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FAMILY BLOWUP FORMULA, ADMISSIBLE GRAPHS AND THE ENUMERATION OF SINGULAR CURVES, I

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Abstract

In this paper, we discuss the scheme of enumerating the singular holomorphic curves in a linear system on an algebraic surface. Our approach is based on the usage of the family Seiberg-Witten invariant and tools from differential topology and algebraic geometry.

In particular, one shows that the number of δ-nodes nodal curves in a generic δ dimensional sub-linear system can be expressed as a universal degree δ polynomial in terms of the four basic numerical invariants of the linear system and the algebraic surface. The result enables us to study in detail the structure of these enumerative invariants.

1. Introduction

Counting the numbers of nodal curves of a linear system on an algebraic surface is a well-recognized problem in enumerative geometry. Counting nodal curves in $\mathbf{CP}^2$, classically known as the Severi degree, attracts the attention from a lot of algebraic geometers. Recently, recursive formulas for the Severi degrees were derived by Z. Ran [Ran] and later by Harris-Caporaso [7]. Both groups had used methods in algebraic geometry. Inspired by the work of several physicists, S.T. Yau and E. Zaslow made the Yau-Zaslow conjecture regarding the modularity of the generating function of nodal curves on an algebraic $K3$ surface [63].

Despite the diverse interest in this type of problem and approaches from different perspectives, there still lacks systematic understanding of the general phenomenon on a general algebraic surface.

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In this paper, I would like to develop an unified theory for counting nodal curves on a general algebraic surface. More specifically, I would extend my discussion to include counting singular curves of a fixed topological type (for the precise definition, see Section 5). As will be demonstrated in the process of discussion, the counting of nodal curves becomes a rather special case of the main machinery. In anticipation of a sequel to the present paper, I will discuss the generalizations and extensions of the results to higher dimensions.

For the counting of nodal curves, the following two main theorems will be presented:

Main Theorem 1.1. *Let M be an algebraic surface. Let L be a sufficiently very ample line bundle on M. Let the number $n_L(\delta)$ denote the number of δ-nodes nodal curves in the linear system $|L|$, then the number can be expressed as an universal polynomial in terms of $L \cdot L$, $L \cdot c_1(M)$, $c_1(M)^2$ and $c_2(M)$.*

By combining the perturbation argument with Göttsche's argument, Main Theorem 1.1 has an effective version, that is, by taking L to be $(3\delta - 1)$-very ample, the n_L can be understood in the classical sense.

If one does not adopt the technique from differential topology, it is sufficient to take L to be $(3\delta - 1)$-very ample and the main theorem holds for algebraic closed fields of characteristic zero.

The $(3\delta - 1)$-very ampleness result may not be the strongest effective version when one applies to the particular algebraic surface. Later, in my solution of the Di Francesco-Itzykson conjecture of $\mathbf{CP}^2$, it shows that the optimal L is $\left(\left[\frac{\delta+2}{2}\right] + 1\right)$-very ample.

In this paper, I would like to focus upon Kähler surfaces from both angles of differential topology and algebraic geometry. The corresponding theory of symplectic four-manifolds will be attempted in a separate article.

In my nomenclature, the number n_L will be interpreted as the nodal curve invariant (or modified family invariant) discussed below. The phrase, "number of curves" can be better understood as virtual numbers or "equivalence." As will be briefly addressed, it also represents the actual number of pseudo-holomorphic curves when L is a high power of an ample line bundle.

Subsequently, a second main theorem will be devoted to singular curves of a fixed topological type:

Main Theorem 1.2. *Let M be a Kähler surface and L be a sufficiently very ample divisor on M. Let $n_L(\Gamma, L - \sum m_i E_i)$ denote the "number of singular curves" in a generic $\sum \frac{m_i^2+m_i-4}{2}$ dimensional linear subsystem of $|L|$ with a fixed topological type of plane curve singularities specified by Γ and m_i, etc.*

Then the virtual number can be expressed as an universal polynomial in $L \cdot L, L \cdot c_1(M), c_1^2(M)$ and $c_2(M)$. The universal polynomial depends explicitly on Γ, the admissible graph, and m_i, the multiplicities of the singularities.

For the definition of "admissible graph" and "topological type of a singular curve", please refer to Section 4.2 and Section 5.

In an earlier communication, the main theorem of counting nodal curve was formulated for $p_g = 0$ Kähler surfaces. Later the author devised a way of defining the algebraic family Seiberg-Witten "invariants," which coincide with the usual family Seiberg-Witten invariants [29] when the geometric genus p_g vanishes. This object has the benefit of counting directly from algebraic geometry, yet it is an actual topological invariant only when $p_g = 0$. Nevertheless, it enjoys the same family blowup formula and family switching formula as were discussed in [38], [39] in detail.

The key insight of the previous paper [36] was that there exists a correspondence between $p_g = 0$ Kähler surfaces and $b_2^+ = 1$ symplectic four-manifolds. Likewise, the $p_g > 0$ Kähler surfaces are related to $b_2^+ > 1$ symplectic four-manifolds. For the purpose of discussion here, I will adopt two versions of family Seiberg-Witten invariants. The first is the topological family Seiberg-Witten invariants defined in [29], using the technique of differential topology. Secondly, there is another version of "invariant" namely the "algebraic family Seiberg-Witten invariants," which can be defined for algebraic closed fields with $char(\mathbf{k}) = 0$ as well. These two versions of "invariants" are equal to each other only when p_g is equal to zero. By adopting the second version of invariants, the proof of the main theorems can be extended to the category of algebraic surfaces over an algebraic closed field $\mathbf{k}$ with $char(\mathbf{k}) = 0$.

Given an integral second cohomology class on a Kähler (symplectic) surface, it is interesting to ask whether it can be represented by (pseudo)-holomorphic curves. One can calculate its expected genus through the usual adjunction equality. The special role of the nodal curves in Gromov-Ruan-Tian theory is that the curves counted by Gromov-Ruan-Tian theory [21], [48] are nodal under a generic perturbation of almost complex structures. My result can be also interpreted as the counting of Gromov-Ruan-Tian invariants.

Be aware that even though statements of the main theorems have been phrased completely in terms of algebraic geometry, the foundation of the main theorems lie in using the novel concept of the family Seiberg-Witten invariants (and the family Gromov-Taubes invariants) previously discussed and studied in a different paper [29].

As recent as the past decade, serious attempts have been directed at the universal solution for counting nodal curves on a general algebraic surface.

The first progress was made by Vainsencher [59], [24], who determined and computed the universal formula for $n \leq 6$ (the number n stands for the number of nodes).

Recently, a further attempt was made by S. Kleiman and R. Piene [24] to generalize to $n \leq 7, 8$ cases. Given the general belief that the above theorem should be true [24], Göttsche [19] formulated it as a conjecture and gave a precise conjecture on what the universal formula should look like. The author has also conjectured the existence of the universal formula independently.

On the other hand, following several physicists' work, S. T. Yau and Zaslow [63] formulated the conjecture regarding the number of nodal curves in a linear system of $K3$. It gave the present author and T. J. Li a strong motivation (1995–1997) to study this problem along the line of Seiberg-Witten theory and Gromov theory. The relationship between nodal curves on $K3$ and families was discussed extensively by Yau-Zaslow [63]. The concept of family Gromov Invariants was discussed in Ruan's paper [46] to define the equivariant Gromov Invariants.

The Yau-Zaslow question was one of the motivations for T. J. Li and the present author [29] to develop the theory of family Seiberg-Witten invariants. The concept of family Seiberg-Witten invariants was first introduced by Professor S. T. Yau soon after the formulation of Seiberg-Witten theory.

Later the concept was also addressed by S. K. Donaldson [9] and G. Tian [58] et al. The author also appreciates their inspiration.

In the aforementioned paper [29], this author derived a version of the family wall crossing formula to discuss the nodal curve counting. Without knowing Vainsencher's work [59], [24] during that time, it was proposed that one could use the Fulton-McPherson spaces [17] to construct a sequence of fiber bundles in handling the counting of nodal curves.

In retrospect, philosophically the family invariants [29] could be relevant to the nodal curve counting, it was not very clear how to produce the precise argument due to various conceptual difficulties during spring 1997 [29].

The purpose of the present paper is to address these points and to address the possible applications of the new technique. The key progress was made in January–February, 1998, after a detailed study of the scheme generalizing McDuff's proposal of the ordinary Seiberg-Witten-Gromov-Taubes theory [37]. A number of ideas and significant insights were gained when the author was a member of the Institute for Advanced Study (1996–1997) and at Park city (IAS) in the summer of 1997. The author would like to express special thanks for the possible inspiration that T. J. Li had provided during the preparation of the earlier work [29].

The author wants to thank Prof. Shing-Tung Yau for his unreserved support and advice. His joint work with E. Zaslow and B. Lian have been the central inspiration for this piece of work.

The author also would like to make note of assistance from Prof. Taubes, who kindly listened to the author's reports and also for his long-time encouragement. Without being inspired by his machinery [51], [52], [53], it would have been impossible to finish this work.

The author thanks S. Kleiman and Piene for informing him their explicit formulas for $n \leq 8$.

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Even though the Fulton-MacPherson space [17] will not be explicitly used in the present paper, I still want to point out the indirect influence of Fulton-MacPhersons' paper [17] upon the scheme I adopt. The observation between the symmetry group $G_\Gamma \subset \mathbf{S}_n$ and the strata was borrowed from an old idea of the author when the Fulton-McPherson spaces were chosen to be the base manifolds [29].

The organization of the paper is as the following: in §2, I survey the main tools used in the paper briefly. For details of the family Seiberg-Witten theory, one can consult the paper [29]. The concept of algebraic family Seiberg-Witten "invariant" will be only briefly addressed here. The details can be found in the paper [38]. In the same section, I state (without proof) the family blowup formula and the family switching formula [38], [39]. As these two formulas have other geometric applications beyond the scope of the present paper, they are discussed in two individual papers [38], [39]. In §3, I study the sequence of manifolds M_n, which will be the base spaces of the fiber bundles. Their cohomology rings and several useful structures will be reviewed. The cohomology rings of M_n are well known to algebraic geometers. I survey it merely for the sake of completeness. The sequence of manifolds will be called the "Universal spaces" informally.

In §4, I survey the algebraic family scheme regarding curve counting in details. The concept of admissible decomposition classes of a cohomology class will be introduced briefly. The admissible decompositions pick up the possible contribution to the total algebraic family invariants other than the smooth curve representatives expected from Taubes' theory [51], [52], [53]. Instead of giving a fully abstract formulation, I will specify to the admissible decomposition classes involved in the family over M_n from time to time.

In §4, I introduce the concept of admissible decomposition classes. Then I specify to the so called type I exceptional curves. As a foundation of the theory of type I exceptional curves, I introduce the concept of admissible graphs and use them in parameterizing the special stratification of M_n. It turns out that the geometric properties regarding M_n can be translated into the combinatorial properties of the graphs in a neat way. Because the stratification, along with the admissible graphs, will play a crucial role in the enumeration problem discussed later, I spend a greater amount of time going through the combinatorial theory. I also study the relationship between the stratifications of M_n and the automorphism group $G_\Gamma \subset S_n$, which acts on the various strata. Even though the material in this section is not particularly difficult, it provides a conceptual link between the various notions developed in this paper which was probably missed elsewhere.

In short, this paper gives the M_n structures similar to Schubert cycles in the flag manifolds; where the strata bijectively correspond to Schubert cells, while the admissible graphs correspond to the Young diagrams. One of my discoveries is that, deeper meaning of the adjective "universal," used ubiquitously in Theorems 1.1 and 1.2 takes root in the fact that the spaces M_n are actually universal objects with an intersection theory parallel to a "Schubert"-like calculus on them.

In §5, I introduce the concept of "topological type" of a singular (pseudo)-holomorphic curve. Using this concept, the topological data can be extracted from the curve singularities. The word "topological type" of a plane curve singularity is not typically used in the standard text. However, it is possible to show that the different definitions coincide [3].

Another concept—"the core of a topological type" will be introduced. This combinatorial concept relates implicitly to the standard resolutions in singular curve resolution theory [3]. It certainly plays an important role in the explicit enumerations of the invariants, as will be shown explicitly near the end of the paper.

Nevertheless, I will not address the full scope of enumerations in a direct manner, as limited by our computation power. A new form of modified invariant FSW^* will therefore be introduced in the section by using the concept of admissible decomposition classes. The reason for introducing the modified invariants will be clarified only after the proof of the main theorem is given in the succeeding sections.

The residual relative obstruction bundle κ will be introduced. I will address its properties in certain detail.

In §7, I combine the various tools introduced in the previous sections to study the relationship between the modified invariants and the counting of singular curves. Then I prove the main theorem in the same section, based on the tools developed in the previous sections, along with the family blowup formula and family switching formula [38], [39]. I remark here that the current proof is simplified because of the implicit usage of $SW = Gr$ in the Kähler category. I also discuss the algebraic set up of the family obstruction bundle associated with $\mathcal{ASW}^*$. It plays an important role in the algebraic proof of Theorems 1.1 and 1.2 as well as their extension to higher dimensions.

In §8, I address the question of the structure of the universal formula, following an idea of Göttsche [19]. However, my formulation allows me to extend his idea to a more general context, which leads to new insights to the cases of nonnodal singularities as well. As a result, the generating functions of the universal polynomials of fixed topological types (which are required to be self-repeated) factorize and are governed by four different power series. In my current formulation, it works for counting duplicated singularities of arbitrary kind. The orders of the automorphism groups G_Γ and $G(\Gamma)$ will play essential roles in my discussion.

Firstly, we restrict ourselves to the enumeration of the nodal curves invariants. I prove that these not-yet-identified power series constructed in §7 can be identified using the result on the calculation of genus g curves in the primitive classes of $K3$, Göttsche's result on genus 2 curves in the primitive classes of T^4 (see also [6]) and Harris-Caporaso's [7] result on the recursive formulas of the Severi degrees, which calculate the numbers of nodal curves in $\mathbf{CP}^2$. The identification of the power series with the Harris-Caporaso calculation was first proposed by Göttsche [19], based on some conjecture of Di Francesco-Itzykson [8]. I show the validity of the conjecture by using almost complex perturbations and Taubes' gluing argument [53].

The significance of the theorem presented here is in that it not only implies the Yau-Zaslow conjecture in its full generality, it also introduces a brand new point of view on calculating the number of nodal curves for general algebraic surfaces, without the full knowledge of their diffeomorphism types. I also indicate at the end that the power series differs by the actual nodal curves counting up to some correction terms. These unknown correction terms involve the contribution from type II exceptional curves.

Even though the nodal curve power series can be completely identified, my machinery proves that a similar pattern holds for a much more general situation. Specifically, if one replaces the nodal singularity with the other curve singularities and "count the curves" with δ identical singularities, then a similar argument also produces some different power series. The possible conjecture regarding these unknown power series will be the subject of further investigation. I hope to return to this topic later. The quotation marks here indicate that the counting is not interpreted in the most naive way.

In the final section of the paper, I offer a comparison between our scheme and that of Vainsencher [59], [24]. I indicate briefly how to use the concept of the core and the family blowup (switching) formula to re-derive the formula of Vainsencher [59] in low nodes cases. I also indicate in the same section how to deal with the $n = 8$ case and compare it with the scheme by the excess intersection formula [16]. I also point out briefly the main difficulty in directly applying Vainsencher-Kleiman-Piene's scheme to higher nodes cases. At the end I also compare the $n = 8$ case with the result from the "ideal answer" of residual intersection theory. Details on how to give an algebraic proof based on excess intersection theory will be given in part II of the paper.

Last, I list the direct applications of my general theorem, before the detailed discussion.

Theorem-Corollary 1.1 (Yau-Zaslow Conjecture). *Let M be an algebraic $K3$ surface. Suppose that C is an integral cohomology class in* $\mathrm{Pic}(M) \subset H^2(M, Z)$ *of square $C^2 \geq 0$ with the additional primitive assumption when $C^2 = 0$. Let $N_n(C^2)$ be the number of n-nodes nodal curves in class C. Consider the generating function*

$$F_k(q)^{K3} = \sum_{r \geq 2k-2} N_{\frac{r}{2}-k+1}(r) q^r,$$

then it is a quasi modular form, given by $(DG_2)^k \cdot q/\Delta(q)$, where

$$\Delta(q) = q \prod_{s>0} (1-q^s)^{24} = \eta(q)^{24}$$

is the classical modular form of weight 12 with respect to $SL_2(\mathbf{Z})$ and $G_2(q)$ denotes the q expansion of the quasi-modular form

$$\frac{-1}{24} + \sum_{s>0} \sigma_1(s) q^s.$$

In the above formula, D denotes the differential operator $q\frac{d}{dq}$.

And similarly, for T^4, we have:

Theorem-Corollary 1.2. *Let $M = T^4$ be an abelian surface. Suppose C is an integral cohomology class representing an element of* $\mathrm{Pic}(T^4) \subset H^2(M, Z)$ *of square $C^2 \geq 0$ with the additional primitive assumption when $C^2 = 0$. Let $N_n(C^2)$ be the number of n-nodal curves in class C and consider the generating function*

$$F_k(q)^{T^4} = \sum_{r \geq 2k-2} N_{\frac{r}{2}-k+1}(r) q^{\frac{r}{2}},$$

then it is a quasi modular form, given by $(DG_2)^k \cdot D^2 G_2(q)$.

The reader might have noticed that it is not clear for the holomorphic genus g curves in a $K3$ or T^4 ample linear system to be always nodal. Otherwise, the original approach of Yau-Zaslow would have been confirmed directly. In terms of Gromov-Witten invariants, the special cases of these two theorems were finished by C. Leung-J. Bryan [5], [6] and later by Parker-Ionel [23] under the assumption that the class is a primitive cohomology class; as a continuation of the author's joint approach with T. J. Li (cf. [4]). It was during a discussion between T. J. Li and the author that they initiated (December 1995) the study of this subject using Gromov-Witten invariants [35]. In these special cases, the arithmetic of the enumerations were based on Nakajima's calculation of the Euler numbers of Hilbert schemes while the algebraic surface had an elliptic surface structure—following the main theme of Yau-Zaslow correspondence between nodal curves and the Euler numbers of Hilbert schemes.

The next theorem is about the identification of the full power series conjectured by Göttsche and Göttsche-Yau-Zaslow, respectively. The symbol $\mathcal{F}^{for}(q)$ in the following statement denotes the generating function of the normalized modified family invariant.

Theorem 1.1 (Göttsche-Yau-Zaslow Conjecture). *By substituting $DG_2(q)$ for q, $\mathcal{F}^{for}(DG_2(q))$ can be identified with*

$$\frac{(DG_2(q)/q)^{\chi(L)} B_1(q)^{K_M^2} B_2(q)^{L\cdot K_M}}{(\Delta(q)D^2G_2(q)/q^2)^{\frac{\chi(\mathcal{O}_M)}{2}}},$$

where B_1 and B_2 are the two power series derived by Göttsche starting as

$$B_1(q) = 1 - q - 5q^2 + 30q^3 - 345q^4 + 2961q^5 \dots ,$$

and

$$B_2(q) = 1 + 5q + 2q^2 + 35q^3 - 140q^4 + 986q^5 + \dots .$$

An important consequence of the theorem is the following blowup formula of nodal curves invariants:

Theorem 1.2 (Blowup Formula of The Nodal Invariants). *The blowup formula relates the generating function in the following way*

$$\mathcal{F}^{for}_{\widetilde{M}}(DG_2) = \mathcal{F}^{for}_{M}(DG_2) \cdot \left(\frac{B_2(q)}{B_1(q)}\right) \cdot \left(\frac{DG_2}{q}\right)^{-1} .$$

The first 26 terms of the power series B_1 and B_2 appearing in Theorems 1.1 and 1.2 are determined by Göttsche, assuming the validity of Di Francesco-Itzykson's conjecture. The higher order terms can be determined similarly as the hypothesis of Göttsche will be justified in the paper. For simplicity, I identify the answers through perturbing the family theory to the almost complex category. In part II of this paper, I introduce the concept of type II exceptional curves and use it to give an intrinsic determination of the full power series without referring to the results in Gromov-Witten theory.

The appropriate definitions and proofs will be given in the following sections. I list here the key ingredients of my proof to give the reader an overview.

A. The explicit form of the family blowup formula [38] (and the family Wall Crossing Formula [29]) without imposing any transversality condition on the appropriate moduli space.

B. The existence of the family curve counting scheme, which compares the expected dimensions of different curves. The key notion of the admissible decomposition classes follows from the scheme.

C. The new structure of the universal spaces M_n and the concept of admissible graphs, which parameterize a special stratification of M_n.

D. The vanishing result on the family Seiberg-Witten invariants which guarantees the vanishing of a certain type of mixed or pure invariants.

E. The family switching formula, which relates the multiple coverings with different multiplicities which are compatible with the family curve counting scheme in B. The possibility of changing the multiplicities of exceptional rational curves between different decompositions was formulated in the announcement version by the name of "cluster decomposition property." As in the algebraic case, we do not really need a version of Gromov invariants; the role of cluster decomposition property is replaced and simplified by the usage of the family switching formula [39].

F. The explicit construction of the $SW(\&Gr)$ obstruction bundle for the multiple coverings of type I exceptional (negative square) rational curves.

G. The nested Kuranishi model and the parallel theory of excess intersection theory in algebraic geometry.

The usage of the concept "family invariant" helps to relax the transversality conditions on the moduli space of curves. Even though the situation may not satisfy the transversality conditions necessary for curve counting in algebraic geometric terms, one is able to identify the invariant contributions by the obstruction bundle technique [38], [39]. This is one of the major differences from previous approaches where regularity problem needs to be dealt with. The possibility of bypassing this issue was the driving force for our approach. A simple comparison between my method (by the switching formula) and the residual intersection theory will be outlined in a special case at the end of the paper.

Another major difference in strategy is that we focused on resolved smooth curves instead of singular curves, as was adopted by Vainsencher (cf. [59], [24]). It turns out that the former format fits more naturally with the family blowup formula. Otherwise, the appearance of numerous different types of singular curves in the counting of nodal curves couples with the problem of transversality seriously which easily blocks out forming any clean geometric picture. The first evidence that my approach might shed light on this problem dated back to the proof of the wall crossing formula for four-manifolds with $b_2^+ = 1$; see for example [29] or [35].

In principle, one can derive the explicit formula using the scheme we adopt. However, it is beyond finite calculation to determine the whole universal formula in this way. Before I can have more geometric understanding about the meaning of Yau-Zaslow-Göttsche conjecture, the direct approach of enumerating the modified invariants seem to be rather difficult. Instead, I prove the existence of the universal formula using an abstract induction method. In this sense my proof does not give as much direct numerical information in the flavor similar to [59], [24], unless one is willing to struggle with endless calculation. As my interest is rather theoretical than numerical, I hope that the reader will not view this as a serious defect.

I cannot help but emphasizing at this point that the proof of the main theorem was not done in a way to generalize the work of Vainsencher. In fact, it surprised us to learn that Vainsencher's approach turned out to have a natural extension—by adopting our theoretical scheme under specific setting. Conceptually, the author followed the lead of Taubes' [51], [52], [53], which identified the Seiberg-Witten invariants and Gromov invariants beautifully. This line of thinking was already apparent in [29].

2. A survey of principal tools

Let me review the principal tools used later in this paper:

Recall that, on any smooth four-manifold with $b_2^+ > 0$, one is able to define the Seiberg-Witten invariants in terms of the datum of the appropriate moduli spaces. Given a smooth Riemannian metric g on the four-manifold M, let $\mathcal{L}$ be a *spin*$_c$ structure on M. The Seiberg-Witten equations consist of a pair of equations,

$$+F_A = \sigma(\Psi, \Psi) + i\mu$$
$$D_A\Psi = 0.$$

The operator D_A is the *spin*$_c$ Dirac operator and $\sigma(\Psi, \Psi)$ is given by the quadratic map $\mathcal{S}_{\mathcal{L}} \otimes \mathcal{S}_{\mathcal{L}} \mapsto \Lambda^2_+$.

Then all the pairs $A \in \mathrm{Conn}(\mathcal{L})$ and $\Psi \in \Gamma(M, \mathcal{S}_{\mathcal{L}})$ constitute a solution space which forms a smooth manifold of dimension

$$\frac{c_1(\mathcal{L})^2 - 2\chi - 3\sigma}{4},$$

(after the gauge group quotient) under the generic perturbation of the self dual two form μ.

If one replaces the generic two form μ by a fixed two form $+F_{A_0}$ and a multiple of self dual symplectic two form $r\omega$ (assuming M to be symplectic), then the solution space (as $r \mapsto \infty$) is closely related to the Gromov moduli space of curves in the cohomology class C with $K_M^{-1} \otimes C^2 = \det(\mathcal{S}_{\mathcal{L}})$ [51], [52], [53].

Remark 2.1. In the paper, the symbol C denotes the cohomology class dual to a (pseudo)-holomorphic curve in M. On the other hand, I also use the same symbol to denote the divisor class associated to the holomorphic curve. In particular, $\mathcal{O}_M(C)$ would denote the invertible sheaf associated to such a divisor class. In general, a class C in $H^2(M, \mathbf{Z})$ determines a smooth complex line bundle on M. If the irregularity $q(M) \neq 0$, the line bundle can be given more than one holomorphic structure. The holomorphic structures on the underlying smooth complex line bundle is parameterized by the dual Albanese torus, which appears in the wall crossing formula [30] explicitly. By inserting a $b_1(M)$ dimensional class on T^{b_1}, one fixes a holomorphic structure on the line bundle implicitly. In the algebraic geometric setting of our paper, the choice has been made implicitly and I use the same symbol C to denote the corresponding Weil divisor class.

Given the moduli space $\mathcal{M}_{\mathcal{L}}$ which collects the solutions (A, Ψ) modulo gauge transformations, one constructs the tautological S^1 bundle on $\mathcal{M}_{\mathcal{L}}$ by replacing the abelian gauge group by the corresponding based gauge group. Suppose that the Euler class of the S^1 bundle is denoted by $e_{\mathcal{L}}$, then one takes

$$SW(\mathcal{L}) = \int_{\mathcal{M}_{\mathcal{L}}} e_{\mathcal{L}}^{\frac{d}{2}}, \qquad d = \frac{c_1(\mathcal{L})^2 - 2\chi - 3\sigma}{4}.$$

On the other hand, let C be the cohomology class which is related to the $spin_c$ structure by $c_1(\det(\mathcal{S}_{\mathcal{L}})) = c_1(K_M^{-1}) + 2C$. The Gromov moduli space $\mathcal{M}(C)$ has an expected real dimension $\mathrm{d}_{\mathbf{R}}(C) = C^2 - C \cdot K_M$. One considers the pseudo-holomorphic curves passing through generic $\frac{\mathrm{d}_{\mathbf{R}}(C)}{2}$ points. This number can be proved to be a symplectic invariant and is defined to be $Gr(C)$.

Theorem 2.1 (Taubes). *Let M be a symplectic four-manifold with $b_2^+ > 1$. Let SW denote the Seiberg-Witten invariants and Gr denote the Gromov-Taubes invariants. Given a $spin_c$ structure $\mathcal{L}$ on M, one can associate a cohomology class $C = \frac{(c_1(\mathcal{L}) + c_1(K))}{2}$. Then one has $SW(\mathcal{L}) = Gr(C)$.*

Taubes' fundamental theorem in [51], [52], [53], [54], [55], [56] relates the smooth topological invariant to the curve counting explicitly. If one considers $b_2^+ = 1$ symplectic four-manifolds, the theorem holds with some additional assumption on the class C.

In the general cases, the curves counted by Taubes' scheme are disjoint unions of embedded smooth pseudo-holomorphic curves, while tori are allowed to form multiple coverings. The picture will be totally different in the family case. The author's main motivation here is to introduce the concept of admissible decompositions classes [37] in studying the new behavior.

In the special case that M is Kähler, the equivalence of the invariants becomes manifestly clear, [51], [52], [53], [54], [13], [14]. It is because (A, Ψ) can be shown to be equivalent to the information of a holomorphic connection on the line bundle C along with a holomorphic section Ψ of the holomorphic line bundle C. Under this identification, $\Psi^{-1}(0)$ gives a holomorphic curve on M Poincare dual to C.

An earlier joint work of T. J. Li and the present author that I generalized (spring 1997) [29] Seiberg-Witten theory to the family version. Let $\mathcal{X} \mapsto B$ be a fiber bundle of smooth four-manifolds over the base manifold B, where B and the fibers are both oriented. Given a family of Riemannian metrics along the fibers, one considers the family Seiberg-Witten equations

$$+F_A = \sigma(\Psi, \Psi) + \mu$$
$$D_A \Psi = 0.$$

Similar to the ordinary case, one can consider the family moduli space whose expected family dimension is equal to

$$\left(\frac{c_1(\mathcal{L})^2 - 2\chi - 3\sigma}{4}\right) + \dim B$$

The definition of the invariant can be generalized in a straightforward manner [29]. Let us recall the following definitions:

Given a homotopical class of sections $\sigma : B \mapsto \mathcal{X}$, one considers the principal $\mathcal{G}$ bundle of gauge transformations which acts on the space of connections. There is a short exact sequence of principal bundles by considering the subgroup $\mathcal{G}_\sigma \subset \mathcal{G}$. The group $\mathcal{G}_\sigma$ consists of group elements trivial along the chosen section. The quotient $\mathcal{G}/\mathcal{G}_\sigma$ is isomorphic to a principal S^1 bundle over B.

Let us consider the space of solutions of the family Seiberg-Witten equations. Instead of making quotient by $\mathcal{G}$, one quotients the solution space of the Seiberg-Witten equations by $\mathcal{G}_\sigma$. The resulting manifold is the based family Seiberg-Witten moduli space. Similar to the ordinary SW theory, the based moduli space has a circle bundle structure over the family moduli space. Let e denote the Euler class of the circle bundle. It depends on the homotopic type of the section σ.

There are two types of topological invariants I will consider in this paper: the so-called pure invariant and the mixed invariants defined in [29].

The pure invariant, denoted by $FSW_B\langle 1, \mathcal{L}\rangle$, consists of raising e to the highest power and integrating over $\mathcal{M} \mapsto B$. On the other hand, one can pull back nontrivial classes c from B and consider

$$FSW_B\langle c, \mathcal{L}\rangle = \int_{\mathcal{M}} \pi^* c \cdot e^{\frac{\dim \mathcal{M} - \deg(c)}{2}},$$

an integer. As defined in [29], the mixed invariant involves the insertion by some cohomology class from the base. I will use the terminology "base class insertion" frequently. A simple but important ramification is that when c is equal to $[B]$, the fundamental class of B; the mixed invariant reduces to the ordinary Seiberg-Witten invariants. Functorially, one can view the family invariant as a sequence of morphisms in $\mathbf{Hom}(H^{\cdot}(B,\mathbf{Z}),\mathbf{Z})$ parameterized by $\mathcal{L}$.

In this paper, the language of differential topology and differential geometry will be used most frequently. Let us explain briefly the link between my machinery and the terminology of algebraic geometry. Exactly as in the ordinary Seiberg-Witten theory, the family Seiberg-Witten moduli spaces are manifestly linked to the moduli spaces of curves. There is a nice correspondence between the language I use and the language used by the algebraic geometers. A certain version of (SW or Gromov) moduli space should be associated with the linear system (projective space) of curves, while the Euler class e of the S^1 bundle should correspond to the tautological line bundle on the projective spaces. Keeping this in mind, my discussion can be translated in a straightforward manner to the language of algebraic geometry.

Another key insight is the observation made in [36] that the complex dimension of the Gromov moduli space was given by [51], [52], [53], [54]

$$\mathrm{d}_{\mathbf{C}}(C) = \frac{C^2 - C\cdot K_M}{2},$$

while surface Riemann-Roch theorem gives

$$\chi(\mathcal{O}_C) = p_g + 1 + \frac{C^2 - C\cdot K_M}{2},$$

for simply connected surfaces.

Assuming $p_g = 0$ and the vanishing of the higher $\bar{\partial}$ cohomology, the linear system $\mathbf{P}(H^0(M,\mathcal{O}_C))$ is exactly of real dimension $2\mathrm{d}_{\mathbf{C}}(C)$. The link between the Gromov moduli space dimension and the surface Riemann-Roch formula was the key motivation for the present author to generalize the Enriques Criterion to the $b_2^+ = 1$ symplectic four-manifolds [36]. It turns out that it also plays a crucial role in this paper.

Suppose that the class c is dual to the compact oriented submanifold $B' \subset B$, then the mixed invariant can be identified with the pure invariants of the new fiber bundle constructed by pulling back $\mathcal{X} \mapsto B$ by $B' \subset B$.

Interesting to note that if one chooses a family of Kähler metrics, the family Seiberg-Witten invariants are manifestly linked to the curve counting in algebraic geometry. The corresponding Gromov moduli space has the expected dimension

$$2\mathrm{d}_{\mathbf{C}}(C) + \dim_{\mathbf{R}} B = C^2 - C \cdot K_M + \dim_{\mathbf{R}} B.$$

In the paper [38], a different version of "invariant" was introduced. The algebraic family Seiberg-Witten "invariants" coincide with the usual family Seiberg-Witten invariants in the algebraic families under the $p_g = 0$ condition. The construction was carried out in detail there. In this paper, I will use interchangeably between these two set ups. While the first version [29] is manifestly topological invariant, and the second version [38] is defined over characteristic zero. By using the algebraic version, which is based on intersection theory [16], the similar statement can also be proved for the characteristic zero case. I only address those arguments which are significantly different from the differential topological argument in $\mathcal{C}^\infty$ category.

Main Theorem 2.1. *Let M be an algebraic surface over an algebraically closed field of characteristic zero. Given an ample cohomology class C', then the "virtual number of singular curves" of a fixed topological type in $L = kC'$, with k sufficiently large is given by a universal polynomial expression in $L^2, L \cdot c_1(K_M), c_2(M), c_1(M)^2$, multiplied by $\mathcal{ASW}(L)$.*

The symbol $\mathcal{ASW}$ denotes the algebraic family Seiberg-Witten invariants, which virtually corresponds to the usual FSW in a $b_2^+ - 1 = 2p_g$ dimensional family [38]. In the general situation, one expects that the virtual numbers to count the pseudo-holomorphic curves with prescribed singularities. As the corresponding theory of pseudo-holomorphic curves is yet to be developed, I leave it as a conjecture.

The formulation of the algebraic "invariant" is rather important to us. Framing in terms of topological Seiberg-Witten invariants, I need to add the $p_g = 0$ condition to the main theorems. For $p_g > 0$ algebraic surfaces, the usual invariants are all zero, while the $\mathcal{ASW}$ are not.

Next I spend some effort in introducing the family blowup formula and family switching formula [38], [39]. These two formulas turn out to be the key ingredients to our proof. With the magical usage of these two formulas, the main obstacle in algebraic geometry is bypassed in a systematic way. This makes proof of Göttsche's conjecture accessible.

The first principal tool for the curve counting is the family blowup formula of a fiber bundle. The usual blowup formula for the SW invariant was derived by essentially all the experts right after its birth [25], [51], [12], [13].

Let $\mathcal{X} \mapsto B$ be a smooth fiber bundle whose fibers are diffeomorphic to the symplectic four-manifold M with $b_2^+ > 0$. Let $s : B \mapsto \mathcal{X}$ be a cross section of the fiber bundle whose tubular neighborhood (identified with a real rank four bundle through a diffeomorphism) is almost complex. Let $\mathcal{N}$ denote the associated complex rank two bundle. Let $\mathcal{X}\sharp\overline{\mathbf{CP}}_B{}^2(\mathbf{C}\oplus\mathcal{N}^*)$ be the family fiber sum to a new fiber bundle $\mathcal{X}'$.

Let $FSW\langle c, \mathcal{L}\rangle$ be the family Seiberg-Witten invariant with base class insertion c. Let E be the tautological exceptional class in $\overline{\mathbf{CP}}_B{}^2$. Given $\mathcal{L} \in spin_c(\mathcal{X})$, one can associate $\mathcal{L}' = \mathcal{L} + (2k+1)E \in spin_c(\mathcal{X}')$ such that $\mathrm{d}(\mathcal{L}') + \dim B \geq 0$.

Theorem 2.2 (A. Liu, 1997). (Family Blowup Formula). *Under the previous assumption, it follows that*

$$FSW\langle c, \mathcal{L}'\rangle = \sum_i FSW\langle c \cup c_i(\mathcal{V}), \mathcal{L}\rangle,$$

over B. The bundle $\mathcal{V}$ is a complex rank $\frac{k(k+1)}{2}$ vector bundle. Letting C be the fiberwise cohomology class which corresponds to $\mathcal{L}$, one can pull back C to B by the section $B \mapsto \mathcal{X}$, denoted by s^*C. Then the obstruction bundle $\mathcal{V}$ is given by

$$s^*C \otimes S^k(\mathbf{C}\oplus\mathcal{N}^*) = \oplus_{i\leq k} S^i(\mathcal{N}^*) \otimes s^*C,$$

where S^i represents the i-th complex symmetric power of vector bundles.

It is important that one derives the formula without using any transversality condition on the moduli spaces. The family blowup formula was initially considered by the author to discuss McDuff's proposal [32]. It turned out to be a big surprise to me that my study of the multiple covering exceptional curves has strong implications to enumerative problems as well. The family blowup formula will be the key tool to enumerate family invariants. An analogue of the formula was discussed in the paper [29]. Over there a wall crossing formula was derived by the author. It was shown to be compatible with the family blowup formula in the special cases $B = M_n$ or $M[n]$.

Next I review briefly the family switching formula. It is slightly different from the family blowup formula that the switching formula does not involve any surgical operation on the fiber bundle, but merely on the $spin_c$ structures.

Theorem 2.3. *Let $\mathcal{X} \mapsto B$ be a smooth fiber bundle of oriented four-manifolds such that the fibers are of $b_2^+ > 0$. Let $\mathcal{C} \mapsto B$ be a relative $\mathbf{S}^2$ fiber bundle embedded into $\mathcal{X} \mapsto B$ as a sub-fiber bundle such that the Poincare dual of $\mathcal{C}$ is nontorsion, and the fiberwise class is denoted by $\mathcal{C}$ again. Assume additionally that the relative tubular neighborhood of $\mathcal{C}$ carries fiberwise almost complex structures such that $\mathcal{C} \mapsto \mathcal{X}$ gives $\mathcal{C}$ a $\mathbf{CP}^1$ bundle structure over B. When the self intersection number $\mathcal{C}^2$ is negative, then the fibers of $\mathcal{C} \mapsto B$ are negative self-intersecting spheres in $\mathcal{X}$. Suppose given two $spin_c$ structures $\mathcal{L}$ and $\mathcal{L} + 2k\mathcal{C}$ such that their family dimensions are both nonnegative.*

Then their family invariants are related by the formula

$$FSW\langle \eta, \mathcal{L}\rangle = FSW\langle \eta \cup c, \mathcal{L} + 2k\mathcal{C}\rangle,$$

where $\eta \in H^*(B, \mathbf{Z})$ and $c = c_*(\mathcal{V})$ is the total chern class of the obstruction bundle constructed in [39].

The explicit form of the obstruction bundle does not play a key role in proving the Göttsche-Yau-Zaslow conjecture. Nevertheless, the family switching formula is relevant in defining the residual relative obstruction bundle κ which is crucial in defining FSW^* or $\mathcal{ASW}^*$.

For the current application, it is important for us to notice that it depends on $\mathcal{L}$ and $\mathcal{C}$ and the multiplicity $2k$ in a functorial way. The obstruction bundle can be decomposed in the K group as two pieces, $\mathcal{V} \equiv \mathcal{V}_1 \oplus \mathcal{V}_2$. The first piece involves the obstruction bundle of the $-l$ curve with κ being a direct factor. The second piece involves the obstruction bundle derived from its multiple coverings. The $\mathcal{V}_1$ will show up in the family Kuranishi model, and $\mathcal{V}_2$ will affect the enumeration of the invariants. The switching formula will be applied to the multiple covering of type I exceptional rational curves inductively. The existence of such a formula plays an essential role in proving the universality of the conjectured formula. Again, the nonnegativity condition on the family dimensions of the $spin_c$ structures is necessary.

If one replaces the FSW by $\mathcal{ASW}$, the above theorems of family blowup formulas and the family switching formulas are still valid. However, it is crucial to notice that in the algebraic context, one needs to impose the extra "simpleness" condition on the cohomology classes L to validate the formulas. As a side remark, one notices that the simpleness condition is automatically satisfied if the algebraic surface is of $p_g = 0$. In the $p_g > 0$ case, one needs to fulfill the simpleness condition in order to apply the family blowup and switching formula.

3. The universal space M_n

I would like to recall a sequence of fiber bundles which were first introduced by Vainsencher [59]. Since the spaces are constructed by successive blowups from $M^n = M \times M \times \cdots \times M$, the family blowup formula will be used repeatedly in these situations. A different family of spaces $M[n]$, the Fulton-MacPherson spaces, were introduced to the family Seiberg-Witten theory in the paper [29] to discuss the family invariants of nodal curves. The wall crossing formula for this particular family was studied in great detail in the previous paper. Although the family blowup formula is conceptually more suitable in the current context, the reader should also notice the importance of the impact of the wall crossing version [29] to my current theory.

Let M be an almost complex four-manifold (or smooth algebraic surface). I would like to define a sequence of spaces M_l of complex dimensions $2l$ such that the fibers of the fiber bundle $M_{l+1} \mapsto M_l$ are diffeomorphic to $M\sharp^l\overline{\mathbf{CP}}^2$ (or isomorphic to M with l (may be nondistinct) points blown up). The current set up works either in the almost complex or in the algebraic category. As the discussions are parallel, I will concentrate on the differential topological aspect while keeping in mind that a parallel theory can easily be built up for algebraically closed fields with characteristic zero. Occasionally, I remind the reader of the difference between the two different foundations and address the issues if necessary. I apologize to the more algebraic geometric background reader for this mild inconvenience.

Inductively, one takes $M_0 = pt$ and $M_1 = M$. Then $M_1 \mapsto M_0$ in a natural way. Suppose $f_l : M_l \mapsto M_{l-1}$ has been defined and forms a nice fiber bundle as described, I consider the fiber product of the map, $P_l = M_l \times_{f_l} M_l \mapsto M_{l-1}$. The relative diagonal $\Delta_l : M_l \mapsto P_l$ defines a complex codimension two submanifold in P_l. One defines M_{l+1} to be the new manifold formed by blowing up $\Delta_l(M_l) \subset P_l$. By composing the blowing down map and the second projection map I construct the projection map $f_{l+1} : M_{l+1} \mapsto M_l$. Inductively, define the M_l for all $l \geq 0$. Notice that M_{l+1} can be constructed alternatively from $M \times M_l$ by blowing up l times consecutively from the fiber bundle $M \times M_l \longrightarrow M_l$, each blowup center is a section of the intermediate fiber bundles.

First observe that M_l can be projected onto $M^l = M \times M \times \cdots \times M$ by induction. Let π_i denote the projection from M_l to the i-th factor of M^l. Given a point $p \in M_l$, $\pi_i(p)$ defines a point in M, its "i"-th coordinate. Therefore, it defines canonically a section $M_l \longrightarrow M \times M_l$

that we call the i-th tautological section in the following. Blowing up consecutively the proper transformations of the i-th ($i \leq l$) tautological sections results in the fiber bundle $M_{l+1} \mapsto M_l$. It can be checked by viewing the relative diagonal Δ_l as a section and by applying the induction hypothesis. I give a brief argument as follows:

Lemma 3.1. *The fiber bundle $M_{l+1} \mapsto M_l$ can be constructed from the product bundle $M \times M_l$ by l consecutive complex codimension 2 blowing ups. The blowing up center of the i-th blowing up is the proper transformation of the i-th tautological section under the previous $i-1$ blowing ups.*

Proof. For $l = 0$ the statement is trivial. The second universal space M_2 can be viewed as the blowing up of $M \times M$ along the diagonal, which can be viewed as the tautological section.

By induction hypothesis, suppose $M_l \mapsto M_{l-1}$ is the result of $l-1$ blowing ups from $M \times M_{l-1}$. Then prove that it also holds for $M_{l+1} \mapsto M_l$.

First, consider the fiber product $M_l \times_{M_{l-1}} M_l \mapsto M_{l-1}$ and the associated commutative diagram,

$$\begin{array}{ccc} M_l \times_{M_{l-1}} M_l & \longrightarrow & M_l \\ \downarrow & & \downarrow \\ M_l & \longrightarrow & M_{l-1}. \end{array}$$

As $M_l \mapsto M_{l-1}$ has been assumed to be constructed by $l-1$ blowing ups from $M \times M_{l-1}$, $M_l \times_{M_{l-1}} M_l \mapsto M_l$, the pull back of $M_l \mapsto M_{l-1}$ by $M_l \mapsto M_{l-1}$, can be

$$\begin{array}{ccc} M \times M_l & \longrightarrow & M \times M_{l-1} \\ \downarrow & & \downarrow \\ M_l & \longrightarrow & M_{l-1} \end{array}$$

constructed from $M \times M_l$ by $l-1$ blowing ups, too. The relative diagonal $\Delta_l : M_l \mapsto M_l \times_{M_{l-1}} M_l$ is a section of $M_l \times_{M_{l-1}} M_l \mapsto M_l$ which maps onto the l-th tautological section of $M \times M_l \mapsto M_l$. The birational morphism between these two sections is isomorphic outside the exceptional locus. Thus it can be alternatively viewed as the proper transformation of the l-th tautological section in $M_l \times_{M_{l-1}} M_l$ under the previous blowing ups.

Thus, M_{l+1}, the blowing up of $M_l \times_{M_{l-1}} M_l$ by the smooth center $\Delta_l : M_l \mapsto M_l \times_{M_{l-1}} M_l$ can be viewed as the $1 + (l-1)$ times blowing ups of $M \times M_l$. q.e.d.

This interpretation is rather useful when one discusses the properties of the admissible strata. The blowing ups can be parameterized by a pair of natural numbers (a, b), $0 \leq a < b \leq l$. They are ordered as follows:

$$(1,2); (1,3), (2,3); (1,4), (2,4), (3,4); \cdots .$$

There are l effective exceptional classes in $H^2(M_{l+1}, \mathbf{Z})$ dual to the exceptional divisors of the l consecutive blowing ups. As they show up in constructing M_{l+1} from M_l, denote them by $E_i(l), 1 \leq i \leq l$. Then one has the following proposition,

Proposition 3.1. *Let $\mathbf{H} = H^{\cdot}(M_{l+1}, \mathbf{Z})$ be the cohomology ring of M_{l+1} over $\mathbf{Z}$. Suppose that g_s, $(1 \leq s \leq \dim H^*(M, \mathbf{Z}))$ form a collection of ring generators of $H^*(M, \mathbf{Z})$. Then it follows that $\pi_i^*(g_s)$ and $E_i(j)$, $i \leq j, 1 \leq j \leq l$ form a collection of ring generators of $\mathbf{H}$.*

I slightly abuse the notation by using the same symbol to denote the exceptional divisors.

Sketch of Proof. According to Lemma 3.1 M_{l+1} can be constructed from M^{l+1} by $\frac{l+2\cdot(l+1)}{2}$ different complex codimension two blowing ups. It is easy to see that it coincides with the total number of $E_i(j)'s$ in the previous proposition. One should also notice that the same number coincides with the number of two-element subsets of $\{1, 2, 3, 4, \ldots, l+1\}$. It was interpreted that the space M_{l+1} is obtained by M^{l+1} by blowing up the various proper transformations of the partial diagonals $\Delta_{ab}, \{a, b\} \subset \{1, 2, 3, \cdots l+1\}$ in some designated order. Thus, Proposition 3.1 is a consequence of the standard fact about the cohomology rings of varieties under blowing ups and Lemma 3.1. q.e.d.

This proposition has an analogue in Fulton-MacPherson's paper [17]. In the algebraic context, one replaces $\mathbf{H}$ by the Chow ring, and the similar conclusion holds.

The proper transforms of the divisors $E_i(l)$ under the subsequential blowing ups have $\mathbf{P}^1$ fibration structure over M_l. Birationally they come from projectifying the normal bundles of these tautological sections. The divisors $E_i(l)$ also have $\mathbf{P}^1$ fibration structures under the same projection map. The combinatorial structures of the singular fibers are rather complicated. They will be analyzed after introduction of the admissible graphs.

Let us consider the fiber bundle by blowing up the first $i-1$ sections. It is not hard to see that the normal bundle of the i-th section is the pull back of the relative tangent bundle $M_{i+1} \mapsto M_i$ by $M_l \mapsto M_i$. Since $f_i : M_{i+1} \mapsto M_i$ is a fiber bundle map, there is a short exact sequence of vector bundles

$$0 \mapsto \mathcal{R}\mathbf{T}(\mathbf{M}_{i+1}/\mathbf{M}_i) \mapsto \mathbf{TM}_{i+1} \mapsto f_{i+1}^* \mathbf{TM}_i \mapsto 0,$$

where $\mathcal{R}\mathbf{T}(\mathbf{M}_{i+1}/\mathbf{M}_i)$ is the relative tangent bundle of the fiber bundle map $M_{i+1} \mapsto M_i$.

Using the property that the total chern classes are multiplicative under the short exact sequences, one can relate the chern classes of the relative tangent bundle to those of $\mathbf{TM}_{i+1}$ and $\mathbf{TM}_i$. On the other hand, M_{i+1} comes from $M \times M_i$ by blowing up i different sections. There is a canonical way to relate the chern classes of $\mathbf{TM}_{i+1}$ to $\mathbf{TM} \times \mathbf{TM}_i$, too. As a result, the chern classes of $\mathbf{TM}_{i+1}$ can be written completely in terms of the various copies of $c_1(TM)$, $c_2(TM)$ and the cohomology classes $E_r(i)$.

The previous assertion plays a role in enumerating the invariants as well as the proof of the main theorem. Its role in enumeration geometry had been noticed in Vainsencher's work [59].

About the structure of the Vainsencher's spaces, along with the admissible graphs, I will discuss it slightly later in a systematic way.

Let us adopt a convention on the family Seiberg-Witten theory. Because it is inconvenient to work with the $spin_c$ structures over a Kähler or algebraic fibration, one redefines the new notation such that the family invariants of the class $2C_0 - K_M$ is denoted by $FSW(\eta, C_0)$ and $\mathcal{ASW}(\eta, C_0)$, respectively. The element η is an element in $H^*(B, \mathbf{Z})$ or $\mathcal{A}(B)$, depending on whether I are working over the topological or algebraic situation. Likewise, the notations in family blowup formula and family switching formula should be changed accordingly.

Notice that there exists a canonical map $M[n] \longrightarrow M_n$, where $M[n]$ denotes the Fulton-MacPherson space. While $M[n]$ enjoys an $\mathbf{S}_n$ action, the space M_n does not allow a natural $\mathbf{S}_n$ action. Only after the appropriate notions are introduced, can one really tell why the family over M_n is easier to deal with than the one over $M[n]$.

4. The admissible decomposition classes and the family Seiberg-Witten invariants

4.1 The exceptional cone and the stratification

The key ingredient to identify FSW and FGr in the Kähler or algebraic context is to give the general scheme [37] of characterizing the types of curves contributing to the family invariants. The scheme generalizes McDuff's proposal [43] when B is degenerated to a point.

Let us define the concept of exceptional pseudo-holomorphic curve first.

Let (M, J) be an almost complex four-manifold with the almost complex structure J. A connected irreducible pseudo-holomorphic curve is the image of the following datum. Let Σ be a connected compact Riemann Surface and $f : \Sigma \mapsto M$ be a pseudo-holomorphic map into M such that f is generically a $1-1$ immersion. A pseudo-holomorphic curve is the image of a pseudo-holomorphic map from an arbitrary compact Riemann surface Σ (which may not be irreducible) to M. A pseudo-holomorphic map $f' : \Sigma' \mapsto M$ (where Σ' is not necessarily connected) is said to be a multiple covering of a connected irreducible pseudo-holomorphic curve if there exists a connected compact Riemann Surface Σ such that $f' : \Sigma' \mapsto M$ factors through $f : \Sigma \mapsto M$ by the finite covering map $\Sigma' \mapsto \Sigma$.

Definition 4.1. A connected irreducible pseudo-holomorphic curve $f : \Sigma \mapsto M$ is said to be exceptional (an exceptional curve) if the fundamental class $f_*(\Sigma) \in H_2(M, \mathbf{Z})$ has a negative self-intersection number. A multiple covering of an exceptional curve is a multiple covering of a connected irreducible pseudo-holomorphic curve which is exceptional.

Given any irreducible algebraic curve $\underline{C}$ in an algebraic surface M, one can take $\widetilde{\underline{C}} = \Sigma$ to be the normalization of $\underline{C}$. The natural projection map induces a holomorphic map $\Sigma \mapsto M$. Thus, any irreducible algebraic curve can be viewed as a special case of the connected irreducible pseudo-holomorphic curve defined above.

In the paper, I consider multiple coverings of exceptional curves which are of special type. Let $f : \Sigma \mapsto M$ be an exceptional pseudo-holomorphic curve. Take $\Sigma' = \Sigma \coprod \Sigma \coprod \cdots \coprod \Sigma$ (m copies), then the natural projection map $\Sigma' \mapsto \Sigma$ induces a m-multiple covering of the given exceptional curve.

When M is algebraic, the image of Σ under f can be given a reduced scheme structure which becomes an algebraic curve in M. The special type of m-multiple covering of $\Sigma \mapsto M$ can be thought to be corresponding to putting the nonreduced scheme structure on $f(\Sigma)$ with a multiplicity m.

Consider a symplectic fibration $(\mathcal{X} \mapsto B, \omega)$ of smooth four-manifolds such that the fibers $\mathcal{X}_b$ are given fiberwise almost complex structures compatible with the given family of fiberwise symplectic forms ω.

Given a fiber-wise cohomology class C_0, one considers the exceptional cone with respect to a class C_0 which collects the exceptional curves which are non-nef with respect to C_0. Throughout the section, I take C_0 to be an effective class in either the symplectic or the holomorphic category. In deriving the main theorems of the paper, C_0 will be taken to be of the special form $C - \sum_i m_i E_i$, $m_i \in \mathbf{N} \cup \{0\}$.

Definition 4.2. Given a point $b \in B$, the exceptional cone of C_0 over b consists of all the exceptional curves in $\mathcal{X}_b$ which have negative cohomological pairings with C_0. The cone is denoted as $\mathbf{EC}_b(C_0)$.

It is particularly interesting to consider the case that C_0 is effective. Namely, C_0 is represented by a (pseudo) holomorphic curve.

The main conclusion in the note [37] implies that if the class C_0 is effective in the pseudo-holomorphic or algebraic category (it means that the class is represented by pseudo-holomorphic or algebraic curves), the exceptional cone $\mathbf{EC}_b(C_0)$ is generated freely by $\dim_{\mathbf{R}} \mathbf{EC}_b(C_0) \otimes_{\mathbf{Z}} \mathbf{R}$ irreducible exceptional curves. Thus, the cone is a simplicial cone with the irreducible exceptional curves generating the extremal rays. This conclusion can be checked directly for the $C_0 = C - \sum_i m_i E_i, m_i \in \mathbf{Z}$ discussed in the paper. The primitive effective generators of the cone are called the extremal generators. I denote the extremal generators by e_i which satisfy $e_i^2 < 0$. Moreover, one has the following conclusion [37]:

Proposition 4.1. *Suppose C_0 is an effective class over b, then the restriction of the quadratic intersection form of $\mathcal{X}_b$ to the exceptional cone $\mathbf{EC}_b(C_0)$ is negative definite.*

Notice that it does not imply the stronger condition that the quadratic form is negative definite on the vector space $\mathbf{EC}_b(C_0) \otimes_{\mathbf{Z}} \mathbf{R}$.

By considering the case that $\mathbf{EC}_b(C_0)$ is generated by n different -1 curves, e_i, with $e_i^2 = -1$, $i \leq n$, one finds that the previous proposition implies that $e_i \cdot e_j = 0$ unless $i = j$. In this sense, the previous proposition generalizes the simple picture of McDuff to the family version.

Under the assumption that C_0 is not numerically effective with respect to e_i, the curve Poincare dual to C_0 must split off a certain amount of energy to the class dual to e_i such that the class C_0 decomposes as $C_0 = C_0' + \sum_i m_i' e_i, m_i' \in \mathbf{N} \cup \{0\}$ in the second cohomology group.

Remark 4.1. The energy of a class e_i is its pairing with the symplectic form (or the ample polarization). When the class is effective, the positive number is equal the energy of the harmonic map representing e_i.

Under such circumstance, we say that the curve dual to C_0' is the good part of the total curve in C_0 and $\mathrm{d}_{\mathbf{R}}(C_0') = C_0' \cdot C_0' - C_0' \cdot K_M$ is the expected dimension of the good part. Likewise C_0' is said to be the good part of the class C_0 in the decomposition $(C_0', \sum m_i' e_i)$. In the algebraic set up, one usually divides the real dimension by two and denotes it as the algebraic dimension $\mathrm{d}_{\mathbf{C}}(C_0')$. One should be cautious that the term "good part" does not necessarily mean that the curves in C_0' form a based point free nonlinear system. In reality the curves dual to C_0', viewed as divisors on the algebraic surface, can still have nonfree components or other based points. Yet the usage of the family switching formula justifies the formal interpretation.

Remark 4.2. Following the convention in [37], the curve C_0' is called the good part (free part) of the curve C_0. Intuitively it is the portion of C_0 which can move on M under the ideal situation. Fixing a holomorphic structure on the smooth line bundle associated to C_0, we abuse the notation by denoting the corresponding linear system by $|C_0|$. Under the analogue $\sum m_i' e_i$ corresponds roughly to the base divisor of the linear system $|C_0|$ over b in terms of algebraic geometry terminology.

Conceptually, it is rather crucial that one views the object $\sum m_i' e_i$ as the base locus of the linear system of divisors $|C_0|$. The concept is intuitively clear in dimension four as the curves are also divisors of the surface. On the other hand, special cases of similar type of decomposition correspond to "bubbling off" some exceptional bubbles in the set up of Ruan-Tian theory. Here I offer a Seiberg-Witten style description of the "bubbling off" phenomenon which is slightly different from what one would expect from Ruan-Tian theory. As a result, the problem typically associated with exceptional "bubbling off" phenomenon is handled by a different methodology in comparison with the more general Ruan-Tian framework. The details of the discussion in the concrete case constitutes the definition of the modified invariants.

It continues that, $d_{\mathbf{R}}(C_0') + \dim_{\mathbf{R}} B$ denotes the real expected family dimension of the good part. The pair $(C_0', \sum m_i e_i)$ will be called a decomposition of C_0, and $d_{\mathbf{R}}(C_0') + \sum(e_i^2 - e_i \cdot K_{\mathcal{X}_b}) + \dim_{\mathbf{R}} B$ is called the expected family dimension of the decomposition.

By attaching an exceptional cone to each point $b \in B$, one gains a huge family of simplicial cones over the base manifold B, which may change if b moves in B. If one specializes from the generic point to a special point (in either topological or algebraic setting), the cones also degenerate. The phenomenon should be understood in terms of the Gromov-Sachs-Uhlenbeck compactness theorem in the almost complex setting. In particular, the specialization of a generic exceptional cone should be always contained in the exceptional cone over the specialized point. This phenomenon plays a crucial role in understanding the degenerations of the admissible graphs. If one works in a fully abstract situation, there might be a monodromy action on the cones. In the current paper, one ignores the possibility and considers the exceptional cone generated by the type I exceptional curves (refer to Definition 4.10 in Subsection 4.3) whose monodromy action is trivial.

