $\sum_{\mathbf{o}} d_{\mathbf{o}}(\alpha_{\mathbf{o}} \otimes \alpha_{\mathbf{o}})$, and this completes the proof of Lemma 5.6.1. □

5.7 Completion of the proof of Theorem 5.1.1

We have identified a divisor $\mathbb{D}_1$ on $\mathbb{WP}(\eta_0)$ and computed its top intersection. We have identified a divisor $\mathbb{D}_2$ on $\mathcal{M}(G,c)$, or rather its pullback to a divisor $\mathbb{D}_3$ on a finite cover of $\mathcal{M}(G,c)$. To find the top power of $\mathbb{D}_3$, we use the next lemma:

Lemma 5.7.1. *Let J be a quadratic form on Λ^*, and view J as an element of $H^2(E \otimes_{\mathbb{Z}} \Lambda; \mathbb{Z})$ via the inclusion $\operatorname{Sym}^2 \Lambda^* \subset H^2(E;\mathbb{Z})$. If the rank of Λ is r, then the top power of J is $(r!)\det J$.*

Proof. Let Ω be the 2-form corresponding to J. First suppose that J is diagonalizable with respect to some $\mathbb{Z}$-basis for Λ, corresponding to a given isomorphism $\Lambda \cong \mathbb{Z}^r$. Then Ω is of the form

$$\sum_{i=1}^{r} a_i[(e \wedge f) \otimes (\pi_i)^2,$$

where $J = \sum_{i=1}^r a_i(\pi_i)^2$, say, $\pi_i \colon \Lambda \cong \mathbb{Z}^r \to \mathbb{Z}$ is projection onto the i^{th} factor, and $e \wedge f$ is the positive generator for $H^2(E;\mathbb{Z})$. In this case, we can write $\Omega = \sum_{i=1}^r a_i(e_i \wedge f_i)$, where $e_i \wedge f_i$ is the generator on the i^{th} factor of $E \otimes_{\mathbb{Z}} \Lambda \cong E^r$. Clearly

$$\Omega^r = (r!)a_1 \cdots a_r = (r!)\det J.$$

In general, the statement makes sense for $\mathbb{Q}$-coefficients. Note that

$$\dim_{\mathbb{Q}} \bigwedge^{2r} \big(H^1(E;\mathbb{Q}) \otimes (\Lambda^* \otimes \mathbb{Q})\big) = 1,$$

and a basis element b is given by choosing the standard positive generator for $H^2(E \otimes_{\mathbb{Z}} \Lambda; \mathbb{Z})$, together with a $\mathbb{Z}$-basis for Λ. Changing the $\mathbb{Z}$-basis for Λ to some new $\mathbb{Q}$-basis changes the element b by $(\det X)^2$, where X is the change of basis matrix. In particular, if X has determinant 1, then b is unchanged. Now every quadratic form on Λ can be diagonalized via a $\mathbb{Q}$-basis such that the change of basis matrix relating the new $\mathbb{Q}$-basis to a $\mathbb{Z}$-basis has determinant 1. Thus we may reduce to the case where J is diagonalizable, where we have already checked the result. ∎

Corollary 5.7.2. $\int_{\mathcal{M}(G,c)} c_1(\mathbb{D}_2)^{r_c} = (-2g)^{r_c} n_0 \Big/ \prod_{\overline{\beta}} g_{\overline{\beta}}$.

Proof. Let e be the degree of the covering $T_0 \times T_0 \to \mathcal{M}(G,c)$. We see by Lemma 5.5.2 and Lemma 5.6.1 that it suffices to prove that

$$(2g)^{r_c} n_0 \Big/ \prod_{\overline{\beta}} g_{\overline{\beta}} = (2g)^{r_c}(r_c)! \det(I_0|\Lambda^{w_c})/e,$$

which we can rewrite as

$$e = \frac{(r_c)!}{n_0} \det(I_0|\Lambda^{w_c}) \prod_{\overline{\beta}} g_{\overline{\beta}}.$$

This is exactly the statement of Theorem 1.5.2. □

To complete the proof of Theorem 5.1.1, it suffices to show that $\Psi^*[\mathbb{D}_2] = [\mathbb{D}_1]$ in $H^2(\mathbb{WP}(\eta_0), \mathbb{Q})$. There is a Zariski open and dense subset $\mathcal{M}_0$ of $\mathcal{M}(G,c)$ consisting of semistable G-bundles for which the regular representative also carries a flat connection [10, Corollary 6.2]. Let $\widetilde{\mathcal{M}}_0$ be the preimage of this subset in $E \otimes \Lambda^{w_c} = \widetilde{\mathcal{M}}$.