Over each point b in the space B (which may be chosen to be the universal space M_n in the paper), there are a finite number of decompositions of C, whose expected family dimensions are not less than the family expected dimension of the class $d_{\mathbf{R}}(C_0) + \dim_{\mathbf{R}} B = C_0^2 - C_0 \cdot K_M + \dim_{\mathbf{R}} B$, while the high multiples of the exceptional curves in $\mathbf{EC}_b(C_0)$ contribute negatively to the expected dimension. In general, the situation can be rather complicated, and the cones can jump randomly.

It turns out that the understanding of the seemingly complicated phenomenon has a rather unexpected bonus. Namely, it provides me the key idea of proving the main theorems in this paper. As it is rather important to the later sections, let us outline the approach. The picture will be realized through the construction of the admissible graphs over M_n.

Following the "ideology" of Gromov-Taubes theory, one perturbs the fiberwise almost complex structures of the fiber bundle $\mathcal{X} \mapsto B$ to simplify the picture and stratify the base manifold B into strata $\mathcal{S}_r$, $B = \cup_r \mathcal{S}_r$ such that the exceptional cones are kept constant under parallel transports over each stratum. One requires that each stratum has to be smooth of correct dimension. By this one means that the number $\dim_{\mathbf{R}} B - \dim_{\mathbf{R}} \mathcal{S}_r$ should be equal to $\sum_i (e_i \cdot K_M - e_i^2)$ if $\mathbf{EC}_b(C_0)$ is generated by the extremal generators e_i. Moreover, the closure of $\mathcal{S}_r$

in B is a stratified compact set such that the boundary components are at least of real codimension two. It will be discussed in detail in Subsection 4.3.1 that the current theoretical scheme is indeed satisfied by the exceptional curve cones of the type I exceptional curves and the strata associated with the admissible graphs which one discusses later. Let us denote the exceptional cones $\mathbf{EC}_b, b \in \mathcal{S}_r$ by $\mathbf{EC}_{\mathcal{S}_r} = \mathbf{EC}[r]$. The class C_0 will be ignored if it does not cause ambiguity.

Definition 4.3. The stratification of B is right (of right dimension) if the space $\mathcal{S}_r$ is smooth and its real codimension is calculated by the Gromov-Taubes theory formula $\sum_i (K \cdot e_i - e_i^2)$, where e_i denote the various extremal generators of the simplicial cone $\mathbf{EC}_{\mathcal{S}_r} = \mathbf{EC}[r]$.

In the following, one works under the assumption that the stratification has been chosen to meet this condition.

4.2 The introduction of admissible decomposition classes

One defines a partial ordering on the exceptional cones (and on the corresponding strata) by saying that $\mathcal{S}_{r_1} \geq \mathcal{S}_{r_2}$ if $\mathcal{S}_{r_2}$ sits on the topological boundary of $\mathcal{S}_{r_1}$, i.e., $\mathcal{S}_{r_2} \cap \overline{\mathcal{S}}_{r_1} \neq \emptyset$. This automatically implies that the exceptional cone associated with $\mathcal{S}_{r_2}$ contains the degenerations of the exceptional cone of $\mathcal{S}_{r_1}$.

The partial ordering can be encoded in a finite graph whose vertexes are bijective to all the strata $\mathcal{S}_r$.

In the following, we define several terminologies which we will use frequently.

Definition 4.4. A decomposition of C_0 over $\mathcal{S}_r$ is by definition a pair $(C_0', \sum m_i' e_i)$ satisfying $C_0 = C_0' + \sum m_i' e_i$, $m_i' \in \mathbf{N}$, with e_i being the extremal generators of $\mathbf{EC}(r)$.

The locally closed stratum $\mathcal{S}_r$ is said to be the support of the corresponding decomposition.

A decomposition is said to be allowable if:

(1) Its expected family dimension,

$$\mathrm{d}_{\mathbf{R}}(C_0 - \sum m_i' e_i) + \sum_i \mathrm{d}_{\mathbf{R}}(e_i) + \dim_{\mathbf{R}} B,$$

is not less than the expected family dimension of the original C_0, $\mathrm{d}_{\mathbf{R}}(C_0) + \dim_{\mathbf{R}} B$.

(2) The expected family dimension of the decomposition $(C_0 - \sum e_i, \sum e_i)$ is not less than $\mathrm{d}_{\mathbf{R}}(C_0) + \dim_{\mathbf{R}} B$.

(3) C_0' has positive cohomological energy, i.e., the class C_0' has a positive pairing with the symplectic form (or an ample polarization).

A decomposition class $\mathbf{D}$ of C_0 over $\mathcal{S}_r$ is by definition the collection of all the allowable decompositions over $\mathcal{S}_r$. A decomposition class is said to be empty if there are no allowable decompositions in the class.

Obviously, an empty decomposition class is not interesting to us. If one drops the condition upon the positivity of the pairing with the polarization, then the finiteness of the allowable decompositions in a given decomposition class follows from a different restriction induced by condition (1) on expected dimension. One should be cautious as the condition affects the maximal level s (which is the maximum value of the numerical "level" one attaches to the decomposition classes) of admissible decomposition classes. In the following, only nonempty decomposition classes will be discussed. In the actual application, I do not specify whether C_0' has positive energy. If not, the associated invariant will automatically be set to zero as there are no curves at all.

Requiring the decomposition class to be allowable put severe numerical constraints on the relationship among the type exceptional curves and the class C_0.

Given a decomposition class of C_0 (which consists of a finite number of allowable decompositions by the energy boundedness constraint), it was canonically associated with some stratum and the exceptional cone over the stratum.

The support of a decomposition class $\mathbf{D}$ associated to $(C_0 - \sum_{i\in I} e_i, \sum_{i\in I} e_i)$ is the defined to be the top admissible strata $\mathcal{S}_{\mathbf{D}}$ characterized by the co-existence of the $e_i, i \in I$.

The discussion offered here will be relevant to the definition of the map Φ.

Definition 4.5. A decomposition class $\mathbf{D}_2$ is said to be subordinate to another decomposition class $\mathbf{D}_1$ if first, these two corresponding strata, $\mathcal{S}_{\mathbf{D}_1}$ and $\mathcal{S}_{\mathbf{D}_2}$, are related $\mathcal{S}_{\mathbf{D}_1} \geq \mathcal{S}_{\mathbf{D}_2}$ by the previous partial ordering defined among different strata. Moreover, given the strata $\mathcal{S}_{\mathbf{D}_i}, i = 1, 2$, let $e_{j,i}, i = 1, 2$ denote the extremal generators of the preexceptional cones $\mathbf{C}_{\mathbf{D}_i}, (i = 1, 2)$.

One requires that $\{e_{j,1} | e_{j,1} \cdot C_0 < 0\} \subset \{e_{j,2} | e_{j,2} \cdot C_0 < 0\}$. When $\mathbf{D}_2$ is subordinate to $\mathbf{D}_1$, one denotes by $\mathbf{D}_1 \gg \mathbf{D}_2$.

It is clear that this subordinate relationship defines a partial ordering among these decomposition classes. This partial ordering will be used in defining the modified family invariant.

For the convenience of the latter application, one also defines,

The support of a decomposition class $\mathbf{D}$ associated to $(C_0 - \sum_{i\in I} e_i, \sum_{i\in I} e_i)$ is defined to be the top strata $\mathcal{S}_{\mathbf{D}}$ of the right codimension characterized by the co-existence of the $e_i, i \in I$. Over $b \in \mathcal{S}_{\mathbf{D}}$, the exceptional cone $\mathbf{EC}_b(C_0)$ remains constant.

The points in the boundary $\bar{\mathcal{S}}_{\mathbf{D}} - \mathcal{S}_{\mathbf{D}}$ can be classified into two types; those over which all $e_i, i \in I$ remain irreducible and those over which some $e_i, i \in I$ degenerates and breaks up into some other irreducible exceptional curve class. Some new decomposition class may appear over $\bar{\mathcal{S}}_{\mathbf{D}} - \mathcal{S}_{\mathbf{D}}$.

Definition 4.6. A decomposition class is said to be generic if its support is of top dimensional $(= \dim B)$.

The generic decomposition classes are on the top of the pyramid of the collection of decomposition classes. These types of decomposition classes will play crucial roles in the definition of the modified family invariants.

The main reason for introducing these concepts is to characterize the decomposition classes which can contribute nontrivially to the family Seiberg-Witten invariants in the Kähler families. To achieve this, I would like to define the concept of admissible decompositions. Before doing so, I define the concept of admissible decomposition classes.

Axiom 4.1. An allowable decomposition class $\mathbf{D}$ is said to be admissible of level n $(n \geq 0)$ if:

1. There exists a descending chain of allowable decomposition classes $\mathbf{D}_i$, $0 \leq i \leq n$ such that the corresponding supports $\mathcal{S}_{\mathbf{D}_i}$ form a monotonically linearly ordered chain of length $n+1$. The given allowable decomposition class is the minimal element in the given chain.

2. Let $\mathbf{D}_i, 0 \leq i \leq m; \mathbf{D}_i \neq \mathbf{D}_j$ be a descending chain of allowable decomposition classes ending at $\mathbf{D}$, then $m \leq n$.

The generic admissible decompositions are of level zero. The converse of the statement also holds.

Discarding the level, an admissible decomposition class with an arbitrary level (≥ 0) is simply called an admissible decomposition class.

The finite number of decompositions in an admissible decomposition class are called the admissible decompositions in the class.

In some simple situation, there is a unique admissible decomposition in an admissible decomposition class. In these cases, sometimes I may abuse the notation by mixing up the decomposition class and the corresponding decomposition. However, this possibility is not guaranteed by the abstract definition.

I consider some examples of the admissible decompositions other than the one studied by Taubes [51], [52], [53].

Example 4.1. Consider $B = pt$ in the context of ordinary Seiberg-Witten theory. Let $C_0 = C_0' + \sum m_i e_i$. Then the decomposition $(C_0', \sum m_i e_i)$ is admissible iff $e_i^2 = -1$ (e_i are -1 curves) and $e_i \cdot e_j = 0$, $e_i \cdot C' = 0$. Under this condition, $m_i = -C_0 \cdot e_i > 0$. This corresponds to the decompositions in McDuff's proposal. Let $B = \Sigma$, a two dimensional surface. Let $e, e^2 = -2$ be a -2 rational curve. Consider a two dimensional family over B. Then $C_0 = C_0' + me$ is admissible only when $C_0' \cdot e = 0, 1$. In this situation, one stratifies the manifold Σ into some two (top) dimensional and zero dimensional strata such that the admissible decompositions lie above some zero dimensional stratum. It is the locus that the -2 curve e exists.

If $e, e^2 = -n$ is a $-n$ rational curve which survives on a generic $2n-2$ dimensional family, then (C_0', me) is admissible only when $0 \leq C_0' \cdot e < n$. Generalization of this special case leads to the general scheme described above. The so called moving lemma was discussed in detail in [37] and will be a key ingredient in applying the family switching formula to this context.

The scheme deals with the nonrational e_i as well. However, they will not be used in the proof of the main theorem. They are nevertheless crucial in understanding the contribution of type II exceptional curves.

Applying the previous general curve counting scheme to the fiber bundle $M_{l+1} \mapsto M_l$, the functorial setup suggests us to stratify the space $B = M_l$ according to the variations of the various exceptional cones of a class C. It is rather surprising to us that it forms the main idea of proving my main theorems as it was not originally designed for this particular purpose.

4.3 The definition of the admissible graph and the admissible strata of M_n

4.3.1. The axioms of the admissible graph and admissible degenerations

In this section, we introduce the axioms which characterize the admissible graphs. We also describe the procedure that admissible graphs degenerate.

Definition 4.7. Let Γ be a finite graph with l vertexes. Γ is said to be admissible if it satisfies the following axioms:

Axiom 4.1. There is a $1-1$ correspondence between the vertexes of Γ and the positive integers smaller or equal to l. An association of this type is called a marking of the graph. More generally, one can mark the graph by any finite subset of $\mathbf{N}$. If $\mathbf{I}$ is the index set, the graph is called $\mathbf{I}$-admissible.

Because of the existence of the marking, one usually names the vertexes by their markings.

Axiom 4.2. The edges are oriented by arrows from the vertex associated with a smaller integer to the vertex associated with a larger integer.

For example, the arrow can start from the 1st vertex to any other vertex, while the arrow can only start from the other vertex to the l-th vertex.

The vertexes which are linked to the i-th vertex by arrows leaving i are called the direct descendents of i. The vertexes which are linked to the i-th vertex by arrows entering i are called the direct ascendents of i. If the j-th vertex is related to the i-th one, by an oriented arrowed path, then j is called the descendent of i, while i is called the ascendent of j.

Axiom 4.3. The only loops allowed in the graph are formed by three vertexes. Suppose $a < b < c$ are the three different vertexes, then b,c must be the direct descendents of a, while a, b must be the direct ascendents of c. The vertexes a,b,c form a triangle.

This axiom rules out the topological types of a great number of graphs. It puts a severe condition on the graph.

Axiom 4.4. Any vertex can have at most two direct ascendents. The vertex having exactly two direct ascendents forms a triangle (loop) with its direct ascendents.

For some reasons which will become clear later, the axiom is called the normal crossing axiom. A vertex in an admissible graph is allowed to have a lot of direct descendents while at most two direct ascendents. This axiom is in a sense dual to the previous one.

Axiom 4.5. Suppose there are two adjacent triangles formed by the vertexes a, b, c and d. Suppose these two triangles share a common edge $\overrightarrow{ab}$, then the end vertex b of this edge has exactly one direct descendent among the two vertexes c and d.

Notice that the admissible graphs satisfying these axioms usually are not trees. The fact that only one and exactly one direct descendent is allowed in the Axiom 4.5 rules out the possibility that there are two or more arrows starting from the end vertex. In other words, if there are m adjacent triangles sharing the same vertex, then the end vertexes of the common edges form a linear chain. Later the author learned that there had been standard combinatorial devices–the resolution graphs used to describe resolutions. It was widely used by algebraic geometers including Vainsencher [59]. The reader will comprehend the benefit of my approach.

Next I would like to study the degenerations of the admissible graphs. The combinatorics I discuss here are not particularly hard. However, they play a crucial role in understanding the stratification I would like to define on M_l. The usage of the admissible graphs simplifies the language quite a bit. It will be shown later that these graphs encode the information of type I exceptional curves on M_l.

Definition 4.8. Let Γ_1 and Γ_2 be two admissible graphs with l vertexes. Γ_1 is said to be a degeneration of Γ_2 if Γ_1 is constructed from Γ_2 in one of the following ways:

(1) Adding a finite number of edges which preserve the admissibility conditions.

(2) A finite sequence of elementary moves preserving the admissibility conditions. By elementary moves I mean: Let a be the a-th vertex with some of its direct descendents marked as j_a. Let b be a vertex such that $a < b < j_b$ and b is not among those j_a. The elementary move from a to b is by constructing a new admissible graph adding

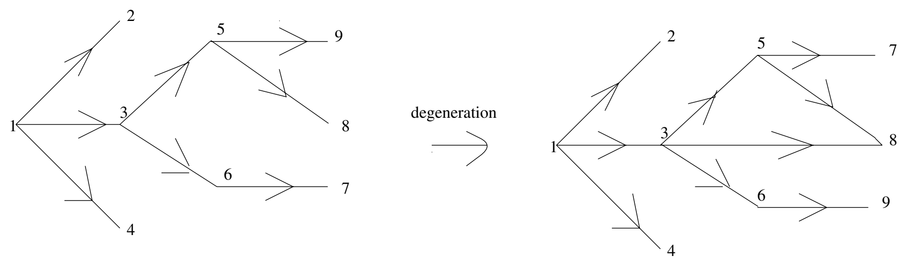

FIGURE 3. The old graph is of codimension 8 while the new graph is of codimension 9. A closed loop is formed by adding the new edge.

an edge from a to b while replacing the aj_a edges by the bj_a edges. That is to say, the vertex b inherits some of the direct descendents from the vertex a while declaring to be the direct descendent of a.

The simplest elementary move is to consider the vertexes marked $a < b < c$ and replace an edge from a to c with two edges from a to b, then from b to c. I denote it by $\Gamma_1 < \Gamma_2$ if Γ_1 is a degeneration of Γ_2.

In the definition of the elementary moves, one requires the existence of these j_a vertexes which are some of the direct descendents of a. If one allows the vertexes j_a to be vicious, then the elementary move simply reduces to the edges adding process described in (1). In this broader sense, the elementary moves contain the first type of edges adding processes as their degenerated situation. In the elementary moves defined above, the vertex a loses some of its direct descendents j_a by claiming that b is its direct descendent. However, it is not as painful as it might look. Because j_a becomes the direct descendents of b, they become the second generation descendents of a now.

Notice that the constraint of preserving the admissibility conditions is crucial here. An arbitrary edge adding or elementary move needs not preserve the admissibility conditions. Only those which preserve them are allowed in my definition of the degeneration of graphs.

Let us introduce some more notations for the latter usage.

Definition 4.9. The set Edge(Γ) is defined to be the set of all 1-edges of the admissible graph Γ. The set Ver(Γ) is defined to be the set of vertexes of Γ. By this notation, the codimension of an admissible graph is simply $|\text{Edge}(\Gamma)|$, the number of 1-edges in Γ.

The definition will be justified in a moment.

4.3.2. The admissible Stratification on M_l

In the following, I introduce and summarize in Proposition 4.2, Proposition 4.3 and Theorem 4.1 several important properties of the admissible stratification. The proofs of Proposition 4.2, Proposition 4.3 and Theorem 4.1 will appear in Section 4.4.

Let $\mathrm{adm}(l)$ denote the set of admissible graphs with l vertexes. Likewise let $\mathrm{adm}(\mathbf{I})$ denote the set of $\mathbf{I}$-admissible graphs. Then one has the following important assertions:

Proposition 4.2. *There exists a finite stratification of M_l, parameterized by the set $\mathrm{adm}(l)$ such that each strata is a locally closed (almost complex) algebraic set.*

Let the stratum corresponding to $\Gamma \in \mathrm{adm}(l)$ be denoted by Y_Γ, then it follows that $M_l = \cup_{\Gamma\in \mathrm{adm}(l)} Y_\Gamma$.

Theorem 4.1. *Let $f_l : M_{l+1} \mapsto M_l$ be the fiber-bundle formed by the tautological map of the universal spaces. Set $\mathbf{C}_b, b \in M_l$ to be the cone of exceptional curves for the l different blowing ups. Then $\mathbf{C}_b$ is a constant cone for all $b \in Y_\Gamma$.*

Moreover, the strata are of the right codimension with respect to the cones. More precisely, if $e_j, j \in J$ are the extremal basis generating the cone, then the equality $\dim_{\mathbf{R}} B - \dim_{\mathbf{R}} Y_\Gamma = \sum_{j\in J}(e_j \cdot K_X - e_j^2)$ holds.

The manifold X in the statement is diffeomorphic to $M\sharp^l\overline{\mathbf{P}}^2$ and the equality is valid for all $\Gamma \in \mathrm{adm}(l)$. Denote the constant cone over a stratum Y_Γ by $\mathbf{C}_\Gamma$ and define it to be the preexceptional cone attached to Y_Γ.

Notice that smooth -1 curves contribute 0 to the codimension. The stratification of M_l exists even if M is a high dimensional (almost complex) algebraic manifold. The key observation of the paper is that when $\dim_{\mathbf{R}} M = 4$, the current discussion has direct contact with Gromov-Taubes Theory.

Definition 4.10. The extremal effective generators of the preexceptional cones are called type I exceptional curves.

Those curves are special cases of the exceptional curves defined earlier in Section 4.1. Type I exceptional curves are characterized by the property of being the irreducible exceptional curves whose projection from the blown up manifold X to M is a point.

Definition 4.11. An exceptional curve whose projection from X to M is not trivial homologically is called a type II exceptional curve.

Proposition 4.3. *Let Y_Γ be the stratum corresponding to Γ. Then the compactification (closure) of Y_Γ in M_l is a smooth (almost complex) algebraic submanifold of M_l whose complex codimension is given by the codimension of the graph Γ. One denotes this compactification by $Y(\Gamma)$. Moreover, one has*

$$Y(\Gamma) = \coprod_{\Gamma' < \Gamma} Y_{\Gamma'},$$

the smooth stratification by the admissible graphs degenerated from Γ.

This proposition characterizes the boundary components of Y_Γ in terms of the degenerations of admissible graphs.

The proofs of Proposition 4.2, Proposition 4.3 and Theorem 4.1 will appear in Section 4.4.

By definition the codimension of a graph is formally defined by counting the number of edges. If Γ_1 is a degeneration of Γ_2, then it contains more edges then Γ_2 does, and is therefore of higher codimension.

From Proposition 4.3, the codimension of a graph is the same as the complex codimension of the stratum in M_l. This justifies the usage of the terminology.

Given the locally closed space Y_Γ, it forms the top stratum of $Y(\Gamma)$. I consider the preexceptional cone $\mathbf{C}_\Gamma$.

Let $E_i, 1 \leq i \leq l$ be the exceptional classes of the l different blowing ups in $X = M\sharp^l \overline{\mathbf{CP}}^2$, and let e_i (I slightly abuse the notation with the previous section about the extremal generators of an exceptional cone) be the generators of the cone of exceptional curves $\mathbf{C}_\Gamma$.

Lemma 4.1. *Given the admissible graph Γ, there exists a bijection between the vertexes of Γ with the extremal generators e_i. In particular, there are exactly l different e_i which generate the simplicial cone $\mathbf{C}_\Gamma$. Let e_i be associated with the ith vertex, and let j_i be the direct descendents of the i-th vertex. Then the class e_i is equal to $E_i - \sum_{j_i} E_{j_i}$.*

By using the lemma, one figures that the cones, along with the stratification, satisfy the key property of being the expected dimension. Namely, the stratum Y_Γ is of right codimension with respect to $\mathbf{C}_\Gamma$.

$$\mathrm{codim}(\Gamma) = -\sum_i \frac{(e_i^2 - e_i \cdot K)}{2},$$

where e_i are the extremal edges of $\mathbf{C}_\Gamma$.

Recall that the Fulton-McPherson space $M[l]$ also allows a stratification parameterized by some other types of graphs (trees) consisting of l marked vertexes and some indefinite number of unmarked vertexes. It turns out that there is a natural surjective morphism from $M[l]$ to M_l which carries the various strata into the certain union of the admissible strata. Different from the Fulton-MacPherson space which is $\mathbf{S}_l$ equivariant, the various strata of M_l are acted by different subgroup of $\mathbf{S}_l$, respectively. I will study this in detail as these automorphism groups appear naturally in the scheme of counting singular curves.

The fact that there exists a stratification on M_l does not surprise us. Because M_l can be described by $\frac{l(l-1)}{2}$ different blowing ups from $M^l = M \times M \times \cdots \times M$ along the various proper transformations of the codimension two partial diagonals, surely there is a stratification associated with it. The surprising thing is the crucial property of this particular stratification which satisfies my need from the previous subsection. Even though the Fulton-MacPherson spaces also carry a stratification, they do not constitute good candidates for my purpose. This property distinguishes the two families and thus "falsifies" the previous proposal in [29].

To describe how the stratification Y_Γ is constructed explicitly, I employ induction. It is not clear from this induction process that each $Y(\Gamma)$ is smooth. One answers this question from an alternative description of the stratification.

Given a vertex i, one considers its direct descendents in Γ and connects the edges between them. Then it forms a graph of the type described in Figure 4. The starting vertex might differ from 1. The subgraph associated with the vertex i is denoted symbolically by Γ^i. Given a Γ^i, one extends it to an element in $\mathrm{adm}(l)$ by the stabilization procedure. Namely, I add the free vertexes to Γ^i and I abuse the notation by denoting them as the same symbol. The resulting $Y(\Gamma^i)$ are given a special name $D(\Gamma^i)$. Then it follows that:

Proposition 4.4. *The space $Y(\Gamma)$ can be canonically identified with the transversal intersection $\cap_{i\in \mathrm{Ver}(\Gamma)} D(\Gamma^i)$. As each $D(\Gamma^i)$ is a complete intersection in M_l, the space $Y(\Gamma)$ is also a complete intersection.*

4.4 Inductive construction of Y_Γ

4.4.1. Proof of Propositions 4.2, 4.3 and Theorem 4.1

In this subsection, we construct the admissible stratification and prove Proposition 4.2, 4.3 and Theorem 4.1.

The first step is to give an explicit construction of the stratification and relate it with the admissible graphs. To construct the stratification, I define $Y(\Gamma)$ and Y_Γ by induction on l. The purpose is to understand the geometric structures of the strata. The geometric structure will be used explicitly if one wants to enumerate the family invariants explicitly.

First, let us consider $l = 1$ case. The only element in adm(1) is $\gamma = \gamma_1$, the trivial graph with one vertex. One defines $Y_\gamma = Y(\gamma) = M_1 = M$.

By forgetting the l-th vertex and the edges ending at it, a new graph $\Gamma(-1) \in \mathrm{adm}(l-1)$ is constructed from $\Gamma \in \mathrm{adm}(l)$. It is not hard to check that this process does not ruin the admissibility condition in ignoring the l-th vertex.

Suppose that for $l \mapsto l-1$, the space $Y(\Gamma(-1))$ has been defined already. I define $Y(\Gamma)$ inductively.

Consider $f_l : M_l \mapsto M_{l-1}$; the fiber bundle map. Given $Y(\Gamma(-1)) \subset M_l$, the space $Y(\Gamma)$ is defined to be a certain compact subvariety of $f_l^{-1}(Y(\Gamma(-1)))$. From the axioms characterizing the admissible graphs, the vertex l can have at most two direct ascendents. If l has no direct ascendent at all, then one says that l is a free vertex. One simply defines $Y(\Gamma)$ to be $f_l^{-1}(Y(\Gamma(-1)))$ in this case.

I slightly abuse the notation by denoting an exceptional curve and its cohomology class by the same symbol.

If the vertex has exactly one direct ascendent, say $p < l$, then one defines $Y(\Gamma)$ to be the set of the points $x \in f_l^{-1}(Y(\Gamma(-1)))$ which lie on the proper transforms of the exceptional set $E_p(l)$. On the other hand, I define $Y(\Gamma)$ to be the intersection of the $f_l^{-1}(Y(\Gamma(-1)))$ with the proper transformations of $E_p(l)$ and $E_q(l)$ if p and q are the direct ascendents of l. In this case $Y(\Gamma)$ is isomorphic to $Y(\Gamma(-1))$ by the projection map. I remark briefly on this point. When p and q are the direct ascendents of l, the three vertexes p, q and l form a triangle (loop). Suppose $p < q$, then p must be the direct ascendent of both q and l. It is obvious from the definition that the spaces $Y(\Gamma)$ are compact (proper).

According to the induction construction and $E_p(l) \cdot E_q(l) = 1$, the proper transformation of $E_p(l)$ and $E_q(l)$ must intersect each other transversally throughout $Y(\Gamma(-1))$.

Therefore, the spaces $Y(\Gamma)$ and $Y(\Gamma(-1))$ are isomorphic through the projection map.

To define the locally closed space $Y_\Gamma \subset Y(\Gamma)$, I follow a similar induction process. Let Γ and $\Gamma(-1)$ be defined as before. Suppose $Y_{\Gamma(-1)}$ is already defined, the space Y_Γ is chosen to be a certain open submanifold of $Y(\Gamma) \cap f_l^{-1}(Y(\Gamma(-1)))$.

If l is a free vertex, then one defines Y_Γ as the open submanifold of $f_l^{-1}(Y_{\Gamma(-1)})$ which consists of the complement of all the other exceptional sets.

If p is the only direct ascendent of l, I consider all the direct descendents of p, not including l. Suppose they are given by $p_1, p_2, \cdots p_s$, then I define Y_Γ by deleting from $f_l^{-1}(Y_{\Gamma(-1)}) \cap Y(\Gamma)$ its intersections with the proper transformations of the various other exceptional set $E_{p_i}(l)$. Suppose p and q are the direct ascendents of l and $p < q$. Let $p_1, p_2, \cdots p_s$ be the direct descendents of p other than q and l, then I define Y_Γ to be the intersection of $f_l^{-1}(Y_{\Gamma(-1)}) \cap Y(\Gamma)$.

By using mathematical induction, I define the spaces Y_Γ and $Y(\Gamma)$ for all $\Gamma \in \mathrm{adm}(l)$. It is rather easy to see that Y_Γ is a smooth, locally closed manifold. However, it is not transparent from the present construction that the compact space $Y(\Gamma)$ is always smooth. We postpone slightly the issue of smoothness by introducing another point of view of characterizing $Y(\Gamma)$, following the lead of Proposition 4.4.

The second step is to establish the relationship between geometric degenerations and graphical degenerations.

Now one is ready to prove the statement that $Y(\Gamma) = \cup_{\Gamma' \leq \Gamma} Y_{\Gamma'}$. First, one needs to show that the various strata $Y_{\Gamma'}, \Gamma' < \Gamma$ lie inside $Y(\Gamma)$. Then one shows that every point in $Y(\Gamma) - Y_\Gamma$ lies in some strata $Y_{\Gamma'}, \Gamma' < \Gamma$.

The graph Γ can be degenerated to a new graph by either repeatedly adding new edges between vertexes or by the elementary moves introduced before. Let us denote the new graph by Γ'.

If one introduces one new edge or some elementary move without involving the $l - th$ vertex, then it has corresponded to a degeneration of $\Gamma(-1)$ already. As by induction one can assume that $Y(\Gamma(-1))$ is the compactification of $Y_{\Gamma(-1)}$ by the strata associated the various degenerations of $\Gamma(-1)$. These degenerations of Γ are included in the space $Y(\Gamma)$.

From now on, let us consider the degenerations which involve the last vertex. One simple possibility involves adding new edges ending at

l. By the axioms of admissible graphs, the vertex l has at most two direct ascendents. Therefore, there are two cases to consider. Either one adds an edge from some p to l or one adds two edges from $p < q$ to l. In both cases, it can be shown easily that the stratum associated with the new graph lies in $Y(\Gamma)$.

A slightly subtle issue is about the elementary moves. As l has been the vertex with a largest marking, an elementary move involving l must consist of the following operations. Let $a < b < l$ be the markings of the three different vertexes and l is the direct descendent of a. Then one eliminates the edge al along with the edges from a to some of its direct descendents, then forms a edges ab along with the edges from b to these direct descendents of a (including l). Let the resulting graph be denoted by Γ'. As this particular process is assumed to preserve the admissible conditions, then the vertexes a and b must be the only two possible candidates for the direct ascendent vertexes of l. This follows from the Axiom 4.4 of defining the admissible graphs. From here one deduces that the edge al must be the only edge in the graph Γ linking to l. Erasing the edge al and the vertex l simply moves Γ to $\Gamma(-1)$. As one recalls from the construction, the stratum Y_Γ can be constructed from $f_l^{-1}(Y_{\Gamma(-1)}) \cap Y(\Gamma)$ by removing the intersections with the various other exceptional curves. By adding the new edge ab and the edges from b to some of the direct descendent vertexes of a other than l, it has corresponded to some degeneration of $\Gamma(-1)$, denoted by $\Gamma(-1)'$. As one has inductively assumed that the space $Y(\Gamma(-1))$ consists of the finite unions of strata associated with all the admissible degenerations of $\Gamma(-1)$'s, one finds that $f_l(Y_{\Gamma(-1)'})$ lies in $Y(\Gamma(-1)) - Y_{\Gamma(-1)}$. Moreover, by adding the new edge bl, the space $Y_{\Gamma'}$ is formed by restricting fiber-wise to the open curve; the proper transform of $E_b(l) - \cup_{j_b \neq l} E_{j_b}(l)$. As $f_l(Y(\Gamma'))$ has already been assumed to lie in $Y(\Gamma(-1))$, the fact that $Y_{\Gamma'}$ lies in $Y(\Gamma)$ is derived by noticing that the proper transformation of $E_a(l)$ splits into two components, one involving the proper transformation of $E_a(l)$ by $E_b(l)$, etc. ; and another involving the proper transformation of $E_b(l)$ by $E_l(l)$, etc. The other elementary moves succeeding the one can involve the l-th vertex as well. Then one argues in a similar manner.

This finishes the proof that $\cup_{\Gamma' < \Gamma} Y_{\Gamma'} \subset Y(\Gamma)$. To prove the other direction, let x be an arbitrary point lying inside $Y(\Gamma) - Y_\Gamma$, one would like to show that it lies inside some $Y_{\Gamma'}$, with $\Gamma' < \Gamma$.

Let us consider the image point $f_l(x)$. There are two possibilities according to my earlier construction. Either $f_l(x) \in Y_{\Gamma(-1)}$ or $f_l(x) \in$

$Y(\Gamma(-1)) - Y_{\Gamma(-1)}$.

Consider the possibility $f_l(x) \in Y_{\Gamma(-1)}$ first. If the vertex l is a free vertex in Γ, then x must lie on some of the exceptional curves of the fibers of $M_{l+1} \mapsto M_l$ or x would have been in Y_Γ already. If it lies in a single exceptional curve, then it lies in the stratum given by the admissible graph adding from $\Gamma(-1)$ an edge from the corresponding vertex to the l-th vertex. Likewise, if it lies at the intersection of two exceptional curves, then it lies in the stratum associated with the graph described below. One simply adds two edges from the two corresponding vertexes to l from $\Gamma(-1)$. As the curves are all of normal crossing (which is associated with the fact that the exceptional divisors in M_l are of simple normal crossing), two is the maximum number of edges one can add at once. Both types of graphs can be degenerated from the original Γ.

Suppose that $f_l(x) \in Y(\Gamma(-1)) - Y_{\Gamma(-1)}$, then $f_l(x)$ must lie in some stratum which corresponds to certain degeneration of $\Gamma(-1)$. This follows from the induction hypothesis. If l is a free vertex in Γ, then x lies in the stratum formed by adding the l-th vertex to the given degeneration of $\Gamma(-1)$, denoted by $\Gamma(-1)'$.

If the vertex l in the graph Γ has exactly one direct ascendent, then x can sit on an arbitrary point in the exceptional curve lying above $f_l(x)$. Let p denote the direct ascendent of l. If the point x lies in the p-th component of the exceptional curve (which is associated with the p-th vertex), then x should lie in the stratum associated with the graph by adding one or two edges ending at l. It depends on whether x lies at the intersection of the two different exceptional curves. In the current situation, it is easy to construct the degeneration of graphs.

If the point x lies in a component other than the p-th one, e.g., the k-th one, then there must be a chain of rational curves linking the p-th one to the k-th one. This fact introduces a sequence of vertexes lying in-between the p-th and the l-th vertexes. From this fact and the construction of the various strata, the point x is in a stratum associated with the new graph Γ', with a linear chain from p to l.

My goal is to prove that the graph Γ' does come from a certain degeneration of Γ. As we know, $\Gamma(-1)'$ does come from the degenerations of $\Gamma(-1)$. That is to say, there exists a finite sequence of edges adding or elementary moves which transform $\Gamma(-1)$ to $\Gamma(-1)'$ in $\mathrm{adm}(l-1)$.

Suppose that the vertexes of $\Gamma(-1)'$ among the linear chain between p and l are denoted by $p = p_1, p_2, p_3, \cdots p_r = l$. It is not clear that the chain is unique. One can still argue that one can always choose the chain such that additionally p_2 is not the direct descendent of $p_1 = p$ in Γ. To prove this statement, one uses the following lemma, which follows from Sacks-Uhlenbeck-Gromov compactness theorem as well as the specialization theorem in algebraic geometry.

Lemma 4.2. *Let $\mathcal{X} \mapsto B$ be a fiber-wise almost complex fiber bundle tamed by a family of fiberwise symplectic form. The base space B is a compact base space. Suppose a fiberwise class T is represented by compact pseudo-holomorphic curves over any set $U \subset B$, then it can be represented by pseudo-holomorphic curves over $\overline{U}$.*

Proof. The lemma follows directly from Sacks-Uhlenbeck-Gromov compactness theorem as well as the specialization theorem in algebraic geometry. q.e.d.

The lemma provides us an effective way to check that whether a point $b \in B$ can be the limit of a sequence of points $b_n, n \in N$.

Consider the exceptional curve dual to a cohomology class of the following form $E_p - \sum E_{j_p}$, where the indexes j_p run through all the direct descendents of p in Γ. According to the assumption, the vertex l is in the list.

By the previous lemma, the same cohomology class should be represented by certain pseudo-holomorphic curves over x, as x is assumed to lie in the closure of Y_Γ. However, as l is not the direct descendent of p in Γ', the term $-E_l$ would show up only through adding up some other exceptional class where the term E_l shows up with a -1 coefficient. Given the exceptional curve such that the term $-E_l$ shows up in its cohomology class, the class E_a, with a being the direct ascendent of l, must show up with a positive sign.

On the other hand, p is not the direct ascendent of l in Γ', neither. Therefore, there must be some other exceptional curve showing up such that the term E_a is cancelled out. Arguing in this way, one traces back to form a chain in the graph Γ'. As Γ' is a finite graph, the process must be terminated somewhere. The last vertex must be p as E_p is the only term with positive coefficient in $E_p - \sum E_{j_p}$. As a result, one constructs a chain in Γ'. The chain can be shown to be non self-intersecting. If one writes down the effective combination, it follows easily that the net coefficient in front of E_{p_2} must be zero. Otherwise, it leads to certain contradiction on the coefficients of the other E_{p_i}. On the other hand, as $E_p - \sum_{j_p} E_{j_p}$ is equal to this effective expression in terms of the original exceptional basis E_i, etc. , none of the j_p can be equal to p_2. As a result,

p_2 is not the direct descendent of $p = p_1$ in Γ. Hence, one has succeeded in finding a chain of vertexes in Γ', starting from p and ending at l, such that p_2 is not the direct descendent of p in the original graph Γ.

As it was assumed at the initial stage that l is not the direct descendent of p in Γ', then l can not be equal to p_2. Then p_2 is not the direct descendent of p in $\Gamma(-1)$ either.

By the induction hypothesis, the graph $\Gamma(-1)$ degenerates into $\Gamma(-1)'$ in $\mathrm{adm}(l-1)$. Consider that the first time that the edge pp_2 is added to an intermediate graph. For every admissible graph proceeding this, one prolongs it to an admissible graph in $\mathrm{adm}(l)$ by adding the edge pl. In adding the pp_2 edge, either one simply adds it or it is an elementary move from p to p_2. In either case, one can prolong the graph alternatively by adding the edge p_2l instead. As a result, the transformation is interpreted as an elementary move from p to p_2 in $\mathrm{adm}(l)$.

Now, one replaces the vertex p by p_2 and discuss similarly as above. Each time one replaces the vertex p by its direct descendent in the linear chain. As there is only a finite number of vertexes involved, the process must be terminated and thus one shows that the new graph Γ' can be reached from Γ by some edges adding or some sequences of elementary moves. q.e.d.

4.4.2. The alternative interpretation of $Y(\Gamma)$ as transversal intersection

As was mentioned, it is not obvious from the construction in Section 4.4.1 that the compact spaces $Y(\Gamma)$ for the various $\Gamma \in \mathrm{adm}(l)$ are smooth. It is my next goal to give an alternative construction to clarify this.

In the set $\mathrm{adm}(l)$, there is always an element called γ_l, or simply γ, which consists of l free vertexes.

It is clear that Y_γ parameterizes all the l distinct points in M. It is also clear that $Y(\gamma) = M_l$ is smooth.

Let us consider the next simplest admissible graphs containing an edge between a pair of vertexes in γ. The resulting graph has exactly one edge and there are exactly C_2^l of them.

Proposition 4.5. *Let $\Gamma_{a,b}$ be the admissible graph described above and $a < b$ are the two vertexes involved, then $Y(\Gamma_{a,b})$ is the proper transformation of the exceptional divisor which corresponds to blowing up the diagonal $\Delta_{ab}; x_a = x_b$ in M^l, denoted as D_{ab}.*

$Y(\Gamma_{a,b})$ are the divisors in M_l which form the building block of all the exceptional strata. It is not hard to derive that these divisors are Poincare dual to the class $E_a(b)$ for $a < b$. Given an edge ϵ of an admissible graph, one uses $s(\epsilon)$ to denote the starting marking and $e(\epsilon)$ to denote the ending mark.

Let us introduce a linear ordering between different edges. Let ϵ_1 and ϵ_2 be two different edges of an admissible graph Γ, then ϵ_1 is said to be greater than ϵ_2 if either $e(\epsilon_1) < e(\epsilon_2)$ or if $e(\epsilon_1) = e(\epsilon_2)$ one requires that $s(\epsilon_1) < s(\epsilon_2)$.

Naively, one might expect that the following relationship holds:

$$Y(\Gamma) \approx \cap_{\epsilon\in \mathrm{Edge}(\Gamma)} D_{s(\epsilon)e(\epsilon)}.$$

However, as can be easily seen, the space on the right hand side is usually reducible and, therefore, cannot be equal to the smooth manifold expected from the left hand side of the equality. Instead, the following proposition is introduced for a more generalized treatment:

Proposition 4.6. *Let Γ_1 and Γ_2 be two different admissible graphs such that Γ_2 is obtained from Γ_1 by adding a new edge ϵ smaller than the others in* Edge(Γ_1). *Then $D_{s(\epsilon)e(\epsilon)} \cap Y(\Gamma_1)$ is an irreducible divisor in $Y(\Gamma_1)$ which is also the total transformation of a smooth divisor under blowing ups. The closed subspace $Y(\Gamma_2)$ is identified with the proper transformation of $D_{s(\epsilon)e(\epsilon)} \cap Y(\Gamma_1)$ in $Y(\Gamma_1)$, which is also one of the $D_{s(\epsilon)e(\epsilon)} \cap Y(\Gamma_1)'s$ irreducible components.*

As it is the proper transformation of a smooth exceptional divisor under the successive blowing ups with smooth centers, the smoothness follows by induction easily. Let us consider the admissible graph $\Gamma \in \mathrm{adm}(n)$ as in Figure 4. Then one can see that graphs of this type form the basic building blocks of an arbitrary admissible graph.

In the following, we prove Proposition 4.4.

Proof. The set theoretical identity $Y(\Gamma) = \cap_{i\in \mathrm{Ver}(\Gamma)} D(\Gamma^i)$ can be proved by arguing that $Y_\Gamma \mapsto \cap_{i\in \mathrm{Ver}(\Gamma)} D(\Gamma^i)$ is dense. By the inductive construction of Y_Γ, $Y_\Gamma \subset D(\Gamma^i), i \in \mathrm{Ver}(\Gamma)$. I.e., the combinatorial condition j_i being a descendent of i corresponds to the geometric construction of requiring $f_{l-1} \circ f_{l-2} \cdots \circ f_{j_i-1}(Y_\Gamma) \subset M_{j_i}$ to lie within the i-th exceptional divisor of $M_{j_i} \mapsto M \times M_{j_i-1}$.

The combinatorial fact that j_i is a direct descendent of i imposes a relative open condition whose compliment is of higher codimension. Thus, Y_Γ is dense in $\cap_{i\in \mathrm{Ver}(\Gamma)} D(\Gamma^i)$.

Next, I prove the transversality of the intersections. The proof is based on induction and the blowup construction of M_l.

Consider the smallest edge ϵ in Γ according to the partial ordering; $\epsilon_1 < \epsilon_2$ if $(i).e(\epsilon_1) < e(\epsilon_2)$, or $(ii).e(\epsilon_1) = e(\epsilon_2)$, yet $s(\epsilon_1) < s(\epsilon_2)$. If the ending vertex $e(\epsilon)$ is not marked by l, then the l-th vertex must be a free vertex. The transversality of $\cap Y(\Gamma^i)$ follows from the induction assumption in M_{l-1}.

Thus, I can assume that $e(\epsilon)$ is marked by the integer l. According to the axioms of admissible graphs, either

(a) l has one direct ascendent, or

(b) l has two direct ascendents.

Consider the admissible graph $\Gamma(-1)$ by removing the l-th vertex and the edges linking to its direct ascendents. By induction hypothesis, $Y(\Gamma(-1))$ has been the transversal intersection of the $D(\Gamma(-1)^i)$. If l has exactly two direct ascendents, then the construction of $Y(\Gamma)$ implies that $Y(\Gamma) \mapsto Y(\Gamma(-1)) \subset M_{l-1}$ induces an isomorphism. In particular $Y(\Gamma)$ is smooth of the correct dimension. To prove the transversality of the intersection, one reduces it to the case of (a): Consider $\Gamma^{(0)}$ and $\Gamma^{(1)}$ to be the one edge extension of $\Gamma(-1)$ by adjoining either of the edges linking l. Consider $Y(\Gamma) \subset f_{l-1}^{-1}(Y(\Gamma(-1)))$ as the intersection of $Y(\Gamma^{(0)})$ and $Y(\Gamma^{(1)})$. Suppose (a) has been handled and both $Y(\Gamma^{(i)}), i = 0, 1$ have been known to be transversal intersections, then $Y(\Gamma)$ is also a transversal intersection once one checks that the intersection $Y(\Gamma^{(0)}) \cap Y(\Gamma^{(1)})$ is transversal. This can be seen as the fiberwise exceptional $\mathbf{P}^1$ are of simple normal crossing.

Thus, I reduce to the case (a) Suppose ϵ is the unique edge in Γ ending at l. I use the fact that M_l is constructed by $M_{l-1} \times M$ through a sequence of $l - 1$ complex codimension two blowing ups marked by $(j, l), j \leq l - 1$.

Suppose that $s(\epsilon)$ is marked by $l - 1$, then $Y(\Gamma)$ has a $\mathbf{P}^1$ fiber bundle structure over $Y(\Gamma(-1))$. Because it is the exceptional divisor $\subset f_{l-1}^{-1}(Y(\Gamma(-1)))$ of the last blowing up marked $(l-1, l)$. The transversality of the intersection $\cap Y(\Gamma^i)$ follows from the induction hypothesis on $Y(\Gamma(-1))$. If $s(\epsilon)$ is marked by $j_0 < l-1$, then one has to discuss the effect of the $(j, l), l - 1 \leq j > j_0$ blowing ups on the $\mathbf{P}^1$ bundle structure. Take Y_{j_0+1} to be the $\mathbf{P}^1$ bundle which is the exceptional divisor in $f_{l-1}^{-1}(Y(\Gamma(-1)))$ determined by the (j_0, l) blowing up. Let $Y_k; k \geq j_0 + 2$ denote the proper transformation of Y_{k-1} under the $(k - 1, l)$ blowing ups. Then Y_l is $Y(\Gamma)$.

It is enough to prove that $Y(\Gamma)$ is a smooth divisor in $f_{l-1}^{-1}(Y(\Gamma(-1)))$. The smoothness of $Y(\Gamma)$ follows from the fact that the centers of the subsequential blowing ups $(j,l), l-1 \leq j > j_0$ are all smooth. It is argued as below. The blowing ups centers are sections of the intermediate fiber bundles which are all birational to $f_{l-1}^{-1}(Y(\Gamma(-1)))$. The (j,l) blowup center intersects each of the fibers of $Y_{j-1} \mapsto Y(\Gamma(-1))$ in at most a single point. I characterize the intersection as the following.

The projection map $Y_{j-1} \mapsto Y(\Gamma(-1))$ induces an isomorphism from the intersection to its image. On the other hand, its image in $Y(\Gamma(-1))$ can be identified as $Y(\Gamma(-1)'), \Gamma(-1)' < \Gamma(-1)$ or the empty set $\emptyset$. The intersection condition forces the $j-1$-th vertex to be the direct descendent of j_0-th vertex. If $p = j-1$-th vertex is free or if j_0 has been its unique direct ascendent in $\Gamma(-1)$, the process stops and it ends up in an admissible $\Gamma(-1)'$. If $p = j-1$-th vertex has other direct ascendents other than j_0, the admissibility condition implies immediately that the direct ascendents of $j-1$-th vertex in $\Gamma(-1)$ (other than j_0) to be the direct descendents of j_0. Either the process will terminate which results in a unique $\Gamma(-1)'$ or it will eventually lead to a new graph violating some axiom of admissibility conditions. In the latter case, the intersection is empty.

By induction hypothesis, $Y(\Gamma(-1)')$ has known to be a transversal intersection of $D(\Gamma(-1)'^{i})$ and therefore it is smooth. Therefore, the intersection of the blowing up centers with Y_{j-1} is smooth. Thus, Y_j, the proper transformation, is smooth, too.

$Y(\Gamma)$ is a smooth divisor in $f_{l-1}^{-1}(Y(\Gamma(-1)))$, which is known to be the preimage of a transversal intersection by applying induction hypothesis. Then $Y(\Gamma)$ is also a transversal intersection. q.e.d.

The locus $D(\Gamma^i)$ plays a special role in curve theory. Let us consider the cohomology class denoted by $E_i - \sum_{j_i} E_{j_i}$. Then $D(\Gamma^i)$ is the locus that the class is represented by (pseudo)-holomorphic curves. The interior D_{Γ^i} is the locus that the same class is represented by a smooth irreducible (pseudo)-holomorphic curve.

By combining this interpretation and Proposition 4.4, one finds the following interpretation for the locus $Y(\Gamma)$.

Proposition 4.7. *Let Γ^i denote the admissible graph derived from Γ as was explained before. Through the identification*

$$Y(\Gamma) = \cap_{i \in \mathrm{Ver}(\Gamma)} D(\Gamma^i),$$

the space $Y(\Gamma)$ *is the locus that these different cohomology classes* $E_i(l) - \sum_{j_i} E_{j_i}(l), i \in \mathrm{Ver}(\Gamma)$ *are simultaneously represented by pseudo-holomorphic curves. Similarly, the interior* Y_Γ *of* $Y(\Gamma)$ *is the locus that the type I curves are all smooth and irreducible. The boundary of* Y_Γ *describes the various degenerations of the exceptional curves.*

On the other hand, any $\Gamma \in \mathrm{adm}(l)$ can be viewed as a degeneration from the trivial graph γ_l without any edges, it follows that $M_l = Y(\gamma) = \cup_{\Gamma<\gamma} Y_\Gamma$ defines a stratification of M_l.

By the induction hypothesis, $\mathrm{codim}(Y(\Gamma)) = \mathrm{codim}(Y(\Gamma(-1))) + 1$ if l has only one direct ascendent. $\mathrm{codim}(Y(\Gamma)) = \mathrm{codim}(Y(\Gamma(-1))) + 2$ if l has exactly two direct ascendents. On the other hand, $1 + \mathrm{codim}(\Gamma(-1)) = \mathrm{codim}(\Gamma)$ in the former case while $\mathrm{codim}(\Gamma) = \mathrm{codim}(\Gamma(-1)) + 2$ in the latter case. Then $\mathrm{codim}_{\mathbf{C}}(Y(\Gamma))$ must be equal to $\mathrm{codim}(\Gamma)$ for all $\Gamma \in \mathrm{adm}(l)$.

It follows that the space Y_Γ is the top strata of $Y(\Gamma)$. According to Proposition 4.3, the codimension of the strata should be counted by $\sum_{i\in Ver(\Gamma)} n_i$. The n_i is the number of direct descendents of i-th vertex, which is also the number of edges initiating from i. Geometrically, the equality can be understood as follows. Each Γ_i is associated with a $-n_i - 1$ rational curve. According to the general fact of rational curves in the algebraic surfaces, its expected dimension is given by $-n_i$.

Next let us consider the preexceptional cone $\mathbf{C}_\Gamma$.

Let $E_i(l), 1 \leq i \leq l$ be the exceptional classes of the l different blowing up in $X = M\sharp^l\overline{\mathbf{CP}^2}$. Let e_i be the extremal generators of the cone of exceptional curves in $\mathbf{C}_\Gamma$.

Lemma 4.3. *Given the admissible graph* Γ*, there exists a bijection between the vertexes of* Γ *with the extremal rays* e_i*. In particular there are exactly* l *different* e_i *which generate the cone* $\mathbf{C}_\Gamma$*. The cone* $\mathbf{C}_\Gamma$ *is simplicial and is generated by* l *different edges. Let* e_i *be associated with the* i*-th vertex and let* j_i *be all the direct descendents of the vertex* i*. Then the class* e_i *is dual to to* $E_i - \sum_{j_i} E_{j_i}$*.*

As the number of direct descendents of i is the same as the number of edges starting from i, the sum $\sum_i \left(\frac{e_i\cdot K_X - e_i^2}{2}\right)$ is equal to the number of edges in Γ^i. From here I conclude that the stratification Y_Γ is of right dimension with respect to $\mathbf{C}_\Gamma$. The exceptional cones discussed in my paper will be some simplicial subcones of the preexceptional cones $\mathbf{C}\Gamma$.

4.5 The automorphism groups of the strata Y_Γ

Given an element $g \in \mathbf{S}_n$, it permutes the vertexes with different markings. It is clear from some simple examples that it does not extend to an action on $\mathrm{adm}(n)$. Likewise, the manifold M_n does not allow a natural $\mathbf{S}_n$ action. To study how does the $\mathbf{S}_n$ action fails to be extended, one can compare it with the Fulton-MacPherson space $M[n]$ which is known to admit a natural $\mathbf{S}_n$ action.

The reason that the $\mathbf{S}_n$ action does not extend on M_n is due to the asymmetry of the blowing ups. Notice that $M[n]$ and M_n can be constructed from M^n by respectively $2^n - n - 1$ and $\frac{n(n-1)}{2}$ blowing ups.

Given an admissible graph $\Gamma \in \mathrm{adm}(n)$, I define two subgroups of $\mathbf{S}_n$ attached to Γ. They show up in the curve counting scheme naturally.

Definition 4.12. Let G_Γ denote the subgroup of $\mathbf{S}_n$ which preserves the admissible graph Γ. Define the group $G(\Gamma)$ to be the subgroup of $\mathbf{S}_n$ which permutes the graph while preserving the admissibility condition. The group $G(\Gamma)$ contains G_Γ as its subgroup such that $G(\Gamma) \cdot \Gamma \subset \mathrm{adm}(n)$.

Moreover, the next proposition summarizes the basic property of the action of G_Γ on Y_Γ.

Proposition 4.8. *The group G_Γ acts naturally and freely on the locally closed space Y_Γ. Similarly, the group element $g \in G(\Gamma)$ acts on Y_Γ which maps the space Y_Γ to the space $Y_{g\Gamma}$.*

The action of G_Γ does not always extend to an action on the closure $G(\Gamma)$. Likewise, the action of $G(\Gamma)$ does not always extend to an action from $Y(\Gamma)$ to $Y(g\Gamma)$.

The introduction of these groups simplifies dramatically the complicated counting of the combinatorial factors. Proposition 4.8 will be used to correct the multiplicity of modified invariants.

Proof. To prove this proposition, one introduces the concept of order function on $\mathrm{Ver}(\Gamma)$.

Let $\Gamma \in \mathrm{adm}(n)$ be an admissible graph with n vertexes. There are always some vertexes which have no direct ascendents.

Definition 4.13. Define the order function on $\mathrm{Ver}(\Gamma)$ first by requiring that the order function takes value 0 on these vertexes.They are called the order zero vertexes. Then the order function takes value $s \in N$ if s is equal to the minimal number of edges in the oriented paths linking an order zero vertex to this specific vertex.

A vertex is said to be an order s vertex if the order function takes value s on the vertex. Alternatively, the vertexes of order s are the direct descendents of vertexes of order $s-1$. I will use the order function to "stratify" the graph. A crucial property of the order function is its invariance under the G_Γ action, preserving the graph Γ.

Given an admissible graph Γ, one defines a sequence of increasing subgraphs by restricting to the s order vertexes $s \leq k$ and the edges linking them. Thus, there is a sequence of monotonously increasing graphs $\Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_n = \Gamma$. It is also easy to see that one can assign a sequence of spaces X_{Γ_i}, $X_{\Gamma_n} = Y_\Gamma$ to these graphs such that $X_{\Gamma_i} \mapsto X_{\Gamma_{i-1}}$ has a fibration structure for all $i \leq n$. The fibration structure will be used in the explicit enumeration as being shown later. The graphs Γ_i are all G_Γ invariant due to the G_Γ invariance of the order function. Notice that Γ^i and Γ_i are different symbols, representing totally different objects.

Suppose that the graph Γ has d different order zero vertexes, then one chooses X_{Γ_0} to be isomorphic to Y_γ in M_d, with $\gamma \in \mathrm{adm}(d)$.

Based on the simple observation that a order s, ($s > 0$) vertex can always be connected to Γ_{s-1} by an edge in Γ, one inductively constructs the fiber bundle structure similar to what was done in Subsection 4.4. After the fiber bundles are constructed, it can be easily shown that all the spaces X_{Γ_i} allow G_Γ actions. In particular, the space $Y_\Gamma = X_{\Gamma_n}$ also allows a G_Γ action.

The construction of the spaces X_{Γ_i} is basically identical to the induction construction of the spaces Y_Γ in Subsection 4.4. Instead of doing induction based on the markings, one inductively constructs X_{Γ_i} based on the usage of the order function. As it does not involve any new idea, I skip the details of the construction. Let us briefly address the freeness of the G_Γ action. I prove it by induction on i.

Let $\phi_i : G_\Gamma \mapsto \mathbf{S}_{|\mathrm{Ver}(\Gamma_i)|}$ be the natural representation acting on $\mathrm{Ver}(\Gamma_i)$. Then one proves inductively that $\phi_i(G_\Gamma)$ acts on X_{Γ_i} freely. It is clear that the image group $\phi_0(G_\Gamma)$ acts on X_{Γ_0} freely. Assume that the action of $\phi_i(G_\Gamma)$ on X_{Γ_i} is free, one inductively proves it for $i \mapsto i+1$. Suppose that there is an element $e \neq g \in G_\Gamma$ which fixes a point $x \in X_{\Gamma_{i+1}}$, then the image of x under the fiber bundle map $X_{\Gamma_{i+1}} \mapsto X_{\Gamma_i}$ is also fixed by g. By the free action assumption on the $\phi_i(G_\Gamma)$ action, the element g must lie in $ker(\phi_i)$. Thus, the element g fixes $\mathrm{Ver}(\Gamma_i)$ while it permutes nontrivially some of the vertexes in $\mathrm{Ver}(\Gamma_{i+1}) - \mathrm{Ver}(\Gamma_i)$.

Choose a vertex v in $\mathrm{Ver}(\Gamma_i)$ such that some of its direct descendents lie in $\mathrm{Ver}(\Gamma_{i+1})-\mathrm{Ver}(\Gamma_i)-\mathrm{Ver}(\Gamma_{i+1})^{G_\Gamma} \neq \emptyset$; the vertexes not being fixed by G_Γ.

As v is fixed under g and the whole graph Γ is also invariant under g, the direct descendents of v must be permuted among each other. One classifies the direct descendent vertexes into two classes. First, one considers these vertexes which have only one direct ascendent v. Then one considers the class of vertexes which consists of those descendents having another direct ascendent other than v. The action of g upon $\mathrm{Ver}(\Gamma_{i+1})$ must not mix up the two different classes. Let the set of the vertexes in the first (second) class be denoted by $\triangle_1$ ($\triangle_2$), respectively. It is crucial to notice that $\triangle_1$ is nonempty by the axioms of the admissible graphs and by induction.

Consider the fiber of $X_{\Gamma_{i+1}} \mapsto X_{\Gamma_i}$ passing through x. Then this fiber is preserved by G_Γ, as is its image. From the inductive construction, this fiber can be decomposed into the direct products of different factors. Among the different factors, I am particularly interested in a single direct factor which is of complex $|\triangle_1|$ dimension. It can be easily shown that the factor is isomorphic to an open subset of the space $(\mathbf{CP}^1)^{|\triangle_1|}$ such that the element g acts by permuting the elements in $\triangle_1$.

Let $\bar{x}$ be the image of x projecting into this direct factor. As x is fixed by g, $\bar{x}$ must be fixed by $\bar{g}$ also. Therefore, it indicates that the image point $\bar{x}$ must lie in the fixed point set of $(\mathbf{CP}^1)^{|\triangle_1|}$ under the action of g. Thus, it must lie in the big diagonal of $(\mathbf{CP}^1)^{|\triangle_1|}$. However, this indicates that among the elements in $\triangle_1$, there is some vertex which is the direct ascendent of the others. This immediately implies that some element in $\triangle_1$ has more than one direct ascendent. This creates a contradiction as by definition the vertexes in $\triangle_1$ can only have one direct ascendent, namely v. The previous argument relies on the assumption $|\triangle_1| > 1$, which can be proved by contradiction and by Axiom 4.5 of the admissible graphs.

This contradiction rules out the possibility of the existence of g and x, and therefore, the action of $\phi_{i+1}(G_\Gamma)$ on $X_{\Gamma_{i+1}}$ is free. By induction process, this applies to $X_{\Gamma_m} = Y(\Gamma)$ and $\phi_n(G_\Gamma) = G_\Gamma$ as well.

This completes the proof of the proposition. q.e.d.

In the following, I give an example to illustrate the difference between G_Γ and the automorphism group of Y_Γ.

Remark 4.3. Even though G_Γ is a subgroup $\subset \mathbf{S}_n$ which acts upon Y_Γ freely, it may not be the largest subgroup of $\mathbf{S}_n$ that can act on Y_Γ. Let $\Gamma \in \mathrm{adm}(2)$ be the admissible graph formed by the arrow from 1 to 2. As M_2 can be constructed by blowing up $\Delta_M \subset M \times M$, the space M_2 admits an $\mathbf{S}_2 = \mathbf{Z}_2$ action. The stratum Y_Γ corresponds to the exceptional divisor under the blowing up. The group $\mathbf{Z}_2$ acts on M_2, its action nevertheless fixes the whole space Y_Γ completely. Because the arrow from 1 to 2 is oriented, the action which switches these two vertexes violates the admissibility condition. Therefore, $G_\Gamma = \{id\}$.

Let us address the geometric meaning of the admissible graphs in terms of pseudo-holomorphic curve theory. Let $\mathcal{X} \mapsto Y_\Gamma$ be the induced fiber bundle by pulling back $M_{n+1} \mapsto M_n$ through the embedding $Y_\Gamma \subset M_n$. Then the combinatorial information of the admissible graph can be translated into the dual graph of the exceptional curves of the fibers. Degenerations of the admissible graphs correspond to the degenerations of the exceptional curves. As was remarked before, the preexceptional cone $\mathbf{C}_\Gamma$ is a constant cone throughout Y_Γ. Through this identification, the various graphs $\Gamma^i, i \in \mathrm{Ver}(\Gamma)$ correspond to the irreducible exceptional curves which form the extremal generators of the cone $\mathbf{C}_\Gamma$. It is clear that the group action of G_Γ and $G(\Gamma)$ upon Y_Γ can be lifted to be the actions upon the preexceptional cones. The various Γ^i are permuted under G_Γ.

An arbitrary bi-holomorphic automorphism on Y_Γ may not necessarily induce an automorphism on the fiber bundle $\mathcal{X}|_{Y_\Gamma} \mapsto Y_\Gamma$. I am interested in identifying the automorphism group, which acts on the fiber bundle making the projection map an equivariant map.

Recall that there is a blowing down morphism from $\mathcal{X}|_{Y_\Gamma}$ to $Y_\Gamma \times M$, which is denoted by π.

Proposition 4.9. *The projection map π induces a homomorphism from $Aut_f(\mathcal{X})|_{Y_\Gamma}$ to $Aut_f(Y_\Gamma \times M)$ denoted by $\pi_\sharp$. The preimage of $Aut_f(Y_\Gamma) \times id$ by $\pi_\sharp^{-1}$ is said to be the exceptional automorphism group of the fiber bundle. The subgroup of the exceptional automorphism group of $\mathcal{X}|_{Y_\Gamma} \mapsto Y_\Gamma$, which preserves the fiber bundle structure $\mathcal{X} \mapsto Y_\Gamma$, is isomorphic to G_Γ.*

Proof. To prove the statement, one notices that the group acts upon the total space $\mathcal{X}$, which lifts the group action upon Y_Γ. Then the group induces an action on the fiber-wise H_2 and, therefore, the effective preexceptional cone, $\mathcal{C}_\Gamma$. By the assumption in Proposition 4.9, the elements in the exceptional automorphism group act trivially outside the exceptional locus. Thus, one is able to concentrate to study its action on the exceptional curves. As the preexceptional cone is kept unchanged throughout Y_Γ, the induced action of the group on the cone must keep the cone invariant while permuting the individual extremal generators. Through the identification between these extremal exceptional curves and the graphs Γ^i, it induces an action on the set of Γ^i's. As these various Γ^i form the building block of the graph Γ, the graph itself is also kept invariant , with different vertexes being permuted. This is a consequence of the fact that the diffeomorphisms induce isometries on the preexceptional cones.

By this argument one constructs a surjection from the specific subgroup of the exceptional automorphism group to G_Γ. To see that the kernel of the group morphism is trivial, it is sufficient to show the following statement: If the element induces a trivial action on Y_Γ, as well as the preexceptional cone, it must be the identity.

I omit the proof as it follows from the standard theory on 2 dimensional blowing ups and mathematical induction. This ends the proof of the proposition. q.e.d.

Next, I will introduce several terminologies and notations which will be used in the later sections:

Definition 4.14. Let G_Γ be the subgroup of $G(\Gamma)$ which leaves Γ invariant. The symbol $\sigma(\Gamma) = |G(\Gamma)/G_\Gamma|$ is defined to be the cardinality of Γ's orbit under $G(\Gamma)$.

I notice that, in general, the group $G(\Gamma)$ does not act on Y_Γ. The element $g \in G(\Gamma)$ maps Y_Γ to $Y_{g\Gamma}$. The group G_Γ is the stabilizer of Γ under the group action on a certain subset of $\mathrm{adm}(n)$.

One says that Γ_1 and Γ_2 are equivalent if one can get Γ_2 by renaming vertexes in Γ_1 while preserving the admissibility condition. In fact, Γ_2 is equivalent to Γ_1 if and only if $\Gamma_2 = g\Gamma_1, g \in G(\Gamma_1) = G(\Gamma_2)$.

Example 4.2. Consider the graph on the left hand side of Figure 3. It is not hard to see that $G_\Gamma = \mathbf{S}_2 \times \mathbf{S}_2$ while $|G(\Gamma)| = 1120$ and $\sigma(\Gamma) = 280$. Consider the new graph by degeneration. $G_\Gamma = Z_2$, $|G(\Gamma)| = 560$ and $\sigma(\Gamma) = 280$.

These numbers are relevant to the calculations in the final section. Consider the Fulton-MacPherson space $M[n]$, it admits a natural $\mathbf{S}_n$ action. Quite different from the Fulton-MacPherson space, the space M_n does not admit any natural $\mathbf{S}_n$ action. Given the special admissible graph γ, the space Y_γ , the top stratum of the stratification admits an $\mathbf{S}_n$ action. However, the action cannot be extended to its compactification,

M_n. Instead, the symmetric group breaks up into its various subgroups. They are the G_Γ defined in this section. Moreover, the groups G_Γ can vary when one moves from a generic stratum to its boundary strata. If the degenerations of graphs break up the symmetry, the groups shrink. Sometimes the degenerations of graphs enhance the original symmetry, the groups get larger. In general, it is the combined behavior that complicates the general picture. On the other hand, the introduction of the modified family invariants can be viewed as an attempt to regulate the complicated situation.

Proposition 4.10. *Let* $\mathrm{adm}(n)'$ *be the equivalence classes of admissible graphs under the previous equivalence relationship. Let* Γ *be a representative in the class* $[\Gamma]$. *Then the stratification* $M_n = \cup_{\Gamma\in \mathrm{adm}(n)} Y_\Gamma$ *can be rephrased as* $M_n = \cup_{[\Gamma]\in \mathrm{adm}(n)'} G(\Gamma)\cdot Y_\Gamma$.

Proposition 4.10 will be used as foundation for enumerations in section 9.2, 9.3, 9.4, and 9.5. The group G_Γ acts on the preexceptional cone $\mathbf{C}(\Gamma)$ while permuting the extremal rays. The elements in the group $g\in G(\Gamma)$ intertwine the preexceptional cones $\mathbf{C}_\Gamma$ and $\mathbf{C}_{g\Gamma}$ under the action.

When we count of nodal curves, the groups G_Γ and $G(\Gamma)$ would be enough for my purpose. The main reason is that $m_i = 2$ for all i and it does not put any extra constraints on the symmetric group. In general, there is no reason to expect that the multiplicity function $\mathbf{M} : \mathrm{Ver}(\Gamma) \mapsto \mathbf{Z}$ is G_Γ invariant. The precise definition of $\mathbf{M}$ and the topological types of curve singularities will be given in the next section.

Let us define two new objects which are useful in discussing the general situation.

Definition 4.15. The action of G_Γ on $\mathrm{Ver}(\Gamma)$ induces an action on the multiplicity functions. Given a multiplicity function $\mathbf{M} : \mathrm{Ver}(\Gamma) \mapsto \mathbf{N}\cup\{0\}$, the group $G_{\Gamma,\mathbf{M}}$ is the subgroup of G_Γ which leaves the function $\mathbf{M}$ invariant.

Similarly, two different topological types $(\Gamma_1, C - \mathbf{M}_1(E)E)$ and $(\Gamma_2, C - \mathbf{M}_2(E)E)$ determines two different multiplicity functions $\mathbf{M}_1$ and $\mathbf{M}_2$. The group $G(\Gamma, \mathbf{M}_1, \mathbf{M}_2)$ is the subgroup of $G(\Gamma)$ which bring $\mathbf{M}_1$ to $\mathbf{M}_2$ under the action.

The reader should notice that $G_{\Gamma,\mathbf{M}}$ preserves the exceptional as well as the preexceptional cones. Likewise, the group $G(\Gamma, \mathbf{M}_1, \mathbf{M}_2)$ intertwines them.

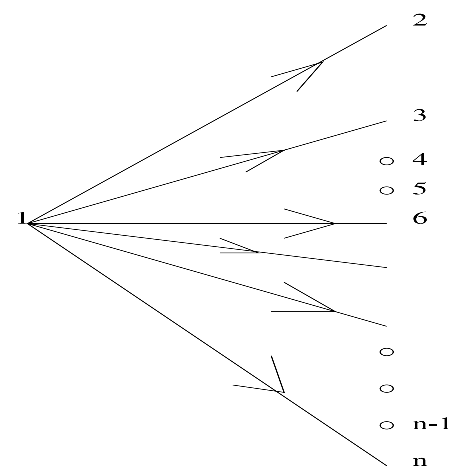

FIGURE 4

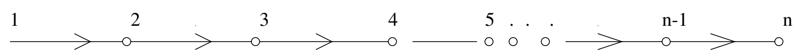

FIGURE 5

5. The topological type of a singular curve and the core of a topological type

Recall the well known theorem in surface theory [1]:

Theorem 5.1. *Let M be an algebraic surface, and C a holomorphic curve on M. Then there exists a sequence of a finite number of blowing ups, after which the proper transformation of the curve C becomes smooth.*

Notice that the blowing up processes are generally nonunique. Especially given such a finite sequence of blowing ups, one can perform an indefinite number of redundant blowing ups and still arrive at the same result. As the singularities of pseudo-holomorphic curves are locally bi-diffeomorphic to the algebraic singularities [43], the similar conclusion can be drawn for pseudo-holomorphic curves in a symplectic four-manifolds.

Let us fix a special sequence of blowing ups of M, under which the proper transformation of C becomes nonsingular.

Suppose that there are n individual blowing ups involved in this process. The combinatorial type of the blowing ups can be described by an admissible graph $\Gamma \in \mathrm{adm}(n)$ as was extensively studied in the previous section. The vertexes of the corresponding admissible graph correspond to the blown up points, while the markings order the blowing ups.

Let E_i be the cohomology classes associated with the i-exceptional curves. Suppose that the proper transformation of C lies in the cohomology class $C - \sum m_i E_i$, then it follows that $m_i - \sum_{j_i} m_{j_i} \geq 0$. As usual, the indexes j_i denote the direct descendents of the index i. This simple fact can be understood and derived in two different ways. First, it indicates that when the centers of the j_i's blowing ups are located at the exceptional locus of the i-th blowing up, then the sum of the multiplicities of these singular points must be less than or equal to the multiplicity of the i-th singular point. On the other hand, the same conclusion can be derived by using the fact that $E_i - \sum_{j_i} E_{j_i}$ and $C - \sum_{k \leq n} m_k E_k$ are dual to the two distinct (pseudo)-holomorphic curves in the same manifold. Then $m_i - \sum m_{j_i}$ can be realized as $[E_i - \sum_{j_i} E_{j_i}] \cdot (C - \sum_k m_k E_k)$, and it is nonnegative by using the fact that distinct irreducible (pseudo)-holomorphic curves intersect positively.

Given a singular curve in the class C, consider one sequence of blowing up process (nonunique) and thus the admissible graph Γ which desingularizes the curve. The pair $(\Gamma, C - \sum m_i E_i)$ is defined to be the topological type of the singular curve. For the convenience of the later discussion, I may impose some additional conditions on the markings and the multiplicities m_i. Suppose $i < j$ are the markings of two vertexes such that j is not a descendent of i, then the vertexes marked $k > j$ are not the descendents of i. This condition is imposed in order to reduce the ambiguity of blowing ups. I require additionally that $m_j \geq m_i$ for $j > i$ if either of the following conditions holds:

(1) The j-th vertex has no direct ascendent vertex, i.e., it is a free vertex.

(2) The i-th and j-th vertexes share the common direct ascendent vertex, yet the j-th vertex is not a direct descendent of the i-th vertex in Γ.

Geometrically, it corresponds to the rule of resolving the singularities with the lower multiplicities first.

Two different singular curves in C are of the same topological type if the associated pairs are the same. The topological type of a curve is equivalent to the datum of the topological types of the singularities and the genera of the resolved curves. Notice that the definition offered here does not coincide with the standard one in the text, but they can be shown to be equivalent. A proof can be found in [3].

One of the goals of the paper is to calculate the "number of singular curves" of a fixed topological type. Before doing so, let us introduce some definitions.

Given a topological type of singular curves $(\Gamma, C - \sum m_i E_i)$, the multiplicities m_i define a $\mathbf{Z}$-valued function from $\mathrm{Ver}(\Gamma)$ to $\mathbf{Z}$ by associating each vertex to the corresponding multiplicity. Let us denote the multiplicity function by $\mathbf{M}(E)$.

Definition 5.1. A vertex of an admissible graph is said to be redundant if the corresponding multiplicity m_i is equal to zero. An edge is said to be redundant if it connects a redundant vertex with the others.

From the previous inequality $m_i \geq \sum m_{j_i}$ it follows that if a vertex is redundant, then all of its descendent vertexes are also redundant.

Definition 5.2. A vertex of an admissible graph is said to be subredundant if first, the multiplicity function $\mathbf{M}$ takes value one on the vertex and the vertex has exactly one direct ascendent. Second, it does not have any descendent vertex which has more than one direct ascendent. An edge is said to be subredundant if it connects a subredundant vertex with the others.

The purpose of introducing these concepts will be clarified momentarily after the introduction of the modified invariants.

Definition 5.3. An admissible graph is said to be nonredundant if it does not contain any redundant or subredundant vertex. Given any topological type $(\Gamma, C - \sum m_i E_i)$, there exists a unique maximal nonredundant subgraph $\Gamma' \subset \Gamma$, which is constructed by removing from the original graph Γ all the redundant and subredundant vertexes and the edges connecting them. The resulting subgraph is called the core of the admissible graph Γ with respect to the topological type. The markings of the subgraph is generally given by a certain subset of $\{1, 2, 3, \ldots, n\}$. One uses the notation $\mathrm{core}(\Gamma, \mathbf{M})$ to denote the core of an admissible graph.

Notice that the concept of cores depends not only on the graph itself, but on the pair $(\Gamma, C - \sum_i m_i E_i)$. These definitions translate the geometry of singularities into a combinatorial language. The concept of "cores" will be used in Section 9 in conjunction with Proposition 5.4, which tells us how the modified invariants (Section 5.1) change under the reduction process.

5.1 The definition of the modified family Seiberg-Witten invariants

Given a topological type $(\Gamma, C-\sum m_iE_i)$, one attaches the space $Y(\Gamma)$ to the admissible graph. We consider the mixed family Seiberg-Witten invariant $FSW(c, C-\sum m_iE_i)$ over the base $B = Y(\Gamma)$.

The first step is to define a version of the modified invariants, denoted by FSW^*. It will be shown that the definition of the modified Seiberg-Witten invariant $FSW^*(c, C-\sum m_iE_i)$ is closely related to the counting of singular pseudo-holomorphic curves of a fixed topological type.

The special properties of FSW^* will be proved in the following section. Namely, if Γ_1 and Γ_2 are congruent to each other through an element g in $\mathbf{S}_n$, then the modified invariants $FSW^*_{\Gamma_1}(1, C-\sum m_iE_i)$ and $FSW^*_{\Gamma_2}(1, C-\sum m_{g(i)}E_{g(i)})$ are equal. Notice that this particular property is not shared by the original family invariant FSW.

First one notices that $m_i - \sum_{j_i} m_{j_i} \geq 0$ for all i. This indicates that the class $C - \mathbf{M}(E)E$ has nonnegative pairings with all type I exceptional curves in the preexceptional cone $\mathbf{C}_\Gamma$. From the discussion in Section 4.2, $C-\mathbf{M}(E)E = C-\sum m_iE_i$ constitutes an admissible decomposition (class) of level zero over $Y(\Gamma)$. Namely, it is a generic admissible decomposition. Following the general wisdom, the family invariant itself is not proportional to the "number" of singular curves, even though these two numbers are closely related. The detailed analysis will be done later. At this moment, I give a formal discussion.

Viewing $C-\mathbf{M}(E)E$ as the total cohomology class, one is interested in the other admissible decomposition classes other than the trivial one. Given the cohomology class $C-\mathbf{M}(E)E$, let $\mathcal{ADM}(s)$ denote the level s admissible decomposition classes of $C-\mathbf{M}(E)E$, with the dependence on $C-\mathbf{M}(E)E$ being omitted. Then it follows from the energy boundedness property of the pseudo-holomorphic curves that $\mathcal{ADM}(s) = \emptyset$ when s is large enough.

By definition, $\mathcal{ADM} = \coprod_{s\geq 0} \mathcal{ADM}(s)$ is the disjoint union of the various admissible decomposition classes of different levels. By the previous remark, it must be a finite set. One should be careful that these sets depend on $C - \mathbf{M}(E)E$ and Γ explicitly. In discussing the corresponding objects for different topological types, one adopts the notation $\mathcal{ADM}(s)(\Gamma, C - \mathbf{M}(E)E)$ instead of $\mathcal{ADM}(s)$.

The Φ map which associates an admissible decomposition class to the associated admissible graph establishes a morphism $\Phi : \mathcal{ADM} \longrightarrow \mathrm{adm}(n)$.

Given an admissible decomposition class, my goal is to attach certain types of mixed invariants to some admissible decomposition (which is allowable by definition) in the same decomposition class. It can be achieved by the application of the family switching formula. If the admissible decompositions in the same class are not unique, then the associated mixed invariants may look different.

The family switching formula assigns mixed invariants to the decomposition class. In general, the uniqueness of the expression is not ensured. However, the family switching formula relates different expressions, and those different mixed invariants always resemble the same numerical value. I state it in the following remark.

Example 5.1. Assuming that there is more than one admissible decomposition in the given decomposition class and a mixed invariant has been assigned to each of them, then the numerical values of the mixed invariants attached to the different admissible decompositions in the same class are equal.

This is a consequence of the repeated applications of the family switching formula, Theorem 2.3. q.e.d.

Notice that it is the expression of mixed invariants which are apparently different. The numerical values are the same. In fact, let $H = F + \sum m_i e_i = F' + \sum m'_i e'_i$ be two admissible decompositions in the same decomposition class. The F and F' are respectively the free parts of them. The classes e_i or e'_j are the type I exceptional curves in the decompositions. As they belong to the same decomposition class, e_i and e'_j must be permutations of each other, and e_i and e'_j are characterized to be the extremal elements in the preexceptional cone which have negative pairings with the total class. There is no guarantee that m_i and m'_i are co-related. Let $\overline{\Gamma}$ be the admissible graph under the Φ map. By using the special case of family switching formula along with the nested Kuranishi model technique (please consult the Section 6.1), the invariant contribution of both decompositions are expressed as $FSW_{Y(\Gamma)}(c_*(\kappa), H - \sum e_i)$ when the decomposition is allowable. The symbol $c_*(\kappa)$ denotes the top chern class of the residual relative obstruction bundle relating H and $H - \sum e_i$. Otherwise it is taken to be zero automatically. In the ensuing discussion, I assume that the decomposition is allowable.

5.2 The mixed family invariant associated with the type I admissible decompositions

We begin by discussing the various numerical properties of the type I admissible decompositions. As any nonempty type I admissible decomposition class always contains a decomposition of the form $(C_0 - \sum e_i, \sum e_i)$, we investigate its degenerations by using the geometrical properties of type I curves.

Two type I classes e_i and e_j are said to be directly connected if $e_i \cdot e_j > 0$. Two type I classes e_i and e_j are said to be connected if there exists a chain of e starting at e_i and ending at e_j such that the adjacent e are directly connected.

Given a finite sequence of type I exceptional classes $e_i, i \in I$ such that $e_i \cdot e_j \geq 0$, one collects them into different "connected components" such that the type I exceptional classes from different " connected components" have trivial intersection numbers.

Lemma 5.1. *Let $(C_0 - \sum_{i \in I} e_i, \sum_{i\in I} e_i)$ be an allowable decomposition, then among the "connected components" of $\cup_{i\in I} e_i$, there exists at least one "connected component", determined by $J \subset I$, such that $C_0 \cdot (\sum_{j \in J\subset I} e_j) \leq (\sum_{j\in J} e_j)^2 < 0$.*

Proof. Suppose I has been decomposed into different "connected components" $I = \coprod_r J_r$ following the definition.

Then it follows that

$$\begin{aligned} \mathrm{d}_{\mathbf{R}}(C_0) &\leq \mathrm{d}_{\mathbf{R}}\Big(C_0 - \sum_{i\in I} e_i\Big) + \sum_{i \in I} \mathrm{d}_{\mathbf{R}}(e_i) \\ &\leq \mathrm{d}_{\mathbf{R}}\Big(C_0 - \sum_r \Big\{\sum_{j\in J_r} e_j\Big\}\Big) + \sum_r \mathrm{d}_{\mathbf{R}}\Big(\sum_{j \in J_r} e_j\Big). \end{aligned}$$

Because the classes from different J_r intersect trivially, it implies that

$$\mathrm{d}_{\mathbf{R}}(C_0) \leq \mathrm{d}_{\mathbf{R}}(C_0) - 2\sum_r \Big\{\sum_{j\in J_r} e_j\Big\} \cdot C_0 + 2 \sum_r \Big\{\sum_{j \in J_r} e_j\Big\}^2.$$

It follows that $\sum_r \{\sum_{j\in J_r} e_j\}\cdot C_0 \le \sum_r\{\sum_{j\in J_r} e_j\}^2$. It implies that for at least one J_r, $\{\sum_{j\in J_r} e_j\}\cdot C \le \{\sum_{j\in J_r} e_j\}^2$.

By using the fact that the quadratic intersection form is negative definite on the cone generated by $e_j, j\in J_r$, it follows that

$$C_0\cdot\Big\{\sum_{j\in J_r} e_j\Big\} \le \Big\{\sum_{j\in J_r} e_j\Big\}^2 < 0. \qquad \text{q.e.d.}$$

If $e_i\cdot e_j = 1, i<j$, it implies that when both are written in the standard basis $e_i = E_i - \sum_{j_i} E_{j_i}$; $e_j = E_j - \sum_{j_j} E_{j_j}$, j is the i's direct descendent and $E_i - \sum_{j_i} E_{j_i}$. The two expressions $E_i - \sum_{j_i} E_{j_i}$ and $E_j - \sum_{j_j} E_{j_j}$ share exactly one E_j in common. This implies that $e_i + e_j = E_i - \sum_{j_i\neq j} E_{j_i} - \sum_{j_j} E_{j_j}$ can be thought to be a new type I exceptional class and can be thought to be the smoothing of $e_i \& e_j$. By induction, $\sum_{j\in J_r} e_j$ can be "smoothed" into a new type I exceptional class. In particular, the previous lemma implies that one can group the type I exceptional classes and smooth them separately. At least one among the new type I classes, e, would satisfy $e\cdot C_0 \le e^2 < 0$.

The condition will play a crucial role in constructing the nested Kuranishi model.

Conversely, suppose C_0 is a class over b and $e_i, \in I, e_i\cdot C_0 < 0$ are all the smooth type I classes over b which are not numerically effective with respect to C_0. One has the following lemma:

Lemma 5.2. *If* $0 > e_i\cdot C_0 > e_i^2$ *for all* $i\in I$*, then the decomposition* $(C_0 - \sum_i e_i, \sum e_i)$ *is not allowable.*

Proof. Suppose that it is allowable. A direct calculation shows that

$$\mathrm{d}_{\mathbf{R}}\Big(C_0 - \sum_{i\in I} e_i\Big) + \sum_{i\in I}\mathrm{d}_{\mathbf{R}}(e_i) - \mathrm{d}_{\mathbf{R}}(C_0) = \\ 2\Big(\sum_{i\in I} e_i^2 + \sum_{i<j;i,j\in I} e_i\cdot e_j\Big) - 2\Big(\sum_{i\in I} e_i\Big)\cdot C_0 \ge 0.$$

On the other hand, the assumption implies $e_i\cdot C_0 - e_i^2 \ge 1$ for all $i\in I$. Summing over i and one gets $\sum_{i\in I} e_i\cdot C_0 - \sum_{i\in I} e_i^2 \ge |I|$.

Combining the two inequalities one derives that $\sum_{i<j;i,j\in I} e_i\cdot e_j \ge |I|$. One would like to prove that this leads to a contradiction.

As in the previous discussion, the collection of $e_i, i\in I$ can be regrouped into connected trees such that e_i in the distinct groups do not intersect each other. Within the same group J_r, the type I classes may intersect each other. Given a J_r, the dual graph of the $e_i, i \in J_r$ forms a connected tree and the total number of intersections $\sum_{i<j;i,j\in J_r} e_i \cdot e_j$ is equal to the number of one dimensional edges of the tree. As a connected tree always has Euler number 1 which is also equal to "number of vertexes-number of one edges", one may identify $\sum_{i<j;i,j\in J_r} e_i \cdot e_j$ with $|J_r| - 1$. Thus the total number of intersections can be identified with $\sum(|J_r| - 1) = |I|-$number of components. In particular, it is always smaller than $|I|$. q.e.d.

One can categorize the allowable decompositions $(C - \mathbf{M}(E)E - \sum e_i, \sum e_i)$ into two types.

Definition 5.4. Let $(C - \mathbf{M}(E)E - \sum e_i, \sum e_i)$ be an allowable decomposition. Then e_i is a type I exceptional class which satisfies $e_i \cdot (C - \mathbf{M}(E)E < 0$. If $e_i \cdot (C - \mathbf{M}(E)E) \leq e_i^2$ for all $e_i, e_i \cdot (C - \mathbf{M}(E)E) < 0$, then it is called a type A allowable decomposition class.

If there exists at least one $e_i, e_i \cdot (C - \mathbf{M}(E)E > e_i^2$, then it is called a type B allowable decomposition class.

Given a type B admissible decomposition class $\mathbf{D}_2$, it is easy to see that there exists a type A admissible decomposition class $\mathbf{D}_1$, such that $\mathbf{D}_1 \gg \mathbf{D}_2$. The type A decomposition classes are essential as they contribute to the family invariants. Yet a vanishing argument by dimension counting implies that a type B decomposition class always contributes trivially to the family invariant. I will focus mostly on the type A admissible decomposition classes in this subsection.

Given an arbitrary, nongeneric, admissible decomposition class associated to Γ, I would like to discuss in certain detail the mixed family invariant associated with it. I will focus on the class $c_*(\kappa)$ and explain the subtlety involved in the construction.

In order to apply the family switching formula directly, one has to be careful about the extra condition imposed in the theorem. Namely, the $\mathbf{S}^2$ bundle is embedded in $\mathcal{X} \mapsto B$ in a way that the normal bundle can be given a structure of relative complex line bundle.

In our situation, this condition is not always met. In fact, in our setup, the naive candidates of the $\mathbf{S}^2$ bundle will be the various holomorphic curves dual to e_i, $e_i \cdot (C - \mathbf{M}(E)E) < 0$. However, those rational curves usually do not form $\mathbf{S}^2$ bundles. Rather, they degenerate into a tree of rational curves whose patterns are encoded in the combinatorial data of the admissible graphs.

A direct consequence of the appearance of the singular fibers is that the algebraic geometric data of "obstruction sheaf" may not be locally free. In other words, the appropriate index bundles jump when the smooth curves degenerate into singular one. On the other hand, it is an indirect indication that the other type of admissible decomposition classes have supported over some $Y(\Gamma'), \Gamma' < \Gamma$, a lower dimensional stratum of Y_Γ.

Instead of analyzing the jumping phenomenon directly, one uses the concept of admissible decomposition classes of different levels to decompose the family moduli space into various "topological components" such that one constructs a specific family Kuranishi model for each "topological component".

I will show that the obstruction semi-bundle, restricted to each "component", contains a vector subbundle canonically associated with the type I exceptional curves. The mixed family invariant associated with each "component" can be analyzed by using the regularly obstructedness condition and the nested family Kuranishi model.

Fix an extremal generator of the exceptional cone, i.e., a type I exceptional class e_i such that $e_i \cdot (C - \mathbf{M}(E)E) < 0$. Let Γ_{e_i} be the admissible graph with $-e_i^2 - 1$ edges which characterizes the effectiveness of e_i.

Notice that $Y(\Gamma_{e_i})$ is the moduli space of the exceptional curve in e_i.

The total space Ξ of the universal curve over $Y(\Gamma_{e_i})$ carries a $\mathbf{P}^1$ fibration over $Y(\Gamma_{e_i})$. The restriction of the $\mathbf{P}^1$ fibration to $Y_{\Gamma_{e_i}}$ gives rise to a $\mathbf{P}^1$ bundle structure.

I state the following proposition regarding the existence of the $\mathbf{P}^1$ bundle over $Y(\Gamma_{e_i})$.

Proposition 5.1. *There exists a relatively minimal $\mathbf{P}^1$ fiber bundle $\tilde{\tilde{\Xi}}$ over each $Y(\Gamma_{e_i})$, denoted by $\tilde{\tilde{\Xi}}_i$ such that $\Xi_i \mapsto \tilde{\tilde{\Xi}}_i$ are birational fiber preserving morphisms.*

Suppose that the indexes j_i are the markings of the direct descendents of the i-th vertex. Then the smooth fibers (dual to e_i) of Ξ_i degenerate into singular curves over the various $Y_{\Gamma'_{e_i}}, \Gamma'_{e_i} < \Gamma_{e_i}$ such that for some $k \neq i, j_i$, etc., the k-th vertex becomes the direct descendent of i-th vertex.

Those $Y(\Gamma'_{e_i})$ are of smooth complex codimension one in $Y(\Gamma_{e_i})$. The two fibrations $\Xi_i \mapsto Y(\Gamma_{e_i})$ and $\tilde{\tilde{\Xi}}_i \mapsto Y(\Gamma_{e_i})$ are identical outside those $Y_{\Gamma'_{e_i}} \subset Y(\Gamma_{e_i})$.

Proof. To construct such a fiber bundle, I recall that $M_{n+1} \mapsto M_n$ can be constructed from $M \times M_n$ by blowing up n different sections. One simply ignores those indexes k mentioned above and blows up the other sections according to their natural orders. The restriction of the new four-manifold fiber bundle to $Y(\Gamma_{e_i})$ carries a family of relative exceptional curves dual to e_i. In the newly constructed family these particular curves do not degenerate into singular curves as I have avoided to blow up the corresponding k-th sections of $M \times M_n \mapsto M_n$. q.e.d.

Given the admissible graph Γ_{e_i}, one repeats the process and constructs a similar $\mathbf{P}^1$ fiber bundle $\tilde{\tilde{\Xi}}_i$ to each e_i. Choose a specific order to move the class $C - \mathbf{M}(E)E$ to $C - \mathbf{M}(E)E - \sum e_i$ through $C - \mathbf{M}(E)E - e_a$, $C - \mathbf{M}(E)E - e_a - e_b$, etc.

In each step, I may apply the family switching formula to the classes and calculate the family obstruction bundle by using the $\mathbf{P}^1$ bundle $\tilde{\tilde{\Xi}}_a$. If some e_i satisfies $0 > e_i \cdot (C - \mathbf{M}(E)E) > e_i^2$, then $(C - \mathbf{M}(E)E - e_i, e_i)$ would not be allowable. It also implies the existence of $\sum_{i; e_i \cdot (C - \mathbf{M}(E)E) > e_i^2} e_i \cdot \{C - \mathbf{M}(E)E - e_i\}$ smoothing directions of $C - \mathbf{M}(E)E$ which are obstructed over Γ. The vanishing argument based on dimension counting implies that the modified family invariant attached to $(C - \mathbf{M}(E)E - \sum e_i, \sum e_i)$ should be zero. Thus define the class $c_*(\kappa)$ to be zero. From now on, we assume that $e_i \cdot (C - \mathbf{M}(E)E) \leq e_i^2 < 0$ hold for all $e_i \cdot (C - \mathbf{M}(E)E) < 0$. Denote the corresponding relative obstruction bundle by $\mathcal{V}_i$. The explicit construction of $\mathcal{V}_i$ is reviewed in the proof of Proposition 5.2.

Then the class $c_*(\kappa)$ is defined to be the Euler class of the underlying real vector bundle $\kappa = \oplus_{a, e_a \cdot (C - \mathbf{M}(E)E) \leq e_a^2 < 0} \mathcal{V}_a$, which is also the top Chern class of the complex vector bundle.

In the following, I prove that the class $c_*(\kappa)$ is independent of the choices of the orders of exceptional curves e_i.

In general, if one starts with a different path to get from $C - \mathbf{M}(E)E$ to $C - \mathbf{M}(E)E - \sum e_i$, one will encounter different vector bundles $\mathcal{V}_a$, etc. , resulting in a possibly different κ. I show that the class $c_*(\kappa) \in H^*(\cap Y(\Gamma_{e_i}), \mathbf{Z})$ is independent of the choices of the different orders.

Proposition 5.2. *The construction of the class $c_*(\kappa)$ is independent of the path of moving $C - \mathbf{M}(E)E$ to $C - \mathbf{M}(E)E - \sum e_i$, where e_i are those extremal generators of the type I exceptional cone.*

Proof. Fixing a switching process $C - \mathbf{M}(E)E - \sum_{b \in J} e_b \mapsto C - \mathbf{M}(E)E - \sum_{b \in J} e_b - e_a$, I first review how are the vector bundles $\mathcal{V}_a$

constructed.

Given the type I exceptional class $e_a, e_a \cdot (C - \mathbf{M}(E)E) < 0$, I have constructed the $\mathbf{P}^1$ fiber bundle $\widetilde{\Xi}_a$. Suppose one switches from $C - \mathbf{M}(E)E - \sum_{b\in J} e_b$ to $C - \mathbf{M}(E)E - \sum_{b\in J\cup\{a\}} e_b$, the relative obstruction bundle can be easily described in term of curve theory.

To make the discussion compatible to algebraic geometry, I adopt the algebraic language here. One can easily translate it into the language of topology by applying index theory of the $\bar{\partial}$ operators. I would slightly abuse the notation in representing the locally free sheaf and the corresponding bundle by the same symbol.

As before, let us denote the relative $\mathbf{P}^1$ bundle by $\widetilde{\widetilde{\Xi}}_a$. Let p_a denote the projection map from $\widetilde{\widetilde{\Xi}}_a$ to $Y(\Gamma_{e_a})$. Let $\mathcal{K} = \omega_{\widetilde{\widetilde{\Xi}}_a/Y(\Gamma_{e_a})}$ be the relative canonical bundle(sheaf) of $\widetilde{\widetilde{\Xi}}_a$ over $Y(\Gamma_{e_a})$. Then the relative obstruction bundle between $C - \mathbf{M}(E)E - \sum_{b\in J} e_b$ to $C - \mathbf{M}(E)E - \sum_{b\in J\cup\{a\}}$ is given by the following expression:

$$\overline{\mathcal{V}}_a = \mathcal{R}^1(p_a)_*(\mathcal{Q}_{C-\mathbf{M}(E)E-\sum_{b\in J} e_b}) = \mathcal{R}^0(p_a)_*(\mathcal{Q}^*_{C-\mathbf{M}(E)E-\sum_{b\in J} e_b} \otimes \mathcal{K})^*.$$

I have used the relative Serre duality to derive the second equality. The invertible sheaf $\mathcal{Q}_{C-\mathbf{M}(E)E-\sum_{b\in J} e_b}$ is constructed as follows:

First notice that because $\widetilde{\widetilde{\Xi}}_a$ has a $\mathbf{P}^1$ bundle structure, the relative Picard group of the fibers are of rank 1. The fibers $\mathbf{P}^1$ are embedded into some birational model of M which are in the class $E_a - \sum E_{j_a}$. The class C is trivial along the fibers of $\widetilde{\widetilde{\Xi}}_a$. Thus, it can be pulled back from the base. The class $-\mathbf{M}(E)E - \sum_{b\in J} e_b$ can be written in a form $k_a E_a + \sum_{j_a} k_{j_a} E_{j_a} + others$.

The term "others" refers to the exceptional classes which are independent of E_a or E_{j_a}. Those classes are trivial along the fibers $\mathbf{P}^1$, too. Throw away those terms and focus upon $k_a E_a + \sum_{j_a} k_{j_a} E_{j_a}$. The class E_a is of relative degree -1 on $\widetilde{\widetilde{\Xi}}_a$ while E_{j_a} all have positive degree 1 on the fibers. The class $k_a E_a + \sum_{j_a} k_{j_a} E_{j_a}$ can be expressed as the $-\sum_{j_a} k_{j_a} + k_a$ multiple of the tautological class of the $\mathbf{P}^1$ bundle plus a class pulled back from the base. This construction determines uniquely an invertible sheaf (holomorphic line bundle) which is denoted by $\mathcal{Q}_{C-\mathbf{M}(E)E-\sum_{b\in J} e_b}$.

The direct image sheaf $\overline{\mathcal{V}}_a$ is locally free, as

$$\dim_{\mathbf{C}} \mathcal{R}^0(p_a)_*(\mathcal{Q}_{C-\mathbf{M}(E)E-\sum_{b\in J} e_b}) \otimes \mathbf{C} = 0.$$

It is because the degree

$$\Big(C - \mathbf{M}(E)E - \sum_{b\in J} e_b\Big) \cdot e_a \leq (C - \mathbf{M}(E)E) \cdot e_a$$

is negative.

On the other hand, if I consider the invertible sheaf associated with $C - \mathbf{M}(E)$, denoted by $\mathcal{Q}$ (following the previous construction for $\mathcal{Q}_{C-\mathbf{M}(E)E}$, then there is a short exact sequence of sheaves over $\tilde{\tilde{\Xi}}_a \times_{M_n} Y(\Gamma_{\mathbf{D}})$, namely the restriction of the fibration to $Y(\Gamma_{\mathbf{D}})$; $\mathbf{D} \in \mathcal{ADM}$. When $\{C - \mathbf{M}(E)E - \sum_{b\in J} e_b - e_a\} \cdot e_a < 0$, the effective divisor Υ_a is taken to be the negation of the anti-effective divisor constructed from $C - \mathbf{M}(E)E - \sum_{b\in J} e_b - e_a$.

$$0 \mapsto \mathcal{O}_{\tilde{\tilde{\Xi}}_a}\Big(\Big[-\sum_{b\in J} e_b\Big]\Big) \otimes \mathcal{Q} \mapsto \mathcal{O}_{\tilde{\tilde{\Xi}}_a}([e_a]) \mapsto \mathcal{O}_{\Upsilon_a}([e_a]) \mapsto 0.$$

The notations in this short exact sequence deserve special explanation. As $(C - \mathbf{M}(E)E - \sum_{b\in J} e_b) \cdot e_a < 0$, one can use the explicit expression of $C - \mathbf{M}(E)E$ to write it as fiberwise anti-effective divisors of $\tilde{\tilde{\Xi}}'$ tensored by a rank one locally free sheaf from the base $Y(\Gamma_{\mathbf{D}})$. It follows from the fact that the divisor classes of any two different sections of $p_a : \tilde{\tilde{\Xi}}'_a \mapsto Y(\Gamma_{\mathbf{D}})$ differ by a divisor class pulled back from the base.

Remark 5.1. If $\{C - \mathbf{M}(E)E - \sum_{b\in J} e_b\} \cdot e_a < 0$, $0 \geq \{C - \mathbf{M}(E)E - \sum_{b\in J} e_b - e_a\} \cdot e_a$, then the short exact sequence should be replaced by

$$0 \mapsto \mathcal{O}_{\tilde{\tilde{\Xi}}_a}([e_a]) \mapsto \mathcal{O}_{\tilde{\tilde{\Xi}}_a}\Big(\Big[-\sum_{b\in J} e_b\Big]\Big) \otimes \mathcal{Q} \mapsto \mathcal{O}_{-\Upsilon_a}\Big(\Big[-\sum_{b\in J} e_b\Big]\Big) \otimes \mathcal{Q} \mapsto 0.$$

Here Υ_a is the anti-effective divisor in $\tilde{\tilde{\Xi}}_a$ by restricting the divisor class $-\{C - \mathbf{M}(E)E - \sum_{b\in J} e_b - e_a\}$ onto $\tilde{\tilde{\Xi}}_a$.

By taking the right derived long exact sequence along $p_a : \tilde{\tilde{\Xi}}'_a \mapsto Y(\Gamma_{\mathbf{D}})$ we find that the relative obstruction sheaf $\overline{\mathcal{V}}_a$ can be decomposed into two parts. The first part is isomorphic to the obstruction sheaf to the deformation of the type I exceptional curve e_a, which is known to be isomorphic to the restriction of the normal sheaf of $Y(\Gamma_{e_a}) \subset Y(\gamma)$.

The second part is of complex $(e_a - (C - \mathbf{M}(E)E - \sum_{b\in J} e_b)) \cdot e_a$ dimension, which is the direct image sheaf $\mathcal{R}^0(p_a)_*(\mathcal{O}_{\Upsilon_a}([e_a]))$ of the degree $(e_a - (C - \mathbf{M}(E)E - \sum_{b\in J} e_b)) \cdot e_a$ divisor Υ_a in $\tilde{\tilde{\Xi}}_a$.

This particular piece of vector bundle (sheaf) $\mathcal{V}_a$ is named to be the residual relative obstruction bundle (sheaf).

After reviewing the construction of the relative obstruction bundle $\mathcal{V}_a$ whose associated locally free sheaf is

$$\mathcal{SH}(\mathcal{V}_a) \equiv \overline{\mathcal{V}}_a = \mathcal{R}^1(p_a)_*(\mathcal{Q}_{C-\mathbf{M}(E)\sum_{b\in J} e_b}),$$

I continue the proof of Proposition 5.2 by using the following lemma:

Lemma 5.3. *Given the two classes $C - \mathbf{M}(E)E - \sum_{b\in J} e_b$ and $C - \mathbf{M}(E)E - \sum_{b\in J} e_b - e_{a_1} - e_{a_2}$ with $i, j \notin J$, the switching processes $C - \mathbf{M}(E) - \sum_{b\in J} e_b \mapsto C - \mathbf{M}(E)E - \sum_{b\in J} e_b - e_{a_1} \mapsto C - \mathbf{M}(E)E - \sum_{b\in J} e_b - e_{a_1} - e_{a_2}$ and $C - \mathbf{M}(E)E - \sum_{b\in J} e_b \mapsto C - \mathbf{M}(E)E - \sum_{b\in J} e_b - e_{a_2} \mapsto C - \mathbf{M}(E)E - \sum_{b\in J} e_b - e_{a_1} - e_{a_2}$ defines equivalent residual relative obstruction virtual bundles in the K group. In particular, they give rise to identical total chern classes.*

From now on we will call the interchange of the shifting orders by e_i and e_j an elementary operation. After proving Lemma 5.3, the proposition can be derived easily. As $(C - \mathbf{M}(E)E) \cdot e_i \le e_i^2 < 0, i \in I$, all the decompositions $(C - \mathbf{M}(E)E - \sum_{b\in J} e_b, \sum e_b)$ are allowable. Any two different effective moves from $C - \mathbf{M}(E)E$ to $C - \mathbf{M}(E)E - \sum_{i; e_i\cdot(C-\mathbf{M}(E)E)\le e_i^2} e_i$ can be related to each other by a sequence of elementary operations as was done in Lemma 5.3. q.e.d.

Proof of Lemma 5.3 . In the previous construction, I have decomposed the relative obstruction bundle (sheaf) into two parts. As the canonical piece is universally independent of the class $C - \mathbf{M}(E)E$, I can focus upon the second piece, which comes from the local contribution of e_{a_1} and e_{a_2}. As in each fiber e_i and e_j are among the components of a tree of $\mathbf{P}^1$, their intersection pairing is either 0 or 1.

By repeating the previous construction, one attaches a $\mathbf{P}^1$ fiber bundle over $Y(\Gamma_{\mathbf{D}})$ to each of e_{a_1} and e_{a_2}. If $e_{a_1} \cdot e_{a_2} = 0$, one mimics the previous construction for $\mathcal{Q}$ and the exceptional curves in e_{a_1} and e_{a_2} restrict to relative degree zero divisors on $\widetilde{\Xi}'_{a_1}$ and $\widetilde{\Xi}'_{a_2}$, and the isomorphism between the two different relative obstruction bundles can be manifestly constructed. The total spaces of the universal curves Ξ_{a_1} and Ξ_{a_2} may have singular fibers where the direct images of invertible sheaves fail to be locally free. This is why one has adopted the relatively minimal birational models $\widetilde{\Xi}_{a_1}$ and $\widetilde{\Xi}_{a_2}$, where the pathetic symptom does not occur.

Suppose that $e_{a_1} \cdot e_{a_2} = 1, a_1 < a_2$, then in Γ the a_2-th vertex is the direct descendent of the a_1-th one. The divisors σ_{a_1} and σ_{a_2} denote the effective sections induced by e_{a_2} and e_{a_1} in the $\mathbf{P}^1$ bundles $\widetilde{\Xi'}_{a_2}$ and $\widetilde{\Xi'}_{a_1}$, respectively.

I first deal with the case all Υ divisors are effective.

Let $\Upsilon_{a_1,1}$ and $\Upsilon_{a_2,2}$ be the effective divisors on $\widetilde{\Xi'}_{a_1}$ and $\widetilde{\Xi'}_{a_2}$, which are constructed from $-C + M(E)E + \sum_{b\in J} e_b$ and $-C + M(E)E + \sum_{b\in J\cup\{a_1\}} e_b$ through the previous process.

On the other hand, let $\Upsilon_{a_2,1}$ and $\Upsilon_{a_1,2}$ be the effective divisors on $\widetilde{\Xi'}_{a_2}$ and $\widetilde{\Xi'}_{a_1}$ constructed from $-C + M(E)E + \sum_{b\in J} e_b$ and $-C + M(E)E + \sum_{b\in J\cup\{a_2\}} e_b$, respectively.

By using the short exact sequence

$$0 \mapsto \mathcal{O}_{\sigma_{a_2}}([e_{a_1}] - \Upsilon_{a_1,1}) \mapsto \mathcal{O}_{\Upsilon_{a_1,2}}([e_{a_1}]) \mapsto \mathcal{O}_{\Upsilon_{a_1,1}}([e_{a_1}]) \mapsto 0,$$

it is easy to see that locally free $\mathcal{R}^0(p_{a_1})_*\mathcal{O}_{\Upsilon_{a_1,1}}([e_{a_1}])$ and $\mathcal{R}^0(p_{a_1})_*\mathcal{O}_{\Upsilon_{a_1,2}}([e_{a_1}])$ differ by $\mathcal{R}^0(p_{a_1})_*\mathcal{O}_{\sigma_{a_2}}([e_{a_1}] - \Upsilon_{a_1,1})$ in the K group of coherent sheaves over $Y(\Gamma_{\mathbf{D}})$. Similarly, $\mathcal{R}^0(p_{a_2})_*\mathcal{O}_{\Upsilon_{a_2,2}}([e_{a_2}])$ and $\mathcal{R}^0(p_{a_2})_*\mathcal{O}_{\Upsilon_{a_2,1}}([e_{a_2}])$ differ by $\mathcal{R}^0(p_{a_2})_*\mathcal{O}_{\sigma_{a_1}}([e_{a_2}] - \Upsilon_{a_2,1})$ in the K group of coherent sheaves over the same space.

Because $Y(\Gamma_{\mathbf{D}})$ is smooth and compact, the K group of coherent sheaves is naturally isomorphic to the K group of vector bundles.

Observe that $\mathcal{O}(-\Upsilon_{a_1,1})$ and $\mathcal{O}(-\Upsilon_{a_2,1})$ restrict to identical locally free sheaves over $\sigma_{a_1} = \sigma_{a_2}$ and can be factorized out from the relatively zero dimensional push-forward $\mathcal{R}^0(p_{a_1})_*$ and $\mathcal{R}^0(p_{a_2})_*$. Thus they do not affect the comparison.

By using the following lemma, one finds that the residual relative obstruction bundles from the two different processes give rise to the same $c.(\kappa)$. q.e.d.

Lemma 5.4. *The two locally free sheaves* $\mathcal{R}^0(p_{a_1})_*\mathcal{O}_{\sigma_{a_2}}([e_{a_1}])$ *and* $\mathcal{R}^0(p_{a_2})_*\mathcal{O}_{\sigma_{a_1}}([e_{a_2}])$ *are isomorphic.*

Proof. From the fact that σ_j and σ_i are the zero loci of of e_j and e_i over $\widetilde{\Xi}'_i$ and $\widetilde{\Xi}'_j$, one finds easily that the locally free invertible sheaves $\mathcal{R}^0(p_i)_*\mathcal{O}_{\sigma_j}([e_i])$ and $\mathcal{R}^0(p_j)_*\mathcal{O}_{\sigma_i}([e_j])$ are the normal sheaves of $\sigma_j \subset \widetilde{\Xi}'_i$ and $\sigma_i \subset \widetilde{\Xi}'_j$, respectively. Because the $\mathbf{P}^1$ bundle structures are invariant under blowing ups, one can focus upon the j-th blowing up and ignore all the direct descendents of j, l_j. Likewise, one can also ignore all the direct descendents of i except j.

It is sufficient to prove that the normal bundles $\sigma_j \subset \widetilde{\Xi}'_i$ and $\sigma_i \subset \widetilde{\Xi}'_j$ are isomorphic. Let $\mathbf{K}_{\sigma_j}$ denote the restriction of the relative canonical bundle of $M_{j+1} \mapsto M_j$ to $\sigma_j = \sigma_i$. Then the normal bundle of $\sigma_j \subset \widetilde{\Xi}'_i$ can be identified with $\mathbf{K}^\star \otimes_{\mathbf{C}} \mathbf{L}^\star_{e_i}$.

On the other hand, the section $\sigma_i \subset \widetilde{\Xi}'_j$ of the $\mathbf{P}^1$ bundle can be lifted to the bundle inclusion $\mathbf{L}_{e_i} \subset \mathbf{T}M_{j+1}/\mathbf{T}M_j|_{\sigma_i}$. Again, the normal bundle of $\sigma_i \subset \widetilde{\Xi}'_j$ can be identified with $\wedge^2(\mathbf{T}M_{j+1}/\mathbf{T}M_j|_{\sigma_i}) \otimes \mathbf{L}^\star_{e_i} = \mathbf{K}^\star \otimes \mathbf{L}^\star_{e_i}$.

Thus, these two normal bundles are isomorphic and Lemma 5.4 has been proved. q.e.d.

5.3 The family invariants associated with the admissible decomposition classes

Having addressed the uniqueness of the class $c_*(\kappa)$, one can associate a canonical mixed family invariant to the given admissible decomposition class. Yet this decomposition $(C - \mathbf{M}(E)E - \sum_{i \in I} e_i, \sum_{i \in I} e_i)$ is not the decomposition which admits an algebraic geometric interpretation for curve counting. In this subsection, I would like to transform the given mixed invariant by an equivalent mixed invariant.

Starting from the original mixed invariant, one can apply the family switching formula repeatedly to get to the appropriate mixed invariants of F or F'. The base class insertions depend on multiplicity function $\mathbf{M}(E)$ explicitly yet the numerical values of the invariants before and after the "switching" the e_i-multiplicities are not changed.

Because of this proposition, it makes perfect sense to talk about the mixed invariants attached to an admissible decomposition class even though the explicit forms of them may not be manifestly independent to the choices of decompositions. To simplify my notation, the mixed invariant attached to the decomposition class $\mathbf{D} \in \mathcal{ADM}(s)$ is denoted by $FSW(\mathbf{D})$.

Let s be the smallest number such that $\mathcal{ADM}(s+1) = \emptyset$. Then one defines the corresponding modified mixed family Seiberg-Witten invariants to be the mixed family invariants without the correction terms. Namely,

$$FSW^*(\mathbf{D}) \equiv FSW(\mathbf{D}).$$

Suppose the modified invariants have been defined for the levels $s \geq 1 + p$; one defines the level p modified mixed family Seiberg-Witten invariants by the following recipe. Let $\mathbf{D} \in \mathcal{ADM}(p)$, then we define,

$$FSW^*(\mathbf{D}) \equiv FSW(\mathbf{D}) - \sum_{s>p} \sum_{\mathbf{D}_i \in \mathcal{ADM}(s); \mathbf{D} \gg \mathbf{D}_i} FSW^*(\mathbf{D}_i),$$

where the partial ordering $\gg$ has been defined in Definition 4.5. By the reversed induction on the integer p, one can eventually decrease p to 0 and define the modified family invariants of the level zero admissible decomposition class, which is unique and is $(C - \mathbf{M}(E)E, 0)$ in the current situation.

Notice that for level zero decomposition class $\mathbf{D}$, all $\mathbf{D}_i$ satisfy $\mathbf{D} \gg \mathbf{D}_i$ automatically. Thus, one has to subtract all the modified family invariants of the nonzero levels in order to define modified invariant. Also notice that in defining the admissible decomposition classes, we have restricted ourselves to consider the exceptional cones, which are the subcones of the preexceptional cones $\mathbf{C}_\Gamma$. They are characterized by the property that the intersection pairing with $C - \mathbf{M}(E)E$ is negative. In this process, one ignores the contribution from type II exceptional curves. I leave the justification of the procedure to the next section where I will clarify the relationship between this procedure and the assumptions in the main theorem.

Suppose that Γ' is in the image of $\Phi(\mathcal{ADM}(s))$. One considers $\mathbf{EC}(C - \mathbf{M}(E)E) \cap \mathbf{C}_{\Gamma'}$ as the effective type I exceptional cone when one determines the admissible decomposition classes. A priori it is not clear from the definition that this cone is simplicial. But it follows from the fact that the intersection form is negative restricted on $\mathbf{EC}$ that the subcone is again simplicial. As we know that the extremal generators are the only irreducible exceptional curves in the cone, they must be some of the type I exceptional curves in $\mathbf{C}_{\Gamma'}$. It follows that the type I effective exceptional curve cone is a simplicial subcone of $_{\Gamma'}$ generated by some of the extremal generators of $\mathbf{C}_{\Gamma'}$.

As the reader may have noticed, the definition offered here is rather formal. Only in the proof of the main theorem, I will clarify the reason to make such a definition.

Remark 5.2. In the previous discussion, one should be aware that the explicit value of s determined by the energy boundedness property may also depend on the manifold M. At first it may look like a potential exception that endangers the "universality" property of our main theorem. However, one can always enlarge the value of s without affecting the definition of the modified invariants. This is because the modified invariants associated with the noneffective classes are automatically zero.

5.4 The invariance of the modified family invariants under the reduction process

Having defined the modified invariants, the first step of reduction is to remove the redundant vertexes and prove the invariance of the modified invariants under this process. By using the family blowup formula [38] repeatedly and the fact that $m_i = 0$ for these redundant vertexes, one reduces the original modified invariant $FSW^*(c, C - \sum m_i E_i)$ by removing these redundant vertexes and edges. Suppose i is a redundant vertex, then $E_i \cdot (C - \mathbf{M}(E)E) = 0$. In particular, E_i is not in the type I effective exceptional cone of the class $C - \mathbf{M}(E)E$ over the whole space $Y(\Gamma)$. In particular, E_i can never be among an admissible decomposition over $Y(\Gamma)$. Then some simple calculation shows that the modified invariant is invariant under the reduction. As the flavor of the argument is similar to that of the subredundant case, the easier case is skipped here. In the following, one can assume that the redundant vertexes have been taken care of and that there is no redundant vertex in the original admissible graph Γ.

Similarly, one would like to eliminate the subredundant vertexes as well as the subredundant edges.

First I show that it is possible to permute the markings of the vertexes such that the vertexes in the core are marked by the index set $I = \{1, 2, \cdots m\}$ and the subredundant vertexes are marked from m+1 to n. A priori it is not quite clear that the permutations always preserve the admissibility condition. In the following proposition, I clarify the situation.

Proposition 5.3. *Let $(\Gamma, C - \sum_i m_i E_i)$ be a topological type of a singular curve such that Γ does not contain any redundant vertex. Then there exists a permutation $g \in G(\Gamma)$ which moves Γ to a new admissible graph $g(\Gamma)$ such that the subredundant vertexes of $g(\Gamma)$ are marked by $\{m + 1, \cdots n\}$ for some m.*

Proof. To prove Proposition 5.3, first we notice that the subredundant vertexes form linear chains. As $m_i - \sum m_{j_i} \geq 0$ and there are no redundant vertexes, each subredundant vertex has at most one direct descendent. On the other hand, by definition, a subredundant vertex can have no descendent vertex with more than one direct ascendent.

This forces the subredundant vertexes to form linear chains, and they are isolated from the other parts of the admissible graph. Since they form linear chains, there is a unique maximal element in every chain. One calls it the leading vertex in the chain.

Because the chains become disconnected from the admissible graph after removing the leading vertexes, one can mark the vertexes such that the vertexes within the core are marked from 1 to m, while the subredundant vertexes are marked from $m+1$ to n. Moreover we can arrange that the direct descendent vertex of a subredundant vertex is marked by the consecutive integer. The permutation defined in this way preserves the admissibility conditions, and therefore, belongs to $G(\Gamma)$.

q.e.d.

The following proposition describes the relationship between the modified invariants before and after the reduction.

Proposition 5.4. (The Reduction Proposition) *Let Γ be an admissible graph without any redundant vertex. Let J be the index set parameterizing the subredundant vertexes in Γ, then the modified invariant $FSW^*_{Y(\Gamma)}(1, C - \sum m_i E_i)$ and $FSW^*_{Y(\mathrm{core}(\Gamma,\mathbf{M}))}(1, C - \sum_{i \notin J} m_i E_i)$ are equal to each other.*

Proof. To prove this proposition, one first shows by induction that the unmodified family invariants FSW are unchanged under this process. One should notice that the $k = 1$ version of the family blowup formula gives a complex one dimensional obstruction bundle.