Lemma 5.7.3. *Let $x \in \widetilde{\mathcal{M}}_0$, and let ξ_x be the corresponding G-bundle. Then the tautological bundle Ξ_1 constructed above identifies an analytic neighborhood of $x \in \widetilde{\mathcal{M}}_0$ with the local semi-universal deformation space of ξ_x, which is locally universal.*

Proof. By the definition of $\widetilde{\mathcal{M}}_0$, there is a unique representative up to isomorphism for the S-equivalence class of ξ_x and it is both flat and regular. Hence the map $H^1(E; \mathfrak{h}^{w_c} \otimes \mathcal{O}_E) \to H^1(E; \operatorname{ad}\xi_x)$ is an isomorphism. Then the tautological bundle Ξ_1 constructed above identifies an analytic neighborhood of $x \in \widetilde{\mathcal{M}}_0$ with the local semi-universal deformation space of ξ_x. Since $\operatorname{Lie}\operatorname{Aut}\xi_x = \mathfrak{h}^{w_c}$, it acts trivially on $H^1(E; \operatorname{ad}\xi_x)$ and thus the local semi-universal deformation of ξ_x is in fact locally universal. □

(See [10, Theorem 6.12] for a more general result along these lines.) We now claim:

Lemma 5.7.4. *Let X be irreducible, and let $\overline{\Xi} \to E \times X$ be an $\operatorname{ad} G$-bundle which lifts to a G-bundle on every slice $E \times \{x\}$. Let $f\colon X \to \mathcal{M}(G,c)$ be the corresponding morphism, and suppose that $f(X) \cap \mathcal{M}_0 \neq \emptyset$. Let $\mathbb{D}_X = \det R\pi_{2*} \operatorname{ad}\overline{\Xi}$. Then $[\mathbb{D}_X] = f^*[\mathbb{D}_2]$ in $H^2(X;\mathbb{Q})$.*

Proof. Choose a component $\widetilde{X}$ of the fiber product $X \times_{\mathcal{M}(G,c)} \widetilde{\mathcal{M}}$, and let $\tilde{f}\colon \widetilde{X} \to \widetilde{\mathcal{M}}$ be the induced map. It suffices to show that

$\widetilde{f}^*[\mathbb{D}_3] = [\mathbb{D}_{\widetilde{X}}]$ in the obvious notation. Since both sides are algebraic, it suffices to show the following: let $\Sigma \subseteq \widetilde{X}$ be an irreducible curve such that $\widetilde{f}(\Sigma) \cap \widetilde{\mathcal{M}}_0 \neq \emptyset$. Then $\widetilde{f}_*[\Sigma] \cdot \mathbb{D}_3 = \Sigma \cdot \mathbb{D}_{\widetilde{X}}$. Now choose a line bundle λ of degree zero on E such that, if div is the canonical section of $\det R\pi_{2*}(\operatorname{ad}\Xi_1 \otimes \pi_1^*\lambda)^{-1}$, then $\operatorname{div} \cap \widetilde{f}(\Sigma) \subseteq \widetilde{\mathcal{M}}_0$ is a finite set of points. Let $p \in \Sigma$ be a point such that $\widetilde{f}(p) \in \operatorname{div} \cap f(\Sigma)$. Since $\widetilde{f}(p) \in \widetilde{\mathcal{M}}_0$, it follows from Lemma 5.7.3 that, in an analytic neighborhood Ω of p, the bundle $\operatorname{ad}\overline{\Xi}$ is pulled back via $\widetilde{f}$ from $\operatorname{ad}\Xi_1$. It follows that the line bundle $\big(\det R\pi_{2*}(\operatorname{ad}\overline{\Xi} \otimes \pi_1^*\lambda)\big)^{-1}$ and its canonical section are also pulled back via $\widetilde{f}$. Thus $\widetilde{f}_*[\Sigma] \cdot \mathbb{D}_3 = \Sigma \cdot \mathbb{D}_{\widetilde{X}}$ as claimed. □