Recall that Δ_l denotes the relative diagonal map from M_l to the $M_l \times_{M_{l-1}} M_l$. Suppose that a and $a+1$ are the markings of the two consecutive subredundant vertexes. When the $(a+1)$-th vertex is removed, family blowup formula relates the family invariants before and after the blowing ups by an insertion class of the form $\Delta^*_{a+1}(C - \sum_{i<a} m_i E_i - E_a)$, while the elimination of the $a+1$-th vertex gives the base space a $\mathbf{CP}^1$ bundle structure whose fibers are dual to $\Delta^*_{a+1} E_a$. The fact that the multiplicity of $-E_{a+1}$ is one implies that the obstruction vector bundle is a line bundle while the powers of relative tangent bundles $\mathcal{R}\mathbf{TM}_i$ do not show up.

Let us summarize as follows: Let Γ be the original admissible graph and $\widetilde{\Gamma}$ be the reduction by removing the subredundant vertex at one of the ends. According to the previous paragraph, there is a canonical map from $Y(\Gamma)$ surjectively to $Y(\widetilde{\Gamma})$, which has the $\mathbf{P}^1$ bundle structure. The bundle structure can be seen easily from the inductive construction displayed in the previous section. The fiber bundle can be embedded into $M_{|\mathrm{Ver}(\Gamma)|} \mapsto M_{|\mathrm{Ver}(\widetilde{\Gamma})|}$ such that the fibers are dual to $\Delta^*_{a+1}E_a$.

If we push forward the cohomology class $\Delta^*_{a+1}(C - \sum +i < am_iE_i - E_a)$ along the $\mathbf{P}^1$ bundle, it is equal to the pairing between $\Delta^*_{a+1}(C - \sum_{i<a} m_iE_i - E_a)$ and $\Delta^*_{a+1}E_a$ which is equal to 1. This shows that FSW is invariant under the single reduction. Then the general situation follows from mathematical induction and the repeated applications of the family blowup formula. To show that the modified invariant is unchanged as well, let us study how the reduction of the subredundant vertexes affects the admissible decompositions. As we know that the subredundant vertexes form distinct linear chains, the Poincare dual of the exceptional curves associated with these subredundant vertexes are -2 curves of the forms $E_a - E_{a+1}$ and -1 curves E_a, respectively. The second case appears only when the vertex a does not have any direct descendent, i.e., it is at the end of an isolated linear chain.

According to the general rule of determining the admissible decomposition classes, the curve class E_a can appear in the list only when they have negative pairings with the class $C - \sum m_iE_i = C - \mathbf{M}(E)E$; which is impossible as $m_a \geq 0$. Suppose that the good part of the hypothetical admissible decomposition is written as $C - \sum n_iE_i$, then the multiplicities $n_v = m_v = 1$ for the ends of the linear chains. Let v be the subredundant vertex at the end of one linear chain, then $n_v = (C - \sum n_iE_i) \cdot E_v \geq 0$ as $C - \sum n_iE_i$ and E_v coexist, and the distinct (pseudo)-holomorphic curves intersect positively. On the other hand, if $n_v > 1$, then some multiple of E_v must have shown up among the exceptional parts of the admissible decomposition. This is not allowed as we have discussed. Therefore, $n_v = 1$ for all these ends of linear chains of subredundant vertexes. Next, suppose that the particular admissible decomposition reexpresses $C - \mathbf{M}(E)E$ as

$$C - \sum n_iE_i + \sum_{i \in J} q_i(E_i - E_{i+1}) + \mathit{others},$$

where the term *others* represents certain collections of type I exceptional curves which I am not interested in at this moment. The middle terms come from the -2 curves in the various linear chains. By setting $q_v = 0$, one derives a collection of linear equations,

$$1 = n_i + q_{i-1} - q_i, i \in J.$$

Again, as $C - \sum n_iE_i$ and $E_i - E_{i+1}$ coexist and $C - \sum n_iE_i$ is the good part, it implies that $n_i - n_{i+1} \geq 0$. As $n_v = 1$ for these ends of the linear subredundant chains, $1 - n_i \leq 0, i \in J$. According to the previous equations, the sequence q_i must be nonincreasing when the indexes run through the chains according to their increasing orders on the markings. On the other hand, $q_v = 0$ for these ending vertexes. It implies that q_i are all zero for these subredundant vertexes. In other words, the hypothetical admissible decomposition cannot possibly involve these $E_i - E_{i+1}$.

When one compares the admissible decompositions with respect to these two pairs $(\Gamma, C - \sum m_i E_i)$, $(\mathrm{core}(\Gamma, \mathbf{M}), C - \sum_{i \notin J} m_i E_i)$, the collections of the admissible decomposition classes are in one to one correspondence. Because we have shown that the exceptional -2 and -1 curves associated with the subredundant vertexes cannot appear among the admissible decompositions of the class $C - \sum m_i E_i$, the good part $C - \sum n_i E_i$ carries the same properties $n_i = 1$, for all $i \in J$. By exactly the same reasonings and by induction as in the case of the un-modified family invariant, one can show that the modified invariants FSW^* are unaltered as well.

A concrete argument involves the backward induction on the levels of the admissible decomposition classes and the usage of Proposition 5.3.

As the admissible decompositions are un-altered, the cohomology classes c which appear in the insertions of the mixed invariants are pulled back from the base manifold $Y(\mathrm{core}(\Gamma, \mathbf{M}))$ by the compositions of the projection maps. Moreover the insertions of the new cohomologies class given by the family blowup formula are always of degree two which "cancel" with the $\mathbf{P}^1$ fibration structures, simply keeping the answer unchanged. q.e.d.

As a result, one has the following conclusion:

Proposition 5.5. *For the purpose of invariants calculation, the admissible graph in the topological type* $(\Gamma, C - \mathbf{M}(E)E)$ *can be replaced by its core and the multiplicity function* $\mathbf{M}$ *by its restriction to the* $\mathrm{core}(\Gamma, \mathbf{M})$.

The proposition can be explicitly used to simplify the evaluation of the contributions of the level $s, s > 0$ admissible decompositions to the family invariants.

6. The family Kuranishi model and the regular obstructedness

6.1 The nested Kuranishi model

I have spent a great deal of effort in defining the concept of admissible graphs, admissible decomposition classes, and the admissible stratification of the universal spaces. The purpose of developing these machineries is to prove the main theorem in the introduction. Before we move on, the following is a simple terminology which I will use frequently.

Definition 6.1. Let B be a stratified manifold and $\mathcal{X} \mapsto B$ is a fiber bundle of almost complex four-manifolds. A curve C in $\mathcal{X}$ is said to support over a point $b \in B$ if its image under $\mathcal{X} \mapsto B$ is the point b. A curve is said to support over a stratum of B if the point b lies in the stratum of B.

In the latter case, I only require the curve to support upon some point in the strata. Having defined the family Seiberg-Witten invariants $FSW_{Y(\Gamma)}(1, C-\mathbf{M}(E)E)$, its relationship with the family Gromov-Taubes theory suggests that the numerical invariant should be related to the counting of smooth curves in the class $C - \mathbf{M}(E)E$. However, it does not count the number of singular curves in C with the prescribed topological types of singularities assigned by $\mathbf{M}(E)$. There are several evidences indicating that the naive curve counting idea does not work directly. Before giving a proof of the main theorem, it is very important to have a deep digestion regarding these issues.

First, the restriction of the family moduli space to Y_Γ admits the action of a possibly nonempty group $G_{\Gamma,\mathbf{M}}$. Therefore, the counting problem is automatically (at least partially) symmetric with respect to the group action. To get the counting of curves, formally one should expect to divide the family invariant by the order of the group. Very unluckily, the original family invariant does not carry the necessary divisibility property.

Second, the small n example in Section 9 (cf. [59]) has shown that it is possible that some other types of curves contribute to the invariants as well. Even though a naive argument may suggest that one can move the generic points (sections) on the fiber bundle with the intersected curves supporting over Y_Γ, the first observation shows that this is not the case. In fact, there are topological obstructions to move the counted solutions to support merely over Y_Γ. In other words, the (pseudo-) holomorphic curves can support over $Y(\Gamma) - Y_\Gamma$ as well. Simple examples of this type lead to some Taubes type admissible decompositions involving certain number of type I exceptional curves.

Third and noteworthy, there might be some multiple coverings of the exceptional curves appearing in the counting problem. In other words, a high multiple of exceptional curves can show up, and the counting scheme detects the singularities as well as the smooth points with multiplicity more than one. Therefore, the appearance of these exceptional curves can be somewhat detrimental to the counting scheme.

As was discussed briefly, the exceptional curves can be divided into two different types. The so-called type I exceptional curves and type II exceptional curves. They are distinguished by their different behavior under the blowing down projection maps. The former type of curves get mapped to points while the latter get mapped to nontrivial curves in M.

In general, there is no constraint about the support of type II exceptional curves. They can show up freely over the top stratum Y_Γ. The appearance of the type II exceptional curve is no doubt detrimental to the formulation.

In the paper [38], [39], I gave an algebraic formalism to handle the type II exceptional curves under some conditions on the associated moduli spaces.

By twisting with a very high power of ample line bundles (which definitely weakens the range of the validity of the theorem), the type II exceptional curves can be gotten rid of. From our point of view, it was the type I exceptional curve along with the complicated behavior of the other singular curvesthat made the original program look hopeless.

Within our framework these various difficulties and discrepancies can be understood and handled from a unified point of view. The existence of the family blowup formula, as well as its cousin, family switching formula, make this somewhat difficult problem transparent and solvable.

The key concept is the "admissible decomposition classes", introduced in the previous section. When this concept was introduced by the author, there was not yet clear indication that it had anything to do with any known mathematical structure. It turns out that, very surprisingly, the purely family Gromov-Taubes theoretical concept roots deeply in the excess intersection theory developed by the algebraic geometers. Historically residual intersection theory has played an important role in the study of enumerative geometry. The theory developed here, which apparently has nothing to do with the well known structure, turns to be tied to the theory [16] very closely. For a demonstration of this phenomenon, in Section 9 we will employ the $n = 8$ case as a concrete example. In that example, I will explain briefly how the multiple coverings of the type I exceptional curves relate to the old problem that might puzzle Vainsencher [59] and probably the other algebraic geometers a lot. In general, the so called "reduced" family moduli spaces over $Y(\Gamma) - Y_\Gamma$ is nonempty. Even worse, they are generally very complicated, nonreduced geometric objects. The greatest problem is that these structures actually depend on the geometry of the manifolds M and C strongly. In two respects it is difficult to extract information from these singular objects:

(1) It is not clear what type of objects one is counting.

(2) Even if the counting interpretation is available, it "might" depend on M and the class C explicitly and ruin the keyword "universal" in the main theorem.

These are the questions I intend to answer.

To begin this discussion, let me prove a version of Kuranishi model suitable to my purpose. The Kuranishi models were extensively discussed by Taubes [53], Li-Tian [28], Ruan [46], Siebert [50], etc. in the context of either Seiberg-Witten theory or Gromov-Witten invariants. Historically, it was frequently used in Donaldson theory [9].

If one adopts the algebraic family Seiberg-Witten "invariants," then the invariants are defined even for nongeneric moduli spaces. The differential topological argument is of perturbation nature. Thus, they should be viewed as the two sides of the same story.

In an old fashion discussion, people started from the perturbation on elliptic PDEs to produce the transversal moduli objects in order to define the suitable invariants. The key idea of the Kuranishi model is to perform the perturbation in a broader context. Specifically, the perturbation needs not come from the perturbation of the elliptic equations, which is much more restrictive. In many situations, there is either difficulty in performing the geometric perturbation or the desired generic geometric perturbation is out of reach. Then one considers some nongeometric perturbation under which the corresponding zero locus become smooth, yet the smooth object usually does not share the same geometric meaning as the original problem. In our situation, the original geometric question is regarding curve counting. By perturbing the family Kuranishi model, one gains a means of counting the "invariants," but loses the original symplectic or algebraic geometric meaning. However, the extremal power of family blowup formula and the family switching formula bring everything back to the algebraic geometric context at least at a formal level.

Let $\mathcal{M}$ be the "zero locus/moding out gauge equivalence" of the family Seiberg-Witten equations. It is usually called the family Seiberg-Witten moduli space over B. If the space is already smooth, there is no difficulty in defining the invariants. On the other hand, a singular $\mathcal{M}$ does not give rise to a meaningful invariant count directly. Following Taubes [53], the insertions of powers of $e(\mathbf{e})$ can be replaced by imposing the extra conditions requiring the spinors sections to vanish when restricted on suitable sections of the fiber bundle $\mathcal{X} \mapsto B$. In this way, one can always assume that the family dimension of the moduli space is zero. It is clear that the extra conditions do not harm the discussion as they are of algebraic nature. Let us review briefly the analytical setup.

Given a family of fiberwise Riemannian metric and self-dual two forms on the fiber bundle, let us denote the family of metrics and two forms by $g(b)$ and $\mu(b)$, respectively, where b denotes a point in B. Then the family Seiberg-Witten equations can be written down immediately. Let us write down the deformation complex accordingly. To simplify the notation, let us assume that a local trivialization of $\mathcal{X} \mapsto B$ has been chosen such that $g(b)$ and $\mu(b)$ are viewed as a local family of metrics and two forms on the same manifold M. Before writing down the complex, I introduce some notations. Let $*$ denote the usual Hodge star operation $\Omega^2(M) \mapsto \Omega^2(M)$. Let v denote a tangent vector in T_bB. Then $\dot{*}_{g(b)}(v) : \Omega^2(M) \mapsto \Omega^2(M)$ is the linearized map along the v direction. Likewise, $\dot{\mu}(v)$ denotes the first jet of variations of self-dual two forms along the v direction. Suppose $2d$ is the real family dimension of the fiber-wise $spin_c$ structure, let $x_i = s_i(b)$ be d different points on M with s_i being the sections of the original fiber bundle $\mathcal{X} \mapsto B$.

Let (A_0, Ψ_0, b) denote a Seiberg-Witten solution supporting over b. Then the triple (a, ψ, v) denotes a first order infinitesimal deformation of (A_0, Ψ_0, b). The family deformation complex reads as,

$$\mathbf{T}_bB \oplus \Omega^1(M) \oplus \Gamma(\mathcal{S}^+) \mapsto \Omega^0(M) \oplus \Omega^2_+(M) \oplus \Gamma(\mathcal{S}^-) \oplus \mathbf{R}^{2d},$$

where the linearized map is given by assigning (v, a, ψ) to

$$(\delta(a), P_+F_a - 2Re(\sigma(\psi, \Psi_0)) - i(P_+\dot{\mu}(v)) +$$
$$\dot{*}_{g(b)}(v)P_-F_{A_0}, D_{A_0}\psi + a \cdot \Psi_0, \alpha(x_1), \dots, \alpha(x_d)).$$

The symbol $\alpha(x_i)$ denotes the α component of the spinor Ψ_0 at the point x_i. The pairing $\cdot$ between a and Ψ_0 is the usual Clifford multiplication. Notice that some reduction has been done to ensure the vanishing of the anti-self-dual projection. The fact that (A_0, Ψ_0, b) is a solution has been used to ensure

$$\dot{*}_{g(b)}(v)(F_{A_0} - \sigma(\Psi_0, \Psi_0) - i\mu) \equiv 0.$$

If the linearized map is surjective, then the solution (A_0, Ψ_0, b) is a smooth point in the family moduli space. Otherwise, there will be a nontrivial cokernel. Even though the reduced family moduli space is "expected" to be of zero dimensional. This is hardly what usually happens in real life. Suppose that the point (A_0, Ψ_0, b) moves along the reduced family moduli space and kernels and cokernels of the complex form semi-bundles. A new subtlety in the family theory is that, even though (A_0, Ψ_0, b) is a smooth point of the family moduli space, (A_0, Ψ_0) at b may not be a smooth point in the fiberwise moduli space over b. This discrepancy opens up the possibility of introducing the nested family Kuranishi model.

In the following, the extended cokernel semi-bundle

$$\Omega^0(M) \oplus \Omega^2_+(M) \oplus \Gamma(\mathcal{S}^-) \oplus \mathbf{R}^{2d}/\mathrm{Im}(T_bB \oplus \Omega^1(M) \oplus \Gamma(\mathcal{S}^+)),$$

will be abbreviated as **Obs**.

In the algebraic geometric description of the Kähler-Seiberg-Witten theory, it can be identified with more recognizable objects through family blowup formula. Suppose $p_g(M) = 0$ and C is very ample, $B = Y(\Gamma)$ and $x_1, x_2, \cdots x_d$ are d cross sections of $\mathcal{X} \mapsto B$, the map

$$T_bB \mapsto H^1(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}(C - \mathbf{M}(E)E)) \oplus \mathbf{C}^d$$

at $(s, b) \in (H^0(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}(C - \mathbf{M}(E)E)), Y(\Gamma))$, $s(x_i) = 0$ can be determined as follows:

First, the deformation of the complex structure at b determines the Kodaira-Spencer map $T_bB \mapsto H^1(\mathcal{X}_b, \Theta_{\mathcal{X}_b})$, where $\Theta_{\mathcal{X}_b}$ denotes the holomorphic tangent sheaf of $\mathcal{X}_b$. The infinitesimal deformation of holomorphic structures on $C - \mathbf{M}(E)E$ prolongs the infinitesimaldeformation of complex structures on $\mathcal{X}_b$ which determines a covariant derivative ∇.

A covariant derivative of $s \in H^0(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}(C - \mathbf{M}(E)E))$, ∇s, determines a morphism from $\Theta_{\mathcal{X}_b}$ to the locally free sheaf $\mathcal{O}_{\mathcal{X}_b}(C - \mathbf{M}(E)E)$. Then

$$T_bB \mapsto H^1(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}(C - \mathbf{M}(E)E))$$

is identified with the composition of the Kodaira-Spencer map with some covariant derivative map ∇s. As usual, the factor $\mathbf{C}^d$ is identified with $(L|_{x_1}, L|_{x_2}, \cdots L|_{x_d})$, where L is the holomorphic line bundle associated to $\mathcal{O}_{\mathcal{X}_b}(C - \mathbf{M}(E)E)$.

We consider some sublocus of the family moduli space as the following. Given a solution (α, β), one requires that the α component vanishes on d sections of $\mathcal{X} \mapsto B$. The sublocus defined by imposing the algebraic constraints will be called a reduced family moduli space and is denoted by $\mathcal{M}_{red}$. If it consists of a finite number of smooth points, the signed points count the invariant [53], [48], [49].

Warning 1. The reduced family moduli space $\mathcal{M}_{red}$ is the reduction of the family moduli space by imposing a finite number (determined by its family dimension) of section passing conditions on the solutions. It has nothing to do with the reduced scheme structure in algebraic geometry.

In the following, I review briefly the standard Kuranishi model construction and give a formulation in the family version.

Proposition 6.1. *Let B be a finite dimensional oriented compact manifold and let $\mathcal{X} \mapsto B$ be a smooth fiber bundle of oriented smooth symplectic four-manifolds whose fibers are diffeomorphic to M. Given a $spin_c$ structure $\mathcal{L}$ and a reduced family Seiberg-Witten moduli space of the $spin_c$ structure $\mathcal{M}_{red} \mapsto B$ with expected dimension zero; there exists a family Kuranishi model of $\mathcal{M}_{red}$ which consists of a smooth section $\mathbf{s} : \mathcal{O} \times \mathbf{R}^k \mapsto \mathbf{R}^{k'}$ satisfying:*

(1) *The finite dimensional smooth manifold $\mathcal{O}$, has a compact closure.*

(2) *The zero locus $\mathbf{s}^{-1}(0)$ is diffeomorphic to $\mathcal{M}_{red}$, which is compact.*

(3) *The image of $\mathcal{O} \mapsto B$ can be arranged to lie in an arbitrarily small prechosen open neighborhood of the projection image $\mathcal{M}_{red} \mapsto B$, by shrinking $\mathcal{O}$ if necessary.*

(4) *For generic choice of $\eta \neq 0$ in the neighborhood of $0 \in \mathbf{R}^{k'}$, the preimages $\mathbf{s}^{-1}(\eta), \eta \in \mathbf{R}^{k'}$ are smooth, compact, oriented and are of zero dimension.*

The signed sum of the finite number of points in the preimage $\mathbf{s}^{-1}(\eta)$ of a generic η defines the family Seiberg-Witten invariant of $\mathcal{L}$ over B, which is independent of the choices of the Kuranishi models satisfying (1), (2), (3), (4).

The construction is standard. For the convenience of the readers, we outline the key points of the construction.

Sketch of Construction. Given a fiberwise $spin_c$ structure $\mathcal{L}$ on $\mathcal{X} \mapsto B$, an oriented fiber bundle of smooth oriented Riemannian four-manifolds over a compact smooth oriented base manifold B, the same argument as in the ordinary Seiberg-Witten theory [61] implies that the family moduli space $\mathcal{M}$ of the solutions of family Seiberg-Witten equations over B is compact and there is a natural projection morphism from $\mathcal{M}$ to B. Suppose that the realfiberwise expected dimension of the moduli space is $d(\mathcal{L})$, then the family expected dimension is $\dim_{\mathbf{R}} B + d(\mathcal{L})$. For simplicity, one assumes that the base manifold B is even dimensional. Consider the submoduli space of the family moduli space requiring the spinor sections $\psi \in \Gamma(\mathcal{S}^+)$ to vanish at $\frac{\dim_{\mathbf{R}} B + d(\mathcal{L})}{2}$ different cross sections $x_i : B \mapsto \mathcal{X}, 1 \leq i \leq \frac{\dim_{\mathbf{R}} B + d(\mathcal{L})}{2}$ of $\mathcal{X} \mapsto B$. The dependence on the choices of the cross sections has been suppressed and it is denoted by $\mathcal{M}_{red}$. The moduli space with $\dim_{\mathbf{R}} B + d(\mathcal{L})$ additional algebraic section-passing constraints is called a reduced family moduli space. When $\dim_{\mathbf{R}} B + d(\mathcal{L}) \geq 0$, the expected family dimension of $\mathcal{M}_{red}$ is zero.

Suppose a family of fiberwise self-dual symplectic forms ω has been given on $\mathcal{X} \mapsto B$ which split the fiberwise $spin_c$ spinor Ψ into $(\alpha, \beta) \in \Gamma(L_{C_0}) \oplus \Gamma(L_{C_0} \otimes \mathcal{K}^{-1}_{\mathcal{X}/B})$. The reduced family moduli space $\mathcal{M}_{red}$ is defined to be the zero locus of the smooth section

$$([A, \Psi], b) \longrightarrow (P_+F_A - \sigma(\Psi, \Psi) - i\mu + ir\omega, D_A\psi,$$

$$\alpha(x_1(b)), \cdots \alpha(x_d(b)), b),$$

of the Banach vector bundle

$$(\Omega^2_+)_{\mathcal{X}/B} \oplus \Gamma(\mathcal{S}^-_{\mathcal{L}}) \oplus \mathbf{R}^{\dim_{\mathbf{R}} B + d(\mathcal{L})}$$

over the family configuration space $\mathcal{C}_{\mathcal{L}} \mapsto B$ of the gauge equivalence classes of $([A, \Psi], b)$ tuples. A trivialization of $L_C|_{x_i(b)}$ has been chosen implicitly.

It is well known that one can construct some Kuranishi models of $\mathcal{M}_{red}$ which reembed $\mathcal{M}_{red}$ into a finite dimensional manifold $\mathcal{O}' \mapsto B$ noncanonically. By shrinking the neighborhood $\mathcal{O}'$ of $\mathcal{M}_{red}$ if necessary, it can be chosen to have compact closure in $\mathcal{C}_{\mathcal{L}}$. Moreover, it maps onto an open neighborhood of the image $\mathcal{M}_{red} \mapsto B$ under the canonical projection map. The existence of Kuranishi model implies the existence of a finite rank vector bundle $\mathbf{W}' = \mathcal{OBS}$ and $\mathcal{M}_{red} \mapsto \mathcal{O}'$ can be realized as the zero locus of a smooth section $\mathbf{s}' \in \Gamma(\mathbf{W}')$ in the neighborhood $\mathcal{O}'$. The datum $(\mathcal{O}', \mathbf{s}', \mathbf{W}')$ defines a Kuranishi model of $\mathcal{M}_{red} = \mathbf{s}'^{-1}(0)$.

The compactness of the closure $\overline{\mathcal{O}}'$ implies that the vector bundle $\mathbf{W}'$ can be generated by a finite number of $\mathcal{C}^\infty$ global sections $s_i \in \Gamma(\mathcal{O}', \mathbf{W}')$ over $\mathcal{O}'$. Then the datum of Kuranishi model $(\mathcal{O}', \mathbf{s}', \mathbf{W}')$ can be stabilized by $\mathbf{W}$ and can be replaced by the one, $(\mathcal{O}' \times_B \mathbf{W} \times \mathbf{R}^k, \mathbf{s} = \mathbf{s}' \oplus \mathrm{Id}_{\mathbf{W}} + \sum_{i \leq k} t_i s_i \oplus (t_1, t_2, t_3, \cdots, t_k), \mathbf{W}' \oplus \mathbf{W} \oplus \mathbf{R}^k)$.

By choosing a suitable $\mathbf{W}$; $\mathbf{W} \oplus \mathbf{W}' \equiv \mathbf{R}^h$, and by renaming an open neighborhood in $\mathcal{O}' \times_B \mathbf{W}$ as $\mathcal{O}$, the resulting family Kuranishi model is isomorphic to one which is of the particularly simple form $(\mathcal{O} \times \mathbf{R}^k, \mathbf{s}, \mathbf{R}^{k'})$.

As $\mathcal{M}_{red} = \mathbf{s}^{-1}(0)$ is compact, there is a small open neighborhood $\mathcal{N}_0$ of $0 \subset \mathbf{R}^{k'}$ such that the preimages $\mathbf{s}^{-1}(\eta), \eta \in \mathcal{N}_0$ is compact. By shrinking $\mathcal{O}$ and $\mathbf{R}^{k'}$ if necessary, this additional property can be assumed to hold on the original model. Then the finite dimensional Sard theorem implies that the regular values are open dense and the generic fibers are smooth, compact of zero dimensional. It is suitable to define the invariants by using the compact regular fibers. Taking any $\eta \in \mathbf{R}^{k'}$ belonging to this dense subset, the sign count on $\mathbf{s}^{-1}(\eta)$ defines the invariant.

The fact that the invariants defined in this way are independent of the way of choosing the Kuranishi family has been proved by the experts mentioned before. I do not plan to repeat their wisdom here. q.e.d.

In the family theory, it is necessary to construct a version of Kuranishi model which enables us to separate the invariant contribution away from a smooth embedded submanifold.

Let us formulate a sufficient condition to achieve this goal. Let $\mathcal{T} \mapsto \mathcal{OBS}$ be the family tangent-obstruction complex of the reduced family Seiberg-Witten moduli space $\mathcal{M}$. As one is discussing the family invariants, there is always a horizontal TB factor in $\mathcal{T}$. At this moment, one does not impose any regularity condition on $\mathcal{T}$ or $\mathcal{OBS}$. Let **Obs** denote the cokernel semi-bundle of the fiberwise tangent obstruction complex.

Definition 6.2. Suppose that $\mathcal{M} \times_B S = \mathcal{M}_S \subset \mathcal{M}_{red}$ is the compact subspace of the reduced family moduli space which is supported over a compact smooth oriented manifold S. The pair ($\mathcal{M}_S$,S) is said to be regularly obstructed if:

(1) There exists a family Kuranishi model of $\mathcal{M}_S$ such that the cokernel obstruction semi-bundle of the fiberwise tangent-obstruction complex **Obs** splits off a subbundle $\mathbf{V}$ over $\mathcal{M}_S$.

(2) The restriction of the family tangent-obstruction map to $\pi_S^*(\mathbf{T}B|_S) \mapsto \mathbf{V}$, constructed by differentiating the fiber-wise deformation complex with respect to the base, is surjective onto $\mathbf{V}$ over $\mathcal{M}_S$.

(3) The kernel of the map in (2). is $\mathbf{T}S \subset \mathbf{T}B|_S$.

The explicit form of the map $\pi^*(\mathbf{T}B|_S) \mapsto \mathbf{V}$ can be found in the proof of Proposition 7.2.

One has to be cautious that the regular obstructedness condition is not an intrinsic differential topology concept independent of Kuranishi perturbation. It is possible that after a slight perturbation, the regular obstructedness condition is easily destroyed. In actual application, the regular obstructedness is closely related to the appearance of some type I exceptional curves. Thus, it should be viewed as a concept in the Gromov-Taubes theory.

In the following, I discuss the tool I use to calculate the "excess" contribution of family invariant. To illustrate the main idea, let us discuss an important special case first. Later I generalize the picture to the "nested" situation based on induction.

Proposition 6.2. *Under the assumption that the pair $(\mathcal{M}_S, S)$ is regularly obstructed, the family invariants attached to $\mathcal{M}_S$ and the family invariants over $\mathcal{M}_{red} - \mathcal{M}_S$ are well defined and they satisfy the excision property. Namely, The sum of the invariant contributions over $\mathcal{M}_S$ and over $\mathcal{M}_{red} - \mathcal{M}_S$ are equal to the invariant defined by $\mathcal{M}_{red}$. Moreover, if the reduced family moduli space $\mathcal{M}_{red} - \mathcal{M}_S$ consists of a finite number of smooth points, then the excess invariant contribution is calculated by the sum of signed numbers counting these finite number of points.*

In general, one does not expect $\mathcal{M}_{red} - \mathcal{M}_S$ to be compact. In fact, its closure $\overline{\mathcal{M}_{red} - \mathcal{M}_S}$ can intersect with $\mathcal{M}_S$. The proposition guarantees that once the "lucky" situation is met, the invariant count can be proceeded in an intuitive way. In terms of algebraic geometric language, it is the "equivalence" of the invariant we are going to determine (cf. [16]).

At first, one may naively try to perturb arbitrarily the family Seiberg-Witten equations. However, there is a potential danger in that the solutions over S can be perturbed into $B-S$ and the counting over S and the counting over $B-S$ mess up. In our explicit application, the compact manifold S will be chosen to be the various $Y(\Gamma), \Gamma \in \Phi(\mathcal{ADM})$, which are the existence loci of certain type I exceptional curves. A random perturbation of the equations will alter the loci as well. In fact, one needs to make an effective use of the fact that, effectively, the pair $(\mathcal{M}_S, S)$ is regularly obstructed.

Following the regular obstructedness assumption, the fiberwise obstruction semi-bundle $\mathbf{Obs}$ can be split in the $\mathcal{C}^\infty$ category into $\mathbf{Obs}|_S = \mathbf{Obs}' \oplus \mathbf{V}$ over $\mathcal{M}_S$. The $\mathbf{Obs}'$ denotes the residual cokernel semi-bundle whose explicit form is not relevant to us at this moment.

In other words, restriction of the family tangent obstruction complex over $\mathcal{M}_S$ is reduced to a subcomplex formally identifiable as one coming from the family theory over S. By passing from the family theory over B to one over S, one removes from $\mathcal{OBS}$ the bundle $\mathbf{V}$, which is identified with the normal bundle of S, $\mathbf{N}_S$ in $\mathbf{T}B$, through the restriction of the tangent obstruction map $\pi_S^*(\mathbf{T}B|_S) \longrightarrow \mathbf{V}$.

Proof of Proposition 6.2. Given an $\mathbf{R}^k$ family of family Kuranishi perturbations of the family tangent obstruction complex over $\mathcal{M}_{red}$, under which the linearized Fredholm operator becomes surjective, the restriction of the Kuranishi map to $\mathcal{M}_S$ defines another $\mathbf{R}^{k_1}$ perturbation family over the subspace $\mathcal{M}_S$. During the process, one has to restrict to those perturbations in the $\mathbf{R}^k$ family with trivial projection onto $\mathbf{V}$.

These two spaces are related by the restriction map $Res_S : \mathbf{R}^k \mapsto \mathbf{R}^{k_1}$. Moreover, the family of finite dimensional perturbation parameterized by $\mathbf{R}^{k_1}$ is surjective to the residual extended obstruction semi-bundle $\mathbf{Obs}'$ over $\mathcal{M}_S$.

One can derive a commutative diagram between : $\mathcal{O} \times \mathbf{R}^k \mapsto \mathbf{R}^{k'}$ and $\mathbf{s}|_S : \mathcal{O}|_S \times \mathbf{R}^{k_1} \mapsto \mathbf{R}^{k_1'}$ as follows:

$$\begin{array}{ccc} \mathcal{O} \times \mathbf{R}^k & \stackrel{\mathrm{Res}_S}{\longrightarrow} & \mathcal{O}|_S \times \mathbf{R}^{k_1} \\ \Big\downarrow \mathbf{s} & & \Big\downarrow \mathbf{s}|_S \\ \mathbf{R}^{k'} & \longrightarrow & \mathbf{R}^{k_1'}. \end{array}$$

Since the restriction map Res_S is surjective, the Baire second category subset of $\mathbf{R}^k$ gets mapped to a Baire second category subset of $\mathbf{R}^{k_1'}$. This enables us to choose the generic elements simultaneously to make the perturbed $\mathcal{M}_{red}$ and $\mathcal{M}_S$ smooth at the same time. It can be achieved by choosing the deformation parameters coherently. Since $\mathbf{V}$ has been removed from the original extended obstruction semi-bundle $\mathbf{Obs}|_S$, the formal family dimensions of the two reduced moduli spaces $\mathcal{M}$ and $\mathcal{M} \cap \pi^{-1}(S)$ are both zero. This follows from the fact that $\dim_{\mathbf{C}} V = \dim_{\mathbf{C}} B - \dim_{\mathbf{C}} S$. Because both the reduced moduli spaces are compact, both of them consist of finite number of signed points. It is crucial that the vector bundle $\mathbf{V}$ has been removed when one considers the restriction morphism. Otherwise, the restricted family over S will have a negative family expected dimension.

When both the perturbed objects are compact, smooth of zero dimension, one can perform the counting simultaneously and get a pair of integers. The second integer is interpreted as the family invariant attached to $\mathcal{M}_S$, and the former one as the total invariant. Their difference is defined to be the invariant contribution over $B - S$. As usual, it does not relate to classical counting directly unless $\mathcal{M}_{red} - \mathcal{M}_S$ has been compact and smooth of dimension zero. This scheme explains why the invariant contributions satisfy the excision property.

On the other hand, once the lucky situation mentioned in the later part of Proposition 6.2 is met, the reduced family moduli space can still be a possibly nonsmooth space over S and a finite number of points over $B - S$. As a finite set is compact, there exist a tubular neighborhood $\mathcal{N}_S B$ of S such that the support of these finite number of solutions do not lie in the neighborhood.

Potentially the tangent obstruction complex may still have nontrivial cokernels over $\mathcal{M}_S$, one can find a finite number of sections of the extended family Seiberg Witten deformation complex such that the restriction of the sections to S generate the cokernel semi-bundles over $\mathcal{M}_S$. By multiplying some bump functions, the sections can be arranged to support within $\mathcal{N}_S B$. Then one can choose the perturbation to support within the neighborhood such that the reduced moduli space over $B - S$ is not changed under the perturbation. Under this special type of perturbation, it is clear that the statement in the proposition is valid. In general, one needs to prove that the definition is independent of the choices of the perturbations as well as the choices of the regular values. The argument is exactly the same as the usual case and is therefore omitted. The only minor difference is that the cobordism also restricts to a subcobordism relating $\mathcal{M}_S \mapsto S$ and one proves the invariance of both objects simultaneously. One can consult [53] or [46] for details. q.e.d.

Having addressed the special case, I move to the general situation. As will be seen in the actual application, I will need to deal with cases that different "components" $\mathcal{M}_S$ coexist at the same time. I will enhance my tool to deal with these situations, too.

The major issue is to deal with the cases that different S_i and S_j intersect each other. Some S_j can even be contained in the other S_i.

The following proposition is therefore named as the nested family Kuranishi model. Let us explain its meaning and formulate it as a new proposition.

To simplify my discussion and make it coherent to the language of algebraic geometry, I make some additional assumptions on S_i and B. I assume that S_i, B are almost complex manifolds. When $S_j \subset S_i$ or $S_i \subset B$, they give rise to pseudo-holomorphic imbeddings. This includes the special cases that B is complex and $S_1, S_2, \cdots$ form a chain of complex submanifolds in B.

Corollary 6.1 (Nested Kuranishi Model). *Let*

$$\emptyset \subset \cdots S_m \subset S_{m-1} \subset S_{m-2} \cdots \subset S_0 = B$$

be a nested family of compact oriented smooth almost complex manifolds such that their dimensions drop down monotonically. Let $\mathcal{M}_{S_i} = \mathcal{M}_{red} \times_B S_i, m \geq i \geq 0$ *be the corresponding "components"of reduced family moduli spaces.*

Suppose that the adjacent $\mathcal{M}_{S_{j+1}} \mapsto S_{j+1}$ *and*

$$\mathcal{M}_{red} \times_B S_j \mapsto S_j; S_{j+1} \subset S_j$$

satisfies the regular obstructedness assumption of Definition 6.2. Then one can define the nested family invariants such that invariant contributions over $S_j - S_{j+1}$ *make sense. They are defined in such a way that the excision property is valid. i.e., The total invariant over* B *can be alternately calculated as the sum of the invariant contributions over the various* $S_j - S_{j+1}$.

Proof. The proof of the corollary follows from Proposition 6.2 by applying induction on the pair $S_{j+1} \subset S_j$. q.e.d.

One can generalize the corollary slightly further to cover the cases that the inclusion pattern of a collection of smooth oriented manifolds is encoded by a finite graph instead of a chain. The manifolds bijectively correspond to the vertexes of the graph, while an oriented 1-edge is constructed when one manifold is a proper submanifold of the other with no other manifolds among the collection lying in-between. This format will be used later. The graphs mentioned here correspond to the subordinate relationships of different levels of admissible decomposition classes.

It is easy to see that the regular obstructedness condition is transitive. Namely, if $S_3 \subset S_2 \subset S_1$ are three oriented smooth manifolds which satisfy the consecutive regularly obstructed conditions, then $S_3 \subset S_1$ is also regularly obstructed.

Let $\mathbf{V}_1$ denote the piece of obstruction subcomplex which splits off over S_2. Let $\mathbf{Obs}'$ be the extended residual obstruction complex (which is in general a semi-bundle) over S_2. By the original assumption, the $\mathbf{Obs}'$ splits off another vector bundle over S_3, which is denoted by $\mathbf{V}_2$. Then the regularly obstructedness of these consecutive pairs are equivalent to the surjectiveness of the maps $\phi_1 : TS_1|_{S_2} \mapsto \mathbf{V}_1$ and $\phi_2 : TS_2|_{S_3} \mapsto \mathbf{V}_2$. Forgetting the manifold S_2, the map $TS_1|_{S_3} \mapsto \mathbf{V}_1|_{S_3} \oplus \mathbf{V}_2$ is induced by first splitting $TS_1|_{S_3}$ into $TS_2|_{S_3} \oplus \mathbf{N}_{S_2/S_1}$ and extending ϕ_2 to a map $\widetilde{\phi}_2 : TS_1|_{S_3} \mapsto \mathbf{V}_2$ by composing with the projection map $TS_1|_{S_3} \mapsto TS_2|_{S_3}$. Then one defines the morphism $TS_1|_{S_3} \mapsto \mathbf{V}_1|_{S_3} \oplus \mathbf{V}_2$ by $\phi_1 \oplus \widetilde{\phi}_2$. The splitting of the tangent bundles and the projection map relies on the introduction of certain Riemannian metrics and is therefore noncanonical. Nevertheless it is an easy exercise in linear algebra to prove the surjectivity of the restriction map.

Proposition 6.2 and Corollary 6.1 are in certain sense the differential topological analogue of the residual intersection theory [16] in the context of family invariants. They will play a crucial role in the proof of my main theorem.

6.2 The nested perturbations preserving the regular obstructedness

The previous discussion is good enough for enumerating the contributions of the admissible decomposition classes supporting over nongeneric strata. However, I need a stronger version in identifying the modified family invariants and the relevant Gromov-Taubes invariants. Namely, under a special sequence of family Kuranishi perturbations, the reduced family moduli space over the top stratum is compact. The reader should be aware that this subsection is not used in the algebraic geometric proof of the main theorems.

We will try to construct such kind of perturbations concretely. First, we set up the theory in a general setting.

As before take $\mathcal{X} \mapsto B$ to be the fiber bundle of four-manifolds over which a fixed fiberwise $spin_c$ structure has been chosen. We focus on a reduced family moduli space over B of the fixed $spin_c$ structure.

Suppose that B is given an almost complex structure and let Y denote the resulting almost complex manifold. Let X be an almost complex submanifold of Y. If there are more than one X involved, I use the double subscripts $j;k$ to parameterize the manifolds. Let J be an index set and let $X_{j;k}, j \in \mathbf{J}$ be a collection of almost complex submanifolds of Y such that different $X_{j;k}$ intersect transversally. The second subscript $k; k \in \mathbf{N} \cup \{0\}$ denotes the level of the set $X_{j;k}$. Namely, it satisfies two properties:

(1) If $X_{l;k_l} \supset X_{j;k_j}$ is a proper inclusion, then $k_l < k_j$.

(2) There exists at least one $X_{l;k-1} \supset X_{j;k}$ with level $k-1$.

The levels discussed in this subsection will correspond to the levels of the admissible decomposition classes, once we apply the general setup to a concrete problem.

As before, I still require the embedding $X \subset Y$ (or in general $X_{j;k} \subset Y$) to induce regularly obstructed reduced family moduli space $\mathcal{M}_X \subset \mathcal{M}_{red}$ ($\mathcal{M}_{X_{j;k}} \subset \mathcal{M}_{red}$).

The regular obstructedness condition requires that the restriction of the cokernel semi-bundle $\mathbf{Obs}|_X$ is isomorphic to $\mathbf{Obs}' \oplus \mathbf{V} \cong \mathbf{Obs}' \oplus \mathbf{N}_X Y$. Through the isomorphism $\mathbf{V}$ inherits a structure of complex vector bundle, because $X \subset Y$ is almost complex. Let $\mathcal{OBS}$ denote the obstruction bundle of a chosen family Kuranishi model. The previous isomorphism induces an isomorphism on the obstruction bundle $\mathcal{OBS} \cong \mathcal{OBS}' \oplus \mathbf{V} \cong \mathcal{OBS}' \oplus \mathbf{N}_X Y$.

The reduced family moduli space $\mathcal{M}_{red}$ can be identified to be $s^{-1}(\mathbf{0})$, where $s \in \Gamma(\mathcal{O}|_Y, \mathcal{OBS})$ is a smooth section of the extended obstruction bundle defining the reduced family moduli space in this particular family Kuranishi model. Through the isomorphism $\mathcal{OBS}|_X \equiv \mathcal{OBS}' \oplus \mathbf{N}_X Y$, the section s induces a smooth section $\perp s$ of $\mathbf{N}_X Y$ viewed as a vector bundle over $\mathcal{N}_X Y \supset X$. One says that the section is regularly obstructed in the strong sense if $\perp s|_{\mathcal{O}|_X}$ is the zero section. Similarly, the section is regular obstructed in the weak sense if the restriction of $\perp s|_{\mathcal{O}|_X}$ to an open neighborhood of the zero locus $s = 0$ is identically zero. It is obvious that the former implies the latter.

Before my discussion, I fix a diffeomorphism between $\mathbf{N}_XY$ and a tubular neighborhood $\mathcal{N}_XY$ of X in Y. Suppose the space Y is stratified into smooth locally closed strata with smooth closures, I requirethe diffeomorphic identification to satisfy some extra constraint.

Let $Y' = X_{j';k'}$ with $X = X_{j;k} = X \subset Y' \subset Y$ be the smooth compactification of an almost complex stratum including X. Then $\mathbf{N}_XY' \subset \mathbf{N}_XY$ denotes the corresponding inclusion of normal bundles. One requires that the tubular neighborhoods and the diffeomorphisms have been chosen in a way such that the image of $\mathbf{N}_XY'$ into $\mathcal{N}_XY$ is compatible with the corresponding tubular neighborhood one has chosen for X into Y'. If for all the possible $Y' = X_{j';k'}$ the condition has been satisfied, then the tubular neighborhoods of $X_{j;k}$ are said to be stratified tubular neighborhoods. Let us fix such a sequence of stratified tubular neighborhoods and diffeomorphisms $\iota_{j;k} : \mathbf{N}_{X_{j;k}}Y \mapsto \mathcal{N}_{X_{j;k}}Y$.

In my actual application, the space Y will be stratified into the union of different strata; each allows the action by a finite group $\subset \mathbf{S}_n$.

Definition 6.3. Let $\mathbf{J}$ be an index set such that $Y = \coprod_{j\in\mathbf{J};k} Y_{j;k}$ be the stratification such that the compactification of all strata are smooth submanifolds of Y. Let $G_{j;k}$ be a finite group acting on $Y_{j;k}$ that preserves the almost complex structure. A set W is said to be $G_{\mathbf{J}}$ equivariant if for all j and k, $W \cap Y_{j:k}$ is equivariant under the $G_{j;k}$ action.

In my setup, I take $Y_{j;k}$ to be the open submanifold $X_{j;k} - \cup_{p;q} X_{p;q} \subset X_{j;k}$.

When Y is stratified by locally closed almost complex submanifolds, we require that the tubular neighborhoods of $X's$ are $G_{\mathbf{J}}$ equivariant. This additional assumption is essential in discussing the divisibility of the family invariants. To simplify the notation, I will implicitly identify $\mathbf{N}_XY$ with $\mathbf{V}$.

Proposition 6.3. *Let $s^{-1}(\mathbf{0}) = \mathcal{M}_{red} \mapsto Y$, $\mathbf{N}_XY$, and $\mathcal{N}_XY$ denote the reduced family moduli space over Y, the normal bundle of X in Y and the $G_{\mathbf{I}}$ equivariant tubular neighborhood, respectively, as have been defined already. Suppose that the reduced family moduli space over X, denoted by the shorthand notation $\mathcal{M}_{red} \times_Y X$, is defined by a regularly obstructed smooth section of the regularly obstructed extended obstruction bundle. Let $\widetilde{Y} = Bl_XY$ denote the almost complex blowup of Y along X, and $p : \widetilde{Y} \mapsto Y$ is the tautological blowing down map. Then there exists a series of nested smooth perturbations of the pulled back sec-*

tion $s^{pert} \in \Gamma(\widetilde{Y}, \mathcal{OBS})$*, such that the perturbed zero loci* $(s^{pert})^{-1}(\mathbf{0})$*, denoted by* $\mathcal{M}_{pert}$*, can be decomposed into* $\mathcal{M}_{pert}^{res} \coprod \mathcal{M}_{red} \times_Y \mathbf{P}(\mathbf{N}_X Y)$*. The support of the perturbation can be chosen to be a smaller* $G_{\mathbf{I}}$ *equivariant tubular neighborhood of* X*. The branch* $\mathcal{M}_{pert}^{res}$ *is compact and is supported away from the exceptional locus* $\mathbf{P}(\mathbf{N}_X Y)$*.*

Here the symbol $\mathcal{M}_{pert}^{res}$ denotes the perturbed version of the residual reduced family moduli space over $\widetilde{Y} - \mathbf{P}(\mathbf{N}_X Y)$, which is disjoint from $\mathcal{M}_{red} \times_Y \mathbf{P}(\mathbf{N}_X Y)$.

When there are more than one X present, I apply the proposition repeatedly to deal with them. I start from the $X's$ with higher levels.The explicit construction of the perturbations allows us to apply an induction argument on the levels k in modifying the k-th perturbed moduli space $\mathcal{M}_{pert;k}, k > 0$ step by step. The subscript k indicates how many times the nested perturbations have been performed. Eventually one ends up with a $\mathcal{M}_{pert}^{res}$, whose support is completely away from all the $X_{j;k}$.

I must emphasize that the perturbations and the perturbed reduced family moduli spaces are not defined over the original base manifold Y but over a birational model.

Proof. Let us begin our proof of the previous proposition. To blow up X in Y, one removes the subspace X from Y and then glues back $E = \mathbf{P}(\mathbf{N}_X Y)$. From now on the blown up tubular neighborhood will be denoted by $\widetilde{\mathcal{N}}_E \widetilde{Y}$.

First one notices that in the blown up manifold $\widetilde{Y} = Bl_X Y$, $\mathbf{P}(\mathbf{N}_X Y)$ is a smooth, almost complex submanifold (of real codimension two).

Take the smooth complex line bundle corresponding to the exceptional submanifold. Let h denote a smooth transversal section of the exceptional line bundle over $\widetilde{Y}$ whose zero locus defines $E = \mathbf{P}(\mathbf{N}_X Y)$. If one works over the holomorphic category, it can be chosen to be the smooth complex line bundle associated with the exceptional divisor. The restriction of the line bundle to E can be identified to be the tautological line bundle of the projectification of $\mathbf{N}_X Y$.

Because the defining section s has been assumed to be regularly obstructed, $p^*s|_X$ projects trivially $(= p^*\widetilde{s}|_X)$ onto the $p^*\mathbf{N}_X Y$ factor under the natural projection map $\mathcal{OBS} \mapsto p^*\mathbf{N}_X Y$.

To choose the small perturbation carefully, one notices that the regular obstructedness of s guarantees that $p^*s|_{\widetilde{\mathcal{N}}_E \widetilde{Y}}$ can be reexpressed schematically as $s_{res} \oplus (\widetilde{s} \otimes_{\mathbf{C}} \bar{h}^{-1}) \otimes \bar{h}$. The formula deserves some special explanation.

In the formula, the section s_{res} is a smooth section of $p^*\mathcal{OBS}'$, the residual extended obstruction bundle.The symbol $\bar{h}$ is defined to be the pulled back version of h under the projection map $\mathcal{M}_{red} \times_Y Bl_X Y \mapsto Bl_X Y$. Because $\bar{h}$ vanishes on E, the expression $\widetilde{s} \otimes_{\mathbf{C}} \bar{h}^{-1}$ has an apparent singularity at E similar to the entire function $\frac{sin(z)}{z}$ in complex analysis. Instead, I interpret this symbol as the nonvanishing smooth section of $p^*\mathbf{N}_X Y \otimes \mathcal{L}_E^\star$ under the isomorphism by the regular obstructedness, which relates to $\widetilde{s}$ by tensoring with $\bar{h}$. The contractability of $\widetilde{\mathcal{N}_E}\widetilde{Y}$ into E implies that the "factorization" can be extended to the tubular neighborhood of E.

Since the zero section s has been splitted into two parts near $E \subset \widetilde{Y}$, so is $\mathcal{M}|_{\widetilde{\mathcal{N}_E}\widetilde{Y}}$. One portion is defined by the zero locus of $s_{res} \oplus \bar{h}$. Another branch is locally defined by $s_{res} \oplus (\widetilde{s} \otimes \bar{h}^{-1})$. These two branches usually are not disjoint without the further perturbations. Since h trivializes $\mathcal{L}_E$ outside E, the factorization operation can be interpreted as a modification of the topological type of $\mathcal{OBS}$. One can extend $s_{res} \oplus \widetilde{s} \otimes_{\mathbf{C}} \bar{h}^{-1}$ to be a smooth section of $\mathcal{OBS}'$ over $\widetilde{Y}$.

Let $\mathcal{H}$ denote the hyperplane line bundle over the projective space fiber bundle $E = \mathbf{P}(\mathbf{N}_X Y) \mapsto X$. As $\widetilde{\mathcal{N}_E}\widetilde{Y}$ can be retracted into $E = \mathbf{P}(\mathbf{N}_X Y)$, we use the same symbol $\mathcal{H}$ to denote its extension from $\mathbf{P}(\mathbf{N}_X Y)$ to $\widetilde{\mathcal{N}_E}\widetilde{Y}$. We have $\mathcal{L}_E^\star = \mathcal{H}$.

The short exact sequence of the vertical tangent bundle $\mathcal{T}_{ver}\mathbf{P}(\mathbf{N}_X Y)$ of $\mathbf{P}(\mathbf{N}_X Y) \mapsto X$ implies that the complex vector bundle $p^*\mathbf{N}_X Y \otimes_{\mathbf{C}} \mathcal{H} = p^*\mathbf{N}_X Y \otimes_{\mathbf{C}} \mathcal{L}_E^\star$ fits into the following short exact sequence,

$$0 \mapsto \mathbf{C} \mapsto p^*\mathbf{N}_X Y \otimes \mathcal{H} \mapsto \mathcal{T}_{ver}\mathbf{P}(\mathbf{N}_X Y) \mapsto 0.$$

The trivial factor $\mathbf{C}$ in the exact sequence is crucial to our construction.

First one considers the constant nonzero section of the trivial line bundle $\mathbf{C}$ over $\widetilde{\mathcal{N}_E}\widetilde{Y}$. One then chooses a $G_{\mathbf{I}}$ equivariant Riemannian metric on $\widetilde{\mathcal{N}_E}\widetilde{Y}$ and then a $G_{\mathbf{I}}$ invariant cut off function $\phi(r)$, which is radially symmetric with respect to the chosen metric. Without losing generality, I assume that $\widetilde{\mathcal{N}_E}\widetilde{Y}$ is within the cut-locus of the smooth metric.

The bump function $\phi(r)$ is identically $\equiv 1$ for all the points near E and is $\equiv 0$ for all points near the collar boundary of $\widetilde{\mathcal{N}_E}\widetilde{Y}$. Then the ϕ-cut off version of the constant section is extended and embedded into a smooth section of the obstruction bundle $\mathcal{OBS}$ over $\widetilde{Y} = Bl_X Y$. Let $\underline{1}_\phi$ denote such a cut off version of the constant section. Then I perform the desired perturbation by adding $\epsilon \cdot \underline{1}_\phi$ to the section $s_{res} \oplus (\widetilde{s} \otimes \bar{h}^{-1})$. One then takes ϵ large enough to exceed $sup_{x\in E}(\| s_{res} \oplus (\widetilde{s} \otimes \bar{h}^{-1}) \| (x))$. After it is carefully chosen, the perturbed section $s_{res} \oplus \widetilde{s} \otimes \bar{h}^{-1} + \epsilon \underline{1}_\phi$ does not vanish in a neighborhood of E.

The construction of the perturbation has ensured that the reduced family moduli space is modified only in the given tubular neighborhood of $E = \mathbf{P}(\mathbf{N}_X Y)$, which is determined by the choice of the cut off function $\phi(x)$.

It is obvious from the construction that, after the specific perturbation, the reduced family moduli space over E defined by $\bar{h} = \mathbf{0}, s_{res} = \mathbf{0}$ is disjoint from the residual part which is defined locally by

$$s_{res} \oplus (\widetilde{s} \otimes \bar{h}^{-1}) + \epsilon \underline{1}_\phi = \mathbf{0}. \qquad \text{q.e.d.}$$

Next, I also need to ensure that the perturbation has not spoiled the nice property of the original reduced family moduli space.

The following lemma is crucial in the inductive construction:

Lemma 6.1. *Let $\mathcal{OBS}$ be the family extended obstructed bundle of a family Kuranishi model $\mathcal{O}$ over the almost complex base manifold Y and $s \in \Gamma(\mathcal{O}, \mathcal{OBS})$ be the defining section of the family moduli space. Let $X_1 \subset X_2 \subset Y$ be the inclusions of almost complex submanifolds of Y. Suppose that the family Kuranishi model is regularly obstructed with respect to both X_1 and X_2. Additionally assuming that the smooth section s is regularly obstructed with respect to X_1 and X_2 in the weak sense, then for sufficiently small ϵ,the ϵ perturbed section of the residual family obstruction bundle over $\widetilde{Y} = Bl_{X_1} Y$ is regularly obstructed with respect to $Bl_{X_1} X_2$ in the weak sense.*

Proof. Notice that after the blowing up, $\mathbf{N}_{Bl_{X_1} X_2} \widetilde{Y}$ is not isomorphic to $\mathbf{N}_{X_2} Y$. The residual extended family obstruction bundle may not be regularly obstructed with respect to $Bl_{X_1} X_2 \subset \widetilde{Y}$, either. As I have shown, the zero locus of the perturbed section is away from $E = \mathbf{P}(\mathbf{N} X_1 Y)$. Thus, by shrinking the family Kuranishi model to a smaller neighborhood of the perturbed zero locus, the regular obstructedness condition still holds.

First I introduce the splitting of the complex normal bundles. $\mathbf{N}_{X_1} Y = \mathbf{N}_{X_1} X_2 \oplus \mathbf{N}_{X_2} Y|_{X_1}$ by using the Riemannian metric. Because the zero locus of the ϵ perturbed version $s_{res} \oplus \perp s \otimes \bar{h}^{-1} + \epsilon \underline{1}_\phi$ is away from $\mathbf{P}(\mathbf{N}_{X_1} X_2)$, I can shrink the family Kuranishi model if necessary.

The new section has been perturbed from $s_{res} \oplus \perp s \otimes \bar{h}^{-1}$ by adding $\epsilon\phi(r)\underline{1}$. As has been remarked, the $\mathcal{OBS}'$, after being restricted to the zero locus of the perturbed section, is regularly obstructed with respect to X_2. I only have to prove that the $\epsilon\phi(r)\underline{1}$ projects trivially into the factor isomorphic to $\mathbf{N}_{X_2}Y$.

Because the cut off function $\phi(r)$ has been chosen, the perturbation supports in a neighborhood of E. Recall that section $\underline{1}$ is constructed by pulling back the tautological section of $\mathbf{N}_{X_1}Y \times_{X_1} (\mathbf{N}_{X_1}Y - \mathbf{0}) \mapsto (\mathbf{N}_{X_1}Y - \mathbf{0})$ by the diffeomorphism $\mathcal{N}_{X_1}Y \cong \mathbf{N}_{X_1}Y$.

By the choice of the stratified tubular neighborhood, $\mathbf{N}_{X_1}X_2$ is diffeomorphic to a tubular neighborhood of X_1 in X_2. Thus, for all x in this tubular neighborhood, the values of $\underline{1}$ all lie in the factor $\mathbf{N}_{X_1}X_2$. Thus, their projection from $\mathbf{N}_{X_1}Y = \mathbf{N}_{X_1}X_2 \oplus \mathbf{N}_{X_2}Y|_{X_1}$ to $\mathbf{N}_{X_2}Y|_{X_1}$ is trivial. This proves the lemma. q.e.d.

Next, I would like to apply this lemma iteratively to the general context. I blow up $X_{j;k}$ according to the descending orders of the levels k. I.e., each $X_{j;k}$ is blown up only after all the $X_{l;k'}$ of higher levels have been blown up already.

Each time during the blowing up, there are three different situations:

(i) If the $X_{l;k_l}$ is totally disjoint from the blowup locus, then the regular obstructedness condition over $X_{l;k_l}$ is not affected.

(ii) If $X_{l;k_l}$ contains $X_{j;k}$ properly, then the previous lemma is applied to guarantee the regular obstructedness of the perturbed section (in the weak sense).

(iii) Here, $X_{l;k_l}$ and $X_{j;k}$ do not contain each other, yet $X_{l;k_l} \cap X_{j;k}$ is nonempty. In this case, one can apply the proof of the lemma to $X_1 = X_{l;k_l} \cap X_{j;k} \subset X_2 = X_{l;k_l}$, after some simple change of notations. Then one proceeds to the following iteration procedures:

(1) Blow up the appropriate loci with level k.

(2) Pull back the family Kuranishi model to the blown up manifold.

(3) Split the defining section into the "excess part" and "residual part."