We cannot apply Lemma 5.7.4 directly to the morphism

$$\Psi\colon \mathbb{WP}(\eta_0) \to \mathcal{M}(G, c),$$

since there is no universal $\operatorname{ad} G$-bundle over $\mathbb{WP}(\eta_0)$. However, as in the proof of Lemma 5.4.3, there is a finite cover of $\mathbb{WP}(\eta_0)$ by a projective space $\mathbb{P}^{r_c}$ and an $\operatorname{ad} G$-bundle over $\mathbb{P}^{r_c}$. Let $\widehat{\Psi}\colon \mathbb{P}^{r_c} \to \mathcal{M}(G, c)$ be the induced morphism. It follows from the proof of Lemma 5.4.3 that $\mathcal{O}_{\mathbb{WP}(\eta_0)}(-2gn_{c,\alpha}/n_0)$ pulls back to $\mathcal{O}_{\mathbb{P}^{r_c}}(-2gn_{c,\alpha}/n_0)$ and that this line bundle is the determinant line bundle on $\mathbb{P}^{r_c}$. By Lemma 5.7.4, $\widehat{\Psi}^*[\mathbb{D}_2] = c_1(\mathcal{O}_{\mathbb{P}^{r_c}}(-2gn_{c,\alpha}/n_0))$, and hence $\Psi^*[\mathbb{D}_2] = [\mathbb{D}_1]$. Together with Corollary 5.7.2 and Corollary 5.4.4, this completes the proof of Theorem 5.1.1.

Appendix: Nonabelian cohomology

We discuss the general formalism, due to Grothendieck [15], for deciding when a principal bundle with structure group a quotient group can be lifted back to the full group, and for classifying all such liftings. In general, the set of liftings (suitably interpreted) is given by a nonabelian cohomology set. We then go on to discuss circumstances under which this cohomology set has a natural scheme structure, and indeed represents an appropriate functor. The arguments here are modifications of arguments due to Deligne and Babbitt-Varadarajan [4], given in a somewhat different context.

A.1 Lifting

We begin with a very general discussion of nonabelian cohomology and liftings of principal bundles. First suppose that G is a linear algebraic group and that ξ is a principal G-bundle over X, where X is a scheme or analytic space. Here it is understood that there is some topology for which ξ is locally trivial, for example, Zariski, étale, or classical, and cohomology will always be computed with respect to the appropriate topology. Throughout this paper, we have always worked with holomorphic bundles and the classical topology. One basic result is that, if X is projective, there is a natural bijection between the set of isomorphism classes of principal holomorphic G-bundle over X in the classical topology and the set of principal G-bundles over X in the étale topology. This follows from the method of proof of Prop. 20 in GAGA [22] and the fact that, if G is a closed subgroup of $GL(n)$, then the quotient $GL(n)/G$ is quasiprojective and admits local cross-sections in the étale topology. Thus the set of isomorphism classes of holomorphic principal G-bundles over X is canonically identified with the set of principal G-bundles over X in the étale topology. However, when we discuss representable functors below and try to put a scheme structure on various cohomology sets, it will be convenient to use the étale topology. For most of the paper, we have only considered the case where X is a smooth projective curve, and the issue of the correct topology is not important. Indeed, it follows from [24] that, if X is a smooth curve and G is linear, then a locally trivial G-bundle in the étale topology is actually Zariski locally trivial. However, we will not use this fact. One fact about cohomology which we shall need is the following: if X is a scheme and V is a coherent sheaf on X, then $H^i(X;V)$ computed for the étale topology is the usual sheaf cohomology computed in the Zariski topology [18], III (3.8). Of course, by GAGA, a similar statement holds in the classical topology for the analytic sheaf associated to V provided that X is projective.

If S is a scheme on which G acts, we can form the associated scheme $\xi \times_G S$ (not to be confused with fiber product). It is fibered over X and the fibers are isomorphic to S. Denote the sheaf (of sets) of cross sections (regular, holomorphic, or étale, depending on the context) of $\xi \times_G S$ by $S(\xi)$. We will usually be interested in the case where S is itself an algebraic group and where G acts on S by homomorphisms. In this case, $\xi \times_G S$ is a group scheme and $S(\xi)$ is a sheaf of (not necessarily abelian) groups. For example, if $S = G$ and the action is by conjugation, then $G(\xi)$ is the automorphism group scheme of ξ and its global sections are the group $\mathrm{Aut}_G\,\xi$. If S is a vector space and G acts on S linearly, then $S(\xi)$ is the vector bundle associated to the corresponding representation on G. Given an algebraic group G and a space X, the sheaf of morphisms from X to G will be denoted $\underline{G}$. Here $\underline{G}$ is a sheaf in the Zariski, étale, or classical topology, depending on the context.

Suppose that G is an algebraic group and that N is a closed normal subgroup. Let $H = G/N$, with $\pi\colon G \to H$ the induced morphism. Let X be a scheme, and let ξ_0 be a principal H-bundle over X. Suppose that ξ is a principal G-bundle lifting ξ_0. Note that G acts on N by conjugation, so that $N(\xi)$ is defined. If moreover N is abelian, then this action of G on N factors through an action of H on N, and so $N(\xi_0)$ is defined.

The group $H^0(X; H(\xi_0))$ acts on the cohomology set $H^1(X; N(\xi))$. We have the following general result [15] or [23]:

Lemma A.1.1. *With the above notation, the set of all principal G-bundles lifting ξ_0, or in other words the fiber of the class $[\xi_0] \in H^1(X; \underline{H})$ under the natural map $H^1(X; \underline{G}) \to H^1(X; \underline{H})$, may be identified with $H^1(X; N(\xi))/H^0(X; H(\xi_0))$.* □

We will also want a slight variant of the above:

Lemma A.1.2. *In the notation of Lemma A.1.1, the set of all isomorphism classes of pairs (η, φ), where η is a principal G-bundle and φ is an isomorphism from η/N to ξ_0, can be identified with $H^1(X; N(\xi))$.* □

Note that $H^0(X; H(\xi_0))$ is the group of global automorphisms of ξ_0, and this group acts naturally on the set of pairs (η, φ) as above. In fact, this action is the same as the coboundary action of $H^0(X; H(\xi_0))$ on $H^1(X; N(\xi))$.

Next, we ask the bundle ξ_0 lifts to an G-bundle. For example, suppose that $G = N \rtimes H$ is a semidirect product of N and H. Then there is a natural lift of ξ_0 given by the choice of an inclusion of H in G. In particular, the map $H^1(X; \underline{G}) \to H^1(X; \underline{H})$ is surjective. In this case, we can see the identification of Lemma A.1.1 quite explicitly: suppose that ξ_0 is defined by the cocycle $\{h_{ij}\}$ with respect to some open cover $\{U_i\}$ of X. Viewing the h_{ij} as taking values in G via the inclusion, it is easy to see that, if ξ is a G-bundle lifting ξ_0 on H, then we can assume that ξ is given by transition functions of the form $h_{ij}n_{ij}$. In order to be a 1-cocycle, the n_{ij} must satisfy

$$\left(h_{jk}^{-1} n_{ij} h_{jk}\right) n_{jk} = n_{ik},$$

which says that $\{n_{ij}\}$ defines an element of $H^1(X; N(\xi))$. If two such cocycles, say $\{h_{ij}n_{ij}\}$ and $\{h'_{ij}n'_{ij}\}$ define isomorphic G-bundles, then we can first arrange by a 1-coboundary that $h_{ij} = h'_{ij}$. In this case, if $\{h_{ij}n_{ij}\}$ and $\{h_{ij}n'_{ij}\}$ are cohomologous, then there exist h_i such that $h_i^{-1}h_{ij}h_j = h_{ij}$, so that $\{h_i\} \in H^0(X; H(\xi_0))$, and moreover $n'_{ij} = h_j^{-1}n_{ij}h_j$, so that the cocycles $\{n_{ij}\}$ and $\{n'_{ij}\}$ differ by the action of $H^0(X; H(\xi_0))$ on $H^1(X; N(\xi))$. Conversely, if $\{n_{ij}\}$ and $\{n'_{ij}\}$ differ by an element of $H^0(X; H(\xi_0))$, then reversing the above argument shows that the corresponding G-bundles are isomorphic.

For another example of the surjectivity of the map $H^1(X; \underline{G}) \to H^1(X; \underline{H})$, we have:

Lemma A.1.3. *Suppose that N is abelian and that $H^2(X; N(\xi_0)) = 0$, for example suppose that N is a vector space and that $\dim X = 1$. Then the bundle ξ_0 lifts to a G-bundle.*

Proof. This follows from [23], Corollary to Prop. 41, p. 70 or [15]. □

One trivial observation which we shall often use is the following:

Lemma A.1.4. *Suppose that we are given an exact sequence*

$$1 \to U \to G \to H \to 1,$$

and that ξ_0 is a principal H-bundle over X. Suppose that U_0 is a closed subgroup of the center of U which is normal in G. Finally suppose that $\widetilde{\xi_0}$ is a lift of ξ_0 to a principal G/U_0-bundle. Then the sheaf $U_0(\xi_0)$ defined by the natural action of H on U_0 is isomorphic to $U_0(\widetilde{\xi_0})$.