(4) Perturb the section by $\epsilon\phi(r)\underline{1}$ supported near the exceptional loci.

(5) Using the lemma repeatedly to guarantee the regular obstructedness (in the weak sense).

(6) Go back to step (1) to blow up the appropriate loci with level $k-1$.

7. The proof of the Main Theorems

In Subsection 7.1, I provide the $\mathcal{C}^\infty$ argument to identify the modified family invariant with the enumeration of smooth curves in the class $C-\mathbf{M}(E)E$. In Subsection 7.2, I prove that the modified family invariant attached to $(\Gamma, C-\mathbf{M}(E)E)$ is a universal polynomial expression of $c_1(C)^2, c_1(C)\cdot K_M, K_M^2, \chi(M)$.

In Subsection 7.3, I combine Göttsche's argument to realize the transversality condition of the reduced family moduli space over an open set in $Y(\gamma)$. As a result, the enumeration of nodal curves in any sufficiently very ample $|C|$ can be identified with a constant multiple of the modified family invariant $FSW_{Y(\gamma)}(1, C-\sum 2E_i)$. In Subsection 7.4 and Subsection 7.4.2, the divisibility and the factorization properties of the modified family invariants FSW^* are studied in detail.

7.1 The proof of the Main Theorems; first step

After all the preparation in the previous subsections, one is ready to state and prove the main theorems in the introduction.

Recall the basic fact that if $\mathrm{d}_{\mathbf{R}}(C_0)+\dim_{\mathbf{R}} B<0$, then all the pure and mixed family invariants of C_0 vanish. Under the usual convention, the vanishing of the invariants under this condition is viewed as a definition rather than a "property." However, I want to emphasize that it is a consequence of the family Kuranishi model, as well as the algebraic Kuranishi model, rather than merely a definition. In terms of family Kuranishi model, the fact is almost trivial. Namely, the generic fibers of the chosen perturbation is compact and smooth of negative dimension, therefore an empty set. The invariants are zero automatically. The corresponding vanishing result in algebraic family Seiberg-Witten theory was explained in [38] under the assumption that the class C_0 is simple.

Given one topological type of singular curves $(\Gamma, C-\mathbf{M}(E)E)$, the primitive object to consider is $FSW_{Y(\Gamma)}(1, C-\mathbf{M}(E)E)$. Notice that when one considers nodal curves, Γ is taken to be γ, and the family invariant is evaluated over $Y(\gamma)=M_n$. To reduce the ambiguity, I choose $\{\Gamma, M(E)\}$ carefully to satisfy the extra conditions on the markings and multiplicities. In general, the situation is rather complicated; the major complication comes from the fact that both type I and type II exceptional curves and their multiple coverings can show up in the counting scheme. In fact, they do show up very frequently.

To simplify this already complicated discussion, type II exceptional curve is temporarily dismissed.

Assumption 1. The family Kähler Seiberg-Witten moduli space is said to be good if the stratum of smooth curve is smooth of right family dimension and the other strata of singular curves are of real codimension at least two.

Under this optimal assumption, the spinors whose zero loci are smooth curves form the top dimension of the moduli space and the counting of the invariants is equivalent to the counting of the smooth holomorphic (algebraic) curves. The concept of "goodness" was first introduced in Donaldson theory and Gromov theory several years ago.

Suppose that the family moduli space has been good and all the curves obtained by cutting down the dimension of the moduli space supporting over Y_Γ, then the contraction of the curves by the blowing down map gives rise to singular curves with singularities exactly of the prescribed types. Under this ideal condition, the family invariant should be proportional to the number of singular curves.

Let us analyze the difficulty of achieving these two assumptions. First, there is some topological obstruction to achieve goodness. Moreover, even if the condition of being a good family moduli space has been achieved, there is still some further serious obstruction for the curves to lie over the open strata Y_Γ. It turns out that the concept of admissible decomposition classes helps to analyze the obstructions in a uniform way.

Let us temporally ignore the first question and concentrate on the second one. Namely, there are occasions that some smooth curves representatives in the moduli space obtained by imposing the points passing condition lie above $Y(\Gamma) - Y_\Gamma$. Let us choose $n = 4$ and $\mathbf{M} \equiv 2$, the 4-nodes nodal curve case, as an example to illustrate the general phenomenon. More examples will be discussed in Section 9.

Let $C-2E_1-2E_2-2E_3-2E_4$ be the cohomology class I would like to count the smooth curves over M_4. It turns out that one can decompose $C-2E_1-2E_2-2E_3-2E_4$ as the sum of $L = C-3E_1-E_2-E_3-E_4$ and $e = E_1-E_2-E_3-E_4$. It is easy to check that these two classes, L and e, are perpendicular to each other.

This indicates that it is possible that the smooth curves we want to count are presented as the disjoint unions of the smooth curves in $C - 3E_1 - E_2 - E_3 - E_4$ and $E_1 - E_2 - E_3 - E_4$. Because the curve $e = E_1 - E_2 - E_3 - E_4$ lies over some subvariety of $Y(\Gamma) - Y_\Gamma$, the total curve does, too.

This gives us the first indication that the family invariants should not count the number of nodal curves—-It also contains the other types of contributions as well. The bad news is that there might be singular curves in $C - 2E_1 - 2E_2 - 2E_3 - 2E_4 - \cdots$ which have a higher family dimension than the expected one. Examples of this type give rise to serious obstruction for a family moduli space to be good.

Even though the curves with lower "expected" family dimensions are actually of lower dimension, cutting down the family moduli space by $\mathrm{d}_{\mathbf{C}}(C) - n$ points can give rise to curves sitting over $Y(\Gamma) - Y_\Gamma$. Ideally, one assumes that the reduced moduli space is a smooth zero dimensional compact manifold, then the pure invariant is counted by the sum of the signed points. To calculate the number of nodal curves, one needs to calculate the other types of contribution and subtract the total contribution from the other. Schematically, one should expect the following equality:

$$FSW_{Y(\Gamma)}(C - \mathbf{M}(E)E) = N(\Gamma, M(E)) + \mathit{others},$$

where $N(\Gamma, M(E))$ denotes the contribution from the smooth curves which are supposed to be proportional to the number of the singular curves we want to count.

In other words, $N(\Gamma, M(E)) = FSW_{Y(\Gamma)}(C - \mathbf{M}(E)E) - \mathit{others}$.

One of the difficulties with the ideal picture is that it is extremely difficult to ensure the transversality of the reduced family moduli space. Therefore, the terms *others* seldom make sense as counting of a finite number of elements in the appropriate moduli spaces. The situation becomes hopeless as the second problem comes into play. This is the main reason that Vainsencher's approach of counting nodal curves only worked for n relatively small ($n \leq 6$) and broke down when n went larger.

It is the goal of the present section to develop a method to bypass the difficulties. The method would be in a sense parallel to the residual intersection theory [16] developed by the algebraic geometers, yet the foundation is the nested Kuranishi model discussed above.

Let us make another working assumption on the family moduli space temporarily.

Assumption 2. The reduced family moduli space is said to be partially good if the strata with expected family dimensions less than the expected one are either empty or of actual dimension less than the expected one.

In general, it is hard to realize the condition in the algebraic category. On the other hand, I will realize the condition in the pseudo-holomorphic category when I consider nodal curves.

Assumption 3. Consider the subspace $\mathcal{M}_{\mathrm{II}}$ in the family moduli space consisting of the curves whose projections into M (under the blowing down map) are not reduced, i.e., they are of multiplicities larger than one. Then the family moduli space is said to be type II nice if the expected dimensions of the types of curves are all less than the expected family dimension of the family moduli space. In other words, the reduced family moduli space does not intersect with $\mathcal{M}_{\mathrm{II}}$.

In particular, if the subset $\mathcal{M}_{\mathrm{II}}$ is empty, then it is viewed as type II nice. If the family moduli space is both partially good and type II nice, then the points of the corresponding reduced family moduli space do not represent curves with multiple coverings of type II exceptional curves.

In the explicit identification of the nodal curve invariants on $K3$, one will encounter this type of situation.

It follows from Göttsche's argument that the reduced moduli space can be made to be type II nice after twisting the line bundle to a high enough power (see below).

Under these working assumptions, the reduced family moduli space consists of points sitting over both Y_Γ and $\cup_{\Gamma'<\Gamma} Y_{\Gamma'}$.

If one does not impose the extra regularity assumption, there is no hope of interpreting the pure family invariant as a counting of curves.

Another working assumption is imposed at this point.

Assumption 4. The intersection of the reduced family moduli space with the preimage of Y_Γ consists of a finite number of (might be nonreduced) points which correspond to smooth curves in the appropriate cohomology classes.

This can be achieved in some particular situation by the perturbation argument. Without this assumption, the curve counting problem should be interpreted as a weaker notion of enumerating the "equivalence" of some type of curves. Only in the pseudo-holomorphic category can it be reinterpreted as a counting of singular pseudo-holomorphic curves.

They are imposed at this moment to clarify and simplify the argument and to give us conceptual guidance. Using the concept of type II exceptional curves, Assumption 4 implies that the admissible decompositions classes involving type II exceptional curves do not contribute to the invariant counting over Y_Γ.

Borrowing language from algebraic geometry, one gets a new interpretation of the previous counting scheme. Namely, the weird term of "the others" in the previous equality should be interpreted as the various "equivalences" of invariants over $\cup_{\Gamma'<\Gamma} Y_{\Gamma'}$.

Let us ignore the technical details at this moment and address how the weakened form of the scheme partially resolves the transversality issue from the other approaches. It buildsup one of the major difficulties in Vainsencher's [59] or Kleiman-Pienes' [24] approach. The curve counting scheme of ours, as well as the algebraic geometric one, relies on the correspondence that relates the singular curves to the singular points lying in the algebraic surface. The family blowup formula establishes a machinery to detect the singular points of the curves. Whenever the curves have nonreduced components, the machinery tends to detect a continuous family of singularities. As the correspondence fails to be a finite to finite correspondence, the naive counting loses its direct geometric meaning. This problem was first recognized by Vainsencher [59] and was the major difficulty of the current scheme to count pseudo-holomorphic curves. It might not be noticed by the reader that these exotic symptoms are dealt with smoothly by the technique of family Seiberg-Witten theory. i.e., under the formulation of family Seiberg-Witten theory, the seemly exotic and un-curable phenomenon becomes a normal phenomenon once the appropriate concepts have been defined and the appropriate tools have been built up.

Because the present assumptions do not guarantee the regularity of the reduced family moduli space lying over $\cup_{\Gamma'<\Gamma} Y_{\Gamma'}$, it is not interpreted as a geometric counting directly. However, the language of the family invariants allows us to interpret these contributions as a certain combination of mixed invariants of some other $spin_c$ structures. Then the repeated applications of family blowup formula and the family switching formula reduce the various mixed family invariants to the topological datum involving cohomology classes $C^2, C\cdot K_M, K_M^2, c_2(M)$.

Let us argue how the "equivalences" of the reduced family moduli space over $Y(\Gamma) - Y_\Gamma$ can be interpreted as the combinations of mixed family invariants. The concept of the admissible decomposition classes plays an essential role here. I first give a argument in some simpler case to demonstrate my point of view. Despite of its simpleness, it serves as a prototype of the general theory.

As I have assumed that the family moduli space is partially good, the image of the reduced moduli space over $Y(\Gamma) - Y_\Gamma$ may not be the whole $Y(\Gamma) - Y_\Gamma$.

Example 7.1. Take $n = 4$ and consider $\Gamma_1 \in \mathrm{adm}(4)$ to be the admissible graph such that the leading vertex is the direct ascendent of the three other vertexes. It is easy to see that, when one considers the case that the unresolved curves have four nodes, the reduced family moduli space which lies over $Y(\Gamma)-Y_\Gamma$ actually lies over $Y(\Gamma_1)$. Without imposing the partial goodness assumption, this may not be the case. To relax the assumption, one has to use the remark at the beginning of Section 6.2. This also constitutes the major reason to impose the allowable condition in defining the admissible decomposition classes.

As has been analyzed, the space $Y(\Gamma_1)$ is the closure of the space over which the type I exceptional curve $e = E_1 - E_2 - E_3 - E_4$ or its degenerations support upon. On the other hand, as was calculated before, the cohomology class $C - 2E_1 - 2E_2 - 2E_3 - 2E_4$ splits into $L = C - 3E_1 - E_2 - E_3 - E_4$ and $e = E_1 - E_2 - E_3 - E_4$. Then the key idea is to notice that $e = E_1 - E_2 - E_3 - E_4$ is a type I exceptional curve lying in the type I preexceptional cone over Y_{Γ_1}. Being exceptional, it cannot move in a single fiber. Therefore, the compactified moduli space of the curve $e = E_1 - E_2 - E_3 - E_4$ (or simply the family Seiberg-Witten moduli space, as it is a compact space) is isomorphic to $Y(\Gamma_1)$. The reduced family moduli space of $C - 2E_1 - 2E_2 - 2E_3 - 2E_4$ over $Y(\Gamma_1)$ is equal to the reduced family moduli space of $C - 3E_1 - E_2 - E_3 - E_4$ with the presence of the curve $e = E_1 - E_2 - E_3 - E_4$. As a result, the "equivalence" of the invariants of $C - 2E_1 - 2E_2 - 2E_3 - 2E_4$ over $Y(\Gamma)$ is reinterpreted as the mixed invariant of $C - 3E_1 - E_2 - E_3 - E_4$ inserting $PD(Y(\Gamma_1))$ into the invariant.

This type of argument is possible due to the nested Kuranishi model developed in Section 6.1. To begin the argument, let us demonstrate the main technique in the special case. As a type I exceptional curve has been splitted off over $Y(\Gamma_1)$, it is not hard to check that the space $Y(\Gamma_1)$ is regularly obstructed in $B = Y(\Gamma)$ with respect to the cohomology class $C - 2E_1 - 2E_2 - 2E_3 - 2E_4$. More explicitly, the bundle $\mathbf{V}$, which is splitted off from the **Obs**, is nothing but the Seiberg-Witten obstruction bundle of the class $e = E_1 - E_2 - E_3 - E_4$. The reader should notice that the smoothness of the stratum $Y(\Gamma_1)$ (Proposition 4.3 and 4.4) is used essentially here.

It follows that, by the machinery of nested Kuranishi model, the invariant contribution over $S = Y(\Gamma_1)$ makes perfect sense even though the moduli space is hardly regular. In family Seiberg-Witten theory, the removal of the bundle $\mathbf{V}$ over $S = Y(\Gamma_1)$ corresponds to replace the cohomology class $C - 2E_1 - 2E_2 - 2E_3 - 2E_4$ by its good part $C - 3E_1 - E_2 - E_3 - E_4$.

In general, the possibility of identifying two nonregular objects directly relies heavily on either the nested Kuranishi models of the family Seiberg-Witten moduli space or the algebraic family Seiberg-Witten invariants defined in [38] based on the idea from [29].

Thus, in the $n = 4$, i.e., the 4 nodes nodal case, the number of nodal curves is related to the family invariants by

$$\begin{aligned}\frac{1}{4!}(&FSW(1, C - 2E_1 - 2E_2 - 2E_3 - 2E_4)\\ &- FSW(PD(Y(\Gamma_1)), C - 3E_1 - E_2 - E_3 - E_4)).\end{aligned}$$

Applying the concept of admissible decomposition classes, it is easy to see that $(C - 3E_1 - E_2 - E_3 - E_4, e = E_1 - E_2 - E_3 - E_4)$ is an admissible decomposition (class) of level one.

One has proved that

$$\begin{aligned}(&FSW(1, C - 2E_1 - 2E_2 - 2E_3 - 2E_4)\\ &- FSW(PD(Y(\Gamma_1)), C - 3E_1 - E_2 - E_3 - E_4))\end{aligned}$$

is proportional to the number of 4-node nodal curves with the proportionality condition 4! (which will be identified more systematically later). This is exactly the modified invariant $FSW^*(1, C - 2E_1 - 2E_2 - 2E_3 - 2E_4)$ defined in the previous section. Without using the nested Kuranishi model technique or the concept of mixed invariants, the portion of reduced moduli space over $Y(\Gamma_1)$ is a nonreduced object depending both on M and C. Algebraic geometers [59], [24] proved that this space can be made to be transversal of right dimensions after imposing conditions on C. Without using the ad hoc argument, it was impossible for them to say anything about it. And the existence of the "universal" formula cannot be achieved then.

As I have cut down the moduli space by imposing points passing condition, the expected family dimension of the reduced family moduli space is zero.

Definition 7.1. A (pseudo) holomorphic curve is said to be type I (or II) free if there is no type I (or II) exceptional curve among its irreducible components.

A priori, there might be some type $I-$ free smooth curves sitting over $Y(\Gamma) - Y_\Gamma$. As Proposition 4.3 asserts that $Y(\Gamma) - Y_\Gamma$ is a divisor in $Y(\Gamma)$ with real codimension two, the family expected dimension of this type of curve drops by at least two. Thus, by choosing the sections of the fiber bundle (which cut down the dimension of the moduli space) to be generic enough, one can assume that the "reduced" family moduli space over $Y(\Gamma) - Y_\Gamma$ do not contain any type $I-$ free smooth curves. Even if it does, one can still analyze them by a detour(see below) that their contribution to the invariant vanishes as they are of lower expected dimension than the original family invariant.

Because the working assumptions guarantee that it is type II nice, it follows from Taubes' [51] calculation of the dimension of pseudo-holomorphic curves that the curves corresponding the reduced family moduli space over $Y(\Gamma) - Y_\Gamma$ must contain type I curves in their irreducible components.

7.1.1. The validity of the regular obstructedness condition

I have formulated the nested Kuranishi model to analyze the reduced family moduli space. I first provide a check that the assumption in the proposition is satisfied. Namely, the regularly obstructed condition is satisfied in the setting.

Suppose that there exists at least one type I exceptional class over b which is non-nef with respect to $C - \mathbf{M}(E)E$. Let us collect all such $e_i \in \mathbf{EC}_b(C - \mathbf{M}(E)E), i \in I$ and consider the decomposition $(C - \mathbf{M}(E)E - \sum_{i\in I} e_i, \sum_{i\in I} e_i)$ and the family moduli space associated to the given decomposition. In order that this component of family moduli space has a nontrivial contribution to the family invariant,

(i) $C - \mathbf{M}(E)E$ must be effective;

(ii) Its formal dimension $d_{\mathbf{R}}(C - \mathbf{M}(E)E - \sum_{i\in I} e_i) + \sum_i d_{\mathbf{R}}(e_i) + \dim_{\mathbf{R}} B$ must be greater or equal to $d_{\mathbf{R}}(C - \mathbf{M}(E)E) + \dim_{\mathbf{R}} B$.

In other words, $(C - \mathbf{M}(E)E - \sum_{i\in I} e_i, \sum_{i\in I} e_i)$ must be allowable. According to Lemma 5.1, one must be able to group $e_i, i \in I$ into "connected components" such that at least one group $J_r \subset I$ satisfies $(\sum_{j\in J_r} e_j) \cdot (C - \mathbf{M}(E)E) \leq (\sum_{j\in J_r} e_j)^2 < 0$.

Geometrically, those $e_j, j \in J_r$ can be smoothed into a type I exceptional curve e_{J_r} such that $\sum_{j\in J_r} e_j$ is viewed as a degenerated configuration of e_{J_r}. Since the lemma guarantees the existence of at least one e_{J_r}, I collect all such $e_{J_r} = \sum_{j\in J_r} e_j$, $e_{J_r} \cdot (C - \mathbf{M}(E)E) \le e_{J_r}^2 < 0$ and discard the others temporally. The coexistence of all such each e_{J_r} defines an admissible stratum over which e_{J_r} appears as an irreducible smooth type I exceptional curve.

Proposition 7.1. *Let $(C - \mathbf{M}(E)E - \sum_{i\in I} e_i, \sum_{i\in I} e_i)$ be an allowable decomposition over b of $C - \mathbf{M}(E)E$ with e_i being the type* I *exceptional classes $e_i\cdot(C-\mathbf{M}(E)E) < 0$. Then there exists a partition of I into $I = \coprod J_r$ and an admissible graph Γ_b, with the subscript indicating its b dependence, which satisfies the following conditions:*

(i) $e_{a_1} \cdot e_{a_2} = 0, a_1 \in J_r, a_2 \notin J_r$. *Any two e_{a_1}, e_{a_2} in the same group, $a_1, a_2 \in J_r$, are connected by a chain of $e_k, k \in J_r$ such that the adjacent e_k's have intersection number* 1.

(ii) $b \in Y(\Gamma_b)$.

(iii) *The locally closed locus Y_{Γ_b} is defined by the coexistence of the smooth irreducible type* I *exceptional curves $e_{J_r}, J_r \subset I, 1 \le r \le r_0$ such that $e_{J_r} \cdot (C - \mathbf{M}(E)E) < e_{J_r}^2 < 0$, for all $r \le r_0$. In other words, $(C - \mathbf{M}(E)E - \sum_{r\le r_0} e_{J_r}, \sum_{r\le r_0} e_{J_r})$ defines a type A allowable decomposition.*

(iv) *When the generic point $z \in Y_{\Gamma_b}$ specializes to b, the smooth curve representing e_{J_r} is broken into the tree of type* I *curves represented by the sum $\sum_{j\in J_r} e_j$.*

(v) *For $r > r_0$, $0 > \{\sum_{j\in J_r} e_j\} \cdot (C - \mathbf{M}(E)E) > \{\sum_{j\in J_r} e_j\}^2$.*

(vi) *The family moduli space associated to the decomposition $(C - \mathbf{M}(E)E - \sum_{i\in I} e_i, \sum_{i\in I} e_i)$ is embedded as the degenerated configurations of the decomposition $(C-\mathbf{M}(E)E-\sum_{r\le r_0} e_{J_r}, \sum_{r\le r_0} e_{J_r})$ such that:*

   (a) *e_{J_r} is degenerated into $\sum_{j\in J_r} e_j$.*

   (b) *$C - \mathbf{M}(E)E - \sum_{r\le r_0} e_{J_r}$ is degenerated into $C - \mathbf{M}(E)E - \sum_{i\in I} e_i + \sum_{r>r_0} \sum_{j\in J_r} e_{J_r}$.*

In particular, any type B allowable decomposition class can be viewed as a degenerated configuration of some type A allowable decomposition class.

Proof. Most of the statements are merely the reformulation of our earlier discussion. The study of the admissible strata and the admissible graphs allows us to compare the topological degenerations of the class $e_{J_r} \mapsto \sum_{j\in J_r} e_j$ and the geometric degenerations of the corresponding curves. (i) follows from the basic property of the "connected components". (ii), (iii), (iv) follow from the construction of the admissible stratum. (v) explains the role of r_0. In (vi), the class $C - \mathbf{M}(E)E$ of any holomorphic curve with $\{C - \mathbf{M}(E)E\} \cdot e_i < 0, i \in I$ must split into $(C - \mathbf{M}(E)E - \sum_{i\in I} e_i, \sum_{i\in I} e_i)$. By grouping $e_i, i \in J_r$ together, the curve dual to $C - \mathbf{M}(E)E$ can be reinterpreted as a curve in $(C - \mathbf{M}(E)E - \sum_{r\leq r_0} e_{J_r}, \sum_{r\leq r_0} e_{J_r})$, where I have taken the union of the curves in $C - \mathbf{M}(E)E - \sum_{i\in I} e_i$ and in $\sum_{r>r_0} e_{J_r}$ and viewed it as $C - \mathbf{M}(E)E - \sum_{r\leq r_0} e_{J_r}$, using the identity $-\sum_{i\in I} e_i + \sum_{r>r_0} e_{J_r} = -\sum_{r\leq r_0} e_{J_r}$. q.e.d.

Conversely, suppose there is an e_i such that $(\mathbf{C} - \mathbf{M}(E)E) \cdot e_i < e_i^2 < 0$. Namely, the strict inequality holds. If the curve poincare dual to e_i is broken into two two components $e_{i;1} + e_{i;2}$ with $e_{i;1} \cdot e_{i;2} = 1$, then the following property holds.

Lemma 7.1. *Under the above assumption, then either* $(C - \mathbf{M}(E)E) \cdot e_{i;1} \leq e_{i;1}^2$ *or* $(C - \mathbf{M}(E)E) \cdot e_{i;2} \leq e_{i;2}^2$.

Proof. If not, I assume both inequalities are violated and derive a contradiction.

Suppose $(C - \mathbf{M}(E)E) \cdot e_{i;1} > e_{i;1}^2$ and $(C - \mathbf{M}(E)E) \cdot e_{i;2} > e_{i;2}^2$. Because all the intersection numbers are integer valued, they imply

$$(C - \mathbf{M}(E)E) \cdot e_{i;1} \geq e_{i;1}^2 + 1$$

and

$$(C - \mathbf{M}(E)E) \cdot e_{i;2} \geq e_{i;2}^2 + 1.$$

Adding them together yields

$$(C - \mathbf{M}(E)E) \cdot \{e_{i;1} + e_{i;2}\} \geq e_{i;1}^2 + e_{i;2}^2 + 2.$$

On the other hand, the original assumption on $e_{i;1}, e_{i;2}$ implies

$$(C - \mathbf{M}(E)E) \cdot (e_{i;1} + e_{i;2}) < e_{i;1}^2 + e_{i;2}^2 + 2e_{i;1} \cdot e_{i;2} \leq e_{i;1}^2 + e_{i;2}^2 + 2,$$

which implies that

$$(C - \mathbf{M}(E)E) \cdot (e_{i;1} + e_{i;2}) \leq e_{i;1}^2 + e_{i;2}^2 + e_{i;1} \cdot e_{i;2} \leq e_{i;1}^2 + e_{i;2}^2 + 1.$$

By combining both inequalities it immediately leads to a contradiction. q.e.d.

When $e_1 \cdot (C - \mathbf{M}(E)E) < e_1^2 < 0$ holds, the switching process $C - \mathbf{M}(E)E \mapsto C - \mathbf{M}(E)E - e_1 + e_1$ produces a residual relative obstruction bundle of complex rank $-e_1 \cdot (C - \mathbf{M}(E)E) + e_1^2$. The lemma implies that any codimension one degeneration of e_1 contains a smooth component e' with $e' \cdot (C - \mathbf{M}(E)E) \leq e'^2$.

Given the class $C - \mathbf{M}(E)E$ which is effective over a point $b \in Y(\Gamma)$, the sheaf short exact sequence

$$0 \mapsto \mathcal{O}(C - \mathbf{M}(E)E) \mapsto \mathcal{O}(C) \mapsto \mathcal{O}_{\mathbf{M}(E)E}(C) \mapsto 0$$

induces a long exact sequence which can be truncated into a four term long exact sequence when C is very ample:

$$\begin{aligned}0 \mapsto \mathcal{R}^0(p)_*(\mathcal{O}(C - \mathbf{M}(E)E)) \mapsto \mathcal{R}^0(p)_*(\mathcal{O}(C)) \mapsto \\ \mathcal{R}^0(p)_*(\mathcal{O}_{\sum m_i E_i}(C)) \mapsto \mathcal{R}^1(p)_*(\mathcal{O}(C - \mathbf{M}(E)E)) \mapsto 0.\end{aligned}$$

The map $\mathcal{R}^0(p)_*(\mathcal{O}(C)) \mapsto \mathcal{R}^0(p)_*(\mathcal{O}_{\sum m_i E_i}(C))$ can be interpreted as the fiberwise restriction map from the sections of $\mathcal{O}(C)$ to a combination of type I exceptional curves. The stalks of the kernel sheaf of the map projectifies to be the family moduli space of $C - \mathbf{M}(E)E$.

Given a point $b \in Y(\Gamma)$, we would like to check that it satisfies the condition in applying nested Kuranishi model.

If b does not support any allowable decomposition, then the reduced family moduli space over b is of negative expected dimension and it does not contribute to the family invariant. Thus, we assume that there exists an allowable decomposition over b. In particular, there are a finite number of smooth type I exceptional curves over b dual to the classes $e_i, i \in I$ such that $e_i \cdot (C - \mathbf{M}(E)E) < 0$. The combinatorial Proposition 7.1 asserts that there exists at least one Γ_b such that the stratum Y_{Γ_b} is defined by the type I curves e_{J_r} with $e_{J_r} \cdot (C - \mathbf{M}(E)E) \leq e_{J_r}^2$, i.e., there exists a type A admissible decomposition class which supports over Y_{Γ_b}. Knowing the existence of such Γ_b, one proves the following statement.

Proposition 7.2. *Let Y_{Γ_b} be an admissible stratum which is the coexistence locus of a finite number of smooth irreducible type* I *exceptional curves e_i with $(C - \mathbf{M}(E)E) \cdot e_i \le e_i^2$, then the closure of the family moduli space over Y_{Γ_b} to $Y(\Gamma_b)$ satisfies the regular obstructedness condition. Namely, there exists a quotient bundle of the cokernel semi-bundle* **Obs** *which is isomorphic to $\mathbf{N}_{Y(\Gamma_b)}(Y(\Gamma))$.*

Proof. From the four term long exact sequence

$$0 \mapsto \mathcal{R}^0(p)_*(\mathcal{O}(C - \mathbf{M}(E)E)) \mapsto \mathcal{R}^0(p)_*(\mathcal{O}(C)) \mapsto \\ \mathcal{R}^0(p)_*(\mathcal{O}_{\sum m_i E_i}(C)) \mapsto \mathcal{R}^1(p)_*(\mathcal{O}(C - \mathbf{M}(E)E) \mapsto 0,$$

it is apparent that the cokernel at $x \in Y(\Gamma_b)$ is isomorphic to $H^1(\mathcal{X}_x, \mathcal{O}(C - \mathbf{M}(E)E))$. As has been addressed, the map

$$T_x(Y(\Gamma)) \mapsto H^1(\mathcal{X}_x, \mathcal{O}(C - \mathbf{M}(E)E))$$

is induced by the Kodaira-Spencer map. To prove the assertion, it suffices to prove that there exists a vector subspace V of $H^1(\mathcal{X}_x, \mathcal{O}(C - \mathbf{M}(E)E))$ and a surjective natural homomorphism from $H^1(\mathcal{X}_x, \mathcal{O}(C - \mathbf{M}(E)E))$ to V.

The vector space V satisfies the property that $T_x(Y(\Gamma)) \mapsto V$ induces an isomorphism on $T_x(Y(\Gamma))/T_x(Y(\Gamma_b))$.

Because we have known that $Y(\Gamma_b)$ is the transversal complete intersection $\cap_{t \in \mathrm{Ver}(\Gamma_b)} Y(\Gamma_b^t)$, thus the normal bundle of $Y(\Gamma_b) \subset Y(\Gamma)$ (being isomorphic to $T_x(Y(\Gamma))/T_x(Y(\Gamma_b)))$, can be decomposed into the sum of the restriction of the normal bundles of $Y(\Gamma_b^t) \subset Y(\Gamma)$. Thus, it suffices to prove that for all t which mark the vertexes of Γ_b, there exists a vector bundle V_t isomorphic to a subbundle of **Obs** such that $T_x(Y(\Gamma))/T_x(Y(\Gamma_b^t))$ maps isomorphic onto $V_t|_x$,

$$T(Y(\Gamma))/T(Y(\Gamma_b^t))|_{Y(\Gamma_b^t)} \cong V_t,$$

while

$$H^1(\mathcal{X}_x, \mathcal{O}_{\mathcal{X}_x}(C - \mathbf{M}(E)E)) \mapsto (V_t)|_x$$

is surjective.

For a given t, the admissible graph Γ_b^t corresponds to a type I curve e_t. Consider the $\mathbf{P}^1$ fiber bundle $\widetilde{\Xi}_t$. One takes $V_t|_x$ to be $H^1(\widetilde{\Xi}_t|_x, \mathcal{O}_{\widetilde{\Xi}_t|_x}(e_t))$. In the following, I define a natural map from $H^1(\mathcal{X}_x, \mathcal{O}_{\mathcal{X}_x}(C - \mathbf{M}(E)E))$ to $H^1(\widetilde{\Xi}_t|_x, \mathcal{O}_{\widetilde{\Xi}_t|_x}(e_t))$.

Step 1. First, because $C - \mathbf{M}(E)E$ is effective, pick an effective divisor $\Sigma_{C-\mathbf{M}(E)E}$ representing $C - \mathbf{M}(E)E$, the short exact sequence

$$0 \mapsto \mathcal{O}_{\mathcal{X}_x} \mapsto \mathcal{O}_{\mathcal{X}_x}(C - \mathbf{M}(E)E) \mapsto \mathcal{O}_{\mathcal{X}_x \cap (C-\mathbf{M}(E)E)}(C - \mathbf{M}(E)E) \mapsto 0$$

induces a map

$$H^1(\mathcal{X}_x, \mathcal{O}_{\mathcal{X}_x}(C - \mathbf{M}(E)E)) \\ \mapsto H^1(\Sigma_{C-\mathbf{M}(E)E}, \mathcal{O}_{\mathcal{X}_x \cap \Sigma_{C-\mathbf{M}(E)E}}(C - \mathbf{M}(E)E)).$$

When C is very ample, the long exact sequence maps the cokernel of the map isomorphically onto $H^2(\mathcal{X}_x, \mathcal{O})$, which is p_g dimensional.

Step 2. By assumption, $e_t \cdot (C - \mathbf{M}(E)E) \leq e_t^2 < 0$. It implies that e_t has a negative pairing with $C - \mathbf{M}(E)E$, which forces any effective representative to break off at least a copy of e_t when the curve representing e_t is smooth irreducible. It remains true if one passes from the family moduli space over Y_{Γ_b} to its closure over $Y(\Gamma_b)$. Thus, $\mathcal{O}_{\Sigma_{(C-\mathbf{M}(E)E)}}(C - \mathbf{M}(E)E)$ can be pulled back to the fiber over x of the universal type I curve Ξ_t, $(\Xi_t)_x = \{x\} \times_{Y(\Gamma_t)} \Xi_t$, and it induces a natural surjection,

$$H^1(\Sigma_{C-\mathbf{M}(E)E}, \mathcal{O}_{\Sigma_{C-\mathbf{M}(E)E}}(C - \mathbf{M}(E)E)) \\ \mapsto H^1((\Xi_t)_x, \mathcal{O}_{(\Xi_t)_x}(C - \mathbf{M}(E)E)) \mapsto 0.$$

Step 3. This step is parallel to the construction of the residual relative obstruction bundle. Suppose e_t is expressed as $E_t - \sum E_{j_t}$ where j_t are the direct descendents of t. Then one considers the forgetting map by reducing $\mathbf{M}(E)E = \sum m_i E_i$ to $\widetilde{\mathbf{M}}_t(E)E = m_t E_t + \sum_{j_t} m_{j_t} E_{j_t}$, i.e., one erases all the multiplicities for $i \neq t, j_t$. Because $\sum_{i \neq t, j_t} m_i E_i$ is an effective divisor on $\mathcal{X}_x$ defined by a holomorphic section $\underline{s}$, then the tensor product with the given section induces a map

$$\cdot \otimes \underline{s} : H^1((\Xi_t)_x, \mathcal{O}_{(\Xi_t)_x}(C - \mathbf{M}(E)E)) \mapsto H^1((\Xi_t)_x, \mathcal{O}_{(\Xi_t)_x}(C - \widetilde{\mathbf{M}}_t(E)E)).$$

Step 4. While $(\Xi_t)_x$ is smooth irreducible over Y_{Γ_b}, it degenerates into some trees of $\mathbf{P}^1$ over $Y(\Gamma_b) - Y_{\Gamma_b}$. Recall that in Section 5.2 I have constructed a universal $\mathbf{P}^1$ bundle $\widetilde{\Xi}_t$ birational to the total space of the universal type I curves Ξ_t over $Y(\Gamma_{e_t}) = Y(\Gamma_b^t)$. Because the reduction of the multiplicity function $\widetilde{\mathbf{M}}_t(E)$ depends on E_t and E_{j_t}

only, the invertible sheaf $\mathcal{O}_{(\Xi_t)_x}(C - \widetilde{\mathbf{M}}_t(E)E)$ is the pull-back of the invertible sheaf $\mathcal{O}_{(\widetilde{\Xi}_t)_x}(C - \mathbf{M}(E)E)$ from $(\widetilde{\Xi}_t)_x$.

The birational map $\Xi_t \mapsto \widetilde{\Xi}_t$ induces the isomorphism,

$$H^1((\Xi_t)_x, \mathcal{O}_{(\Xi_t)_x}(C - \widetilde{\mathbf{M}}_t(E)E)) \mapsto H^1((\widetilde{\Xi}_t)_x, \mathcal{O}_{(\widetilde{\Xi}_t)_x}(C - \widetilde{\mathbf{M}}_t(E)E)).$$

Step 5. According to the construction of the residual relative obstruction bundle in Section 5.2, there exists a surjective map,

$$H^1((\widetilde{\Xi}_t)_x, \mathcal{O}_{(\widetilde{\Xi}_t)_x}(C - \widetilde{\mathbf{M}}_t(E)E)) \mapsto H^1((\widetilde{\Xi}_t)_x, \mathcal{O}_{(\widetilde{\Xi}_t)_x}(e_t)),$$

when the numerical condition $e_t \cdot (C - \mathbf{M}(E)E) \le e_t^2 < 0$ is satisfied. I take $V_t = H^1((\widetilde{\Xi}_t)_x, \mathcal{O}_{(\widetilde{\Xi}_t)_x}(e_t))$.

By composing the maps from Step 1 to Step 5 together one gets a map

$$H^1(\mathcal{X}_x, \mathcal{O}_{\mathcal{X}_x}(C - \mathbf{M}(E)E)) \mapsto H^1((\widetilde{\Xi}_t)_x, \mathcal{O}_{(\widetilde{\Xi}_t)_x}(e_t)).$$

In the following, the surjectivity of the constructed map is proved. Steps 2, 3, 4 and 5 are all surjective. The map constructed in Step 1 has a p_g dimensional cokernel. Thus, it suffices to prove that the connecting homomorphism

$$H^1((\widetilde{\Xi}_t)_x, \mathcal{O}_{(\widetilde{\Xi}_t)_x}(e_t)) \mapsto H^2(\mathcal{X}_x, \mathcal{O}_{\mathcal{X}_x})$$

is trivial. Recall that the connecting homomorphism is a portion of the long exact sequence

$$H^1((\widetilde{\Xi}_t)_x, \mathcal{O}_{(\widetilde{\Xi}_t)_x}(e_t)) \;\mapsto\; H^2(\mathcal{X}_x, \mathcal{O}_{\mathcal{X}_x}) \;\stackrel{\tau}{\to}\; H^2(\mathcal{X}_x, \mathcal{O}_{\mathcal{X}_x}(e_t)) \;\mapsto\; 0,$$

where $H^2(\widetilde{\Xi}_t|_x, \mathcal{O}_{\widetilde{\Xi}_t|_x}(e_t)) = 0$ has been used. On the other hand, Serre duality and the adjunction formula $\mathbf{K}_{\mathcal{X}_x} = \mathbf{K}_M \otimes \{\otimes_{1 \le i \le n} \mathcal{O}(E_i)\}$ implies that

$$H^2(\mathcal{X}_x, \mathcal{O}_{\mathcal{X}_x}(e_t)) = H^0(\mathcal{X}_x, \mathbf{K}_{\mathcal{X}_x} \otimes_{\mathcal{O}} \{\mathcal{O}_{\mathcal{X}_x}(e_t)\}^*)$$

is also of p_g dimensional. In particular, this implies that the τ map is an isomorphism. Therefore,

$$H^1((\widetilde{\Xi}_t)_x, \mathcal{O}_{(\widetilde{\Xi}_t)_x}(e_t)) \mapsto H^2(\mathcal{X}_x, \mathcal{O}_{\mathcal{X}_x})$$

is trivial. By commutativity of the diagram, the cokernel of the map in Step 1 maps trivially to $H^1((\widetilde{\Xi}_t)_x, \mathcal{O}_{(\widetilde{\Xi}_t)_x}(e_t))$.

The proof of the proposition is complete after the following lemma is proved. q.e.d.

Lemma 7.2. *The composite linear map*

$$T_x(Y(\Gamma)) \mapsto H^1((\widetilde{\Xi}_t)_x, \mathcal{O}_{(\widetilde{\Xi}_t)_x}(e_t))$$

is surjective and the subspace $T_x(Y(\Gamma_b))$ *lies in the kernel.*

Proof. I construct a map

$$T_x(Y(\Gamma)) \mapsto H^1((\widetilde{\Xi}_t)_x, \mathcal{O}_{(\widetilde{\Xi}_t)_x}(e_t))$$

by the following procedure and prove that it induces an isomorphism from $T_x(Y(\Gamma))/T_x(Y(\Gamma_b^t))$ to $H^1((\widetilde{\Xi}_t)_x, \mathcal{O}_{(\widetilde{\Xi}_t)_x}(e_t))$.

Because the isomorphism

$$H^1((\Xi_t)_x, \mathcal{O}_{(\Xi_t)_x}(e_t)) \mapsto H^1((\widetilde{\Xi}_t)_x, \mathcal{O}_{(\widetilde{\Xi}_t)_x}(e_t)),$$

I construct a composite map

$$T_x(Y(\Gamma)) \xrightarrow{a} H^1(\mathcal{X}_x, \mathcal{O}(e_t)) \xrightarrow{b} H^1((\Xi_t)_x, \mathcal{O}_{(\Xi_t)_x}(e_t)).$$

The first map a is induced by composing the Kodaira-Spencer map with $\nabla s_t : \Theta_{\mathcal{X}_x} \mapsto \mathcal{O}(e_t)$, with s_t being the holomorphic section defining the divisor $(\Xi_t)_x \subset \mathcal{X}_x$ and ∇ being a connection determined by the infinitesimal deformation of holomorphic structures. The second map b is induced by the sheaf short exact sequence

$$0 \mapsto \mathcal{O} \mapsto \mathcal{O}(e_t) \mapsto \mathcal{O}_{\Xi_t|_x}(e_t) \mapsto 0.$$

To prove the surjectivity of the composite map $b \circ a$, one considers the following exact sequence, which is the datum of an algebraic Kuranishi model of the family Seiberg-Witten theory of e_t,

$$0 \mapsto \mathcal{R}^0(p)_*(\mathcal{O}(e_t)) \mapsto \mathcal{R}^0(p)_*(\mathcal{O}(E_t)) \xrightarrow{ev_t} \mathcal{R}^0(p)_*(\mathcal{O}_{\sum_{j_t} E_{j_t}}(E_t)) \mapsto \mathcal{R}^1(p)_*(\mathcal{O}(e_t)) \mapsto \mathcal{R}^1(p)_*(\mathcal{O}(E_t)) \mapsto 0.$$

From surface Riemann-Roch and Serre duality, $h^1(\mathcal{X}_x, \mathcal{O}(E_t)) = q = \frac{b_1}{2}$. The map $ev_t : s \to \oplus_{j_t} s|_{E_{j_t}}$, $s \in \mathcal{R}^0(p)_*(\mathcal{O}(E_t))$ defines a global section of the locally free sheaf $\mathcal{R}^0(p)_*(\mathcal{O}_{\sum_{j_t} E_{j_t}}(E_t))$ of rank $-e_t^2 - 1$ such that the zero locus of the global section defines the moduli space of the exceptional curves in e_t.

The following three spaces are isomorphic:

(i) The moduli space of the exceptional curves dual to e_t.

(ii) The support of the coherent sheaf $\mathcal{R}^0(p)_*(\mathcal{O}(e_t))$.

(iii) The closure $Y(\Gamma^t)$ of the admissible stratum Y_{Γ^t}.

It follows from the construction of the universal spaces $M_n, n \in \mathbf{N}$ that there is at most one effective curve in e_t over each fiber $\mathcal{X}_x$. The equivalence (i) $\leftrightarrow$ (ii) follows from the fact that $h^0(\mathcal{X}_x, \mathcal{O}(e_t)) \neq 0$ if and only if the class e_t is represented by an effective divisor over $x \in Y(\Gamma)$. The equivalence (i) $\leftrightarrow$ (iii) follows from the interpretation of $Y(\Gamma^t)$ as the existence locus of e_t.

Because $Y(\Gamma^t)$ is known to be smooth of the right dimension, the global section defined by ev_t is transversal. Therefore $c : T_x(Y(\Gamma)) \mapsto H^0(\mathcal{X}_x, \mathcal{O}_{\sum_{j_t} E_{j_t}}(E_t))$ is surjective and its kernel defines $T_x(Y(\Gamma^t))$. It is standard to check that the composition of c with the connecting homomorphism $H^0(\mathcal{X}_x, \mathcal{O}_{\sum_{j_t} E_{j_t}}(E_t)) \mapsto H^1(\mathcal{X}_x, \mathcal{O}(e_t))$ is identical to a, which is induced from the Kodaira-Spencer map of the fiberwise deformation of complex structures.

On the other hand, the composite sheaf morphism

$$\mathcal{R}^0(p)_*(\mathcal{O}_{\sum_{j_t} E_{j_t}}(E_t)) \mapsto \mathcal{R}^1(p)_*(\mathcal{O}(e_t)) \mapsto \mathcal{R}^1(p)_*(\mathcal{O}_{\Xi_t}(e_t))$$

induces a map for all $x \in Y(\Gamma^t)$,

$$H^0(\mathcal{X}_x, \mathcal{O}_{\sum_{j_t} E_{j_t}}(E_t)) \mapsto H^1((\Xi_t)_x, \mathcal{O}_{(\Xi_t)_x}(e_t)).$$

This map is an isomorphism because of the following reasons. Firstly, curve Riemann-Roch implies that these two vector spaces have the same dimension. Secondly, the kernel $H^1(\mathcal{X}_x, \mathcal{O})$ of

$$H^1(\mathcal{X}_x, \mathcal{O}(e_t)) \mapsto H^1((\Xi_t)_x, \mathcal{O}_{(\Xi_t)_x}(e_t))$$

is mapped isomorphically onto the cokernel $H^1(\mathcal{X}_x, \mathcal{O}(E_t))$ of

$$H^0(\mathcal{X}_x, \mathcal{O}_{\sum_{j_t} E_{j_t}}(E_t)) \mapsto H^1(\mathcal{X}_x, \mathcal{O}(e_t))$$

under the composite map

$$H^1(\mathcal{X}_x, \mathcal{O}) \mapsto H^1(\mathcal{X}_x, \mathcal{O}(e_t)) \mapsto H^1(\mathcal{X}_x, \mathcal{O}(E_t)).$$

The composite map $H^1(\mathcal{X}_x, \mathcal{O}) \mapsto H^1(\mathcal{X}_x, \mathcal{O}(E_t))$ is the one defined by tensoring with a nonzero element of $H^0(\mathcal{X}_x, \mathcal{O}(E_t))$. q.e.d.

Given any stratum Y_{Γ_b} which is defined by the coexistence of a finite number of type I curves dual to $e_i; (C - \mathbf{M}(E)E) \cdot e_i \leq e_i^2$, I pass to its closure $Y(\Gamma_b)$ and consider points in the boundary $Y(\Gamma_b) - Y_{\Gamma_b}$. The same analysis also applies to the closure of the family moduli space over Y_{Γ_b} to $Y(\Gamma_b)$. On the other hand, new type of decompositions may appear and dominate the family moduli space over some degenerated strata $Y_{\Gamma'}, \Gamma' < \Gamma_b$. Let us summarize the pattern of their degenerations.

(i) If $e_i \cdot (C - \mathbf{M}(E)E) < 0$, and e_i is degenerated into at least two components $\widetilde{e}_i$. Then at least one among them also satisfies $\widetilde{e}_i \cdot (C - \mathbf{M}(E)E) < 0$.

(ii) If $(C - \mathbf{M}(E)E - \sum e_i, \sum e_i)$ is allowable, then e_i can be grouped and smoothed out into $e_{J_r} = \sum_{j \in J_r} e_j$ such that certain e_{J_r} satisfies $e_{J_r} \cdot (C - \mathbf{M}(E)E) \leq e_{J_r}^2 < 0$.

(iii) Given a class C_0 and an allowable type I decomposition $(C_0 - e, e)$. If the inequality $d(C_0 - e) + d(e) > d(C_0)$ holds strictly and e breaks into two components e_1, e_2, then at least one of them satisfies $d(C_0 - e_i) + d(e_i) \geq d(C_0)$.

In particular it implies that if a decomposition $(C - \mathbf{M}(E)E - \sum e_i, \sum e_i)$ is of Taubes' type and $e_a \cdot e_b = 0, a \neq b$, then any degeneration of the curves in e_i into some other degenerated type I curves leads to a nonallowable decomposition.

(iv) If $(C_0 - \sum e_i, \sum e_i)$ is allowable, then there must be at least one e_i such that $C_0 \cdot e_i \geq e_i^2 < 0$. If one collects these $e_i, e_i \cdot C_0 \leq e_i^2$, $(C_0 - \sum_{i; e_i \cdot C_0 \leq e_i^2} e_i, \sum_{i; e_i \cdot C_0 \leq e_i^2} e_i)$ forms a type A allowable decomposition.

From (i), (ii), (iii), (iv), we conclude the following. If an allowable decomposition $(C - \mathbf{M}(E)E - \sum \widetilde{e}_j, \sum \widetilde{e}_j)$ supports over $Y_{\Gamma'}, \Gamma' < \Gamma_b$, then either these all e_i remain irreducible, $\{e_i\}$ forms a sublist of $\{\widetilde{e}_j\}$ and the family moduli space of $(C - \mathbf{M}(E)E - \sum \widetilde{e}_j, \sum \widetilde{e}_j)$ embeds canonically into the corresponding moduli space of $(C - \mathbf{M}(E)E - \sum e_i, \sum e_i)$, or some e_i breaks up into irreduciblepieces and certain components of e_i are amongthese $\widetilde{e}_j$. In the former case, the decompositions are related by the partial ordering $\gg$. Conversely, if $\mathbf{D}_1 \gg \mathbf{D}_2$, then one can always embed the reduced family moduli space attached to $\mathbf{D}_2$ into that of $\mathbf{D}_1$ as its boundary component.

In particular, by viewing the type B decompositions as the degenerations from some type A allowable decomposition by the partial ordering

$\gg$, the validity of the regular obstructedness condition on the latter can be inherited by the former. Next, we discuss the weaker regular obstructedness condition for type B decompositions.

We have analyzed the validity of the regular obstructedness condition on type A allowable decompositions. Namely, the numerical condition $e \cdot (C - \mathbf{M}(E)E) \leq e^2 < 0$ has been used. If the decomposition is of type B, then the pairing with some e satisfies $0 > e \cdot (C - \mathbf{M}(E)E) > e^2$ and $\dim_{\mathbf{C}} H^1(\tilde{\Xi}_b, \mathcal{O}(C - \mathbf{M}(E)E))$ is smaller than $\dim_{\mathbf{C}} H^1(\tilde{\Xi}_b, \mathcal{O}(e))$. After all, this type of decomposition does not contribute to the invariant, due to a vanishing argument we will discuss below. Yet we still point out the weaker regular obstructedness condition it satisfies.

Let us investigate in detail what happens in this situation.

Let Σ_{C_0} be an effective curve over b poincare dual to $C_0 = C - \mathbf{M}(E)E$. As $e \cdot (C - \mathbf{M}(E)E) < 0$ and Ξ_b is irreducible, Σ_{C_0} breaks off at least a copy of Ξ_b and $\Sigma_{C_0} = \Sigma_{C_0 - e} + \Xi_b$. The kernel of the sheaves surjection $\mathcal{O}_{\Sigma_{C_0}}(C_0) \mapsto \mathcal{O}_{\Xi_b}(C_0)$ is isomorphic to $\mathcal{O}_{\Xi_b}(e)$ and they fit into a short exact sequence,

$$0 \mapsto \mathcal{O}_{\Xi_b}(e) \stackrel{\otimes s_{\Sigma_{C_0-e}}}{\longrightarrow} \mathcal{O}_{\Sigma_{C_0}}(C_0) \mapsto \mathcal{O}_{\Sigma_{C_0-e}}(C_0) \mapsto 0$$

by tensoring with $s_{\Sigma_{C_0-e}}$, the defining holomorphic section of Σ_{C_0-e}.

The cokernel of the connecting homomorphism $\hat{\delta}$ in the derived long exact sequence

$$\hat{\delta} : \mathcal{R}^0(p)_*(\mathcal{O}_{\Sigma_{C_0-e}}(C_0)) \mapsto \mathcal{R}^1(p)_*(\mathcal{O}_{\Xi_b}(e))$$

maps injectively into $\mathcal{R}^1(p)_*(\mathcal{O}_{\Sigma_{C_0}}(C_0))$. The regular obstructedness condition can not be satisfiedbecause the connecting homomorphism $\hat{\delta}$ is not trivial. The numerical condition $e \cdot C_0 > e^2$ implies that $\Xi_b \cdot \Sigma_{C_0-e} = e \cdot (C_0 - e) > 0$.

Lemma 7.3. *The rank of the connecting homomorphism $\hat{\delta}$ is at least $e \cdot (C_0 - e)$.*

Proof. The splitting $C_0 \mapsto (C_0 - e) + e$ induces an isomorphism between the moduli space of curves of C_0 and $C_0 - e$ over b. In particular, it induces an isomorphism between the Zariski tangent space $H^0(\Sigma_{C_0}, \mathcal{O}_{\Sigma_{C_0}}(C_0))$ and $H^0(\Sigma_{C_0-e}, \mathcal{O}_{\Sigma_{C_0-e}}(C_0 - e))$. Then one compares $H^0(\Sigma_{C_0-e}, \mathcal{O}_{\Sigma_{C_0-e}}(C_0 - e))$ and $H^0(\Sigma_{C_0-e}, \mathcal{O}_{\Sigma_{C_0-e}}(C_0))$ by using the sheaf injection,

$$0 \mapsto \mathcal{O}_{\Sigma_{C_0-e}}(C_0 - e) \mapsto \mathcal{O}_{\Sigma_{C_0-e}}(C_0).$$

Because $(C_0 - e) \cdot e$ is the total degree of $\mathcal{O}(e)$ on the components of Σ_{C_0-e}, the results follows. q.e.d.

Notice that the rank of $\hat{\delta}$ is not universal, which depends on the multiplicity of Ξ in the curve Σ_{C_0-e}.

In the following I give a geometric interpretation of the connecting morphism $\hat{\delta}$. Because the curve Σ_{C_0-e} has a positive pairing with Ξ_b, there exists infinitesimal deformations which corresponding to the smoothing of the generic intersection $\Sigma_{C_0-e} \cap \Xi_b$, viewed as the singularities of the total curve Σ_{C_0}.

Because $e \cdot C_0 < 0$, any curve over b in C_0 must split off at least a Ξ_b and these infinitesimal smoothing deformations are obstructed over b. The map $\hat{\delta}$ determines the obstruction classes of those infinitesimal deformations in the obstruction space of e, $H^1(\Xi_b, \mathcal{O}_{\Xi_b}(e))$. The exact sequence in Remark 5.1 singles out a universal $(C_0 - e) \cdot e$ dimensional infinitesimal deformations which allows us to identify $H^1(\Xi, \mathcal{O}_\Xi(C_0))$ as the quotient of $H^1(\Xi, \mathcal{O}_\Xi(e))$.

By discarding those $(C_0 - e) \cdot e$ dimensional universally obstructed infinitesimal deformations, the regular obstructedness condition can be realized over b, not in the cokernel space $H^1(\Sigma_{C_0}, \mathcal{O}_{\Sigma_{C_0}}(C_0))$ of the tangent obstruction complex but in the obstruction vector spaces of the pair $\Sigma_{C_0-e} \coprod \Xi_b$.

One can generalize the previous discussion to the decomposition involving more than one typeI curve as follows: Choose a switching process

$$C_0 \mapsto C_0 - e \cdots \mapsto C_0 - \sum e_i$$

and reenumerate e_i such that $0 > C_0 \cdot e_i > e_i^2$ if and only if $i \leq r$ for some r. The switching process of $C_0 - \sum_{i \leq r'} e_i \mapsto C_0 - \sum_{i \leq r'+1} e_i$ $r' \geq r$ and the construction of its residual relative obstruction class can be done as in Section 5.2. For $i \leq r$, each e_i introduces an $e_i \cdot C_0 - e_i^2$ dimensional obstructed infinitesimal deformations to the deformation complex. The total $\sum_{i \leq r}(e_i \cdot C_0 - e_i^2)$ dimension of the obstructed deformation can be counted alternatively by the following procedure. Consider the corresponding obstruction space for the intermediate class $C_0 - \sum_{i \leq r_0} e_i, r_0 < r$ in the switching process $C_0 \mapsto C_0 - e_1 \mapsto \cdots C_0 - \sum_{i \leq r} e_i$, the dimension should be $e_i \cdot (C_0 - \sum_{j \leq r_0} e_j)$ instead. And the total dimension sums up to $\sum_{0 \leq r_0 \leq r} e_{r_0+1} \cdot (C_0 - \sum_{j \leq r_0} e_j)$. However we have to take into account $\sum_{i<j \leq r} e_i \cdot e_j$ dimensional obstructed deformations from the type I curves $e_i, i \leq r$ which has been ignored implicitly when we

split off $\sum_{i\leq r} e_i$ from C_0. Or equivalently, consider a residual relative obstruction bundle of $e_i\cdot(\sum_{j<i} e_j)$ dimension by taking the direct image of $\mathcal{O}_{\cup_{j<i}\sigma_j}(C_0)$ along the $\mathbf{P}^1$ bundle $\tilde{\tilde{\Xi}}_i$, where σ_j is the effective cross section of $\tilde{\tilde{\Xi}}_i$ defined in the proof of Lemma 5.3. After adjusting by the cross terms of type I curves, the answers sum up again to

$$\sum_{0\leq r_0\leq r} e_{r_0+1}\cdot\Big(C_0-\sum_{j\leq r_0} e_j\Big)+\sum_{i<j\leq r} e_i\cdot e_j=\sum_{i\leq r}(e_i\cdot C_0-e_i^2).$$

Recall that the universal $\sum_{i\leq r}(C_0-e_i)\cdot e_i$ dimensional infinitesimal deformations constructed from the smoothing of $(C_0-e_i, e_i)\mapsto C_0$ are all obstructed. Because the existence of these infinitesimal deformations, the actual degree of the total residual relative obstruction class is of $\sum_{i\leq r}(C_0-e_i)\cdot e_i$ degree higher than the naive expected value $\mathrm{d}_{\mathbf{C}}(C_0-\sum_i e_i)+\sum_i \mathrm{d}_{\mathbf{C}}(e_i)-\mathrm{d}_{\mathbf{C}}(C_0)$.

It implies the vanishing of the mixed invariant attached to the type B decomposition class. This also explains why we had assigned $c.(\kappa)$ to be zero in Section 5.1.

7.1.2. The excision property and the identification with the modified Family invariants

To count the singular curves in C with the prescribed topological types of isolated singularities, one considers a particular resolution process which desingularizes the curve. The combinatorial datum are encoded in Γ and the multiplicity function $\mathbf{M}$. The resolution process is nonunique. One can reduce the ambiguity by the following principle. First, resolve a curve singularity until all the "infinitesimally near" singularities are resolved into smooth germs. Second, always choose to resolve singularities with lowest multiplicities. Suppose a pair $(\Gamma,\mathbf{M})$ has been chosen which satisfies the additional property. Then the singular curves in C with the prescribed topological types of singularities become smooth curves in the class $C-\mathbf{M}(E)E$ over $Y(\Gamma)$.

Take the preexceptional cone $\mathbf{C}_\Gamma$ associated with the admissible graph Γ. It follows from the construction of Γ that the following statement is true: The linear functional of cupping with $C-\mathbf{M}(E)E$ is nonnegative on $\mathbf{C}_\Gamma$.