Proof. The groups H and G/U_0 act on U_0 by conjugation, and the action of G/U_0 factors through the projection to H. Thus, the sheaves $U_0(\xi_0)$ and $U_0(\widetilde{\xi_0})$ are identified as well. □

A.2 Representability

Let $G = LU$, where U is a normal subgroup of G and G is a semidirect product of U and L. Let ξ_0 be a principal L-bundle. We want to find some circumstances under which the points of the cohomology set $H^1(X; U(\xi_0))$ can be identified with the points in an affine space. In fact, it is important to prove a much stronger statement, that a certain functor corresponding to the cohomology set is representable by an affine space. For example, if U is a vector space on which L acts linearly, then $H^1(X; U(\xi_0))$ is an ordinary sheaf cohomology group and thus is itself a vector space, and the corresponding affine space represents a functor. We will encounter nonabelian groups U which are unipotent. Thus, U has a filtration $U_N \subset \cdots \subset U_1 = U$ by normal, L-invariant subgroups U_i with the property that U_i is contained in the inverse image in U of the center of U/U_{i+1}. The idea then, following the general lines of [4], will be to work inductively, starting with the case where U is a vector group. The inductive step depends on the following ([4], Lemma 2.5.3):

Lemma A.2.1. *Let R be a ring. Suppose that $\mathbf{F}$ and $\mathbf{G}$ are two covariant functors from the category of R-algebras to sets and that $\varphi\colon \mathbf{F} \to \mathbf{G}$ is a morphism of functors, with the following property:*

(i) *$\mathbf{G}$ is represented by a polynomial algebra $R[x_1, \dots, x_n]$ over the ring R.*

(ii) *For every R-algebra S and for every object $\xi \in \mathbf{G}(S)$, the functor $\mathbf{F}_{\varphi,\xi}$ from S-algebras to sets defined by*

$$T \mapsto \varphi(T)^{-1}(\xi'),$$

where ξ' is the element of $\mathbf{G}(T)$ induced by ξ, is represented by a polynomial algebra $S[y_1, \dots, y_m]$ over S.

Then the functor $\mathbf{F}$ is represented by $R[x_1, \dots, x_n, y_1, \dots, y_m]$.

Proof. Let $S = R[x_1, \dots, x_n]$ and let $\xi \in \mathbf{G}(S)$ correspond to the identity in $\mathrm{Hom}_R(S, S)$. For an R-algebra T, if $\eta \in \mathbf{F}(T)$, let $\xi' = \varphi(T)(\eta)$. Then there exists a unique homomorphism $f\colon S \to T$ corresponding to $\xi' \in \mathbf{G}(T)$, so that T is an S-algebra. Now ξ' is the image of $\xi \in \mathbf{G}(S)$ under f_*, and the element $\eta \in \varphi(T)^{-1}(\xi')$ defines a unique homomorphism $S[y_1, \dots, y_m] \to T$. Thus η defines a unique homomorphism $S[y_1, \dots, y_m] = R[x_1, \dots, x_n, y_1, \dots, y_m] \to T$. Conversely, suppose given a homomorphism

$$f\colon R[x_1, \dots, x_n, y_1, \dots, y_m] = S[y_1, \dots, y_m] \to T.$$

Then f induces a homomorphism $R[x_1, \dots, x_n] = S \to T$ and thus an element $\xi' \in \mathbf{G}(T)$ induced by ξ, and the homomorphism f then gives an element $\eta \in \mathbf{F}(T)$ mapping to ξ'. Clearly these are inverse constructions. It follows that $\mathbf{F}$ is represented by $R[x_1, \dots, x_n, y_1, \dots, y_m]$. □

The proof shows more generally that, if $\mathbf{G}$ is represented by some R-algebra S, and, for ξ the object of $\mathbf{G}(S)$ corresponding to the identity, if the functor from S-algebras to sets defined by

$$T \mapsto \varphi(T)^{-1}(\xi'),$$

where ξ' is the element of $\mathbf{G}(T)$ induced by ξ, is represented by an S-algebra $\widetilde{S}$, then $\widetilde{S}$ represents $\mathbf{F}$.

We shall apply Lemma A.2.1 as follows. First let us define the functor $\mathbf{F}$ from $\mathbb{C}$-algebras to sets corresponding to the group $H^1(X; U(\xi_0))$. For a $\mathbb{C}$-algebra S, let $\mathbf{F}(S)$ be the set of isomorphism classes of pairs (Ξ, Φ), where Ξ is a principal LU-bundle over $X \times \operatorname{Spec} S$ and $\Phi : \Xi/U \to \pi_1^*\xi_0$ is an isomorphism from the principal L-bundle over $X \times \operatorname{Spec} S$ induced by Ξ to the pulled back bundle $\pi_1^*\xi_0$. Thus,

$$\mathbf{F}(S) = H^1(X \times \operatorname{Spec} S; U(\pi_1^*(\xi_0)).$$

Theorem A.2.2. *Let $G = LU$ be an algebraic group over $\mathbb{C}$, where U is a closed normal unipotent subgroup of G and G is isomorphic to the semidirect product of L and U. Let X be a projective scheme, let ξ_0 be a principal L-bundle over X, and let $U(\xi_0)$ be the corresponding sheaf of unipotent groups. Let $\{U_i\}_{i=1}^N$ be a decreasing filtration of U by normal L-invariant subgroups such that, for every i, U_i/U_{i+1} is contained in the center of U/U_{i+1}. Suppose that, for every i,*

$$H^0(X; (U_i/U_{i+1})(\xi_0)) = H^2(X; (U_i/U_{i+1})(\xi_0)) = 0.$$

Then:

(i) *The cohomology set $H^1(X; U(\xi_0))$ has the structure of affine n-space $\mathbb{A}^n$. More precisely, there is a G-bundle Ξ_0 over $X \times \mathbb{A}^n$ and an isomorphism $\Phi_0 \colon \Xi_0/U \to \pi_1^*\xi_0$ such that the pair (Ξ_0, Φ_0) represents the functor $\mathbf{F}$ defined above.*

(ii) *There is a natural action of the algebraic group $\operatorname{Aut}_L \xi_0$ on the affine n-space $\mathbb{A}^n$ representing $H^1(X; U(\xi_0))$. This action lifts to an action on Ξ_0.*

Proof. We claim that the functor $\mathbf{F}$ is representable by an affine space. The proof is by induction on the length of the filtration $\{U_i\}$. If this length is zero, then $U = \{0\}$ and there is nothing to prove. Suppose that the claim has been verified for every group and filtration of length less than N, and let $\{U_i\}$ be a filtration of U length exactly N satisfying the hypotheses of Theorem A.2.2. If U_N is the first term in $\{U_i\}$, then the filtration $\{U_i/U_N\}$ of U/U_N has length $N-1$. By induction, the functor $\mathbf{G}$ corresponding to LU/U_N is representable. Moreover, there is an obvious morphism of functors $\varphi\colon \mathbf{F} \to \mathbf{G}$. Now suppose that S is a $\mathbb{C}$-algebra and that we are given an object ξ of $\mathbf{G}(S)$, in other words a pair (Ξ, Φ), where Ξ is a principal LU/U_N-bundle over $X \times \operatorname{Spec} S$ and $\Phi : \Xi/U \to \pi_1^*\xi_0$ is an isomorphism from the principal L-bundle over $X \times \operatorname{Spec} S$ induced by Ξ to $\pi_1^*\xi_0$. We define the functor $\mathbf{F}_{\varphi,\xi}$ on S-algebras T as follows: $\mathbf{F}_{\varphi,\xi}(T)$ consists of isomorphism classes of pairs (P, Ψ), such that P is a principal LU-bundle over $X \times \operatorname{Spec} T$ and Ψ is an isomorphism from the principal LU/U_N-bundle over $X \times \operatorname{Spec} T$ induced by P to the pullback $\widetilde{\Xi} = (\operatorname{Id} \times f)^*\Xi$ of Ξ to $X \times \operatorname{Spec} T$, where $f\colon \operatorname{Spec} T \to \operatorname{Spec} S$ is the morphism corresponding to the homomorphism $S \to T$. There is a natural map $\mathbf{F}_{\varphi,\xi}(T) \to \varphi(T)^{-1}(\widetilde{\Xi}, (\operatorname{Id} \times f)^*\Phi) \subseteq \mathbf{F}(T)$. First we claim:

Claim A.2.3. There exists a a principal LU-bundle P over $X \times \operatorname{Spec} T$ lifting the principal LU/U_N-bundle $\widetilde{\Xi} = (\operatorname{Id} \times f)^*\Xi$ over $X \times \operatorname{Spec} T$. In other words, for all T, $\mathbf{F}_{\varphi,\xi}(T) \neq \emptyset$.

Proof. By Lemma A.1.3, the obstruction to finding such a lift lives in the group $H^2(X \times \operatorname{Spec} T; U_N(\widetilde{\Xi}))$. By Lemma A.1.4, we can identify $U_N(\widetilde{\Xi})$ with the sheaf $U_N(\pi_1^*\xi_0)$, which is the pullback via π_1^* of the vector bundle $V = U_N(\xi_0)$ on X. Now since $\operatorname{Spec} T$ is affine,

$$H^2(X \times \operatorname{Spec} T; U_N(\widetilde{\Xi})) = H^2(X \times \operatorname{Spec} T; \pi_1^*V) = H^2(X; V) \otimes_{\mathbb{C}} T.$$

Since $H^2(X;V) = 0$ by hypothesis, we can lift $\widetilde{\Xi}$ to a bundle P. □