To discuss the family invariant of $C-\mathbf{M}(E)E$ over $Y(\Gamma)$, take the fiber bundle $M_{n+1}\times_{M_n} Y(\Gamma)\mapsto Y(\Gamma)$ and consider the pure invariant $FSW_{Y(\Gamma)}(1, C-\mathbf{M}(E)E)$. One can always rewrite this pure invariant as the mixed invariant over M_n,

$$FSW_{Y(\gamma)}(PD(Y(\Gamma)), C - \mathbf{M}(E)E).$$

In this reinterpretation, the family expected dimension is raised by $\operatorname{codim}_{\mathbf{R}} \Gamma$, which is compensated by the base cohomology class insertion of the poincare dual $PD(Y(\Gamma)) \in H^{\operatorname{codim}_{\mathbf{R}} \Gamma}(Y(\gamma), \mathbf{Z})$. As the constant dimension shift applies to all decompositions, it does not affect the concept of allowable decompositions.

Over the support of an allowable decomposition class, Y_{Γ_1}, the reduced family moduli space may contribute to $FSW_{Y(\Gamma)}(1, C - \mathbf{M}(E)E)$ as well. By interpreting the original pure invariant as a mixed invariant over M_n, one keeps track of all the allowable decompositions of $C - \mathbf{M}(E)E$ on M_n. If the closure of the support $Y(\Gamma_1)$ is disjoint from $Y(\Gamma)$, the corresponding allowable decomposition in $Y(\gamma)$ does not appear in the family theory over $Y(\Gamma)$. This is coherent with the observation that under this $Y(\Gamma) \cap Y(\Gamma_1) = \emptyset$ assumption, $PD(Y(\Gamma))$ annihilates cohomology classes from $Y(\Gamma_1)$.

Proposition 7.3. *Let Y_{Γ_1}, $\Gamma_1 < \Gamma$, be the support of an allowable decomposition class. on $Y(\gamma) = M_n$. Then the preexceptional cone $\mathbf{C}_\gamma$ is a proper subcone of $\mathbf{C}_{\Gamma_1}$. Moreover, the extremal generators e_i of $\mathbf{C}_{\Gamma_1} - \mathbf{C}_\gamma$ are characterized by the property that $e_i \cdot (C - \mathbf{M}(E)E) < 0$.*

Proof. The first statement is a direct consequence of the Gromov-Sacks-Uhlenbeck compactness theorem (which is also true over an algebraic closed field of characteristic zero by specialization argument). The second statement follows from the maximality condition on the defining property of the support of an allowable decomposition. We prove by contradiction. Suppose that there exists at least one extremal generator $e \in \mathbf{C}_{\Gamma_1} - \mathbf{C}_\gamma$ such that $e \cdot (C - \mathbf{M}(E)E) \geq 0$ holds. The markings of their leading vertexes determine an ordering among these $e's$. Then one chooses the e with a smallest leading vertex i.

By the choice of e, there must be some direct descendent of i in Γ'. Otherwise, the class e would be a -1 class and it would have been in $\mathbf{C}_\gamma$. List all the direct descendents of i and find the one with the smallest marking j. Removing the edge from i to j leads to a new graph which we claim to be admissible. Among all the axioms of admissible graphs, the only one which is relevant to removing an edge is Axiom 4.4. It is not violated as j is the smallest direct descendent of i. One can compare the preexceptional cones before and after removing this edge. It is easy to see that the only change in their extremal edges is e. In particular, all the extremal $e_i, e_i \cdot (C - \mathbf{M}(E)E) < 0$ have not been touched. Therefore, I can iterate this procedure until all these $e's$ disappear. At the end, it results in a $\Gamma_0', \Gamma_0' > \Gamma'$ such that the extremal edges of $\mathbf{C}_{\Gamma_0'} - \mathbf{C}_\gamma$ have negative pairings with $C - \mathbf{M}(E)E$. This implies that Γ_0', instead of Γ', would have been the support of an allowable decomposition. q.e.d.

Recall that the collection of extremal edges $e_i, e_i \in \mathbf{C}_{\Gamma_1} - \mathbf{C}_\gamma$ generates a low dimensional simplicial subcone of $\mathbf{C}_{\Gamma_1}$, which was given the name "exceptional cone over Y_{Γ_1}" in defining the concept of admissible decomposition classes.

As $e_i \cdot (C - \mathbf{M}(E)E) < 0$, the (pseudo)-holomorphic curves over Y_{Γ_1} are forced to break off certain multiples of e_i spontaneously. Here one should notice that all e_i are represented by exceptional rational curves or their various degenerations.

The type I exceptional curve dual to e_i is irreducible and smooth over Y_{Γ_1}. In general, e_i can be represented by a bunch of rational curves over the higher codimension strata inside $Y(\Gamma_1)$. Even though a (pseudo) holomorphic curve supported over Y_{Γ_1} and dual to $C - \mathbf{M}(E)E$ always splits off those type I curves dual to e_i with $e_i \cdot (C - \mathbf{M}(E)E) < 0$; it is not necessarily the case when we pass to the boundary $Y(\Gamma_1) - Y_{\Gamma_1}$. In fact, the argument merely implies that the (pseudo) holomorphic curves in $C - \mathbf{M}(E)E$ and e_i both contain some type I curve bubbled off from e_i.

Thus, I have introduced the admissible decomposition classes of higher levels to take care of the possible contribution from the higher codimension strata, which parameterize the new topological types of curves popping out.

On the other hand, the reduced family moduli space associated with $(C - \mathbf{M}(E)E - \sum_{i, e_i \cdot (C - \mathbf{M}(E)E) < 0} e_i, \sum_{i, e_i \cdot (C - \mathbf{M}(E)E) < 0} e_i)$ is canonically embedded into the reduced family moduli space associated with $C - \mathbf{M}(E)E$ such that the embedding map (after restricted to Y_{Γ_1}) is an isomorphism over Y_{Γ_1}.

Suppose that the map has extended to an isomorphism over $Y(\Gamma_1)$, the nested Kuranishi model can be applied to calculate the invariant contributions inside and outside $Y(\Gamma_1)$.

This additional assumption implies that all the (pseudo)holomorphic curves of $C - \mathbf{M}(E)E$ supported over $Y(\Gamma_1)$ can be decomposed into holomorphic curves in $C - \mathbf{M}(E)E - \sum_{i, e_i \cdot (C - \mathbf{M}(E)E) < 0} e_i$ and $\sum_{i, e_i \cdot (C - \mathbf{M}(E)E) < 0} e_i$. Violation of this hypothesis indicates the appearance of the higher level admissible decomposition classes.

Recall that one has defined the concept of admissible decomposition classes of level s. It is clear that the map intertwining the admissible decomposition classes and the admissible graphs $\Phi_s : \mathcal{ADM}(s) \mapsto \mathrm{adm}(n)$ is injective. The subscript s here denotes the level of the decomposition classes.

Given such a decomposition, one can canonically attach a reduced family moduli space to it such that the compactified reduced family moduli space contains all the curves associated with this decomposition or its various degenerations. If the decomposition does not satisfy the allowable condition, its Seiberg-Witten invariant contribution must be zero; thanks to the vanishing result for the family invariants with negative family dimensions. This follows from a standard family Kuranishi model argument as well as the dimension restriction. On the other hand, if one decomposition can be cohomologically degenerated from the other, the family moduli space of the former decomposition can be viewed as the boundary component of latter. In paricular, the reduced family moduli space of a type B allowable decomposition class is included in the one of some type A decomposition class.

By using the concept of admissible decomposition classes, one can decompose the original Kähler Seiberg-Witten family moduli space into a union of different closed sub-moduli spaces. It will be shown in the following that the corresponding family invariants enjoy a corresponding decomposition.

Schematically, one can write down the following equality between the (reduced) family Seiberg-Witten moduli spaces

$$\mathcal{M}_{red}^{K\ddot{a}hler} = \cup_{\mathbf{D}\in\mathcal{ADM}}\mathcal{M}_{\mathbf{D}}^{K\ddot{a}hler},$$

where one has summed over all the admissible decomposition classes of different levels. The reader should notice that some admissible decomposition class may support on the boundary strata of an admissible stratum under the Φ map. Yet its Kähler family moduli space is not completely contained in the one attached to a lower level admissible decomposition class. One should take those decomposition classes into consideration, too.

The argument of nested perturbation allows us to find some better representatives other than $\mathcal{M}^{K\ddot{a}hler}$ such that the corresponding decomposition into the perturbed $\mathcal{M}_{\mathbf{D}}^{K\ddot{a}hler}$ becomes a disjoint union. The reader should notice that under the special perturbation the perturbed geometric objects completely lose their special role being the moduli space of unparameterized curves.

Even though each individual piece is usually not transversal, for enumeration purposes one can adopt the family Kuranishi model to enumerate the invariant contribution of each individual piece, with the understanding that the base manifold M_n or $Y(\Gamma)$ has to be blown up in advance to get anenhanced birational model. By taking $X_{j;k} = Y(\Gamma_{\mathbf{D}}), j \equiv \mathbf{D}$ and $k = level(\mathbf{D})$ and $Y = Y(\Gamma)$, one apply the nested perturbation to the blown up space $\widetilde{Y} = \widetilde{Y}(\Gamma)$.

The projection morphism $\widetilde{Y}(\Gamma) \longrightarrow Y(\Gamma)$ induces an isomorphism over Y_Γ. Over a partial compactification $\mathbf{pc}(Y_{\Gamma_{\mathbf{D}}})$ containing $Y_{\Gamma_{\mathbf{D}}} \subset Y(\Gamma)$, the preimage of $\mathbf{pc}(Y_{\Gamma_{\mathbf{D}}})$ under the blowing down morphism $\widetilde{Y}(\Gamma) \mapsto Y(\Gamma)$ is isomorphic to a projective space fiber bundle over $\mathbf{pc}(Y_{\Gamma_{\mathbf{D}}})$.

The space $\mathbf{pc}(Y_{\Gamma_{\mathbf{D}}})$ is taken to be the union of $Y_{\Gamma_{\mathbf{D}}}$ and $Y_{\Gamma'}, \Gamma' < \Gamma_{\mathbf{D}}$ such that the exceptional cone $\mathbf{EC}_b(C - \mathbf{M}(E)E)$ remain constant over $b \in Y_{\Gamma'}$.

The nested perturbation has the crucial property that after the nested perturbations, the perturbation of $\mathcal{M}_{\mathbf{D}}^{Kähler}$ is moved to support over the corresponding projective space bundle over $\mathbf{pc}(Y_{\Gamma_{\mathbf{D}}})$.

Then I proceed as follows:

1. For each $\mathbf{D}$, I project the perturbed version of the reduced family moduli space to $\mathbf{pc}(Y_{\Gamma_{\mathbf{D}}})$.

The stratification of $Y(\Gamma)$ into $\mathbf{pc}(Y_\Gamma) \coprod_{\mathbf{D}} \mathbf{pc}(Y_{\Gamma_{\mathbf{D}}})$ and a union of some strata which support no allowable decompositions unifies the local family Kuranishi models of the perturbed $\mathcal{M}_{\mathbf{D}}$ into a new family Kuranishi model of $\mathcal{M}_{pert}$. Even though the new spaces are not complex analytic, one can choose a small neighborhood of $\mathcal{M}_{pert}$ and set up a family Kuranishi model accordingly.

By the uniqueness of the family invariant, the pure invariant it determines is given by $FSW_{Y(\Gamma)}(1, C - \mathbf{M}(E)E)$. On the other hand, all the nongeneric $Y_{\Gamma_{\mathbf{D}}}$ are supported over by some reduced family moduli spaces associated with $\mathbf{D}$. These reduced family moduli spaces are regularly obstructed, according to Proposition 7.2 and argument right after the proof of Lemma 7.3.

If the decompositions are all of Taubes' type, the analysis of the nested Kuranishi model is rather easy. The major complication is that some Taubes' type admissible decomposition class may support at the boundary stratum of the others. Moreover, the supports of the two different Taubes' type decompositions may intersect and lead to some admissible decomposition of a higher level. Let us illustrate this phenomenon by giving a simple example.

Suppose that $e_i, i = 1, 2$ are two type I exceptional classes and let e_1, e_2 and $C - \mathbf{M}'(E)E$ be perpendicular to one another.Let $Y(\Gamma_{e_1})$ and $Y(\Gamma_{e_2})$ be the closed loci over which the classes e_1 and e_2 are effective. It is easy to see that $\mathbf{D}_1 = (C - \mathbf{M}'(E)E + e_2, e_1)$, $\mathbf{D}_2 = (C - \mathbf{M}'(E)E + e_1, e_2)$ are two decompositions of Taubes' type. The new admissible decomposition $\mathbf{D}_{1,2} = (C - \mathbf{M}'(E)E, e_1 + e_2)$ supporting over $Y(\Gamma_{e_1}) \cap Y(\Gamma_{e_2})$ is admissible because $e_1 \cap e_2 = 0$.

It is easy to see that the nested Kuranishi model can be applied to calculate the invariant contribution of $\mathbf{D}_1, \mathbf{D}_2$ and $\mathbf{D}_{1,2}$ which can be expressed as $FSW_{Y(\Gamma_1)}(1, C - \mathbf{M}'(E)E + e_2) - FSW_{Y(\Gamma_{1,2})}(1, C - \mathbf{M}'(E)E)$, $FSW_{Y(\Gamma_2)}(1, C - \mathbf{M}'(E)E + e_1) - FSW_{Y(\Gamma_{1,2})}(1, C - \mathbf{M}'(E)E)$ and $FSW_{Y(\Gamma_{1,2})}(1, C - \mathbf{M}'(E)E)$, respectively. The first two objects are the modified family invariants in this special context. In general, one has to introduce the combinatorial pattern similar to Cech theory to record the family invariants defined over each stratum. More precisely, if $e_1, e_2, \cdots, e_k, e_i \cap e_j = 0$ are k disjoint type I class supporting over $Y(\Gamma_{e_i}), i \leq k$.

The locus that e_i and e_j coexist is the transversal intersection of $Y(\Gamma_i)$ and $Y(\Gamma_j)$. Similarly, let I be a subset of $\{1, 2, \cdots n-1, n\}$. The space $Y(\Gamma_I) = \cap_{i\in I} Y(\Gamma_{e_i})$ is the locus over which $e_i, i \in I$ coexist. Let $(C - \mathbf{M}(E)E)$ be the generic admissible decomposition to start with. Then $\mathbf{D}_I = (C - \mathbf{M}(E)E - \sum_{i\in I} e_i, \sum_{i\in I} e_i)$ is an admissible decomposition such that the corresponding holomorphic curves in $C - \mathbf{M}(E)E - \sum_{i\in I} e_i$ and e_i, etc. are pairwisely disjoint. After the admissible perturbation, one attaches a family invariant to the perturbed family moduli space over Y_{Γ_I}.

By the excision principle, one can show easily that the invariant attached to the configuration is defined to be the following alternating sum:

$$\begin{aligned}
&FSW_{Y(\Gamma_I)}\left(1, C - \mathbf{M}(E)E - \sum_{i\in I} e_i\right) \\
&\qquad - \sum_{I\subset J\subset\{1,2,\cdots k\}, |J|=|I|+1} FSW_{Y(\Gamma_J)}\left(1, C - \mathbf{M}(E)E - \sum_{j\in J} e_j\right) \\
&\qquad + \sum_{I\subset J\subset\{1,2,\cdots k\}, |J|=|I|+2} \cdots .
\end{aligned}$$

It is apparent that the pattern is parallel to the Cech theory encoding the intersection pattern of these $Y(\Gamma_I)$. Apparently, it may look different from what have appeared in Section 5.3. However, one can cast it into

$$FSW_{Y(\Gamma_I)}\left(1, C - \mathbf{M}(E)E - \sum_{i\in I} e_i\right) \\ - \sum_{I\subset J\subset\{1,2,\cdots k\},|J|=|I|+1} FSW^*_{Y(\Gamma_J)}\left(1, C - \mathbf{M}(E)E - \sum_{j\in J} e_j\right) \\ - \sum_{I\subset J\subset\{1,2,\cdots k\},|J|=|I|+2} FSW^*_{Y(\Gamma_J)}(1,\cdots,$$

if one has defined

$$FSW^*_{Y(\Gamma_J)}\left(1, C - \mathbf{M}(E)E - \sum_{j\in J} e_j, \sum_{j\in J} e_j\right)$$

to be a similar alternating sum for all $J, J \supset I$ which contains I properly. This justifies the definition of the modified invariants in Section 5.3, under the assumption that all the decompositions are of Taubes' type.

Given a decomposition class $\mathbf{D}$, one considers all the admissible $\mathbf{D}', \mathbf{D} \gg \mathbf{D}'$. It is obvious that $\mathcal{M}_{\mathbf{D}}^{Kähler} \supset \mathcal{M}_{\mathbf{D}'}^{Kähler}$ when $\mathbf{D} \gg \mathbf{D}'$. Because the relation $\gg$ is transitive, $\mathbf{D}' \gg \mathbf{D}''$ implies that $\mathbf{D}''$ also satisfies $\mathbf{D} \gg \mathbf{D}''$.

My goal is to argue by induction that the contribution of $\mathcal{M}_{\mathbf{D}}^{Kähler}$ to the family invariant is given by

$$FSW_{Y(\Gamma_{\mathbf{D}})}(c.(\kappa_{\mathbf{D}}), C - \mathbf{M}(E)E - \zeta_{\mathbf{D}}),$$

$\zeta_{\mathbf{D}} = \sum e_i, e_i \cdot (C - \mathbf{M}(E)E) < 0$, $e_i \in \mathbf{EC}(\Gamma_{\mathbf{D}})$ while the excess contribution of $\mathcal{M}_{\mathbf{D}}^{Kähler} - \cup_{\mathbf{D}\gg\mathbf{D}'}\mathcal{M}_{\mathbf{D}'}^{Kähler}$ is given by

$$FSW^*_{Y(\Gamma_{\mathbf{D}})}(c.(\kappa_{\mathbf{D}}), C - \mathbf{M}(E)E - \zeta_{\mathbf{D}}).$$

In Subsection 5.1, the appearance of new admissible decomposition classes in higher codimensional strata has been the main cause for the family moduli space of $C - \mathbf{M}(E)E$ to differ from that of $C - \mathbf{M}(E)E - \zeta_{\mathbf{D}}$. By inductive assumption, all the higher level admissible decomposition classes have been handled already. By blowing up the various lower dimensional strata suitably as in Subsection 6.2, one can separate the reduced family moduli space $\mathcal{M}_{\mathbf{D}}^{K\ddot{a}hler}$ from all those $\mathcal{M}_{\mathbf{D}''}^{K\ddot{a}hler}, \mathbf{D} \not\gg \mathbf{D}''$ no matter $\mathbf{D}''$ is of type A or type B.

The family switching formula between $C - \mathbf{M}(E)E - \zeta_{\mathbf{D}}$ and $C - \mathbf{M}(E)E$ over a birational model of $Y(\Gamma_{\mathbf{D}})$ identifies the family Kuranishi models for these two classes. One can choose the Kuranishi models coherently that their extended family obstruction bundles differ by the relative obstruction bundle. Namely, if $\mathbf{s} : \mathcal{O} \times \mathbf{R}^k \mapsto \mathbf{R}^{k'}$ is one family Kuranishi model of the class $C - \mathbf{M}(E)E - \zeta_{\mathbf{D}}$ with the central fiber $\mathbf{s}^{-1}(\mathbf{0})$ being the reduced Kähler family moduli space, then the corresponding object for $C - \mathbf{M}(E)E$ is $\mathbf{s}' : \mathcal{O} \times \mathbf{R}^k \mapsto \mathbf{R}^{k'} \oplus \mathcal{V}^{rel}$. In fact, both family Kuranishi models can be perturbed such that the zero loci of the defining sections $\mathbf{s}, \mathbf{s}'$ are away from the blown up loci. Thus, one can still assume effectively that these two family Kuranishi models are defined over $Y(\Gamma_{\mathbf{D}})$ instead of some smooth birational model of $Y(\Gamma_{\mathbf{D}})$.

If the type I exceptional curve dual to one of e_i breaks into at least two irreducible components over $b \in Y(\Gamma_{\mathbf{D}})$, then a higher level admissible decomposition class $\mathbf{D}'', \mathbf{D}'' \not\ll \mathbf{D}$ is supported over some degenerated stratum containing b. Thus, one can assume that all the type I curves dual to these e_i in the decomposition class $\mathbf{D}$ are all smooth irreducible away from the blown up exceptional loci of $Y(\Gamma_{\mathbf{D}})$. This conclusion has been adopted in Proposition 5.1 to construct the relative minimal model Ξ_a and $\kappa_{\mathbf{D}}$.

By the computation in Section 5.1, the relative obstruction bundle has been shown to be isomorphic to $\mathbf{N}_{Y(\Gamma_{\mathbf{D}})}Y(\Gamma) \oplus \kappa_{\mathbf{D}}$ in the $\mathcal{C}^\infty$ category. Moreover $\mathbf{s}'$ also maps trivially into $\mathbf{0} \in \mathcal{V}^{rel}$.

The nested perturbation introduces a perturbation of $\mathbf{s}'$ and hence a coherent perturbation of $\mathbf{s}$ such that the perturbed $\mathbf{s}'$ still projects trivially into the $\mathbf{N}_{Y(\Gamma_{\mathbf{D}})}Y(\Gamma)$ factor.

Thus, one has the following,

Proposition 7.4. *For a particular choice of nested perturbation, the perturbed version of the reduced family moduli space attached to $C - \mathbf{M}(E)E$ can be identified to be the solution points in the reduced family moduli space attached to $C - \mathbf{M}(E)E - \zeta_{\mathbf{D}}$ which satisfies one additional constraint: it lies in the zero locus of a smooth section of $\kappa_{\mathbf{D}}$.*

The smooth section is determined by the nested perturbation through the projection of $\mathbf{s}'$ into $\kappa_{\mathbf{D}}$.

By performing further perturbations upon these two family Kuranishi models simultaneously, the regular obstructedness condition implies that the family invariant associated with $\mathcal{M}_{\mathbf{D}}$ should be identified with $FSW(c.(\kappa_{\mathbf{D}}), C - \mathbf{M}(E)E - \zeta_{\mathbf{D}})$. The additional insertion $c.(\kappa_{\mathbf{D}})$ represents the Euler class of $\kappa_{\mathbf{D}}$. Because the top chern class of a complex vector bundle is equal to the Euler class of the underlying even rank real vector bundle, one can replace the zero locus by its Poincare dual $c.(\kappa_{\mathbf{D}})$. Notice that the piece of obstruction bundle $\cong \mathbf{N}_{Y(\Gamma_{\mathbf{D}})}Y(\Gamma)$ has been removed because of the reduction from the family theory over $Y(\Gamma)$ to the family theory over $Y(\Gamma_{\mathbf{D}})$.

When $\mathbf{D}$ is of the highest level, there is no other $\mathbf{D}', \mathbf{D} \gg \mathbf{D}'$. The argument given above implies that the invariant contribution of $Y(\Gamma_{\mathbf{D}})$ is equal to the mixed family invariant $FSW_{Y(\Gamma_{\mathbf{D}})}(c.(\kappa), C - \mathbf{M}(E)E)$, that was why we had defined $FSW^* \equiv FSW$ in this case.

In general, we argue inductively by the descending orders of their levels that the invariant contribution over $Y(\Gamma_{\mathbf{D}}) - \cup_{\mathbf{D} \gg \mathbf{D}'} Y(\Gamma_{\mathbf{D}'})$ is the corresponding modified invariant.

For simplicity, one can employ the nested perturbation to separate $\mathcal{M}_{\mathbf{D}}^{K\ddot{a}hler} - \cup_{\mathbf{D} \gg \mathbf{D}'} \mathcal{M}_{\mathbf{D}'}^{K\ddot{a}hler}$ and the various different $\mathcal{M}_{\mathbf{D}'}^{K\ddot{a}hler} - \cup_{\mathbf{D}' \gg \mathbf{D}''} \mathcal{M}_{\mathbf{D}''}^{K\ddot{a}hler}$ from each other. If $\mathbf{D}'$ is of type B, it does not contribute to the family invariant, as was argued after the proof of Lemma 7.3. Thus, we can focus upon type A decomposition classes $\mathbf{D}'$.

The separated applications of Kuranishi models to each of them implies that one can attach an integral valued invariant to each of them while the total sum is equal to

$$FSW_{Y(\Gamma_{\mathbf{D}})}(c.(\kappa_{\mathbf{D}}), C - \mathbf{M}(E)E).$$

On the other hand for those $\mathbf{D}' \ll \mathbf{D}$, their levels are higher than the level of $\mathbf{D}$. Thus their invariant contributions have been identified to be $FSW^*_{Y(\Gamma_{\mathbf{D}'})}(c.(\kappa_{\mathbf{D}'}), C - \mathbf{M}(E)E)$ through the induction process.

Therefore, the excision property implies that the invariant attached to $\mathcal{M}_{\mathbf{D}}^{K\ddot{a}hler} - \cup_{\mathbf{D}' \ll \mathbf{D}} \mathcal{M}_{\mathbf{D}'}^{K\ddot{a}hler}$ is

$$FSW_{Y(\Gamma_{\mathbf{D}})}(c.(\kappa_{\mathbf{D}}), C - \mathbf{M}(E)E) - \sum_{\mathbf{D}' \ll \mathbf{D}} FSW^*_{Y(\Gamma_{\mathbf{D}'})}(c.(\kappa_{\mathbf{D}'}), C - \mathbf{M}(E)E).$$

Having expressed inductively the invariant contributions of $Y(\Gamma_{\mathbf{D}}) - \cup_{\mathbf{D}' \ll \mathbf{D}} Y(\Gamma_{\mathbf{D}'})$ by $FSW^*_{Y(\Gamma_{\mathbf{D}})}(c.(\kappa_{\mathbf{D}}), C - \mathbf{M}(E)E)$, finally one has the following identity:

$$FSW(1, C - \mathbf{M}(E)E) = unknown + \sum_{\mathbf{D}} FSW^*_{Y(\Gamma_{\mathbf{D}})}(c.(\kappa_{\mathbf{D}}), C - \mathbf{M}(E)E).$$

By comparing with the definition of modified family invariant, one can identify the invariant contribution from the Kähler reduced family moduli space over $Y(\Gamma) - \cup_{\mathbf{D}\in\mathcal{ADM}} Y(\Gamma_{\mathbf{D}})$ with the pure modified family invariant of $C - \mathbf{M}(E)E$. I will address in Section 7.3 how to realize the modified invariants as the curve counting explicitly.

Remark 7.1. In the $\mathcal{C}^\infty$ argument, one can perturb the Kuranishi models in computing the invariant. In an algebraic argument, such operation is not allowed. Instead, one should observe that after blowing up the higher codimensional strata, the top Chern class of the modified family obstruction bundle becomes zero when it is restricted to any blown up exceptional loci. It is because the modified obstruction bundle always has a one dimensional trivial factor over each blown up exceptional locus. Moreover, it implies that the top Chern class of the modified obstruction bundle annihilates any cycle classes of the blown up exceptional loci. This provides an algebraic explanation why the family invariant can still be enumerated on $Y(\Gamma_{\mathbf{D}})$.

7.2 The universal polynomials and the completion of the proofs

Having discussed the algebraic construction, one moves back to the proof of the main theorem. I will use both the $\mathcal{C}^\infty$ and the algebraic arguments in my discussion.

According to family Kuranishi model, if the reduced family moduli space over Y_Γ is known to consist of a finite number of isolated smooth points, then a direct counting is possible.

Theorem 7.1. *Under the previous Assumptions 2 and 3, the number of solutions in the reduced family moduli space over Y_Γ can be identified with the modified pure invariant of $C - \mathbf{M}(E)E$, which can be expressed as the various combinations of different mixed invariants by reversing the induction process adopted on page 500.*

The theorem, along with the repeated application of family blowup formula (Theorem 2.2), and the family switching formula (Theorem 2.3), gives rise to the proof of the main theorem. To see this, one notices that under the condition requiring that the admissible decomposition involving only type I exceptional curves, the decomposition schematically looks like $C - \mathbf{M}'(E)E + \sum m_i e_i$, where $e_i = E_i - \sum_{j_i} E_{j_i}$ and $\mathbf{M}'$ is a new multiplicity function. A modified mixed invariant of this decomposition can be expressed schematically as $FSW^*_{Y(\Gamma')}(\eta, C-\mathbf{M}'(E)E)$ with η being a cohomology class in $H^*(Y(\Gamma'), \mathbf{Z})$, canonically constructed from the various e_i. As e_i are all type I exceptional curves, the topological information extracted from them does not depend on the global geometry of the chosen fiber bundle $M_{n+1}|_{f^{-1}_{n+1}(Y(\Gamma'))} \mapsto Y(\Gamma')$, even though they have a mild but explicit dependence on the class C through its tensor product with the obstruction vector bundles.

It can be demonstrated easily that the cohomology class η depends on $c_1(C)$ and also on the normal bundles of e_i in the fibers explicitly. Through the explicit forms of the family switching formula, it can be traced to the data of the relative tangent bundles $\mathbf{TM}_{i+1}/\mathbf{TM}_i$. The explicit forms of these topological datum are calculated by using either the Grothendieck-Riemann-Roch theorem or the family index theorem.

On the other hand, a cohomology class of the type $C - \mathbf{M}'(E)E$ can be explicitly written as $C - \sum p_i E_i, p_i \geq 0$. To calculate its mixed family invariant, one makes use of the family blowup formula.

At this point, some discussion is required. There are two possible approaches to reduce the invariants. One is geometric and the other is purely algebraic. Suppose that Γ' is the admissible graph that supports the particular decomposition of curves. Then $C - \mathbf{M}'(E)E$ has nonnegative pairings with all elements e_i in $\mathbf{C}_{\Gamma'}$. On the other hand, there can still be redundant and subredundant vertexes in Γ' with respect to $C - \mathbf{M}'(E)E$.

The concept of core discussed earlier (see Section 5) can be used in this situation. Namely, there is a subgraph of Γ' which is the core such that one can reduce $C - \mathbf{M}'(E)E$ to $C - \mathbf{M}'_{red}(E)E$. The vertexes that are removed from Γ' are some of the vertexes over which the multiplicity function $\mathbf{M}'$ takes values 0 or 1. Those are the redundant and subredundant vertexes.

As a result, the modified mixed invariant of $C - \mathbf{M}'(E)E$ can be rewritten as certain modified mixed invariant of $C - \mathbf{M}'_{red}(E)E$ over $Y(\mathrm{core}(\Gamma', \mathbf{M}'))$ such that the class η is pulled back from $Y(\Gamma')$ to $Y(\mathrm{core}(\Gamma', \mathbf{M}'))$ through $H^*(Y(\Gamma'), \mathbf{Z}) \mapsto H^*(Y(\mathrm{core}(\Gamma', \mathbf{M}')), \mathbf{Z})$. For simplicity, we denote it by the same symbol if it does not cause confusion. By comparing the preexceptional cones $\mathbf{C}_{\Gamma'}$ and $\mathbf{C}_{\mathrm{core}(\Gamma', \mathbf{M}')}$, one finds that the latter is a simplicial subcone of the former and the extremal generators lying out of this subcone are the -1 and -2 curves corresponding to the redundant and subredundant vertexes. The reduction to the core depends heavily on the usage of the family blowup formula in a systematic way. If there are no admissible decomposition classes succeeding this, then the mixed invariant can be reinterpreted formally as the invariant attached to the curves of topological type $(\mathrm{core}(\Gamma', \mathbf{M}'), C - \mathbf{M}'_{red}(E)E)$, such that the singularities are required to lie in the Poincare dual of η. Only when $\eta = 1 \in H^0(Y(\mathrm{core}(\Gamma', \mathbf{M}')), \mathbf{Z})$ can one interpret the mixed invariant as (up to a multiple) the virtual number of curves of a fixed topological type. Examples of these types can be found easily in the nodal curves counting of low n (please consult Section 9).

On the other hand, one can discard the geometric meaning of these modified family invariants and view them merely as combinations of mixed family invariants of the form $FSW_{Y(\Gamma')}(\eta, C - \mathbf{M}'(E)E)$ schematically.

To enumerate the answer, one applies the family blowup formula directly. The process is purely algebraic, and the geometric meaning of the mixed invariants is completely lost. To simplify the notation, we assume from now on that Γ' is its own core.

Let r be the cardinality of $\mathrm{Ver}(|\Gamma'|)$. By forgetting the r-th vertex and the edges ending at r, one gets a new admissible graph $\Gamma'(-1)$ with $r-1$ vertexes. From the definition-construction of the strata in Section 4.4, there is a canonical projection morphism $Y(\Gamma') \mapsto Y(\Gamma'(-1))$. Applying the family blowup formula, one relates the mixed invariant of $M_{r+1}|_{f_{r+1}^{-1}(Y(\Gamma'))} \mapsto Y(\Gamma')$ to the mixed invariant of $C - \sum_{i<r} m_i E_i$ over the blown down fiber bundle, contracting the r-th exceptional divisor. As the class $C - \sum_{i<r} m_i E_i$ does not depend on E_r, the family moduli space actually comes from the pulled back of the corresponding moduli space over $Y(\Gamma'(-1))$.

Notice that one has the following vanishing theorem of family invariants:

Theorem 7.2. *Let $\mathcal{X} \mapsto B_1$ be a fiber bundle of four-manifold which we discuss the family Seiberg-Witten theory upon. Let $g : B_2 \mapsto B_1$ be a smooth fiber bundle map of relative dimension a, and $g^*\mathcal{X} \mapsto \mathcal{B}_2$ is the pull back fiber bundle by the fiber product construction. Let $\mathcal{L}$ be a $spin_c$ structure over $\mathcal{X} \mapsto B_1$ and $g^*\mathcal{L}$ be the corresponding $spin_c$ structure by the pull back of g. Then the family invariants* (*in the associated chamber; cf.* [29]) *of these two fiber bundles and $spin_c$ structures are related by*

$$FSW_{B_2}(PD(F) \cup \eta, g^*\mathcal{L}) = FSW_{B_1}(\eta, \mathcal{L}),$$

where η is an arbitrary cohomology class in $H^*(B_1, \mathbf{Z})$ and $PD(F)$ is the Poincare dual of the fibers of $g : B_2 \mapsto B_1$. Or equivalently

$$FSW_{B_2}(\eta', g^*\mathcal{L}) \mapsto FSW_{B_1}(g_*\eta', \mathcal{L}).$$

Thus, if η' is in the kernel of the push forward morphism $g_* : H^*(B_2, \mathbf{Z}) \mapsto H^*(B_1, \mathbf{Z})$, then the mixed invariant $FSW_{B_2}(\eta, g^*\mathcal{L})$ over B_2 vanishes.

In particular, the mixed invariant vanishes automatically if deg(η') $<$ $\dim(F) = a$.

Proof. The theorem is proved by noticing that the family moduli spaces of $g^*(\mathcal{L})$ and $\mathcal{L}$ are related by pulling back. Specifically, let $\mathcal{M}_{\mathcal{L}}$ be the family moduli space over B_1 and $\mathcal{M}_{\mathcal{L}'}$ be the one over B_2. For uncorrelated perturbations, these two objects are not linked. However, if one requires the perturbation of the family Seiberg-Witten equations over B_2 to be pulled back from that of B_1, then one has $\mathcal{M}_{\mathcal{L}'} = \mathcal{M}_{\mathcal{L}} \times_{B_1} B_2$, the fiber product. I have implicitly make choices of the chamber structures for them to be compatible.

In particular, the expected family dimensions of these two different moduli spaces differ by a. The conclusion of the theorem follows from the push-forward formula (integration along fibers) in differential topology or algebraic geometry (by replacing H^* by $\mathcal{A}_*$). q.e.d.

Despite the simplicity of the vanishing theorem, it is rather important in my discussion. It indicates that, potentially, there are admissible decomposition classes whose mixed invariants are zero due to the vanishing theorem. In fact, concrete examples in the nodal curves counting shows that this phenomenon appears frequently.

The vanishing theorem suggests that only certain members of admissible decomposition classes contribute to the family invariants even though they are all allowable.

Let us continue our discussion. By using the push forward formula in the vanishing theorem, one can reduce $FSW_{Y(\Gamma')}(\eta, C - \sum_{i<r} m_i E_i)$ to $FSW_{Y(\Gamma'(-1))}(\eta_p, C - \sum_{i<r} m_i E_i)$, where η_p is constructed from η by first cupping with the various Chern classes of the obstruction bundle and then pushing forward along the morphism $Y(\Gamma') \mapsto Y(\Gamma'(-1))$. One should notice that the morphism is a holomorphic fibration (in the

sense of algebraic geometry) rather than a fiber bundle. But the same argument as in the proof of Vanishing theorem goes through.

In particular, the vanishing theorem implies that certain terms are killed in performing the push-forward operation. By the mathematical induction upon r, one can eventually reduce the original mixed invariant to the form $FSW_{pt}(\mu, C) \cdot \int_{Y(\Gamma')} \varpi$. The class μ is taken to be 1 if M is regular of $q = \frac{b_1}{2} = 0$. Otherwise, one needs to insert the top cohomology class from the Albanese torus counting the curves within a fixed holomorphic structure as was discussed briefly in [29] as well as in my thesis.

If one couples the Vainsencher's families with, e.g., hyperkähler family of $K3$, T^4, or S^1 family of Kodaira surfaces [29], then the subscript pt in FSW_{pt} should be replaced by S^2 and S^1, respectively.

The explicit form of $\int_{Y(\Gamma')} \varpi$ can be rather complicated. To us, the key property it carries is that ϖ is a cohomology class involving the various chern classes of the relative tangent bundles (see Theorem 2.2 and Section 3) $\mathbf{TM}_{i+1}/f_{i+1}^{*}\mathbf{TM}_i = \mathcal{R}\mathbf{T}(M_{i+1}/M_i)$ and $c_1(C)$. On the other hand, the fact that Y_{Γ_I} is a complete intersection of divisors in M_r implies that $\int_{Y(\Gamma')} \nu = \int_{M_r} \nu \cup PD(Y(\Gamma'))$. An arbitrary class ϖ on an arbitrary submanifold can not be automatically pulled back from a class on M_r. In our situation, the cohomology class ϖ is a product of chern classes $c_1(C), c_{\cdot}(\mathbf{TM}_{i+1}/\mathbf{TM}_i)$. Thus it is actually pulled back from M_r to $Y(\Gamma')$.

The key property about these spaces $Y(\Gamma')$ is that its Poincare dual can be formally expressed as the cup products of the various $E_i(s)$, etc.

Take Γ^i (with superscript) to be the admissible subgraph starting from the i-th vertex followed by its direct descendents j_i. List j_i according to their orders, $j_i^1 < j_i^2 < j_i^3 < j_i^4 \dots j_i^u$, where the subscript i stands for the fixed i and the superscripts indicate their orders. The number u denotes the number of direct descendents of the i-th vertex.

Proposition 7.5. *The cohomology class $PD(Y(\Gamma^i))$ is expressible as* $E_i(j_i^1) \cup (E_i(j_i^2) - E_{j_i^1}(j_i^2)) \cup (E_i(j_i^3) - E_{j_i^1}(j_i^3) - E_{j_i^2}(j_i^3)) \cup \dots = \cup_{q \le u}(E_i(j_i^q) - \sum_{t<q} E_{j_i^t}(j_i^q))$.

Sketch of Proof. It follows from Proposition 4.4 that each $Y(\Gamma^i)$ is a complete intersection, and each irreducible divisor is taken to be the proper transformation of the exceptional divisors under the consecutive blowing ups. One should notice that the expression of $PD(Y(\Gamma^i))$ can be calculated cohomologically from the family blowup formula by applying it to the type I class $E_i - \sum_{j_i} E_{j_i}$. In the process of enumerating $FSW_{Y(\gamma)}(1, E_i - \sum_{j_i} E_{j_i})$, family blowing up formula relates $FSW_{Y(\gamma)}(1, E_i - \sum_{j_i} E_{j_i})$ to $FSW_{Y(\gamma)}(1, E_i)$ through inserting a sequence of first Chern classes of one dimensional obstruction bundles. The class

$$\cup_{q \leq u}(E_i(j_i^q) - \sum_{t<q} E_{j_i^t}(j_i^q))$$

is the cup product of these first Chern classes. Or equivalently, the top Chern class of the direct sum of the obstruction line bundles. q.e.d.

By using the various facts, the original mixed invariants can be reexpressed as the integration over M_r of various cohomology classes involving $c_1(C)$, the relative tangent bundles of $f_{i+1} : M_{i+1} \mapsto M_i$and the various exceptional classes $E_a(b), a < b$. By using the explicit cohomology ring structure of M_r, one can eventually reduce the cohomology pairing to a polynomial of $c_1(C), c_1(TM), c_2(TM)$, which is easily seen to be of the form $c_1(C)^2, c_1(TM) \cdot c_1(C), c_1(TM)^2$ and $c_2(TM)$. As $c_1(TM) = -K_M$, and $c_2(TM) = \chi(M)$, the polynomial is of degree r in terms of these basic variables.

This proves the assertions of the main theorem under the working assumptions posed earlier in Section 7.1. The relationship between the modified invariants and the number of singular curves will be addressed in Section 7.3. To explain the disappearance of the term $FSW(\mu, C)$, one notices that it must be ± 1 either by using the following theorem, or by a direct calculation.

Theorem 7.3 (Li-Liu, Kroheimer-Mrowka). *The wall crossing number associated with the special variation of Seiberg-Witten invariant is* ± 1.

If the first Betti number b_1 is zero, one uses directly the usual Seiberg-Witten invariants in the literature. If the first Betti number of M is positive, then the variation of Seiberg-Witten invariant is used here.

Given a four-manifold, there is a Seiberg-Witten μ map

$$\mu : H_1(M, \mathbf{Z}) \to H^1(\mathcal{M}_{\mathcal{L}}, \mathbf{Z})$$

similar to the Donaldson μ map. One can insert a certain number of μ map image classes into the definition of the Seiberg-Witten invariants. This variation of the Seiberg-Witten invariant corresponds to the counting of holomorphic curves fixing the holomorphic structure of the holomorphic line bundle. Their wall crossing formulas were discussed in the author's thesis and in [29].

One would like to remark here that, there is a large distinction between the topological version and the algebraic version of family Seiberg-Witten invariants.

According to the discussion in [38], they match up when $p_g = 0$. On the other hand, the topological family Seiberg-Witten invariants usually vanish for ample classes over $p_g \neq 0$ algebraic surfaces. Only by taking suitable families of real $2p_g$ dimension, can one get the answer ± 1. Otherwise the invariants vanish despite the fact that the beautiful polynomials are still there.

On the other hand, the algebraic version of the "invariant" produces ± 1 if the cohomology class is simple in the sense of [38]. Because the simpleness condition is valid for suitably high powers of very ample line bundles, one gets a nonvanishing result as in the $p_g = 0$ case. In this sense, algebraic geometers would definitely prefer the algebraic "invariants" that are more compatible with the algebraic calculation. However, I need to warn the reader that they are not the real topological invariants living on $B = pt$.

Let X be an irregular algebraic surface. If one considers the "nonlinear system" fixing only the topological type of $\mathcal{L}$, then the Seiberg-Witten invariants $SW(L)$ can be calculated by the wall crossing formula as was done by Li-Liu [29], [30]. I skip the general formula here as the reader can find the general formulas in the papers [29], [30]. In the special case that X is an abelian surface, the wall crossing numbers are given by $\Delta FSW_{S^2}(1, L) = \frac{L^2}{2} + 1$. One should modify the universal polynomials by multiplying with the universal wall crossing numbers.

In the next subsection, I discuss to what extent the working assumptions are realistic. For technical reasons I do so only for curves with nodal or ordinarysingularities. The curves with more general singularities are handled under some additional assumption on the topological types of the curve singularities.

7.3 The verification of the working assumptions for nodal singularities

The goal of this subsection is to verify the four working assumptions in Section 7.1 for nodal curves. In this case, by replacing C with kC, an argument based on Göttsche's observation [19] provides the transversality condition.

The discussion I offer here is well known to some algebraic geometers. In particular, Göttsche offered an argument in his paper [19, Proposition 5.2. and Remark 5.4.]

Göttsche's basic conclusion is that by taking the number k sufficiently large (he gave an effective bound), the singular curves in the k-very ample linear system with a fixed number of multiple points with prescribed multiplicities (of expected dimension $2d$) which pass through d different generic points is a finite set. In particular, if one imposes generic conditions (points) more than half of its real dimension, the set is empty. This implies the type II freeness condition and the projection (blowing down) of the algebraic curves in the reduced moduli space are irreducible. However as will be explained below, Göttsche's argument only implies the partially goodness condition over Y_γ.

To discard the accidental appearance of those non-partially good curves in the holomorphic category, one works in the pseudo-holomorphic category and considers almost complex structures sufficiently closed to the Kählerian complex structures (as will be done in the special cases of $X = K3$ and T^4 later), then the various strata of Gromov moduli spaces can be perturbed to satisfy the partially good condition. By requiring the class C to be sufficiently very ample for the original complex structure, Göttsche's argument [19] implies that the holomorphic curves in the reduced Kähler family moduli space projects irreducibly to M. Even after a small perturbation, it continues to hold for pseudo-holomorphic curves in the family Gromov moduli space. This can be seen alternatively by arguing that if the image of the curve is not reduced or irreducible, the expected dimension will drop below the critical level and is not picked up in the reduced perturbed moduli space. From now on I use the symbol $\mathcal{S}$ to denote the union of the closures of the admissible strata supporting nongeneric type I decompositions. The space $\mathcal{S}$ is the locus over which some type I exceptional curve has a negative pairing with the cohomology class $C - \mathbf{M}(E)E$. If one takes $\mathcal{S}'$ to be the union of all the admissible strata $Y(\Gamma_{\mathbf{D}})$ supporting nongeneric admissible decomposition classes, then $\mathcal{S}'$ is a subset of $\mathcal{S}$. It is because the extra allowable conditions.

Combining these facts, one can choose a generic family sufficiently closed to the original family such that the reduced family Gromov moduli space over $Y(\gamma) - \mathcal{S}$ consists of a finite number of irreducibly smooth curves and over $\mathcal{S}$ some possibly continuous families of singular curves along with the type I multiple coverings. The finiteness of the smooth curves is implied by using the existence of Ruan-Tian invariant [48] for the generic choices of almost complex structure j.

As Taubes' technique only identifies the smooth curves with the solutions of the Seiberg-Witten invariant, I screen away those singular curves by keeping the family over $\mathcal{S}$ unperturbed. One applies the nested family Kuranishi models to analyze the invariant contribution over $\mathcal{S}$.

According to the numerical analysis in the note [37] and the fact that it is partially good in the pseudo-holomorphic category, the perturbed family Gromov-Taubes moduli space over $Y(\gamma)-Y_{\gamma}-\mathcal{S}$ is of at least real codimension two relative to the expected family dimension. Thus, one can assume that the reduced perturbed Gromov-Taubes moduli space supports over Y_{γ}. By screening the nongeneric admissible decompositions, one can identify the modified invariant with (up to a multiple) the number of smooth curves in $C-\sum 2E_i$ by applying Taubes "SW=Gr" ([51], [52], [53]). The same number is also equal to the corresponding Ruan-Tian invariant. The reader can consult Section 8 for additional discussion about the identification.

The appearance of the type II curves usually makes the expected dimensions of the good part higher than the expected dimension of the original class. Once the type II multiple covering shows up in the decomposition, it usually happens that there are high dimensional families of curves satisfying the points passing constraint.

The same assertion also holds for type I multiple coverings. The reason that one wants to get rid of the type II, but not type I, exceptional curves is their spontaneous appearance everywhere on the manifold $Y(\Gamma)$ or M_l. Moreover, they depend on the class C as well as the algebraic surface M. When they appear, the conclusion of the main theorem cannot hold. To simplify the discussion, one imposes conditions to make them go away. In the paper [38], the author has set up some fundamentals to discuss the type II curves. The details of the application of this idea will appear in some separate article. In discussing the general singular curves, the working assumption is NOT known to be true or not. It is one of the obstacles algebraic geometers face even in proving the conjecture for nodal curves. In this paper, the "number of singular curves" is generally interpreted as a kind of equivalence or virtual numbers. I leave the transversality issue as an open problem.

Recall that a line bundle $\mathcal{L}$ is p-very ample if the sheaf morphism

$$H^0(X,\mathcal{L}) \mapsto H^0(X,\mathcal{L}\otimes\mathcal{O}_Z) \mapsto 0$$

is exact for all zero dimensional subschemes Z of X of length $p+1$.

Recall the following theorem of Göttsche, which gives an effective version of the transversality theorem of nodal curves over Y_γ.

Theorem 7.4 (Göttsche). *Assume C is $3n-1$-very ample, then a general n dimensional linear subsystem $\subset |C|$ contains only finitely many curves with $\geq n$ singularities. If, furthermore, C is $5n-1$-very ample (5-very ample if $n=1$), then the curves have precisely n nodes as singularities.*

One can find the proof in [19, Proposition 5.2.], By applying this theorem, one gets an effective condition for the reduced family moduli space over Y_γ to be a finite number of smooth points. It is not clear that it is optimal for a general M. It would be rather interesting to conjecture a precise relationship between k-very ampleness and the numerical type II free condition in the pseudo-holomorphic category.

When C is a $5n-1$-very ample, it only guarantees the transversality of the reduced family moduli space of $C-\sum_{i\leq n} 2E_i$ over Y_γ. As before, $\mathcal{S}'$ denotes the union of all $Y(\Gamma)$ such that Γ supports nongeneric admissible decomposition classes. The space $\mathcal{S}\supset\mathcal{S}'$ is the union of loci over which the class $C-\mathbf{M}(E)E$ is not type I free.

I had explained how to use the nested perturbation to deal with the reduced family moduli space over $\mathcal{S}'$. The set $\mathcal{S}-\mathcal{S}'$ is stratified by admissible strata over which some type I class e has a negative pairing with $C-\mathbf{M}(E)E$. However, the intersection pairing is in the range $e^2 < e\cdot(C-\mathbf{M}(E)) < 0$ and the decomposition is not allowable. The argument on page 491 can be applied to these types of decomposition classes which shows that their contributions to the family invariant vanish, due to dimension reason.

The reduced family moduli space over $Y(\gamma)-Y_\gamma-\mathcal{S}$ has not been analyzed yet. If one chooses to work in the $\mathcal{C}^\infty$ category, one can bypass the part of moduli space by using Taubes' gluing theorem and prove that, effectively, it does not contribute to the curve counting.

This is done by proving that the counting of the isolated number of curves over Y_γ matches up to $n!$ with the Gromov-Ruan-Tian invariant count (if $p_g=0$). Under the condition that the Kähler reduced family moduli space over Y_γ is compact (it is sufficient to assume that C is $3n-1$-very ample), a suitably chosen one parameter family of perturbation [48] of the almost complex structures of M will introduce a compact cobordism between the reduced Kähler family moduli space over Y_γ with the reduced Gromov-Ruan-Tian moduli space over the same space. As we had known that $n!$ of the Gromov-Ruan-Tian invariant count matched up with the modified family Seiberg-Witten invariant, so is the number of algebraic curves over Y_γ.

The following is the main result in this subsection.

Proposition 7.6. *If the class L is $3n$-very ample, then the actual dimension of the algebraic family moduli space of $L - \sum_{i\leq n} 2E_i$ over $Y(\gamma) - Y_\gamma - \mathcal{S}$ matches its expected family dimension.*

In particular, if one requires the curves to pass through $\frac{L^2 - L\cdot K_M}{2} - q + p_g - n$ generic points in M, then there exist no such curves over $Y(\gamma) - Y_\gamma - \mathcal{S}$.

Proof of Proposition 7.6. I begin the proof by characterizing the locus $Y(\gamma) - Y_\gamma - \mathcal{S}$ explicitly.

Lemma 7.4. *A type* I *class e_i has a nonnegative pairing with $L - \sum_{i\leq n} 2E_i$ if and only if e_i is a -1 class E_i or a -2 class $E_i - E_{j_i}$.*

Proof. If i is a vertex with more than one direct descendent, then its pairing with $L - \sum_{i\leq n} 2E_i$ is negative. The argument is still valid if one replaces $L - \sum_{i\leq n} 2E_i$ with $L - \sum_{i\leq n} mE_i, m \geq 2$. This ends the proof of Lemma 7.4 q.e.d.

If a point $z \in Y(\gamma) - Y_\gamma - \mathcal{S}$ is not in the stratum supporting admissible decomposition classes, then every vertex in $\Gamma, z \in Y_\Gamma$ has at most one direct descendent vertex. The admissible graph of this type is a union of linear chains (including isolated vertexes).

Fix an arbitrary point $z \in Y(\gamma) - Y_\gamma - \mathcal{S}$, let L be an ample class on M. We want to prove that, for k large enough, $|[kL - \sum_{i\leq n} 2E_i]|$ over such $z$is based point free. Then it follows from the strong form of Bertini theorem that a generic member of $[|kL - \sum_{i\leq n} 2E_i]|$ is smooth.

In this context, I first choose $p = 3n$ and pick k such that $L = kL$ is $3n$-very ample over $X = M$. Then

$$H^0(M, \mathcal{O}([kL])) \mapsto H^0(M, \mathcal{O}_Z([kL])) \mapsto 0$$

is exact for all length $3n + 1$ subschemes Z of M.

This implies that the restriction map from the global sections of M to its local $3n + 1$ jets at a point in M is always surjective.

An algebraic curve in $|[kL - 2\sum_{i\leq n} E_i]|$ intersects with multiplicity two with any smooth type I -1 curve. By blowing down consecutively, one finds that this curve has at least one multiplicity-two singularity on the previous exceptional loci. Suppose the curve in $|[kL - 2\sum_{i\leq n} E_i]|$

has been smooth, the blown down curve in M has type A double point singularities.

One has the simple observation:

Lemma 7.5. *Let f be a local defining equation $f(x,y) \in \mathbf{k}[x,y]$ having a multiplicity two singularity at $(0,0)$ such that the blown up locus is singular at some point in the exceptional $\mathbf{P}^1_{\mathbf{k}}$. Then the map $\mathbf{k}[x,y] \mapsto \mathbf{k}[\widetilde{x},y]$ or $\mathbf{k}[x,y] \mapsto \mathbf{k}[x,\widetilde{y}]$ with respect to the new singularity in $\mathbf{P}^1_{\mathbf{k}}$ drops the total degree of a monomial at most by 2.*

Proof of Lemma 7.5. As the blowing up is a local phenomenon, an analysis using the local model is enough. Let $\mathbf{k}[x,y]$ be the polynomial ring over an algebraic closed field of characteristic zero.

Algebraically, blowing up the origin in $\mathbf{L}^2$ involves the change of variables $(x,y) \mapsto (x, x\overline{y})$ or $(x,y) \mapsto (y\overline{x}, y)$. Some simple calculation shows that, over a double point, the monomials in a local defining equation get changed by $x^a y^b \mapsto x^{a+b-2}\overline{y}^b$ or $x^a y^b \mapsto \overline{x}^a y^{a+b-2}$. A change of variable $\overline{y} = \widetilde{y} + c$ or $\overline{x} = \widetilde{x} + c$ transforms $x^a y^b$ into terms such as $x^{a+b-2}\widetilde{y}^i, i \leq b$ or $\widetilde{x}^j y^{a+b-2}, j \leq a$. q.e.d.

Because I know that each curve in $|[kL - 2\sum_{i\leq n} E_i]|$ has multiple points with at least multiplicity-two in each of the earlier blowing ups, the generic members ideally would have double points at each step of the blowing ups. Specifically, they carry type A double points. This is always possible as kL is taken to be $3n$-very ample. By applying Lemma 7.5 repeatedly in a linear chain with r vertexes, only the first $2r$-jets can contribute to the second jet in the final blown up algebraic surface. Because the total number of vertexes in Γ is n, a $2n$-very ample class gives us enough control about the local behavior of the algebraic curves in $|[kL - \sum_{i\leq n} 2E_i]|$.

Let Γ be a union of different connected admissible graphs which are all linear chains. For the a-th linear chain ($a \leq s$), $\hat{n}_a$ denotes the number of vertexes in this chain. Then it follows $\sum_{a\leq s} \hat{n}_a = n$. Here s stands for the number of components in Γ.

The vanishing of the 0 and 1-st jets at each blown up point ($\in M$) impose all together $3n$ linear conditions on the direct sum vector space of the local $2\hat{n}_a$ jets at all the initial blowing up point. For general singularities, it is nontrivial to check that these conditions are all linear independent. In this particular case one knows that the $3n = 3\sum_{a\leq s} \hat{n}_a$ linear conditions are all linear independent because the singularities involved are all simple $A_{\hat{n}_a}$ singularities. Their local versal deformation spaces are well known to be of $\hat{n}_a$ dimensional, a different number than one would have obtained if these linear conditions are not independent.

Take Z to be a specific length $3n$ subscheme of M such that it supports over the blown up points with lengths $3\hat{n}_a$. It is not hard to show that the annihilator of those $3\hat{n}_a$ linear conditions form an ideal in $\mathbf{k}[x,y]$. The length $3\sum_a \hat{n}_a$ subscheme of M is defined by the corresponding ideal of co-length $3\sum_{a\leq s}\hat{n}_a = 3n$.

It is not hard to see that the curves that have the prescribed singularities are determined by imposing $\sum_{a\leq s} 3\hat{n}_a = 3n$ linear conditions on $|[kL]|$.

Let $\mathbf{V}$ be the $3n$ dimensional vector space $\subset H^0(X, \mathcal{O}_Z([kL]))^\star$ which corresponds to the linear conditions imposed on the space $H^0(X, \mathcal{O}_Z([kL]))$. Because $H^0(X, \mathcal{O}_Z([kL]))^\star \mapsto H^0(X, \mathcal{O}([kL]))^\star$ is injective, these conditions are linear independent globally.

From this simple argument one is able to estimate the actual fiber-wise dimensions of $|[kL - 2\sum_i E_i]|$ to be less than that of $|[kL]|$by $3n$. Because $\dim_{\mathbf{k}} Y(\Gamma) = n + s$, the family moduli space over $Y(\gamma) - Y_\gamma - \mathcal{S}$ is always of lower dimension than the expected family dimension $\frac{k^2L^2 - kL\cdot K_M}{2} - q + p_g - n$. This ends the proof of Proposition 7.6. q.e.d.

Despite of the possibility to get rid of type II curves by imposing the extra sufficiently very ample condition on L, it is not a natural condition from the point of view of symplectic geometry.

7.4 The structure of the modified family invariant

7.4.1. The divisibility of the modified invariant

In the following, I first offer a proof of the divisibility of the modified family invariants, even if C is not sufficiently very ample. I still assume that $\Gamma = \gamma$ and the curves carry isolated ordinary singularities. Then I formulate a sufficient condition on $(\Gamma_0, \mathbf{M}_0)$ for the modified invariant to have the expected divisibility condition.

Recall that by a repeated application of the family blowing up and switching formulas, the modified family invariants can be expressed as the universal polynomial of four variables $C^2, C\cdot K_M, K_M^2, \chi(M)$, I would like to extrapolate the divisibility to the cases that C is not sufficiently very ample.

Because of some technical reason, the algebraic proof I present here only works for the case of singular curves with ordinary singularities. I first prove the proposition under the optimistic situation that the reduced family moduli space sitting over Y_Γ is smooth of zero dimension.

Notice that one also needs to choose the cutting sections in an equivariant way, which can be realized if the cutting sections of the family invariants are constructed from the trivial sections in $Y(\Gamma) \times M$.

In this situation, the finite number of smooth points must be equivariant under $G_{\Gamma,\mathbf{M}}$. Then the divisibility of the modified invariants follows directly.