Once we know that there exists one lift P as in the claim, it follows from Lemma A.1.2 that the set of all such pairs (P, Ψ) is classified by $H^1(X \times \operatorname{Spec} T; U_N(P))$. Next we claim that the map $\mathbf{F}_{\varphi,\xi}(T) \to \mathbf{F}(T)$ is injective, and thus identifies $\mathbf{F}_{\varphi,\xi}(T)$ with the fiber $\varphi(T)^{-1}(\widetilde{\Xi}, (\operatorname{Id} \times f)^*\Phi)$. To see this, it follows from the general formalism of nonabelian cohomology that there is a transitive action of $H^0(X \times \operatorname{Spec} T; U/U_N(P))$ on the fibers of the map from $\mathbf{F}_{\varphi,\xi}(T) = H^1(X \times \operatorname{Spec} T; U_N(P))$ to $\mathbf{F}(T) = H^1(X \times \operatorname{Spec} T; U(\pi_1^*\xi_0))$, which identifies the fibers with the coset space

$$H^0(X \times \operatorname{Spec} T; U/U_N(P))/\operatorname{Im} H^0(X \times \operatorname{Spec} T; U(P)).$$

In our case, the unipotent group $H^0(X \times \operatorname{Spec} T; U/U_N(P))$ is filtered, with successive quotients contained in

$$H^0(X \times \operatorname{Spec} T; (U_i/U_{i-1})(\pi_1^*\xi_0)) = 0.$$

Hence $\mathbf{F}_{\varphi,\xi}(T) \to \mathbf{F}(T)$ is injective.

Using Lemma A.1.4 and the fact that $\operatorname{Spec} T$ is affine,

$$H^1(X \times \operatorname{Spec} T; U_N(P)) = H^1(X \times \operatorname{Spec} T; \pi_1^*V) = H^1(X; V) \otimes_{\mathbb{C}} T.$$

If $e_1, \dots, e_n$ is a basis for $H^1(X; V)$, with dual basis $x_1, \dots, x_n$, this says that $\mathbf{F}_{\varphi,\xi}(T) \cong \operatorname{Hom}_S(S[x_1, \dots, x_n], T)$. Thus $\mathbf{F}_{\varphi,\xi}$ is representable by an affine space over $\operatorname{Spec} S$. Applying Lemma A.2.1 and induction, the functor $\mathbf{F}$ is then representable by an affine space, in other words there exists an affine space $\mathbb{A}^n = \operatorname{Spec} \mathbb{C}[x_1, \dots, x_n]$ and an object $(\Xi_0, \Phi_0) \in \mathbf{F}(\mathbb{C}[x_1, \dots, x_n])$ such that the couple $(\mathbb{C}[x_1, \dots, x_n], (\Xi_0, \Phi_0))$ represents $\mathbf{F}$.

The fact that the functor $\mathbf{F}$ defined above is representable implies in particular that there is a universal LU-bundle over $X \times \mathbb{A}^n$, where $\mathbb{A}^n$ is the affine space representing $\mathbf{F}$. A formal reduction also shows that obvious extension of $\mathbf{F}$ to a functor from schemes of finite type over $\mathbb{C}$ to sets is also representable by $\mathbb{A}^n$.

Lastly we must analyze the action of $\operatorname{Aut}_L \xi_0$ on $H^1(X; U(\xi_0))$.

Proposition A.2.4. *Suppose that X is proper and that L is a linear algebraic group. Then $\operatorname{Aut}_L \xi_0$ is also a linear algebraic group, and the natural set-theoretic action of $\operatorname{Aut}_L \xi_0$ on $H^1(X; U(\xi_0))$ is an algebraic action. Moreover, this action lifts to an action on Ξ_0.*

Proof. Let $\mathbf{A}$ be the functor from $\mathbb{C}$-algebras to groups corresponding to $\operatorname{Aut}_L \xi_0$: for a $\mathbb{C}$-algebra S, $\mathbf{A}(S)$ is the group of automorphisms of the pullback of ξ_0 to $X \times \operatorname{Spec} S$. Since X is proper and L is affine, $\mathbf{A}$ is representable by a linear algebraic group scheme $\operatorname{Aut}_L \xi_0$ over $\mathbb{C}$. To see this, first assume that $L = GL_n$. If V is the vector bundle corresponding to ξ_0, $\operatorname{Aut}_L \xi_0$ is an affine open subset of the affine space $H^0(X; End\, V)$ and we claim that this linear algebraic group represents the associated functor. Indeed, an automorphism of π_1^*V is the same thing as a section of $\pi_1^* End\, V$, in other words a $\operatorname{Spec} S$-valued point φ of $H^0(X; End\, V)$, such that the determinant of φ is an invertible element of S, and this is the same thing as a morphism from S to the Zariski open subset of $H^0(X; End\, V)$ consisting of elements with nonzero determinant. In general, choose an embedding of L as a closed subgroup of

GL_n for some n, defined by polynomials f_i. It is then straightforward to verify that $\mathbf{A}$ is representable by a closed subgroup of the corresponding group scheme for GL_n.

If $\mathbf{F}$ is the functor associated to $H^1(X; U(\xi_0))$, there is an obvious morphism of functors $\mathbf{A} \times \mathbf{F} \to \mathbf{F}$: if the points of $\mathbf{F}(S)$ correspond to pairs (Ξ, Φ), where Φ is an isomorphism from the principal L-bundle over $X \times \operatorname{Spec} S$ induced by Ξ to $\pi_1^*\xi_0$, then the automorphisms of $\pi_1^*\xi_0$ act by composition with Φ. Since $\mathbf{A}$ and $\mathbf{F}$ are representable by $\operatorname{Aut}_L \xi_0$ and by the affine coordinate ring of $H^1(X; U(\xi_0))$ respectively, there is a corresponding morphism $\operatorname{Aut}_L \xi_0 \times H^1(X; U(\xi_0)) \to H^1(X; U(\xi_0))$, which is easily checked to give an action. It again follows formally by representability that this action lifts to an action on Ξ_0. □

This concludes the proof of Theorem A.2.2. □

Remark A.2.5. In the hypotheses of Theorem A.2.2, suppose we only assume that, for all i, $H^2(X; (U_i/U_{i+1})(\xi_0)) = 0$ and that, for all $i > 1$, $H^1(X; (U_i/U_{i+1})(\xi_0)) = 0$. Then, in the inductive construction of the proof, the fibers $\mathbf{F}_{\varphi,\xi}(T)$ are all a single point and thus the map $\mathbf{F}_{\varphi,\xi}(T) \to \mathbf{F}(T)$ is automatically injective. Thus the proof goes through in this case as well.

There is also a relative version of Theorem A.2.2.

Theorem A.2.6. *Let $G = LU$ and the filtration $\{U_i\}$ be as in Theorem A.2.2. Suppose that $\pi\colon Z \to \operatorname{Spec} R$ is a flat proper morphism, and that ξ_0 is a principal L-bundle over Z, with $V_i = (U_i/U_{i+1})(\xi_0)$ the vector bundle associated to the action of L on U_i/U_{i+1} and the principal L-bundle ξ_0. Suppose that $H^2(\pi^{-1}(t); V_i|\pi^{-1}(t)) = 0$ for every point $t \in X$, that $H^0(\pi^{-1}(t); V_i|\pi^{-1}(t)) = 0$ for every point $t \in X$, and that the R-module $H^1(Z; V_i)$ is locally free and compatible with base change, in the sense that, for every homomorphism $R \to S$ of $\mathbb{C}$-algebras, with corresponding morphism $f\colon \operatorname{Spec} S \to \operatorname{Spec} R$, we have*

$$H^1(Z \times_{\operatorname{Spec} R} \operatorname{Spec} S; f^*V_i) \cong H^1(Z; V_i) \otimes_R S.$$

For example, if π has relative dimension one and, for every i, $\dim H^1(\pi^{-1}(t); V_i|\pi^{-1}(t))$ is independent of $t \in \operatorname{Spec} R$, then $H^1(Z; V_i)$ is locally free and compatible with base change in the above sense. Then

(i) *There exists a locally trivial bundle of affine spaces $\mathbb{A}$ over $\operatorname{Spec} R$, such that the set of sections of $\mathbb{A}$ is isomorphic to the set $H^1(Z; U(\xi_0))$.*

(ii) *There exists a universal bundle* Ξ *over* $Z \times_{\operatorname{Spec} R} \mathbb{A}$ *in the obvious sense.*

(iii) *The automorphism group scheme* $\mathcal{A}$ *of* ξ_0 *acts on the bundle* $\mathbb{A}$ *of affine spaces over* $\operatorname{Spec} R$*, and this action lifts to an action on* Ξ*.*

□

Remark A.2.7. One can also replace $\operatorname{Spec} R$ in the above statements by a scheme of finite type over $\mathbb{C}$. Moreover, in case $R = \mathbb{C}[t_1, \dots, t_j]$, or more generally if every vector bundle over $\operatorname{Spec} R$ is trivial, then the inductive proof of Theorem A.2.2 shows that we can take $\mathbb{A} = \operatorname{Spec} R[x_1, \dots, x_n]$.

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