This type of assumption is realized when the singular curves only carry ordinary singularities. As was repeatedly discussed by Vainsencher [59] and Göttsche [19], the regularity can be achieved by twisting the ample line bundle to a very high power.

By the main Theorem 1.1, the modified invariants are expressible as universal polynomials in the four different variables, $K_M^2, K_M \cdot C, C^2$, $\chi(M)$. Let these four variables be represented by x, y, z and w, then the invariant FSW^* can be expressed as a universal polynomial $F(x, y, z, w) \in \mathbf{Z}[x, y, z, w]$, a polynomial of four variables. The assertion in the special case implies that the polynomial takes values in $|G_{\Gamma,\mathbf{M}}| \cdot \mathbf{Z}$ on each M when C is taken to be a high power of very ample line bundles.

By using Göttsche's [19] argument reviewed in Subsection 7.4.2, one can choose M and the holomorphic line bundles among the four explicit series (which I will apply again in the proof of Theorem 7.5) carefully such that x, y, z, w mod p take arbitrary values in $\mathbf{Z}/p\mathbf{Z}$.

This enables us to prove inductively that the polynomial F is divisible by $p^i, i \leq \mathrm{ord}_p(|G_{\Gamma,\mathbf{M}}|)$ for all p. Then its divisibility by $|G_{\Gamma,\mathbf{M}}|$ follows.

It seems to us that the algebraic argument for the general cases still relies upon some difficult transversality condition that we know very little about.

After dividing the modified invariants by the corresponding orders of finite groups, one gets the equivalences or virtual numbers ($\in \mathbf{Q}$) associated with the singular curves. Only when the reduced moduli space over Y_Γ consists of a finite number of (possibly nonreduced) points, the virtual number can be directly interpreted as a weighted count of singular algebraic curves.

As the analogue of Ruan-Tian invariant is not known to exist for the other singular curves of different topological types of singularities, the divisibility and the equivalence of the various modified invariants of the general singular curves are not ensured in a similar way.

Nevertheless, I prove the divisibility of the modified invariant for a general class of singular curves. As usual, Γ_0 denotes a connected admissible graph and $\lfloor \Gamma_0, \mathbf{M}_0 \rfloor$ encodes the topological type of an isolated curve singularity.

Definition 7.2. A topological type of curve singularity $\Gamma_0, \mathbf{M}_0$ is said to be strictly descending if the value of $\mathbf{M}_0$ is strictly decreasing in the sense that $m_i > m_{j_i}$ for all the direct descendents of i.

If a vertex has no direct descendents, it is called an ending vertex. Otherwise, it is called a nonending vertex.

If all the nonending vertexes of Γ_0 have more than one direct descendents, then the topological type $\lfloor \Gamma_0, \mathbf{M}_0 \rfloor$ automatically becomes strictly descending.

On the other hand, an A_n curve singularity is not strictly descending as all the multiplicities $m_i = 2$.

Fix an admissible graph Γ of this special type, the number of its connected components will be denoted by $\mathrm{comp}(\Gamma) = m$.

Proposition 7.7. *If* $(\Gamma, \mathbf{M})$ *is lifted from a strictly descending topological type* $(\Gamma_0, \mathbf{M}_0)$*, then the modified family invariant associated with* $\Gamma, \mathbf{M}$ *is divisible by* $\mathbf{S}_{|\mathrm{comp}(\Gamma)|}$.

Proof. The top stratum Y_Γ allows a natural $\mathbf{S}_{|\mathrm{comp}(\Gamma)|}$ action. To derive this statement in the proposition, I prove that $Y(\Gamma) - \mathcal{S} \supset Y_\Gamma$ also allows a natural free $\mathbf{S}_{|\mathrm{comp}(\Gamma)|}$ action.

Suppose Γ' is an admissible degeneration of Γ. Suppose $|\mathrm{comp}(\Gamma')| = |\mathrm{comp}(\Gamma)|$, then the degeneration do not fuse any two components together. It is easy to see that the permutations of different components do not destroy the admissibility condition and the $\mathbf{S}_{|\mathrm{comp}(\Gamma)|}$ action intertwines the different components and the action is free.

My goal is to show that for any Γ', $|\mathrm{comp}(\Gamma')| < |\mathrm{comp}(\Gamma)|$, $Y_{\Gamma'} \subset \mathcal{S}$. Therefore, $\mathbf{S}_{|\mathrm{comp}(\Gamma)|}$ acts on $Y(\Gamma) - \mathcal{S}$ freely.

According to my convention of choosing Γ_0, the multiplicity m_i is the minimum among all $m_l, l \geq i$ which are not the descendents of i. I had pointed out earlier, this corresponds to blowing up the singularities with lower multiplicities first.

Suppose that $|\mathrm{comp}(\Gamma')| < |\mathrm{comp}(\Gamma)|$, then there are at least two different components in Γ which are fused into one. In other words, some new edge must relate two vertexes from the different components of Γ. Fixing such pair of components Γ_1 and Γ_2 such that there exists an edge from Γ_i to Γ_j, I argue that there must be an edge ending at the leading vertex of Γ_j.

Consider all the edges connecting Γ_i to Γ_j. One defines an order among them by comparing the markings of the ending vertexes. Choose the smallest edge in this sense. I claim that it must end at the leading vertex of Γ_j. If not, it ends at some other vertex of Γ_j. However, all the vertexes in Γ_j other than the leading one have direct ascendents in Γ_j. Therefore, this vertex must have at least two direct ascendents; one from a vertex in Γ_i, another one from Γ_j. By the axiom of admissible graphs, these two vertexes in Γ_i and Γ_j must be related by a new edge, yet the new edge is smaller than the previous one because the ending vertex of the new edge is the direct ascendent of the chosen one. Contradiction!

Consider the 1-edge and denote its beginning vertex and ending vertex by a and b. I prove that $E_a - E_b$ has a negative intersection pairing with $C - \mathbf{M}(E)E$.

Because b is the leading vertex of Γ_j, the value m_b is larger than the multiplicities for all other descendents; thanks to the strictly descending condition on the topological type. On the other hand, a is congruent to a vertex in Γ_j. Thus, $m_a < m_b$. Because $(E_a - E_b) \cdot C - \mathbf{M}(E)E = m_a - m_b < 0$, $E_a - E_b$ has a negative intersection pairing with $C - \mathbf{M}(E)E$ and $Y_{\Gamma'}$ is in the locus $\mathcal{S}$.

By applying the nested perturbation, one can construct a G_I equivariant smooth section of the modified extended family obstruction bundle whose zero locus supports over $Y(\Gamma) - \mathcal{S}$ completely. Because the noncompact base allows a free $\mathbf{S}_{|\text{comp}(\Gamma)|}$ action, one descents the bundle as well as the section onto the quotient $Y(\Gamma) - \mathcal{S}/\mathbf{S}_{|\text{comp}(\Gamma)|}$.

A generic perturbation and the transversality argument implies that the Euler number of the original bundle is divisible by $|\mathbf{S}_{|\text{comp}(\Gamma)|}| = |\text{comp}(\Gamma)|!$. q.e.d.

7.4.2. The factorization formula of the modified invariants

Next I want to introduce the result that under a quite general assumption, the generating functions of the normalized modified invariants satisfies certain factorization formula. This phenomenon was first observed and proved by Göttsche [19] for the cases of nodal curves under the assumption that the line bundles are high powers of very ample line bundles. But the observation has a wider implication than what it sets out to prove. As the modified invariants are equal to the singular curves counting only under suitable hypothesis, my attempt was to introduce a structure without imposing conditions on C.

Due to the discussion in the previous subsections, the proof of the main theorems is complete. But to apply the conclusion of the main theorems to the concrete cases, a better understanding about the structure of the modified invariants is necessary. It is my goal to provide such a structure. In this subsection, I focus on singularities of some special types. Namely, the singular curves with a finite number of singularities of identical topological types, which cover the nodal curves cases.

First, we state the congruence property of $\mathbf{F}$ in Proposition 7.8, which follows from Proposition 7.7.

Proposition 7.8. *Let $(\Gamma, C-\mathbf{M}(E)E)$ be a topological type of singular curves such that $(\Gamma, \mathbf{M}(E)E)$ is strictly descending. Let $\mathbf{F}(x,y,z,w) \in \mathbf{Z}[x,y,z,w]$ be the universal polynomial such that*

$$FSW^{*}_{Y(\Gamma)}\left(1, C-\sum \mathbf{M}(E)E\right) = \mathbf{F}(C^2, C\cdot K_M, K_M^2, \chi(M))\cdot SW(C).$$

Then $\mathbf{F}(x,y,z,w) \in |G_{\Gamma,\mathbf{M}}|\cdot \mathbf{Z}[x,y,z,w]$.

One should notice again that the divisibility property is usually not shared by the un-modified family Seiberg-Witten invariants.

The following convention and definition are useful in my discussion.

Definition 7.3. An admissible graph Γ is said to be a disjoint union of the $\Gamma_i, 1 \leq i \leq m$, denoted as $\Gamma = \coprod_{i\leq m}\Gamma_i$ if the admissible graph Γ, viewed as an ordinary graph, is the disjoint union of Γ_i. Moreover, through this decomposition, the markings of Γ are inherited by the Γ_i. The various graphs Γ_i are I-admissible for some finite index set I.

I slightly abuse the definition of admissible graphs and still call them admissible graphs (with respect to an index set). Two admissible graphs are congruent if there is a one to one correspondence between their markings under which these two arrowed graphs are isomorphic.

We are interested in some special types of admissible graphs that can be decomposed into "isomorphic" connected components, i.e., Γ is the disjoint union of Γ_i such that each Γ_i is a connected admissible graph, and different Γ_i and Γ_j are congruent to each other.

Suppose that the topological type of the singular curve satisfies the property that the multiplicity functions $\mathbf{M}$ of the congruent components are identified through the isomorphism, one would like to study the modified family invariants of these types.

Definition 7.4. Given a connected admissible graph Γ_0, one considers the set of all admissible graphs whose connected components are congruent to Γ_0 and one denotes it by $\mathrm{adm}(\Gamma_0)$. Then the set can be further decomposed into $\coprod_k \mathrm{adm}_k(\Gamma_0)$ according to the values of the function $\mathrm{comp}(\cdot)$, the cardinality of its connected components.

Given a connected admissible graph Γ_0, one considers the multiplicity function $\mathbf{M}(E)_0 : \mathrm{Ver}(\Gamma_0) \mapsto \mathbf{N} \cup \{0\}$ defined on the vertexes of Γ_0. Through the congruence relationships among the connected components, one can canonically extend $\mathbf{M}(E)$ to a multiplicity function of an arbitrary $\Gamma \in \mathrm{adm}(\Gamma_0)$.

Let Γ and Γ' be arbitrary two different elements in $\mathrm{adm}_k(\Gamma_0)$ and $\mathbf{M}$, $\mathbf{M}'$ denotes the canonical extension of the multiplicity function $\mathbf{M}(E)_0$ to Γ and Γ', respectively.

I would like to argue that the modified family invariants over $Y(\Gamma)$ and $Y(\Gamma')$, as were defined in the previous section, are identical. First, one notices that it is not true if we replace the modified invariants by the corresponding mixed invariants.

As the base manifolds $Y(\Gamma)$ and $Y(\Gamma')$ are merely birational to each other in the complex analytic sense, there is no obvious reason to believe that their associated mixed invariants should be related. On the other hand, the top dimensional strata Y_Γ and $Y_{\Gamma'}$ in $Y(\Gamma)$ and $Y(\Gamma')$ are isomorphic to each other. Let $\mathbf{c}$ and $\mathbf{c}'$ denote the base class insertions used in the mixed as well as the modified family invariants. Then $\mathbf{c}$ and $\mathbf{c}'$ can be identified after being pulled back to $H^*(Y_\Gamma, \mathbf{Z}) \cong H^*(Y_{\Gamma'}, \mathbf{Z})$.

We have the following proposition:

Proposition 7.9. *Under the same convention as above, the modified family invariants of $C - \mathbf{M}(E)$ and $C - \mathbf{M}'(E)$ over $Y(\Gamma)$ and $Y(\Gamma')$ are identical. Namely, the following equality holds:*

$$FSW^*_{Y(\Gamma)}(\mathbf{c}, C - \mathbf{M}(E)) = FSW^*_{Y(\Gamma')}(\mathbf{c}', C - \mathbf{M}(E)).$$

I offer a simple proof of the proposition:

Proof. Because in general $Y(\Gamma) \neq Y(\Gamma')$, a direct identification of the family moduli space is not possible. First, one notices that the reduced family moduli spaces are defined by the zero loci of some finite dimensional vector bundles. In the Kähler category, one can make the canonical choice by using the algebraic geometric data. The obstruction bundles can be calculated by some repeat application of family blowing up formula, etc.

At this moment, I do not care about their detail forms. Let us denote the associated obstruction bundles by $\mathcal{OBS}_\Gamma$ and $\mathcal{OBS}_{\Gamma'}$. It is rather crucial that $\mathcal{OBS}_\Gamma|_{Y_\Gamma} \cong \mathcal{OBS}_{\Gamma'}|_{Y_{\Gamma)}}$.

On the other hand, the reduced perturbed family moduli spaces can be splitted into different components. The dominated components that define the modified family invariants support over some compact subsets of the open manifolds $Y(\Gamma), Y(\Gamma')$. Despite that the $\mathcal{OBS}_\Gamma$ and $\mathcal{OBS}_{\Gamma'}$ are not co-related, their restrictions to Y_Γ and $Y_{\Gamma'}$ are canonically isomorphic. As the dominant components of the reduced perturbed family moduli spaces can be arranged to lie completely over $Y_\Gamma, Y_{\Gamma'}$– after one has perturbed the defining sections carefully, the isomorphism of the vector bundles ensure that they are two different smooth sections of the same vector bundle. Then they can be both interpreted as two different family Kuranishi models for the same moduli problem.

Regarding the issue that the sections can be perturbed to have their zero loci in the top strata, I have used the fact that the dominant components completely avoid the strata which support admissible decomposition classes. By using the uniqueness of the invariants under the family Kuranishi model technique, one can identify the modified invariants accordingly.

Recall that the Φ map associates each admissible decomposition class into the corresponding admissible stratum. In the condition that characterizes the admissible decomposition classes, they have the property that their family expected dimensions are not less than the one of the original class. As one has performed a special perturbation in the first step such that the dominant component is disjoint from these components, the dominant component has the crucial property that the preimage of any stratum of codimension two or higher is of smaller expected dimension than the original expected dimension. Thus, the generically perturbed reduced family moduli space avoidsany stratum $Y(\widetilde{\Gamma})$ with codimension two or higher and consists of a finite number of smooth points. q.e.d.

After proving this proposition, it makes sense to consider the following generating series,

$$\mathcal{F}(\Gamma_0, \mathbf{M}_0; M) = \sum_k \sum_{\Gamma \in \mathrm{adm}_k(\Gamma_0)} \frac{\mathbf{F}_{\Gamma,\mathbf{M}}(M)}{\sigma(\Gamma, \mathbf{M})|G_{\Gamma,\mathbf{M}}|} t^k,$$

where $\mathbf{F}_{\Gamma,\mathbf{M}}(M)$ is the abbreviation of the universal polynomial constructed by the main theorem and the group $G_{\Gamma,\mathbf{M}(E)} \subset G_\Gamma$ is the subgroup of G_Γ which keeps the multiplicity function $\mathbf{M}$ invariant. The symbol $\sigma(\Gamma, \mathbf{M}(E))$ denotes the cardinality of the orbit through Γ under the $G(\Gamma, \mathbf{M}(E))$ action.

My next goal is to study the structure of the generating function. The following theorem is largely inspired by Göttsche's argument [19] on the nodal curves ($\mathbf{M} \equiv 2$) case. My formulation shows that the similar but more general phenomenon holds in a much more general context.

Consider M to be the disjoint union of two algebraic surfaces $M(1)$ and $M(2)$, denoted by $M = M(1) \coprod M(2)$. Then one has the following decomposition result for $Y(\Gamma), \Gamma \in \mathrm{adm}_n(\Gamma_0)$. Notice that the notations $M(i)$ instead of M_i are used in order not to be confused with the i-th universal space.

Proposition 7.10. *The space $Y(\Gamma)$ allows the following decomposition into disjoint unions:*

$$Y(\Gamma) = \coprod_{\Gamma_1 \coprod \Gamma_2 = \Gamma, \Gamma_1, \Gamma_2 \in \mathrm{adm}(\Gamma(0))} Y(\Gamma_1) \times Y(\Gamma_2)$$

where $Y(\Gamma_1)$ denotes the stratum of Γ_1 associated with $M(1)$ while $Y(\Gamma_2)$ is associated with $M(2)$.

Namely, decomposition of the strata follows closely from the decompositions of the corresponding admissible graphs, which runs through all the possibilities.

Proposition 7.10 follows from a simple lemma which asserts that:

Lemma 7.6. *Let Γ be an admissible graph. Suppose that Γ can decomposed into Γ_a and Γ_b, then Y_Γ is canonically isomorphic to an open submanifold of the direct product of Y_{Γ_a} and Y_{Γ_b}.*

Let M be decomposed as $M(1)$ and $M(2)$ and let Γ be a connected admissible graph, then the closure $Y(\Gamma)$ of Y_Γ in M_k, $k = |\mathrm{Ver}(\Gamma)|$, is equal to the disjoint union of those of $M(1)$ and $M(2)$.

Proposition 7.10 is a simple corollary of this lemma. I leave the proofs of these combinatorial lemma and proposition to my reader. The key fact is that the diagonal $\Delta(M^2) = \Delta(M(1)^2) \coprod \Delta(M(2)^2)$. i.e., there is no mixed term among $M(1)$ and $M(2)$.

Let us consider the following situation. As before, suppose $M = M(1) \coprod M(2)$ is the disjoint union of the two different algebraic surfaces. Let $C = C(1) + C(2)$ be the sum of the two different line bundles on $M(1)$ and $M(2)$, respectively. Then one considers the modified family invariant of $C - \mathbf{M}(E)E$ on $Y(\Gamma)$. According to the previous proposition, one finds that the modified invariant satisfies the convolution relation.

Namely,

$$FSW^*_{Y(\Gamma)} = \sum_{\Gamma_1+\Gamma_2=\Gamma} FSW^*_{Y(\Gamma_1)} \cdot FSW^*_{Y(\Gamma_2)}.$$

The invariants are taken with respect to $C-\mathbf{M}(E)E$,$C(1)-\mathbf{M}(E)_1E$ and $C(2) - \mathbf{M}(E)_2E$, respectively. Notice that the class $C - \mathbf{M}(E)E$ decomposes accordingly.

By taking FSW on both sides, it is not hard to see that the identity

$$FSW_{Y(\Gamma)} = \sum_{\Gamma_1+\Gamma_2=\Gamma} FSW_{Y(\Gamma_1)} \cdot FSW_{Y(\Gamma_2)},$$

holds for the original family invariants.

To see that a similar equality holds for FSW^* as well, we interpret $FSW_{Y(\Gamma)}(1, C-\mathbf{M}(E))$ as $FSW_{Y(\gamma)}(PD(Y(\Gamma)), C-\mathbf{M}(E)E)$ and then compare their admissible decompositions on $Y(\gamma)$. In defining the admissible decompositions, one begins with the generic decompositions and then inductively defines the nongeneric decompositions of the various levels.

Since $M(1)$ and $M(2)$ are disjoint, a decomposition over the universal space of $M(1)\coprod M(2)$ can carry curves only when $\mathbf{M}(E)E$ is written as $\mathbf{M}(E)_1E_1 + \mathbf{M}(E)_2E_2$, with $\mathbf{M}(E)_1, \mathbf{M}(E)_2$ being the multiplicity functions over $M(1)$ and $M(2)$, respectively. It assigns a multiplicity function uniquely to each factor $Y(\Gamma_1) \times Y(\Gamma_2)$ of the disjoint union decomposition in Proposition 7.10. Conversely, each pair of decompositions of $C(1) - \mathbf{M}(E)_1E_1$ and of $C(2) - \mathbf{M}(E)_2E_2$ sum up to a decomposition of $M(1)\coprod M(2)$. It is easy to see that the type A allowable condition is preserved as the dimension formula simply adds together. On the other hand, a type B decomposition may split into allowable and nonallowable decompositions. It does not affect our discussion as type B decompositions do not contribute to the modified invariant.

First, the generic admissible decomposition classes are easily seen to be preserved under the splitting. One would like to inductively prove that the level function is additive under the splitting. Namely, if $\mathbf{D}_1(1)$ and $\mathbf{D}_2(2)$ are two admissible decompositions on $M(1)$ and $M(2)$ of level a and b, respectively, then $\mathbf{D}_1(1) + \mathbf{D}_2(2)$ is admissible over $M(1)\coprod M(2)$ of level $a+b$; and vice versa.

In defining the modified invariants, one can express it as the alternating sum of the mixed invariants associated with the type A admissible decomposition classes of different levels. Schematically, one has

$FSW^*(\mathbf{D}) = FSW(\mathbf{D}) - \sum_s \sum_{\mathbf{D} \gg \mathbf{D}' \in \mathcal{ADM}(s)} FSW(\mathbf{D}')$.

Then the splitting of the modified invariant follows from the corresponding splitting of the various mixed invariants on the right hand side.

On the other hand, take $\mathbf{X}$ to be the principal homogeneous space over $\mathbf{S}_{\mathrm{comp}(\Gamma)}$, consisting of a finite number of points. Then one considers the relative trivial $G_{\Gamma_0,\mathbf{M}_0}$ bundle over $\mathbf{X}$, and let $\mathbf{G}_{\mathbf{X},\Gamma}$ be the sections of the relative bundle which inherits the group structure from $G_{\Gamma_0,\mathbf{M}_0}$. Notice that $\mathbf{G}_{\mathbf{X},\Gamma}$ is of order $|G_{\Gamma_0,\mathbf{M}_0}|^{\mathrm{comp}(\Gamma)}$. The symmetric group $\mathbf{S}_{|\mathrm{comp}(\Gamma)|}$ acts upon $G_{\mathbf{X},\Gamma}$ naturally.

Then the group $G_{\Gamma,\mathbf{M}}$ can be factorized by the following finite groups short exact sequence:

$$1 \mapsto \mathbf{G}_{\mathbf{X},\Gamma} \mapsto G_{\Gamma,\mathbf{M}} \mapsto \mathbf{S}_{\mathrm{comp}(\Gamma)} \mapsto 1.$$

Alternatively, $G_{\mathbf{X},\Gamma}$ is described by $|\mathrm{comp}(\Gamma)|$ elements in $G_{\Gamma_0,\mathbf{M}_0}$ such that the multiplication is performed component-wisely. The group $\mathbf{S}_{|\mathrm{comp}(\Gamma)|}$ acts upon $G_{\mathbf{X},\Gamma}$ by permutation. $G_{\Gamma,\mathbf{M}}$ is called the wreath product in the literature.

As was mentioned before, the modified family invariant does not depend on the explicit markings. Therefore, the previous convolution identity implies immediately the product formula of the generating series. Namely,

$$\mathcal{F}(\Gamma_0, \mathbf{M}_0; M) = \mathcal{F}(\Gamma_0, \mathbf{M}_0; M(1)) \times \mathcal{F}(\Gamma_0, \mathbf{M}_0; M(2)).$$

On the other hand, the denominators $|G_{\Gamma,\mathbf{M}}|$ can be factored as $|G_{\Gamma_0,\mathbf{M}}|^{\mathrm{comp}(\Gamma)}$(which is the order of $\mathbf{G}_{\mathbf{X},\Gamma}$) times $|\mathbf{S}_{\mathrm{comp}(\Gamma)}|$. One changes the variable $t \to |G_{\Gamma_0,\mathbf{M}}| \cdot t$ in $\mathcal{F}(\Gamma_0, \mathbf{M}_0)(t)$, then one defines

$$\overline{\mathcal{F}}(\Gamma_0, \mathbf{M}; M)(t) = \mathcal{F}(\Gamma_0, \mathbf{M}_0; M)(\{|G_{\Gamma_0,\mathbf{M}_0}|\} \cdot t).$$

Then $\overline{\mathcal{F}}$ satisfies the following factorization property:

Theorem 7.5. *The power series $\overline{\mathcal{F}}(\Gamma_0, \mathbf{M}; M) \in \mathbf{Z}[[t]]$ defined above can be factored as follows:*

$$\overline{\mathcal{F}}(\Gamma_0, \mathbf{M}; M) = (A_1(\Gamma_0, \mathbf{M}_0))^{\chi(M)} \cdot (A_2(\Gamma_0, \mathbf{M}_0))^{K_M^2} \cdot (A_3(\Gamma_0, \mathbf{M}_0))^{C \cdot K_M} \cdot (A_4(\Gamma_0, \mathbf{M}_0))^{C^2}.$$

Here, $A_i(\Gamma_0, \mathbf{M}_0) \in \mathbf{Q}[[t]], 1 \leq i \leq 4$ are universal, independent of the manifold M or the cohomology class C one chooses. However, the *formal power series depends manifestly on the graph* Γ_0 *and and the multiplicity function* $\mathbf{M}_0$*, and, therefore, the topological type of the curve singularities.*

In the theorem, the power series are formulated to have coefficients in $\mathbf{Q}$. If one knows that FSW^* is always divisible by $|\mathbf{S}_{\mathrm{comp}(\Gamma)}|$, it is enough to restore the integrability. I give the sufficient conditions for the integrability condition to hold. In particular, the power series associated with ordinary singularities are all in $\mathbf{Z}[[t]]$.

The key argument of the proof is essentially due to Göttsche [19]. I follow his idea and go through certain details.

Proof of Theorem 7.5. To prove a theorem of this type, which extracts out the dependence of the invariants on the four different variables $K_M^2, K_M \cdot C, C^2$ and $\chi(M)$, one formally sets $\chi = x, K^2 = y, K \cdot C = z, C^2 = w$ as four formal variables. By the family blowup formula, etc., the power series $\overline{\mathcal{F}}$ can be expressed as an element in $\mathbf{Z}[x, y, z, w][[t]]$.

To show that the equality holds, one first shows the following lemma:

Lemma 7.7 (Göttsche). *Let* $\Psi \in \mathbf{Z}[x, y, z, w][[t]]$ *be a power series in* t *such that the coefficients of all* t^k *are homogeneous in* x, y, z, w*. Then* $\Psi \equiv 0$ *if and only if the power series takes value zero for all* $(x, y, z, w) \in \mathbf{Z}^4$.

Proof. One side of the lemma is trivial. If $\Psi|_{\mathbf{Z}^4}$ is zero, then by homogeneous property, $\Psi|_{\mathbf{Q}\times\mathbf{Q}\times\mathbf{Q}\times\mathbf{Q}} = 0$, too. As the rational numbers $\mathbf{Q}$ are dense in R, $\Psi|_{\mathbf{R}^4} = 0 \in \mathbf{Z}[[t]]$ identically. q.e.d.

Following Göttsche, one considers the following four different algebraic surfaces together with four different very ample line bundles. Consider $M(i), C_i, 1 \leq i \leq 4$ to be

$$(\mathbf{CP}^2, H), (\mathbf{CP}^1 \times \mathbf{CP}^1, H_1 + H_2), (K3, L_4), (T^4, L_{18}).$$

The four variables x, y, z, w take values

$$v_1 = (3, 9, -3, 1), v_2 = (4, 8, -4, 2), v_3 = (24, 0, 0, 4), v_4 = (0, 0, 0, 18),$$

respectively.

It is easy to see that these four column vectors are linearly independent in $\mathbf{Z}^4$. That is to say, they span a rank four sublattice of $\mathbf{Z}^4$. Let the discriminant of the sublattice be r. Then for any element $\mathbf{z} \in \mathbf{Z}^4$, there exists $c_i, 1 \leq i \leq 4$ such that $r\mathbf{z} = \sum c_i \cdot v_i$. By repeatedly applying the product formula for $\overline{\mathcal{F}}$, one finds that

$$\overline{\mathcal{F}}|_{(x,y,z,w)=\zeta} = \prod \overline{\mathcal{F}}^{c_i/r}|_{(x,y,z,w)=v_i},$$

which is valid for all $\zeta \in \mathbf{Z}^4$. In particular, it applies to the standard basis vectors $(1,0,0,0).(0,1,0,0).(0,0,1,0),(0,0,0,1)$. By comparing the identities after taking suitable powers, one proves that the identify $\mathcal{F}|_{(x,y,z,w)=\mathbf{z}} = \prod_i \left(A_i(\Gamma_0, \mathbf{M}_0) \right)^{z_i}$ is valid for all $\mathbf{z} \in \mathbf{Z}^4$. Then the theorem is proved by considering the power series formed by the difference of the left hand side and the right hand side and by application of the previous lemma. This ends the proof of Theorem 7.5. q.e.d.

By considering $\Gamma_0 \in \text{adm}(1)$, which consists of a single vertex, the factorization holds in this particular case. One can choose the unique multiplicity $\mathbf{M}$ to be any positive integer $m_1 > 1$. This corresponds to ordinary singularities of multiplicity m.

Corollary 7.1. *Take $m_1 = 2$, then the factorization identity holds for the nodal curves counting, and the power series is the weighted powers of the four different series determined in Section 8.*

It was derived by Göttsche [19] in a different formulation, assuming the classes are high power of ample line bundles. By taking m_1 to be other positive integers, one gets equalities of similar types, too.

Remark 7.2. In the $m = 2$ case, the explicit formula for A_3, A_4 have been conjectured explicitly by Göttsche-Yau-Zaslow [63], [19]. It is very interesting to figure out the explicit formulas for general m or even more generally, for all $(\Gamma_0, \mathbf{M}_0)$ pairs.

In the next section, the explicit forms of $A_i(\Gamma_0, \mathbf{M}_0), 1 \leq i \leq 4$ will be determined for $\Gamma = \gamma \in \text{adm}(n)$ and $m_i = 2$; namely, the nodal curve cases.

8. The explicit determination of the universal polynomials

In the previous sections, we have proved the existence of the universal polynomials. The generating functions of these universal polynomials exhibit certain interesting factorization properties that merit further examination.

The explicit determination of these polynomials has been done for $n \leq 6$ by Vainsencher [59], [24] and recently for $n \leq 8$ by Kleiman-Piene

[24], respectively. Their method was based on a direct calculation using the computer program Schubert for $n \leq 8$.

It is my goal to remedy this situation. In this section, I offer an alternative scheme to determine the whole power series. First, I will discuss the major difficulty to work over the algebraic category. Then I try to identify the power series $A_i, 1 \leq i \leq 4$ with the known calculation including the Yau-Zaslow conjecture in the special cases ([5], [6] and later [23]) and the Caporaso-Harris calculation of the Severi degree on $\mathbf{CP}^2$, partially following Göttsche's approach [19].

My method is based on a small variant of Taubes' original proof of $SW = Gr$. The idea of the current attempt was developed during my stay at Park City in the summer of 1997. The equivalence of FSW and FGr in some special cases had been discussed in a joint work with T. J. Li already.

The difficulty of working in the algebraic context is that the regularity and transversality of the appropriate reduced moduli space is difficult to achieve. Sometimes it is even impossible to achieve it within the algebraic context. Despite the fact that algebraic geometers have succeeded in developing intersection theory to deal with the nongeneric situation, it is still quite a challenge when one handles the enumeration questions.

One encounters this problem in the picture of singular curves counting, too. My scheme suggests that the universal polynomials we have determined give rise to the "equivalence" of singular curves. Only when the associated reduced moduli space is regular and is of dimension zero, then a direct enumeration is possible.

Let us restrict ourselves to the case of nodal curves. Given a linear system on an algebraic surface, let us investigate the difficulty of identifying the invariants with the number of nodal curves. Under the ideal assumption that the number of nodal curves in the linear system has been determined by other schemes, there is still a big obstruction to claim directly that the invariant is up to a constant($n!$) equal to the number of nodal curves in the linear system.

Resolving the nodal singularities of the finite number of n-nodes nodal curves on M, they give rise to smooth curves (which may not be irreducible) in $C - \sum 2E_i$ supporting over M_n. There is no doubt that they show up in the enumeration of the modified family invariant $FSW(1, C - \sum 2E_i)$. On the other hand, if one does not impose the type II nice condition and the partially goodness condition on the reduced family moduli space, the smooth curves in the class $C - \sum 2E_i$ usually constitute a proper subset of the objects we count.

Even though one expects that the modified invariants and the the number of smooth curves should be identified, it is quite difficult to argue that the other nonsmooth curves contribute zero to the enumeration problem. It is because no transversality condition has been imposed and there can be cancellations between different contribution of nonsmooth curves.

It turns out that the argument in this section relies heavily upon the techniques developed in the C^∞ category. To remedy the major difficulty in the algebraic category, one needs to extend the view to the almost complex category.

As algebraic surfaces over $\mathbf{C}$ carry integrable complex structures, they can be viewed as almost complex manifolds such that their polarizations give rise to symplectic forms. This point of view has been emphasized by [21], [48] and [51], [52], [53], etc.

Given an almost complex structure j over the differentiable four-manifold M, it makes sense to define the pseudo-holomorphic Mori-cone. Take ω to be an symplectic form tamed by j.

Recall the following definition:

Definition 8.1. Let c be a cohomology class in $H^2(M, \mathbf{Z})$. Then c is said to be effective with respect to j (or simply j-effective) if c is represented by j-pseudo-holomorphic curves in M. The collection of all the j-effective classes in $H^2(M, \mathbf{Z})$ is denoted as $\mathcal{C}_j$ and is called the j-Mori cone of M. Likewise, one defines

$$\mathcal{C}_j^Q = \{x \mid x \in \mathcal{C}_j, \text{energy}(x) = x \cup \omega < Q\}$$

if one fixes a symplectic form or ample polarization on M.

The usual homological Mori (effective) cone is defined in $H_2(M, \mathbf{Z})$. As I often use the cohomology than homology, the j-Mori cone is the dual of the usual one when j is integrable.

By using Sacks-Uhlenbeck-Gromov compactness theorem for pseudo-holomorphic curves, it follows that:

Proposition 8.1. *Let j_∞ be a degeneration of a sequence of almost complex structures $j_n, n \in \mathbf{N}$ with $j_n \to j_\infty$ in the appropriate topology. For all fixed positive numbers Q, there exists a large enough N_0 such that the energy bounded subset in the j_n-Mori cone of M, $\mathcal{C}_{j_n}^Q$, with $n \geq N_0$ are embedded into $\mathcal{C}_{j_\infty}^Q$.*

This proposition asserts that the curve cone in the almost complex category is semi-continuous while taking limits of the almost complex structures.

Proof. If not, one can find a subsequence of j_n such that $\mathcal{C}^Q_{j_n} - \mathcal{C}^Q_{j_\infty}$ are nonempty. Because the energy is uniformly bounded by Q, Sacks-Uhlenbeck-Gromov compactness implies that $\mathcal{C}^Q_{j_n}$ and $\mathcal{C}^Q_{j_\infty}$ are all finite sets. By restricting to a subsequence, one can find a sequence of j_n pseudo-holomorphic curves in a fixed class $\notin \mathcal{C}^Q_{j_\infty}$. But by compactness the class must be represented by a j_∞ pseudo-holomorphic curve, a contradiction. q.e.d.

Similarly, one can define the family Mori-cone for a family of almost complex structures parameterized by a compact manifold B to be the cohomology classes which are j effective for some j among the family. The similar conclusion holds for them, too.

The j_∞ will be taken as a single or an S^2 family of integrable complex structures on either $\mathbf{CP}^2$, T^4 or $K3$, etc. in our application. When I discuss the general algebraic surfaces, I restrict myself to algebraic surfaces with $p_g = 0$.

The main conclusion of this simple proposition is that one can control the j-Mori cones by choosing j sufficiently closed to the integrable complex structures whose j_∞-Mori cones are explicitly known by algebraic geometric means.

The key benefit of working in the almost complex category is that the transversality condition can be met in an easier way, as long as the question of multiple coverings of exceptional curves does not come in to play a role.

It is in the papers of Ruan-Tian [48], [49] that they developed the Gromov-Witten invariants for semi-positive symplectic manifolds. The reader should consult their formulations for the details.

Under the condition of being semi-positive, the compactified components of the Gromov moduli spaces can be better controlled which do not appear in the enumeration of their invariants for dimension reason. One reason that I stick to the earlier formulation of Ruan-Tian is that the general invariants [42], [15], [47], [50], defined by the existence of the virtual fundamental cycles, are $\mathbf{Q}$ valued instead of being $\mathbf{Z}$ valued.

Heuristically, the "nodal curve invariants" should correspond to the Ruan-Tian invariants coupled to gravity [49]. However, one should be cautious, as the Ruan-Tian invariants are not always enumerative. Complicated by the appearance of the multiple covering of exceptional curves

(-1 curve in the usual case and -2 curves in the S^2 family), the invariants were defined under an inhomogeneous perturbation on the right hand side of the Cauchy-Riemann equations [48]. As a result, those high genus invariants do not usually count the number of holomorphic curves, but instead, the signed sum of the zero sets of some perturbed Cauchy-Riemann equations.

To justify the argument, let me remark the following for the semi-positive symplectic manifolds:

Remark 8.1. The Ruan-Tian invariants coupled to gravity can be defined as integer valued invariants using the unperturbed equations when the -1 pseudo-holomorphic spheres are known to be extinct or when the class is not multi-toroidal. Similarly, the same conclusion holds if both -1 and -2 pseudo-holomorphic spheres are known to be extinct on a hyperwinding S^2 family of T^4 (which is true automatically) and $K3$.

By hyperwinding S^2 family of T^4 or $K3$, one means the S^2 family of almost complex structures on T^4 or $K3$ which tame an S^2 family of symplectic forms homotopic to the S^2 family of hyperkähler Kähler forms.

The remark is a straightforward observation from Ruan-Tian theory. Let us remark briefly. According to Ruan-Tian [48], in the case that the source complex structure is fixed (without coupled to topological gravity), the reason that the inhomogeneous terms were used is because of the following two phenomena. First, there are usually multiple covering of exceptional curves. Second, the appearance of ghost bubbling in the high genus case.

In dimension four, the only troublesome multiple coverings are over -1 curves. Their Gromov expected dimension is usually negative, but they exist for generic almost complex structures as multiple covering of pseudo-holomorphic -1 spheres. A similar conclusion holds if one considers an S^2 family of almost complex structures and pseudo-holomorphic -1 and -2 spheres. The condition in the lemma gets rid of them. On the other hand, the Gromov moduli space dimension formula for a smooth genus g pseudo-holomorphic curve is given by $2c_1(M) \cap [A] + 4(1-g)$, where $[A]$ denotes the homology class of the curve. It is crucial that the genus g appears with a negative sign. If the smooth curve degenerates into one with a lower genus, the dimension of the moduli space gets promoted, which appears to contradict to the intuition.

On the other hand, it is well known that the Deligne-Mumford moduli space $\mathcal{M}_g$ (and the Teichmüller space $\mathcal{T}_g, g > 1$) are of real $6g-6$ dimension. If one couples the Ruan-Tian theory to gravity [49] and considers the point class insertion, the expected dimension formula for a smooth genus g curve gets corrected into $2c_1(M) \cup [A] + 2(g-1)$ or $2c_1(M) \cup [A] + 2(g-1) + 2$, depending whether one is considering a single or an S^2 family of almost complex structures.

It is crucial that we are working over a four-manifold, in which case the positivity of this term $g-1$ is insured. In other words, the dimension drops down whenever the genus drops down. This can guarantee the compactified strata are at least of real codimension two as was argued in [48], [49]. The remaining argument goes through as in Ruan-Tian [48], [49]. The transversality of the appropriate moduli spaces are argued as usual [44]. The simple calculation offered here explains why, after coupling the theory to the gravity, the "ghost bubbling" phenomenon is no longer a problem.

Let us consider the following situations:

♣: Either $M = \mathbf{CP}^2$, with $B = pt$ or $M = K3$ or T^4 with an S^2 family of almost complex structures homotopic to the hyperkähler structures(which were named as the hyper-winding family in [29]).

In the last two cases, I have chosen the cohomology class C to be a primitive class with $C^2 \geq 0$.

♣: M is an algebraic surface with $p_g = 0$. In this case, one choose C to be sufficiently very ample.

By choosing the (families of) almost complex structures suitably closed to the integral ones, the previous proposition implies that the energy bounded subsets of the j Mori-cones for these almost complex structures are embedded in the corresponding cones of integral complex structures. It is a special property of hyperkähler complex structures that a class C is effective for one and only one complex structure among the twistor family. It is a simple consequence by some Hodge structure consideration.

Because that $\mathbf{CP}^2$ is known to be minimal (it does not contain any -1 curve), a small perturbation of its almost complex structures surely satisfies the condition in the remark. Pick a complex structure on $K3$ or T^4 such that the Picard lattice $Pic = \mathbf{Z}[C]$. Then one considers a hyperkähler S^2 family containing this particular complex structure. Then choose any hyper-winding family of almost complex structures sufficiently closed to some specific hyperkähler families. The existence of this type of hyperkähler families follows from Yau's solution of Calabi conjecture [62].

If a pseudo-holomorphic curve in $[C]$ degenerates, then it is easy to argue that there can be no -2 curves in its connected components.

By the primitiveness assumption of C, and the fact that no other class is effective near the support of C, the family Gromov moduli spaces of all such C are "compact" in the sense that all the pseudo-holomorphic curves are irreducible. In particular, the class is not multiple-toroidal.

In case where M is an algebraic surface with $p_g = 0$, there is some additional assumption on C, as was pointed out earlier. If C is sufficiently very ample, Göttsche's argument implies that any singular curve with a fixed number of singularities (fixing the multiplicities) does not appear in a generic linear system with a dimension lower than its expected one. In particular, it implies that the holomorphic curve in the reduced moduli space must be irreducible and reduced. One can choose the almost complex structure sufficiently closed to the integral one to preserve the property. In particular, the bubbling of exceptional -1 curves does not cause trouble, as these curves carry an infinite number of singularities.

After this preparation, one is ready to prove that the $n!$ times the number of nodal curves in the family can be identified with the modified family Seiberg-Witten invariants of the class $C-\sum_{i\leq n} 2E_i$, for the pairs (M, C), discussed above.

Mimicking the construction in [29], one considers the space $B = S^2 \times M_n$ in considering the hyper-winding family of almost complex structures. The symbol $FSW^*_B(1, C - \sum 2E_i)$ will be often called the nodal invariants.

After perturbing from complex to the almost complex category, the family Seiberg-Witten invariants no longer have a direct contact with Gromov theory. To remedy this, one needs the gluing argument of Taubes which was an important part of his fundamental work "SW=Gr."

To discuss the direct link between family Seiberg-Witten invariants and the family Gromov theory, a version of gluing argument stronger than the one contained in Taubes' paper will be necessary. While it is not my intent to develop the argument in the the present paper, I adopt a "detour" to suit for my derivation. I plan to discuss the general case in the future.

Thus, one needs to insure that the type II multiple coverings do not contribute to the modified invariants. Or the counting scheme would be modified again, and it would depend on the manifolds as well as the cohomology classes involved explicitly.

In the cases of hyper-winding family of $K3$ or T^4, the assumption on the cohomology class guarantees that the pseudo-holomorphic curves must be irreducible. The fact that the j-Mori cone is locally of at most rank one near the support of C insures that the multiple coverings of type II exceptional curves can not show up in the family over $S^2 \times M_n$ for these types of class C.

On the other hand, for $\mathbf{CP}^2$, one is not in such a nice situation. The type II exceptional curves frequently come in to play a role in the universal formula. The contribution of the type II exceptional curves to the invariants is a rather subtle issue. Later it will be addressed briefly.

Conjecture 8.1 (Di Francesco-Itzykson). *Take the algebraic surface M to be $\mathbf{CP}^2$, then the polynomial determined in the main theorem is equal to $n!\times$ the number of n nodes nodal curves in $|dH|$ if $n \leq 2d-2$.*

The conjecture was formulated as a sub-conjecture in Göttsche's paper [19], in which he mentioned that the conjecture was already known to P. Di Francesco and C. Itzykson [8].

I would like to point out that I do not plan to prove the conjecture directly in the algebraic category, which will involve resolving the non-transversality issue of the reduced moduli space. The issue may be out of reach using the current technique. Instead, one shows that the algebraic counting of the nodal curves are in fact symplectic invariants, and they coincide with the Ruan-Tian invariants with appropriate choice of genus. Then one applies Taubes' gluing technique to prove the equivalence between the two invariants. One should notice that I restrict myself to the embedded curves cases without considering the multiple tori, in which case the identification between these two invariants is possible, yet not direct (e.g., in [23]).

Let us summarize the cohomological conditions we put upon the classes in $\mathbf{CP}^2$, $K3$, or T^4. When $M = \mathbf{CP}^2$, one assumes the classes dH to satisfy $d \geq \frac{n+2}{2}$, following Di Francesco-Itzykson. If the algebraic surface is either T^4 or $K3$, one assumes the cohomology classes to be primitive with nonnegative self-intersections.

Proposition 8.2. *Under the previous assumption, the multiple coverings of the type* II *exceptional curves do not show up in the reduced moduli space over M_n if one chooses the (families of) almost complex structures to be sufficiently closed to the chosen integral complex structures.*

Proof of Proposition 8.2. The cases of $K3$ and T^4 have been discussed already. In considering $\mathbf{CP}^2$, one supposes that the conclusion was wrong. Then one would be able to generate by Sacks-Uhlenbeck-Gromov compactness theorem a holomorphic multiple covering type II exceptional curve over the complex manifold M_n.

As it is a type II exceptional curve, it follows that the type II curve is represented by $d_1H - \sum n_iE_i$ such that $m \cdot d_1 \leq d$. The integer m stands for its multiplicity.

The difficulty of dealing with the type II exceptional curves is that they do not show up according to their expected dimensions. It is subtle to bound their actual dimensions or to describe their deformation tangent obstruction complexes.

On the other hand, the reduced moduli space is constructed by requiring the holomorphic curve to pass through $\frac{d^2+3d}{2} - n$ different generic points (sections). In general, it is rather hard to classify the explicit form of all the type II exceptional curves. I leave the systematic study of the type II exceptional curve to a separated article. To argue their disappearance in the reduced moduli space, there are two different approaches; one can either argue geometrically that they do not show up, or one can argue homologically that the appropriate moduli space contributes trivially to the family invariants applying the concept of admissible decomposition classes involving the type II curves. In this paper, I adopt the direct geometric approach as it is elementary. In short, one argues by dimension count and by contradiction.

Suppose there is a configuration involving multiple coverings of type II exceptional curves, then it implies that the class dH can be decomposed into at least two components, $d = \sum m_id_i$ with at least one $m_i \geq 2$. On the other hand, a holomorphic curve of the form $d_jH - \sum n_iE_i$ projects to a possibly singular curve in the class d_jH on $\mathbf{CP}^2$. Denote $f(x) = \frac{x^2+3x}{2}$ to simplify the notation.

It is well known that the moduli space (or linear system) of holomorphic curves representing d_jH is of $f(d_j)$ dimensional. Then one can estimate the actual dimension of this configuration and bound it by $\sum f(d_j)$ from above. In the inequality, one has ignored the singularities which may develop under the contraction map, so the inequality is usually not sharp.

On the other hand, it is easy to see that

$$f(d) = f(\sum m_i d_i) = \sum f(d_i) + \sum_i \frac{(m_i^2 - 1)}{2} d_i^2 + \sum_i \frac{3}{2}(m_i - 1)d_i + \sum_{a \neq b} m_a \cdot m_b d_a \cdot d_b.$$

The existence of the particular configuration implies that $\sum f(d_i) \geq f(d) - n = \frac{d^2+3d}{2} - n$. It follows that

$$\sum_i \frac{(m_i^2 - 1)}{2} d_i^2 + \sum_i \frac{3}{2}(m_i - 1)d_i + \sum_{a \neq b} m_a \cdot m_b d_a \cdot d_b \leq n.$$

To minimize this number, one shows that there is a way to reduce this number if the decomposition has more than two components. In fact, one can reduce the number of d_j by fusing two into one. Suppose d_a and d_{a+1} are the two elements under fusion. Then define $d'_{a+1} = 0$ and $d'_a = m_a \cdot d_a + m_{a+1} \cdot d_{a+1}$ with $m'_a = 1$. This process does not affect the crossing terms with the other classes, yet the terms involving multiplicities m_i and the crossing term between d_a and d_{a+1} disappear. Thus, the previous expression is monotonously decreasing under this process. Finally, one can reduce to the two d_i case. By replacing d_1 by $m_1 d_1$ and d_2 by $m_2 d_2$ with new multiplicity changed to one, the expression still drops down. Then it becomes an elementary exercise to check that all the possibilities violate the previous inequality as far as $n \leq 2d - 2$. q.e.d.

In the proof, the condition $n \leq 2d - 2$ guarantees that the type II multiple covering does not show up in the "reduced" family moduli space. In other words, they do show up in the family moduli space in a lower dimension. The language of type II multiple-coverings cast the original Di Francesco-Itzykson conjecture into a theoretical framework. Not interrupted by the appearance of these type II curves, the enumeration of the nodal curves in the holomorphic category is still blocked by the fact that the other singular curves in $C - \sum 2E_i$ over Y_γ may not behave according to their expected family dimension. There is no real cure about this problem in an easy way. That is why I need to expand to the almost complex category which provides us a larger room to maneuver.

Recall the construction of the universal space M_n which was reviewed earlier. Given an almost complex structure (or an S^2 family of them) on M, the almost complex structure on M_{n+1} is canonically constructed. In particular, $M_{n+1} \mapsto M_n$ is a pseudo-holomorphic fibration. The fibers inherit the almost complex structures from the fibration structure. The construction of M_n in the almost complex category set up a map from the space of almost complex structures of M to the almost complex fibration $M_{n+1} \mapsto M_n$. A perturbation of the almost complex structure on the underlying four-manifold induces a corresponding perturbation on the pseudo-holomorphic fibration $M_{n+1} \mapsto M_n$.

To begin the argument, I first show,

Proposition 8.3. *Let $\mathcal{U}$ be a small neighborhood of the specific integrable complex structures of $M = \mathbf{CP}^2$ (or $M = K3, T^4$, or a $p_g = 0$ algebraic surface) in the space of almost complex structures (or S^2 families of almost complex structures) of M.*

There exists a Baire second category subset in $\mathcal{U}$ of almost complex structures of M such that the restriction of the reduced family Gromov-Taubes moduli space of a class over Y_γ (or $S^2 \times Y_\gamma$) consists of a finite number of smooth points which represent smooth curves supporting over Y_γ.

The reader must be warned that the statement is not true over the whole M_n due to the appearance of the type I exceptional curves, etc.

Proof. It is well known that by perturbing the almost complex structures, one can make sure that the moduli space of pseudo-holomorphic curves of a given type is smooth of a correct dimension. The exception is due to either multiple covering of type I and type II curves. When multiple coverings of either type of exceptional curves are among the irreducible components, they contribute negatively to the family dimension. As a result, the good part's family dimension gets enhanced and sometimes exceeds the expected family dimension of the original curve.

The type I curves do not show up on Y_γ, the open top stratum of M_n, and the type II curves are forbidden to show up either by dimension reason (for $\mathbf{CP}^2$) or by primitiveness assumption on the cohomology class (or for $K3$ and T^4) or by the sufficiently very ample assumption of C using Göttsche's argument.

Then the restriction of the family Gromov-Taubes moduli space over Y_γ gives rise to a partially compactified stratified space of right dimension as far as the almost complex structures are chosen to be generic in $\mathcal{U}$. It is crucial to notice that the space is not compact as one has removed the portion over $\cup_{\Gamma<\gamma} Y_\Gamma$.

The standard theory (e.g., [48], [49]) gives a cobordism between different family moduli spaces if one deforms the almost complex structures in a generic real one-parameter family [48], [44].

However, the un-compactness of these restricted family moduli space makes the usual cobordism argument useless. Namely, it does not follow from the technique that the numbers associated with the reduced restricted family moduli spaces are actual invariants. Even if one makes the reduced family Gromov-Taubes moduli space transversal over Y_γ, the noncompactness nature of the space forbids us to conclude that there is a finite number of solutions on the general ground. It is because the solutions may leak to the boundary due to un-compactness.

The surprising thing is that they really do. The proof is actually beyond the scope of the apparently useless argument and it involves use of the Ruan-Tian invariants [49].

One should notice that we are not free to perturb the almost complex structures of the family $M_{n+1} \mapsto M_n$ in an arbitrarily generic way, or it loses the geometric meaning, being the universal fibration of M. Instead, one is only able to perturb the almost complex structures of the fibration $M_{n+1} \mapsto M_n$ that are induced from the corresponding perturbation of almost complex structures on M. Apparently, one cannot expect the smoothness result under these non-sufficiently generic perturbations. The crucial observation is that one can contract the exceptional curves and the pseudo-holomorphic curves supporting over Y_γ are contracted into pseudo-holomorphic curves in M with at least n distinct singularities. If the reduced family moduli space over Y_γ contains nonsmooth curves, then the corresponding curves in M would either have more than n singularities or with exactly n singularities (some of those are worse than the nodes). In either case, it is easy to estimate their dimension, and they are strictly smaller than the expected Gromov dimension of the nodal curves. Thus, they do not show up in the top dimensional stratum under the generic perturbation of almost complex structures of M.

As a consequence, the generic reduced family Gromov-Taubes moduli space over Y_γ does not contain singular curves. q.e.d.

The previous argument sets up a bijective correspondence of smooth curves supporting over Y_γ and n-nodes nodal singular curves in M. As was explained before, the appearance of G_γ introduces a symmetric factor $n!$, which is hardly relevant to our discussion. In the following, we will vaguely say that the number of curves in the restriction of family Gromov-Taubes moduli space is equal to the number of n-nodes pseudo-holomorphic nodal curves in M.

To make contact with the Ruan-Tian theory [49], one notices that Ruan-Tian invariants (without the inhomogeneous perturbation) count the number of genus g curves and its stable degenerations. In general, these two objects are not identical. In the case of $K3$ and T^4, the assumption that the classes are primitive force the pseudo-holomorphic curves to be irreducible. On the other hand, one has the following simple but crucial observation, which have probably been noticed by other people previously:

Lemma 8.1. *An embedded irreducible genus g pseudo-holomorphic curve with respect to a generic almost complex structure* (*generic in the sense of being in certain Baire second category set*) *develops nodal singularities in M.*

Using this, one proves that, generically, there are only a finite number of curves in the reduced moduli space over Y_γ, as a consequence of the fact that Ruan-Tian invariants are well defined and are finite.

Proof. The lemma is well known to the experts. It comes from comparing the dimension of Gromov-Witten invariants and that of embedded singular curves in a smooth symplectic four-manifold. Let us mention briefly that irreducible curves in C with singularities of multiplicity $m_i, i \leq n$ has an expected complex dimension $\frac{C^2 - K_M \cdot C}{2} - \sum(\frac{m_i(m_i+1)}{2} - 2)$ with $B = pt$. On the other hand, the genus g curve has an expected Ruan-Tian dimension $-K_M \cdot C + (g-1)$ (the genus 0 case can be discussed similarly).

On the other hand, the adjunction formula of embedded genus g curves implies that the geometric genus and the arithmetic genus are related by $2g - 2 + \sum_i \frac{m_i^2 - m_i}{2} = C^2 + C \cdot K_M$.

By comparing the two formulas one gets $\sum_i (m_i - 2) = 0$. As each m_i is assumed to be bigger than 1, the sum can be zero only when $m_i = 2$ for all i. Then the conclusion follows from the fact that, under a generic perturbation, the actual dimensions of the relevant moduli spaces obey the dimension formulas. q.e.d.

On the other hand, the previous discussion has not touched the issue that the modified family invariants should calculate the number of irreducible as well as reducible nodal curves. In fact, if A and B are represented by irreducible nodal curves, respectively, of n and m nodes.

Then the total curve $A+B$ in the relative generic position should carry $m+n+A\cdot B$ nodes.

Keeping this in mind, it follows that one should have the following relationship between the numbers of reducible and irreducible nodal curves. Even though I only need the version for $\mathbf{CP}^2$, I discuss it in the general setting, as the remark is rather important in understanding the relationship between our modified invariants with Ruan-Tian type invariants [49].

The adjunction equality in dimension four predicts a special genus that an embedded pseudo-holomorphic curve should have. One defines the corresponding Gromov invariant to be $Gr(C)$. In general, the symbol $Gr_i(C)$ denotes the Gromov-Witten-Ruan-Tian invariant whose source curve is irreducible of genus $g-i$ (or its various bubbling configurations). Either by dimension reason or by the restriction $i\leq g$, the symbol will represent zero for large enough i.

Define for each cohomology class C the following generating function (cf. [23], [18]):

$$F(C)=\sum_{i\geq 0}\frac{1}{\mathrm{d}_{\mathbf{C}}(C)-i!}Gr_i(C)q^{\mathrm{d}_{\mathbf{C}}(C)-i}.$$

The symbol $Gr_i(C)$ denotes the genus $g-i$ Gromov-Ruan-Tian invariants coupled to gravity. One adopts $\mathrm{d}_{\mathbf{C}}(C)-i$ to parameterize the sequence of invariants because of two reasons: Firstly, it is the dimension of the irreducible genus $g-i$ curve; Secondly, if $C=C_a+C_b$, then $\mathrm{d}_{\mathbf{C}}(C)-i_a-i_b-C_a\cdot C_b=(\mathrm{d}_{\mathbf{C}}(C_a)-i_a)+(\mathrm{d}_{\mathbf{C}}(C_b)-i_b)$. Namely, it is additive under taking unions.

Given an almost complex structure, consider the j-Mori (effective) cone. Given a class C in the cone, a cohomological decomposition is said to be permissible if $C=\sum C_i$, with all the C_i in the j-Mori cone, and $C_i\cdot C_j\geq 0$. Let us denote the permissible decompositions of C by $\mathrm{Perm}(C)$ (please do not confuse it with the admissible decompositions). Notice that I do not exclude the possibility that $C_i=C_j, i\neq j$. When this happens, the previous inequality forces that $C_i^2=C_j^2\geq 0$.

Let us consider the following compound Ruan-Tian power series

$$F_{RT}(q)=\sum_{(C_i)\in\mathrm{Perm}(C)}\prod F(C_i).$$

The power series encodes the number of nodal curves in C while the total number of nodes is recorded as the difference between the exponent of q and $\mathrm{d}_{\mathbf{C}}(C)$. One should notice that $F_{RT}(q)$ is a polynomial of degree $\mathrm{d}_{\mathbf{C}}(C)$, with the leading term being $Gr(C)$, in the sense of Taubes.

A priori, one may suspect that it should be equal to the corresponding normalized generating function $\mathcal{F}_{nor}(q)$ of FSW^*.

$$\mathcal{F}_{nor}(q) = \sum_{\delta} \frac{1}{(\mathrm{d}_{\mathbf{C}}(C)-\delta)!} \frac{FSW^*(C - 2\sum_{i=1}^{\delta} E_i)}{\delta!} q^{\mathrm{d}_{\mathbf{C}}(C)-\delta}.$$

It is indeed correct, as far as the type II exceptional curves do not show up.

It is very interesting to study $\mathcal{F}_{nor}(q) - F_{RT}(q)$, which we hope to discuss in a separated article.

Translated into this formation, the Di Francesco-Itzykson conjecture asserts that the difference, denoted as $\Delta_{\mathrm{II}}(q)$, is a polynomial of degree at most $\frac{d^2-d}{2}+1$, for dH over $\mathbf{CP}^2$.

To identify the two different invariants, one needs to adopt Taubes' gluing theorem "$Gr \mapsto SW$" [53]. However, I would like to make an efficient usage of the fact that I am working on a slight perturbation of Kähler surfaces. To bypass the gluing of the curves involving the type I multiple coverings, I adopt the gluing theory over Y_γ for smooth curves.

Let ω denote the family of symplectic two forms, which is the slight perturbation of the fiberwise Kähler forms on $M_{n+1} \mapsto M_n$.

As I had argued, one first chooses a generic j of M such that all the moduli spaces of j-pseudo holomorphic curves in M are of their expected dimensions. After it is done, one considers the corresponding Vainsencher fibration $M_{n+1} \mapsto M_n$.

One divides the space M_n into three regions: $M_n = Y_\gamma \cup (M_n - Y_\gamma - \mathcal{S}) \cup \mathcal{S}$. The space $\mathcal{S}$ has been defined on page 508 to be the set over which at least one type I class pairs negatively with $C - \mathbf{M}(E)E$. By choosing the intersecting cross-sections to be generic, by Proposition 8.3 one has known that the reduced family Gromov-Taubes moduli space over Y_γ consists of a finite number of smooth curves (notice that the finiteness is not a trivial issue and is the consequence of the Ruan-Tian theory). By the blowing down projection, they give rise to n-nodes nodal curves we would like to count. Over the portion $\mathcal{S}$ the reduced Gromov-Taubes moduli space behaves badly, and it may contain a continuous family of curves along with a certain multiple of type I curves.

I would like to argue that the reduced family moduli space over $(M_n - Y_\gamma - \mathcal{S})$ can be perturbed to be empty.

First, it follows from the choices of the almost complex structures and the class C that the curves in this portion of the reduced family Gromov-Taubes moduli space are reduced complex curves. In other words, no multiple coverings of type II curves are allowed to show up.

Lemma 8.2. *Under the previous convention of (M, C) and the choices of j, the portion of the reduced family Gromov-Taubes moduli space over $(M_n - Y_\gamma - \mathcal{S})$ can be chosen to be empty.*

Proof. If not, there would be a curve in $C - \sum 2E_i$ which passes through the d generic sections. The number d stands for $\mathrm{d}_{\mathbf{C}}(C) - n$, the expected family dimension of $C - \sum 2E_i$. The curve can be used to define a decomposition of cohomology classes. Then it follows that either the decomposition is allowable or it can be not allowable, yet the actual moduli space exceeds the expected dimension.

As was calculated in the note [37] by following Taubes' argument, if a decomposition over $(M_n - Y_\gamma - \mathcal{S})$ is allowable, then it must contain some multiple coverings of type II curves and the type II curves must have a negative pairing with the total class. If this happens, the projection of the total curve would be nonreduced, a contradiction to the choice of C and j. If the decomposition is not allowable, yet its family dimension jumps up, then its projection to M would give a j pseudo-holomorphic curve with the prescribed singularity multiplicities such that its actual dimension exceeds its expected dimension. This violates the choice of j and is impossible. q.e.d.

Following the machinery of Taubes [51], one considers the family Seiberg-Witten equations perturbed by $-r\omega, r \to \infty$. It is crucial at this moment that the family theory only uses the fiber-wise vertical almost complex structures. As a result, Taubes' theorem implies that the large r limit of the solutions will converge to pseudo-holomorphic curves supporting over M_n. As I have shown, the j version of the reduced family Gromov-Taubes moduli space does not support over $M_n - Y_\gamma - \mathcal{S}$; namely the pseudo-holomorphic curves are lying over Y_γ and $\mathcal{S}$. The choice of generic sections and almost complex structures also guarantee that the reduced Gromov-Taubes moduli space over Y_γ consists of a finite number of smooth points (see Proposition 8.3).

Taubes' theorem [51], [52], [53] implies that there is a bijective correspondence between the reduced family Seiberg-Witten moduli space over Y_γ and the reduced family Gromov-Taubes moduli space over Y_γ such that the germs of these two spaces are homeomorphic to each other.

By the reduced family Seiberg-Witten moduli space it means the following: Suppose that d denotes the expected family complex dimension of the specific $spin_c$ structure. Take $s_1, s_2, \ldots, s_d$ to be the d number of sections of the fiber bundle $M_{n+1} \mapsto M_n$. The fiber-wise homotopy class of the sections is specified as was discussed in [29]. As a Seiberg-Witten solution consists of the equivalence class of pairs (A, Ψ, b), $b \in M_n$. The reduced family Seiberg-Witten moduli space collects all the pairs such that the restriction of Ψ to the d points $s_1(b), s_2(b), \ldots, s_d(b)$ vanish.

As Taubes had written down the identification in fully detail, the reader can directly consult [51], [52], [53] for details. His proof works for the smooth curves gluing in the family case with essentially no major change.

The subtle part that the current gluing theorem [53] does not analyze is the family moduli space over $\mathcal{S}$.

It is a well known fact in Kähler Seiberg-Witten theory that the deformation of Kähler forms on a Kähler surface Z does not change the zero loci of the holomorphic spinors. Instead, the number r that appears in Taubes' analysis only changes the scale of a spinor by solving the Kazdan-Warner equation. More explicitly, given a holomorphic section s, one needs to solve $f : Z \mapsto \mathbf{R}$ [13] in the equation such that

$$\Delta(f) - e^f \cdot \frac{|s|^2}{4} - \frac{1}{2} * (iF_A^+ \wedge \omega) = r\omega \wedge \omega.$$

Here $e^f \cdot s$ is viewed as a nonunitary gauge transformation acting on s. The connection A is the background connection.

The explicit form of f is irrelevant to us. Due to the observation, it follows that the Kähler family moduli spaces for different r can be canonically identified.

Proposition 8.4. *Let $\mathcal{M}^{r_1}_{K\ddot{a}hler}$ and $\mathcal{M}^{r_2}_{K\ddot{a}hler}$ be the Kähler family Seiberg-Witten moduli spaces over $X = Y(\gamma)$. Then there exists a canonical isomorphism $\phi_{r_1,r_2} : \mathcal{M}^{r_1}_{K\ddot{a}hler} \mapsto \mathcal{M}^{r_2}_{K\ddot{a}hler}$ between them such that $\phi_{r_2,r_3} \circ \phi_{r_1,r_2} = \phi_{r_1,r_3}$. Through these identifications, the tangent obstruction complex of the moduli spaces are identified, too.*

Proof. The argument has been given in [13] when $B = pt$. The argument of the family version is identical. q.e.d.

This proposition is the mathematical statement which is mainly responsible for $SW = Gr$ in the Kähler category and is well known to the experts. R. Friedman and J. Morgan [13] studied their identification as analytic spaces when the irregularity $q = 0$ and $B = pt$. In other words, the dual picture of holomorphic curves counting and the Seiberg-Witten invariants enumeration is manifest over Kähler surfaces. Thus, one should view the large r version of the family Kähler Seiberg-Witten moduli space over X as the nongeneric family moduli space stabilized under the identifications.

Even though one is not able to count the holomorphic curves over $\mathcal{S}$ as our intuition might have suggested, the Kuranishi model technique (e.g., [53] or [38]) readily defines some integer associated to the apparently nonsmooth moduli object, which can be further analyzed by using the concept of admissible decompositions , family blowup formula and the family switching formula, etc.

Let us continue the previous argument. Consider the version of the family Seiberg-Witten equations perturbed by the $-r\omega$ with r large. Let us take a closer look at the reduced family Seiberg-Witten moduli space over M_n. First, Taubes' theorem $SW \mapsto Gr$ implies that the $r \mapsto \infty$ limits of the solutions (after suitable rescaling by r) gives rise to some pseudo-holomorphic curves passing through the the same tuples of defining sections. On the other hand, we have prepared the family of almost complex structures in a generic way such that the curves support over Y_γ and $\mathcal{S}$. There are a finite number of smooth curves supporting over Y_γ. As the reduced Gromov-Taubes moduli space over Y_γ is $G_\gamma = \mathbf{S}_n$ equivariant, the signed number is divisible by $n!$, and its quotient by $n!$ gives the number of n-nodes nodal curves in C. The large r version of the reduced family Seiberg-Witten moduli space restricted over Y_γ is homeomorphic to the reduced Gromov-Taubes moduli space over Y_γ as a consequence of Taubes' gluing theorem for smooth curves.

Next, let us move to the reduced family moduli space over $\mathcal{S}$, as it is another source which contributes to the family invariant in a nontrivial way. Their invariant contribution can be calculated by using some family Kuranishi model and the switching formula, etc.

As the previous proposition has identified the Kähler family moduli spaces for different r. The diffeomorphisms ϕ_{r_1,r_2} also identify local neighborhoods of the r_1 and r_2 versions of the reduced Kähler Seiberg-Witten moduli spaces over $\mathcal{S}$, denoted by $\mathcal{M}^{r_1}_{K\ddot{a}hler,red} \cap \pi^{-1}(\partial Y_\gamma)$ and $\mathcal{M}^{r_2}_{K\ddot{a}hler,red} \cap \pi^{-1}(\partial Y_\gamma)$, respectively. Thus, one can identify their family Kuranishi models and construct a universal family Kuranishi model working for all large enough r. The previous identification of Kuranishi models automatically normalizes the fact that the solution spinors and the Kähler forms expand with the scale $O(\sqrt{r})$ and $O(r)$ with the r varying.

Let $\mathcal{M}_{red}(\omega, r, Z)$ denote the $r \cdot \omega$ perturbed version of the reduced Seiberg-Witten family moduli space over the set $Z \subset M_n$. When one takes $Z = M_n$, one abbreviates it as $\mathcal{M}_{red}(\omega, r)$.

The primary concern is to study $\mathcal{M}_{red}(\omega, r)$ and its relationship with the Kähler family Seiberg-Witten moduli space. It follows from Taubes' "$SW \mapsto Gr$" that the solutions of the $r\omega$ perturbed($r \mapsto \infty$) Seiberg-Witten equations creates pseudo-holomorphic curves. On the other hand, the reduced family Gromov-Taubes moduli space for the specific family of almost complex structures support over $Y_\gamma \cup \mathcal{S}$. Moreover, its support over Y_γ is a finite number of points p_i invariant under the $\mathbf{S}_n$ action.

Thus, one concludes that if one chooses the real number r to be large enough, the image of $\pi : \mathcal{M}_{red}(\omega, r) \mapsto M_n$ lies in $\mathcal{B} \cup \mathcal{O}$. The set $\mathcal{B}, \mathcal{O}$ with $\mathcal{B} \cap \mathcal{O} = \emptyset$ denote the union of ϵ balls $\cup_i \mathbf{B}(p_i, \epsilon)$ and the ϵ neighborhood of $\mathcal{S}$, $\mathcal{N}(\mathcal{S}, \epsilon)$, respectively. In the mean time we have implicitly chosen a Riemannian metric over M_n.

Even though the compact sets $\mathcal{M}_{red}(\omega, r, \mathcal{B}) = \mathcal{M}_{red}(\omega, r) \times_{Y(\gamma)} \mathcal{B}$ and $\mathcal{M}_{red}(\omega, r, \mathcal{O}) = \mathcal{M}_{red}(\omega, r) \times_{Y(\gamma)} \mathcal{O}$ may not consist of a finite number of points, the standard technique of family Kuranishi model still allows us to read off the invariant contribution through perturbation argument [53]. In particular, it follows from Taubes' argument that the invariant contribution over $\mathcal{M}_{red}(\omega, r, \mathcal{B})$ is identified with the counting of smooth pseudo-holomorphic curves in the class $C - 2\sum E_j$.

The next goal is to compare $\mathcal{M}_{red}(\omega, r, \mathcal{O})$ with $\pi^{-1}(\mathcal{S}) \cap \mathcal{M}^r_{Kähler,red}$.

The nested Kuranishi model and the family switching formula, etc. have been used to read off the invariant contribution from $\pi^{-1}(\mathcal{S}) \cap \mathcal{M}^r_{Kähler,red}$. The next proposition identifies this invariant contribution with the one from $\mathcal{M}_{red}(\omega, r, \mathcal{O})$.

Proposition 8.5. *Let r be a large enough real number and let $n_{\mathcal{O}} \in \mathbf{Z}$ and $n_{\mathcal{S}} \in \mathbf{Z}$ be the family invariants associated with $\mathcal{M}_{red}(\omega, r, \mathcal{O})$ and $\pi^{-1}(\mathcal{S}) \cap \mathcal{M}^r_{Kähler,red}$, respectively. Then $n_{\mathcal{O}} = n_{\mathcal{S}}$.*

If the reduced Kähler family Seiberg-Witten moduli space has been sufficiently good, the proof of the proposition would be quite trivial. In the general situation that no additional assumption has been made upon $\mathcal{M}^r_{Kähler,red}$, I need to apply the technique I developed to separate the excess contributions from the residual contributions.

Proof. In the proof of the main theorem, I argued by induction over the levels of the admissible decomposition classes that the Kähler family Seiberg-Witten moduli space can be perturbed coherently such that the different branches associated with the different admissible decompositions are disjoint from each other.

The benefit of working with the perturbed objects rather than the original Kähler moduli spaces dramatically simplifies the identification. I view the perturbed version family moduli space as the reference space. On the other hand, the large r version of the ω deformed family Seiberg-Witten moduli space can still be viewed as a family Kuranishi model of the former.

Let us follow the previous convention and name the closure of the family Seiberg-Witten moduli space over Y_γ to be the dominant branch. The dominant branch of the Kähler family moduli space may support upon $\mathcal{S}$ as well. It is hard to distinguish its contribution to the family invariants from those from the branches lying completely over $\mathcal{S}$. In the perturbed version, one needs to make sure that the dominant branch supports over a compact set $\subset Y(\gamma)$ totally disjoint from $\mathcal{S}$. Once it is achieved by the specific perturbation, the dominant branch and the nongeneric branches do not intersect each other.

In this way, one can view the large $r\omega$ deformed reduced family moduli space as an alternative family Kuranishi model of the perturbed dominant branch.

As was discussed, the perturbations are performed after the base manifold $Y(\gamma)$ is blown up repeatedly into a sequence of birational manifolds, finally into $\widetilde{Y(\gamma)}$. However, I can always pull back the large $r\omega$-deformed reduced family moduli space to $\widetilde{Y(\gamma)}$ and compare them as family Kuranishi models over $\widetilde{Y(\gamma)}$. As $\widetilde{Y(\gamma)} \mapsto Y(\gamma)$ is isomorphic over Y_γ, the $r\omega$-deformed reduced family moduli space is not altered under the pull back processes as long as I have chosen the almost complex structure generic enough such that the finite number of pseudo-holomorphic curves all lie above Y_γ.

Both family Kuranishi models are of dimension zero. Either of them can be interpreted as small perturbations of the dominant branch of the Kähler family moduli spaces. As the family Kuranishi model defines a unique invariant, the invariant defined by these two different models must coincide. q.e.d.

Because the invariant attached to the former is equal to $n!$ times the number of j pseudo-holomorphic curves counted by the Gromov-Ruan-Tian invariant, finally one has identified up to an $n!$ multiple the number of n-nodes pseudo-holomorphic curves representing C with the family Seiberg-Witten invariant contribution over Y_γ. As the latter object has been defined to be the modified invariant, as is expressible by a universal formula in $C^2, C \cdot K_M, K_M^2$ and $\chi(M)$, one has the following theorem:

Theorem 8.1. *Let C be one of the following:*

(1) *A cohomology class with $C^2 > 0$ on $\mathbf{CP}^2$ satisfying the condition $C = dH$, $d > \frac{n+2}{2}$;*

(2) *A primitive class on $K3$ or T^4;*

(3) *A sufficiently very ample class in a $p_g = 0$ algebraic surface.*

Under the generic choices of (an S^2 family of) almost complex structures in a small neighborhood of the specified integrable complex structures, the modified family invariants FSW^* of $C - \sum 2E_i$ are identified with (up to a $n!$ factor) the $(\mathrm{d}_{\mathbf{C}} - n)$-th power coefficient of the power series $F_{RT}(q)$.

Notice that one imposes the condition $C^2 > 0$ to avoid discussing the multiple toroidal classes, which are the possible sources of the type II exceptional curves. The previous identification still works as far as the following three conditions hold.

(1′) The multiple coverings of type II exceptional curves do not show up in the generic "reduced" moduli space over the M_n family.

(2′) The permissible decomposition of C, $\mathrm{Perm}(C)$ does not contain any multiple covering of exceptional curves.

(3′) The multiple toroidal classes are excluded as they are, from my point of view, curves with infinite numbers of singularities.

The coefficients $F_{RT}(q)$ in the special case (2) is already calculated by Bryan-Leung [5], [6] and later by Parker-Ionel [23] as simple modular forms and in (3) by Harris-Caporaso [7] by some beautiful recursive formulas.

As a corollary of the identification, one has the following corollary, which is known as the Göttsche-Yau-Zaslow conjecture.

Theorem 8.2. *Let M be a $K3$ surface. Suppose that C is an integral cohomology class in* $\mathrm{Pic}(M) \subset H^2(M, \mathbf{Z})$ *of square $C^2 \geq 0$, with the additional primitive assumption when $C^2 = 0$. Let $N_n(C^2)$ be "the virtual number" of n nodes nodal curves" in the class C with square C^2. Consider the generating function*

$$F_k(q)^{K3} = \sum_{r \geq 2k-2} N_{\frac{r}{2}-k+1}(r) q^r.$$

Then it is a quasi modular form, given by $(DG_2)^k \cdot \frac{q}{\Delta(q)}$, where

$$\Delta(q) = q \prod_s (1 - q^s)^{24} = \eta(q)^{24}$$

is the modular form of weight 12 of $SL_2(\mathbf{Z})$.

And similarly, for T^4:

Theorem 8.3. *Let $M = T^4$ be an abelian surface. Suppose that C is an integral cohomology class in* $\mathrm{Pic}(T^4) \subset H^2(M, \mathbf{Z})$ *of square $C^2 \geq 0$, with the additional primitive assumption when $C^2 = 0$. Let $N_n(C^2)$ be the "virtual number" of n-nodal curves in the class C and consider the generating function*

$$F_k(q)^{T^4} = \sum_{r \geq 2k-2} N_{\frac{r}{2}-k+1}(r) q^{\frac{r}{2}}.$$

Then it is a quasi modular form, given by $(DG_2)^k \cdot D^2 G_2(q)$.

The Gromov-Witten analogue of the previous theorems were known previously in the special cases of "primitive cohomology classes". It appeared in the papers of Bryan-Leung [5], [6] and Göttsche [19].

The same theorem also shows that the four different power series can be explicitly identified in the situation that the curves are nodal. Let us recall the formulation in Göttsche [19].

First notice that the modified family invariants are equal to zero if the family dimensions are negative. In this case, one can formally apply the family blowup formula, etc. to generate formally the same polynomial as in my main theorem. Let us give them a name "formal invariants." They are obtained by extrapolating the universal power series to the cases that the expected dimensions are negative. I emphasize this point as the universal formula itself does not include this condition.

Replacing the modified invariants by their formal analogue, one considers the formal power series $\mathcal{F}^{for}(q)$.

Then my main theorem, along with the identification offered here shows that:

Theorem 8.4. *By substituting $DG_2(q)$ for q, $\mathcal{F}^{for}(DG_2(q))$ can be identified with*

$$\frac{(DG_2(q)/q)^{\chi(C)} B_1(q)^{K_M^2} B_2(q)^{C\cdot K_M}}{(\Delta(q) D^2 G_2(q)/q^2)^{\frac{\chi(\mathcal{O}_M)}{2}}},$$

with $B_1(q)$, $B_2(q)$ being power series which are computable from Caporaso-Harris' formulas in [7].

In particular, the first 28 terms have been determined by Göttsche [19]. I want to point out that the insight about the appearance of these modular and nonmodular objects here is completely due to Yau-Zaslow and Göttsche. The previous theorem was stated as a conjecture in Göttsche's paper. I list it here as an application of the technique in gauge theory.

Knowing that the Di Francesco-Itzykson conjecture is valid, one wonders about its generalization toward the general algebraic surfaces. If the algebraic surface is of nonvanishing geometric genus p_g, then it follows from the fact that they are simple type for the usual Seiberg-Witten invariants, all the modified family invariants of the classes $C - \sum m_i E_i$ are zero. Thus, the virtual number of nodal curves are all zero. In other words, the counting of algebraic nodal curves is not an invariant count in the symplectic setting. That is why I restrict myself to $p_g = 0$ algebraic surfaces in the previous discussion.

Given a class C in the Picard group of an algebraic surface, let us consider the subset of Perm(C) which collects all the permissible decompositions containing some nonprimitive classes in its components. Let us define several concepts:

Definition 8.2. The class C is said to be perfect if there are no permissible decompositions of C which contain multiple coverings of exceptional classes generic enough to survive the almost complex perturbation, i.e., they exists for generic almost complex structures.

Given a perfect class C, the permissible decomposition of C' is said to be a reduction of some permissible decomposition of C if there exists a map from the components of C to the components of C' such that each component of C is a multiple of the corresponding decomposition component of C'. If all the multiplicities are equal to one, it is called trivial. Otherwise the reduction is said to be nontrivial.

In $b_2^+ = 1 (p_g = 0)$ category, smooth -1 curves are the only exceptional curves that survive the almost complex perturbation. In $b_2^+ = 1$ category, the nonperfect classes are those that the equality $SW = Gr$ fails naively (cf. [53]).

For obvious reasons, one is only interested in the collections of nontrivial reduction, which I denote by $\mathrm{Perm}_{red}(C)$. Following this convention, $\mathrm{Perm}(C) \cap \mathrm{Perm}_{red}(C) = \emptyset$.

Given a perfect class with nontrivial $\mathrm{Perm}_{red}(C)$, one defines a dimension function $\alpha(C) : \mathrm{Perm}_{red} \mapsto \mathbf{N} \cup \{0\}$ by counting the sum of the Gromov-Taubes formal dimension of each component.

Let us define an important number attached to each C.

Definition 8.3. Let α be the $\mathbf{Z}$ valued function defined above. Then one defines $\alpha^\sharp(C)$ to be $sup_{x \in \mathrm{Perm}_{red}(C)} \alpha(C)(x)$.

The number $\alpha^\sharp(C)$ is the maximum value of the function $\alpha(C)$. If $\alpha^\sharp(C)$ is strictly less than $\mathrm{d}_{\mathbf{C}}(C)$ by m, then the class is said to be m-completely perfect. My scheme of curve counting in Section 4 and 5 does not allow us to say anything explicitly about the nodal curves counting of a class which is not completely perfect. To do so, it involves the modification of the scheme, which will depend on the geometry of the specific manifolds. The machinery built up in [38] is aiming to discuss this issue.

The specialty of the j-ample classes (by j-ample, one means that the class defines a positive linear functional on the j-Mori cone) is that a j-ample class is eventually m-completely perfect, after twisting by a high k power which may depend on m.

Lemma 8.3. *Let C be a j-ample class, then the number $d(kC)$ is greater than $\alpha^\sharp(kC)$ for large enough k.*

The lemma is elementary and is left to the reader. The gap between $d(kC)$ and $\alpha^\sharp(kC)$ gives us an upper bound on the number of nodes that the δ nodes nodal pseudo-holomorphic curves (with node numbers bounded above by this gap) counting can be free from the interruption of multiple coverings of type II exceptional curves.

Let us remark here that even if the m-completely perfect condition is satisfied by a class C, there is still serious difficulty to realize the curve counting in the algebraic or Kähler category. The reduced family moduli space is not partially good or type II free, since the algebraic curves seldom behave according to the dimension formula. As was pioneered by Taubes [51], [52], [53] and Ruan-Tian [48], [49], the almost complex category seems to give the better playground for enumerative geometry of algebraic surfaces.

Let us give some simple examples which are not 1-completely perfect.

Example 8.1. Let C be an effective class on $K3$ with $C^2 = 0$, then kC is never m-completely perfect for any m and k.

Let $C = dH$ on $\mathbf{CP}^2$, then C is m perfect if and only if $m \leq 2d - 2$.

They are both simple translations of my previous observation. This concept of being m-completely perfect gives a sufficient condition for the nodal curves invariants defined by the modified family invariants and the pseudo-holomorphic nodal curves counting to be compatible. In the following, I would like to explain from the point of view of modified invariants the special phenomenon happened on $K3$ and T^4.

Despite the general complications due to the appearance of the type II exceptional curves, these curves are not harmful to the prediction of "nodal curves counting" for algebraic $K3$ and T^4 in the Yau-Zaslow conjecture.

In the original Yau-Zaslow argument, they predicted that the counting of rational curves in a linear system of $K3$ is dependent on the class C through the weaker numerical invariant C^2. As the question of counting algebraic nodal curves in an arbitrary linear system is not a well posed question, one needs to replace it by some other concept. In an earlier approach, T. J. Li and the author tried to interpret it as some Gromov-Witten invariant of pseudo-holomorphic curves. Later, N. C. Leung and J. Bryan followed the lead and calculated the invariant for primitive classes. Their calculation was used in the previous argument to identify the whole power series of nodal curves invariants. In the following, I follow the definition of modified family Seiberg-Witten invariants and prove that the virtual numbers are independent of the geometric details of C.

It turns out that the existence of the hyperkähler structures on these manifolds and a special type of vanishing theorems on the family invariants play rather crucial roles in our discussion.

As was discussed in [29], one needs to consider $X = K3 \times S^2$ or $T^4 \times S^2$ with $B = S^2$. Then one can deform the family Seiberg-Witten equations by the S^2 family of hyperkähler forms. By the family wall crossing formula [29], it follows that the relevant family invariants are nonvanishing. As it has been discussed [29] in detail, I do not plan to repeat it here. It is easy to see that the topological family invariants of the S^2 family are equivalent to the algebraic Seiberg-Witten invariants for algebraic $K3$ or T^4. As usual, I have used the curve class to parameterize the $spin_c$ structures.

It was noticed that the family Seiberg-Witten invariants enjoy a nice functorial property that I plan to discuss. Let C_1 and C_2 be two cohomology classes with $C_i^2 \geq -2$. Also let $\mathcal{M}_{C_i} \mapsto B$ denote the relative family moduli space associated with C_i. Suppose that one is interested in calculating the family invariant of $C_1 \coprod C_2$, then one takes $\mathcal{M}_{C_1 \coprod C_2} = \mathcal{M}_{C_1} \times_f \mathcal{M}_{C_2}$, the fiber product of the relative moduli spaces.

As the fiber product of two fibrations can be viewed as the preimages of the diagonals under $\mathcal{M}_{C_1} \times \mathcal{M}_{C_2} \mapsto B \times B$, one can view the original family invariant as the mixed invariant over a larger family over $B \times B$, by inserting $PD(\Delta(B))$. The symbol $\Delta(B)$ denotes the diagonal embedding $B \mapsto B \times B$.

Let η_i be a chosen basis of $H^{\cdot}(B, \mathbf{Z})$. It is well known that $PD(\Delta(B)) = \sum \eta_i \otimes \eta_i^*$. The elements η_i^* form the dual basis in $H^{\cdot}(B, \mathbf{Z})$.

By combining these facts together, the contribution of the pair of curves $C_1 \coprod C_2$ to the family invariant is given by

$$\sum_i FSW(\eta_i, C_1) \cdot FSW(\eta_i^*, C_2).$$

Even if $\mathcal{M}_{C_i}$ is not smooth, the identity can be proved easily by using the family Kuranishi model technique.

By applying the argument to $B = S^2$, one has the following special vanishing theorem for $K3$ or T^4. A generalization of this theorem has been used by T. J. Li and the author to study some other questions.

Proposition 8.6. *Let M be either $K3$ or T^4, with $B = S^2$. Let C_1, C_2 be two curve classes in $H^2(M, \mathbf{Z})$ such that $C_1 \coprod C_2$ represents the coexistence of C_1 and C_2 among the family. Then the expected contribution to the family invariant of the coexistence of C_1 and C_2, $C_1 \coprod C_2$, always vanishes.*

Proof. It is crucial that S^2 has no middle cohomology. Thus, $H^{\cdot}(S^2, \mathbf{Z})$ is generated by $[pt]$ or $[S^2]$. From the previous formula, it follows that the contribution of $C_1 \coprod C_2$ to the S^2 family is given by

$$FSW(1, C_1) \cdot FSW([S^2], C_2) + FSW([S^2], C_1) \cdot FSW(1, C_2).$$

Then its vanishing follows from the well known fact that the ordinary Seiberg-Witten invariants of $K3$ and T^4 are trivial for all nonzero $spin_c$

classes. Notice that I have reinterpreted $FSW([S^2], C_i)$ as the ordinary Seiberg-Witten invariant, adopting the Gromov type notation. This completes the proof of the proposition. q.e.d.

As I have discussed, the algebraic Seiberg-Witten invariants of $K3$ (or T^4) are equivalent to the family invariants over S^2. As the space S^2 is hidden in the algebraic invariant, the previous argument does not apply directly. I outline an equivalent argument that works for the algebraic "invariants" as well. Let us give a brief discussion as to why it should be trivial. The reason that the Seiberg-Witten invariants of the algebraic $K3$ vanish is because of the appearance of a complex rank one trivial obstruction bundle over the moduli space. When one promotes it into a S^2 family, it is cancelled by the tangent bundle of S^2. The coexistence of two curves, C_1 and C_2, produces a rank two trivial obstruction vector bundle, which cannot be completely cancelled by the one dimensional TS^2. As a result, their contribution to the family invariants is zero.

One can interpret Proposition 8.6 in the following way: if the contribution of $C_1 \coprod C_2$ to the family invariant is nonzero, it provides a topological obstruction for the curve C_1 to disappear completely while C_2 survives. The vanishing of these types of contributions have been reflected by the fact that one can deform the S^2 family of hyperkähler complex structures to one with rank(Pic) = 1 locally. The fact has been used in the previous argument in identifying the nodal invariants.

Consider $B = S^2 \times M_n$, with M being either $K3$ or T^4. It is not easy to pin down the schematic expression of the expected contribution of the type II multiple coverings. However, a slight generalization of the vanishing theorem shows that the expected invariant contribution of any nonirreducible curves vanishes. This includes the contribution of the admissible decomposition classes involving the appearance of the type II exceptional curves.

A complete proof of the fact will involve a longer argument as well as a lot of new notations. We sketch a simplified argument when $Pic = \mathbf{Z}C$. It is enough to capture the main spirit of the argument without losing the generality.

Suppose that C is a primitive class which generates the Picard group locally among the S^2 hyperkähler family. Suppose one is studying the class $kC - 2\sum E_i$ and the decomposition involving type II curves looks like

$$(k_1 C - \dots) + m(k_2 C - \dots) + \dots ,$$

where $\dots$ represent certain terms the details of which we do not care about. They can be either the various E_i, the type I or the other type II exceptional curves, etc. It can happen that the free part $k_1C - \ldots$ is zero while $m > 1$, or $m = 1$ when $k_1C - \ldots$ and $k_2C - \ldots$ coexist. No matter which possibility occurs, let us take a closer look at the tangent obstruction complex of the joint curves. It is easy to see that there is always a trivial rank (≥ 2) obstruction subbundle sitting on the appropriate moduli space by following the previous argument. As either the topological or the algebraic family invariants only have the capacity to remove a trivial obstruction line bundle, the contribution to the family invariants always vanish. It is not hard to reformulate and recast the previous argument into an invariant calculation, based on the family blowup and switching formulas. I do not plan to do so as it will occupy too much un-necessary space in the paper. The set up of virtual admissible decomposition classes will be discussed in the sequel to the paper [41].

I cannot help pointing out the important consequence of the previous result. It shows that only reduced irreducible curves can have nontrivial expected contributions to the family invariants. It gives a philosophical explanation as to why the algebraic geometric counting of nodal curves gives the same result as from the Gromov-Ruan-Tian theory. My result implies that the universal "equivalence" of the nodal invariants is not corrected in terms of what was predicted from the general consideration. This type of miracle only happens for these special manifolds with hyperkähler structures. This also explains why these manifolds are chosen in identifying the universal power series.

I need to warn the reader that the result does not imply the strong statement that the top stratum of the family moduli space over Y_{γ_n} consists of irreducible curves. In fact, the complex structures may be chosen such that all curves in the given linear system are reducible. In this type of situation, the reducible curves do contribute to the invariant (otherwise the invariant would have been simply zero), but their expected contribution to the invariant still vanish.

The reader who feels puzzled about this fact may consult the example in [48]. For a certain choice of perturbation, a certain genus zero boundary component of the Gromov moduli space suddenly grows to a wrong dimension and occupies the whole moduli space. Yet their expected contribution to the Ruan-Tian invariants is zero due to dimension restriction under the generic perturbation.

Next, let us state an interesting corollary of our explicit identification. Namely, one has the following blowup formula of "nodal curves invariants."

Let C be an ample class in M. Consider the one point blowup of M, denoted by $\widetilde{M}$, then $mC - E$ is ample on $\widetilde{M}$ for m large. In fact, the minimum of m is related to the well known Seshadri constant in the literature [11].

One is interested in comparing the nodal curves invariants of mC and $mC - E$. As usual, let us consider the formal analogue instead of the actual modified invariants. Then the generating functions are related by multiplying the original power series by

$$\left(\frac{DG_2}{q}\right)^{-1} \frac{B_2(q)}{B_1(q)}.$$

More precisely:

Corollary 8.1 (Blowup Formula of The Nodal Invariants). *The blowup formula relates the generating function in the following way*

$$\mathcal{F}^{for}_{\widetilde{M}}(DG_2) = \mathcal{F}^{for}_{M}(DG_2) \cdot \left(\frac{B_2(q)}{B_1(q)}\right) \cdot \left(\frac{DG_2}{q}\right)^{-1},$$

where B_1 and B_2 are the two power series derived by Göttsche, starting as

$$B_1(q) = 1 - q - 5q^2 + 30q^3 - 345q^4 + 2961q^5 \dots,$$

and

$$B_2(q) = 1 + 5q + 2q^2 + 35q^3 - 140q^4 + 986q^5 + \dots.$$

Proof. This follows from simply plugging $K^2_{\widetilde{M}} = K^2_M - 1$ and $(mC - E) \cdot K_{\widetilde{M}} = mC \cdot K_M + 1$ into the universal formula. q.e.d.

One should notice that it is exactly the same process which lead Kroheimer-Mrowka to derive their blowup formula of Donaldson invariants for simple type manifolds [26]. It is desirable to have a proof of the Fintushal-Stern style blowup formula, which can cast $B_1(q)$ and $B_2(q)$ into a closed form. Even though the proof is simple once assuming the validity of the main theorem, the corollary admits some interesting interpretations. It implies that the coefficients of the two power series $B_1(q)$ and $B_2(q)$ should have a purely local interpretation independent of the global geometry of the algebraic surface. The identification is from the global point of view which involves the global geometric information of the manifolds. The blowup formula of nodal invariants suggests that the coefficients can be determined from purely local information, which is independent of the geometry of the algebraic surfaces. This interpretation resembles a version of local-global duality in geometry. I hope to come back to this topic in the future.

The following problem is worthy to be further investigated.

Remark 8.2 (Open Problem). Determine the closed forms of $B_1(q)$ and $B_2(q)$ and understand their geometric meanings, e.g., find out the analogue interpretation similar to Yau-Zaslow conjecture. Study the relationship between $B_1(q), B_2(q)$ and modular forms. Prove the nodal curve blow-up formula by local geometry.

The similar factorization formulas for other types of repeated singularities also imply similar blowup formulas for these invariants. In fact, there are infinite numbers of them such that the case of nodal curves $\Gamma_0 \in \mathrm{adm}(1)$ and $\mathbf{M} = 2$ is the lowest among a huge tower.

The detailed structure of these formulas is not well studied at this moment. The formulas of these types probably have no direct link with Ruan-Tian type invariants [49].

9. Case study: some explicit determination of the modified invariants and the relationship with Vainsencher's work

I have exhausted almost all the effort in giving a theoretical yet abstract deduction. Let us discuss briefly the application of the machinery developed above to recover the formula of Vainsencher [59].

We will illustrate by using $n \leq 8$ cases to give the reader certain insight about how my machinery works and how it relates to the argument of Vainsencher. Several examples of the admissible decompositions and the admissible graphs are illustrated.

9.1 The list of Vainsencher's formulas

Even though the language I have used here is quite different from the one used in Vainsencher's paper, the results are parallel for $n \leq 6$. The result for $n \leq 7, 8$ is known by Kleiman-Piene [24] already. For historical interest, the reader can consult Vainsencher's original paper [59]. Notice that in my language, the concept of "singular curves" does not show up at all. My scheme counts the smooth curves in the appropriate resolved classes. The fact that the invariants can be related to the nonnodal singular curves is NOT used in the proof of the main theorem.

Let us begin by listing the recursive formulas of Vainsencher for $n \leq 6$. I follow his notations but rearrange the terms.

$$n!tg_n = \sharp\Sigma((2^{[n]}); S), n = 1, 2, 3;$$

$$\sharp\Sigma((2^{[4]}); S) = 4!tg_4 + 6\sharp\Sigma((3); S);$$

$$\sharp\Sigma((2^{[5]}); S) = 5!tg_5 + 30\sharp\Sigma((3, 2); S);$$

$$\sharp\Sigma((2^{[6]}); S) = 6!tg_6 + 30\sharp\Sigma((3(2)); S) + 90\sharp\Sigma((3, 2, 2); S);$$

$$\sharp\Sigma((2^{[7]}); S) = 7!tg_7 + 210\sharp\Sigma((3(2)), 2; S)$$

$$+1260\sharp\Sigma((3, 2, 2, 2); S) + 30\sharp\Sigma(3(2)'; S)).$$

In his notation, S is the algebraic surface, while tg_n is equal to N_n in my notation. For the definition of $\sharp\Sigma$, please consult his original paper [59].

We will see how these coefficients come out from the admissible graphs and the weights are deduced from the family blowup formula. It will be clear in a moment how his notion $\sharp\Sigma$ can be related to FSW^*.

In the general proof, I have not written down explicitly the terms FSW^* on the right hand side. In practice, it will be rather difficult to identify all the admissible decomposition classes, especially when n goes large. Some of them do not contribute to the invariants by the vanishing theorem deduced before. Therefore, my proof is a theoretical proof rather than an enumerative one. In the following I plan to discuss the small n cases and link the modified invariants FSW^* and the admissible decompositions of different levels with the counting of nonnodal curves.

First one has:

Proposition 9.1. *For $n \leq 2$, the smooth representatives without type* I *components are the only representatives that are admissible. Therefore, the invariant $FSW(1, C - 2E_1)$ or $FSW(1, C - 2E_1 - 2E_2)$ are manifestly equal to FSW^*. In other words, the admissible decompositions are all of level zero (generic).*

Proof. The proposition is proved by a direct calculation, we omit the simple arithmetic here. q.e.d.

Before studying the concrete examples, let us make a brief remark here.

Remark 9.1. The nongeneric admissible decomposition classes appearing in the following examples all have unique representatives. Because they are eventually reduced to some mixed family invariants involving a few nodes, one can easily identify those invariants with the counting of of nonnodal singular curves. One should be cautious that when n goes larger, the family invariants associated with those nongeneric admissible classes may need to be modified before I interpret them as enumerative invariants. I do not go into details here because these interpretations do not affect the proof of my main theorem.

9.2 The n=3, 4 cases

Take $n = 3$. Then there exists an exceptional curve $W = E_1 - E_2 - E_3$, whose pairing with $L = C - 2(E_1 + E_2 + E_3)$ is equal to -2. As a result, when W shows up, L must split as $L - W + W$. As it is of multiplicity one and $(L - W) \cdot W = 1$, the expected dimension of the curve is lower than the smooth representative [51]. It is not admissible.

The first nonnodal contribution shows up when $n = 4$. The case was discussed theoretically in Example 7.1. Let us be brief here. By adding a new vertex "4" to a graph in adm(3), one obtains an injection from adm(3) to adm(4). We are interested in the graphs in adm(4) − adm(3). Let us consider Γ to be the admissible graph $\in$ adm(4), with three edges from 1st vertex to the vertexes marked $2, 3$ and 4. The stratum associated with it has complex codimension 3. The exceptional curves in the preexceptional cone $\mathbf{C}_\Gamma$ is generated by $E_1 - E_2 - E_3 - E_4, E_2, E_3$ and E_4. The strata is the locus that these type I curves coexist.

As has been discussed in Example 7.1 that $L - W + W$ with $(L - W) \cdot W = 0$ is a new admissible decomposition of level 1. We have $L - W = C - 3E_1 - E_2 - E_3 - E_4$.

For this type of graph, we have $G(\Gamma) = G_\Gamma = \mathbf{S}_3$, permuting the last three vertexes. Therefore, $\sigma(\Gamma) = 1$.

Applying the family blowup formula to the situation, notice that the vertexes $2, 3, 4$ are all subredundant. By the construction of cores, one can in principle reduce the invariant to its core, the one vertex graph in the situation. Originally one has $FSW^*(c, L - W) = FSW(c, L - W)$, where c is the cohomology class dual to $Y(\Gamma)$. This follows from the fact that there are no level two admissible decomposition classes. It is easy to calculate by Proposition 4.4 that

$$c = \pi_2^*(E_1(2)) \cup \pi_3^*(E_1(3) - E_2(3)) \cup \pi_4^*(E_1(4) - E_2(4) - E_3(4)).$$

On the other hand, blowing down three times produces

$$c' = \pi_2^*(C - 3E_1(2)) \cup \pi_3^*(C - 3E_1(3) - E_2(3)) \\ \cup \pi_4^*(C - 3E_1(4) - E_2(4) - E_3(4)),$$

By pairing c with c' one finds that the original invariant is equal to $FSW(c \cup c', C - 3E_1)$ evaluated on the fiber bundle formed by the fiber product of $M_2 \mapsto M$ with $M_4 \mapsto M$. Due to the next vanishing result, one reduces to the pure invariant on $M_2 \mapsto M$, $FSW(1, C - 3E_1)$. The coefficient 6 comes from $\int_{M_4/M_1} c \cup c'$. It matches up with the coefficient in front of $\sharp\Sigma((3); S)$ in Vainsencher's formula. Let us remind the reader that the number 6 does not come from $\sigma(\Gamma)$ as one may suspect naively.

9.3 The vanishing theorem and n=5, 6

Let us first recall the vanishing theorem of family Seiberg-Witten invariants.

Theorem 9.1. *Let B_1, B_2 be two smooth base spaces. Let $g : B_1 \mapsto B_2$. Let $\mathcal{X} \mapsto B_2$ be a fiber bundle of four-manifolds with $b_2^+ > 0$. One can consider the fiber product of $\mathcal{X}$ with B_1 by pulling back the fiber bundle $\mathcal{X} \mapsto B_2$ to B_1. Then one can pull back the spin$_c$ structure on $\mathcal{X}$ to a spin$_c$ structure on $\mathcal{X} \times_{B_2} B_1$. One can consider the family Seiberg-Witten invariant on the new fiber bundle in the corresponding pulled back chambers. Then we have*

$FSW(c, \cdot) = 0$ unless c is in the image of

$$[F_z] \cup : H^*(B_2, Z) \mapsto H^*(B_1, Z)$$

by cupping with the Poincare dual of the fibers $F_z = g^{-1}(z), z \in B_2$.

Proof. The proof of the theorem is given in Theorem 7.2. It is similar to the argument of the special case $B_1 = M \times M_n \mapsto M_n = B_2$. q.e.d.

If n is equal to 5, one still has the canonical embedding $ST_4^5 : \text{adm}(4) \mapsto \text{adm}(5)$, by adding a new vertex marked by 5. The image of the graph Γ in adm(5) also gives a nontrivial contribution to FSW. On the other hand, we would like to investigate the elements in $\text{adm}(5) - ST_4^5(\text{adm}(4))$.

Lemma 9.1. *Let Γ be an admissible graph such that every vertex of Γ has less than three direct descendents, then Γ cannot be in $\Phi(\mathcal{ADM}) \subset \text{adm}(n)$ with respect to $C - 2E_1 - 2E_2 - 2E_3 \ldots - 2E_n$.*

This lemma rules out almost all elements in $\mathrm{adm}(5) - ST_4^5(\mathrm{adm}(4))$ from being in the image of Φ map. The only new element that survives is the graph Γ, constructed by adding the new edge from the vertex 1 to the vertex 5.

This particular stratum is of complex codimension 4 in M_5, even though it is an admissible decomposition class. It turns out that the invariant FSW^* does not contribute to the right hand side due to the vanishing theorem, Theorem 9.1.

Remark 9.2. This example indicates that not all the admissible decomposition classes contribute to the invariant. In Vainsencher's formulation, it was directly thrown away by dimension reason. In my approach, it can be ignored only after applying the vanishing result appropriately. The main difference between Vainsencher's approach ([59], see also [24]) and mine is that I consider the class $C - \sum 2E_i$ rather than the singular curves themselves (which can be different from $C - \sum 2E_i$ if nonnodal singularities develop). It turns out that it is the type I exceptional classes which affect the counting in a rather subtle way.

Consider the admissible graph $\in \mathrm{adm}(n)$ with $n-1$ edges linking from 1 to $2, 3, \cdots n$. Let $W = E_1 - E_2 \cdots - E_n$ be the exceptional curve associated with the first vertex. One calculates that $W \cdot L = 4 - 2n$. $W^2 = -n$. Then the decomposition $L = L - 2W + 2W$ with $(L - 2W) \cdot W = 4$ is an allowable decomposition (for $n \geq 5 > -(-4)$). which lies in some admissible decomposition class.

The obstruction bundle can be calculated to be of complex $(n - 4 - 1) + (2n - 4 - 1) = 3n - 10$ dimension. On the other hand, as $L = C - \sum 2E_i$, $L - 2W = C - 4E_1$ does not depend on the class $E_i, 2 \leq i \leq n$, then the cohomology class (and the corresponding $spin_c$ structure) is pulled back from $M_2 \mapsto M_1$ by $B_1 = M_n \mapsto M_1 = B_2$. From the vanishing theorem, the mixed family invariant is zero unless the base class insertion is in the image of $H^*(M_1) \mapsto H^*(M_n)$.

In order that the contribution is nonzero, it is necessary to have $3n - 10 \geq 2n - 2$. The number n must be greater than or equal to 8. This statement replaces the corresponding argument in Vainsencher [59] that the curves with multiplicity$=4$ singularities contribute to the curve counting only when $n \geq 8$.

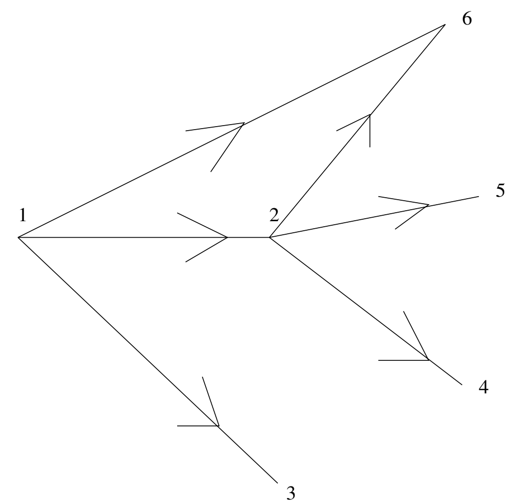

Figure 6a

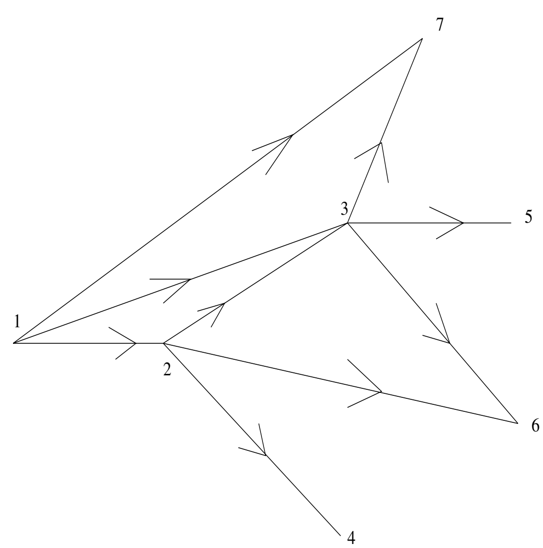

Figure 7a

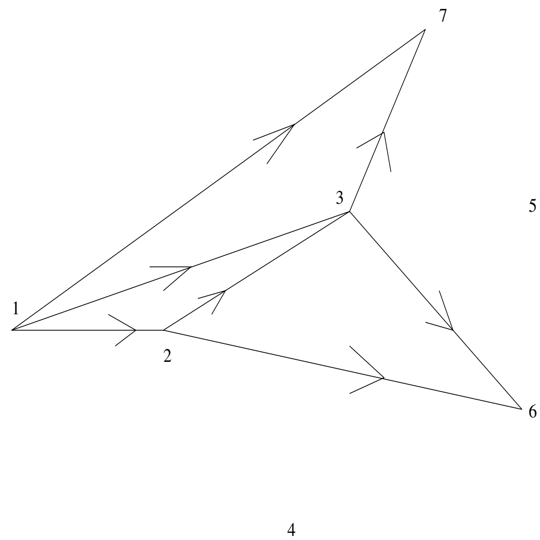

FIGURE 7B

9.4 The n=7 cases

First I consider the special admissible graph linking 1 to three of the remaining 6 vertexes and leave the other three untouched. The $\sigma(\Gamma)$ of this type of graphs is 35. On the other hand, there is a standard factor 6 appearing in the family invariant by the similar derivation I performed in the previous cases. It turns out that the number $210 = 35 \times 6$ shows up in front of the net contribution.

Let us consider the graph connecting edges from the first vertex to the second and the third and the 6th. Then one connects edges between the $2nd$ vertex and the vertexes marked $4, 5, 6$. Finally one leaves the $7th$ vertex un-touched. It is the stable version of the Figure 6a. The number $\sigma(\Gamma)$ is given by 105. On the other hand, there is a standard factor 2 in front of the family invariant after I perform the reduction using family blowup formula. As a result, the coefficient 210 appears in front of the invariant.

Another type of new contribution comes from the graph in Figure 7a.

Let $1, 2, 3, 4, 5, 6, 7$ be the 7 vertexes marked by these numbers. One connects 1 to $2, 4, 7$. Then one connects 2 to $3, 4, 6$. One also connects the $3-rd$ vertex to the vertexes $5, 6, 7$. The graph has 9 edges and three loops. According to Proposition 4.3, the stratum corresponding to this graph is of complex codimension 9. First let us calculate $\sigma(\Gamma)$.

The first two vertexes 1 and 2 are fixed. The vertex 4 can be replaced by any vertex between 3 and 7 (there are exactly 5 different ways to pick it). Once it is fixed, there are 4 vertexes left unmarked. Then one picks the smallest number among them to replace 3. I assign the remaining three numbers to vertexes $5, 6, 7$ according to all possible orders and the resulting graphs are not equivalent to each other. As a result, $\sigma(\Gamma) = 30$. It is also easy to see that $G(\Gamma)$ is the semi-direct product of $\mathbf{Z}/5\mathbf{Z}$ and $\mathbf{S}_3$ while $G_\Gamma = \{1\}$.

Over the compactification of this stratum, the class $L = C - 2(E_1 + \cdots + E_7)$ has negative pairings with $W_1 = E_1 - E_2 - E_3 - E_7$, $W_2 = E_2 - E_3 - E_4 - E_6$ and $W_3 = E_3 - E_5 - E_6 - E_7$. In other words, if the class L is represented by a pseudo-holomorphic curve, certain multiples of W_1, W_2, W_3 must show up. It turns out that $W_i \cdot W_j = 0$ for $i, j = 1, 2, 3$. $L \cdot W_i = -4, 1 \leq i \leq 3$ and $W_i^2 = -4$.

Thus, the new admissible decomposition splits L into

$$L' + W_1 + W_2 + W_3$$

with $L' = C - 3E_1 - 2E_2 - E_3 - E_4 - E_5$.

In this way, the invariant contribution should be $FSW(1, L')$ with the invariant being calculated over $Y(\Gamma)$.

Similarly, one considers the space $Y(\Gamma_{red})$ in M_5. Γ_{red} is obtained from Γ by removing the vertexes 4 and 5 from the graph (please see Figure 7b). The resulting graph contains 5 vertexes. The new graph has 7 edges and three different loops. According to the general concept of cores, it is the core with respect to the class L' that I discussed before.

The space $Y(\Gamma)$ has a $(\mathbf{CP}^1)^2$ fibration structure over the space $Y(\Gamma_{red})$.

Applying the family blowup formula reducing the class L' to $C - 3E_1 - 2E_2 - E_4$, the invariant is then replaced by

$$FSW(c, C - 3E_1 - 2E_2 - E_4),$$

with $c = \pi_4^*(C - 3E_1(4) - 2E_2(4) - E_3(4)) \cup \pi_5^*(C - 3E_1(5) - 2E_2(5) - E_3(5) - E_4(5))$.

Using the fact that the fibers $\mathbf{CP}^1 \times \mathbf{CP}^1$ are dual to $\pi_4^*(E_2(4) - E_3(4)) \cup \pi_5^*(E_4(5))$, one finds that the mixed invariant can be reduced to

$$FSW(1, C - 3E_1 - 2E_2 - E_4),$$

with the invariant being calculated on $Y(\Gamma_{red})$. The $Y(\Gamma_{red})$ is a complex three dimensional submanifold in M_5. From the fact that there are

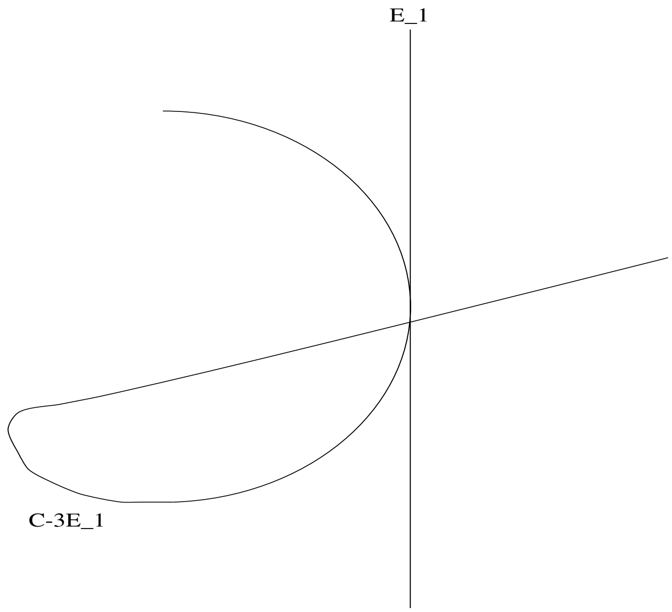

FIGURE 8

three loops in Γ_{red}, it turns out that $Y_{\Gamma_{red}}$ is isomorphic to the stratum corresponding to the graph $1 \mapsto 2$ in M_2. From here, one can calculate the invariant by using either family blowup formula or by identifying this invariant with the counting of curves with a triple point. The proper transformation has two branches passing through E_1, while one of these branches being tangent to E_1. In the old notation of Vainsencher [59] it was denoted by $3(2)'$. If one remarks the graph by switching the order of the 3rd and 4th vertexes, then it is illegal to blow down the third exceptional curve without doing it for the fourth one first. Therefore, our blowup formula formalism does not connect it with the counting of singular curves of type $3(2)'$ directly. If one bears in mind that the new graph is equivalent to the original one by some element in $G(\Gamma)$, then their contributions can be identified by using the group actions of $G(\Gamma)$ and by the existence of the universal formulas. The argument has been outlined in Section 5.4 and Section 7.2.

9.5 The case n=8

If n is equal to 8, the naive counting of Vainsencher failed. As his argument extensively relies on the transversality of the zeros of the obstruction bundle (or in his terminology, the reduceness of the space and the correctness of the dimension), his method breaks down when the assumption does not hold.

Let the curve C be a singular curve with a fourth order point. Then a single resolution of the algebraic surface at the singular point resolves into a smooth curve which intersects the exceptional curve at four different points. However, as I use the cohomology class $C - 2E_1$ instead of $C - 4E_1$, it is represented as $C - 4E_1 + 2E_1$. Namely, the double covering of E_1 shows up among the irreducible components. As there is an infinite number of singularities now, any further blowing ups at the support of E_1 will detect singularities. In other words, the points in M_8 that associate to singular curves become nonisolated! That is why Vainsencher's counting fails. My key observation is that by applying the family Seiberg-Witten theory, the counting still makes sense in the situation. The so-called family switching formula was not originally developed for the nodal curves counting. After applying my tools to the current situation, it resolves the question immediately. It was historically the first indication that the family Seiberg-Witten theory should solve the problem.

There are a few different admissible decomposition classes which could contribute to the family invariant. Define $W_f = E_1 - E_2 - E_3 \cdots - E_f$ to be an $-f$ curve. For each $8 \geq f \geq 3$, any holomorphic curve in $C - \sum_{i\leq 8} 2E_i$ coexists with an irreducible W_f must split off at least one copy of W_f. The existence loci of W_f can be described easily as the admissible strata associated with the admissible graphs Γ_f. Each Γ_f is the union of $8 - f$ free vertexes and a $f - 1$ fan graph.

Consider the admissible decomposition classes of the form $(C - 4E_1 - 2E_{f+1} - 2E_{f+2} \cdots - 2E_8, 2W_f)$ when $f \geq 5$. They are all non-Taubes type decompositions as the double coverings of W_f are involved.

The family dimensions of these decomposition classes are easily calculated. They are all of $\mathrm{d}_{\mathbf{C}}(C) - 1 = \mathrm{d}_{\mathbf{C}}(C - 2\sum_{i\leq 8} E_i) + 7$ dimension. In other words, all these non-Taubes type decompositions are of 7 dimension higher than the expected dimension of nodal curves. A priori, all these decomposition classes can contribute to the family invariants. In fact, it is not the case. The $f = 5, 6, 7$ cases can be proved to contribute trivially due to the vanishing theorem.

One finds out that the only admissible decomposition of $C-2E_1\cdots-2E_8$ that contributes is

$$C-4E_1+2W_8=C-4E_1+2(E_1-E_2-E_3-E_4\cdots-E_8),$$

with

$$W_8\cdot(C-4E_1)=4.$$

I will describe immediately the contribution of this type of curve to the family invariant. For $f=4$, there is a level one admissible decomposition class of the form $(C-3E_1-E_2\cdots-E_4-2(E_5\cdots-E_8),W_4)$; which is a nongeneric Taubes type decomposition. The decomposition $(C-4E_1,2W_8)$ is of level 5 in terms of our terminology.

In terms of my formulation, the nonisolated locus in Vainsencher's observation can be identified as follows. Given a singular curve in $|C|$ singular at the point p. The locus (it does not consist of isolated points any more) in M_8 is the space $Y(\Gamma)\cap f^{-1}(p)$, where $f=f_7\cdot f_6\cdots f_1$ is the projection map $M_8\mapsto M_1$. The graph Γ in the current situation is the $n=8$ version of the graph in Figure 4. The space Y_Γ is the locus that the -8 curve $W=E_1-\sum_{i\geq 2}E_i$ exists. The closure $Y(\Gamma)$ is the locus ($\subset M_8$) that the pseudo-holomorphic curve $E_1-\sum_{i\geq 2}E_i$ or its various degenerations support upon.

According to the general scheme [37] or the proof of my main theorem, the contribution of this type of curve should be given by $FSW(c\cup c',C-4E_1)$, where c represents $PD(Y(\Gamma))$, while the cohomology class $c'\in H^*(M_8)$ is the sum of the various chern classes of $\mathcal{V}$.

$$c'=\sum_i c_i(\mathcal{V}).$$

The bundle $\mathcal{V}$ is a certain complex rank 7 virtual vector bundle on M_8. If one restricts to the sublocus, one can choose an explicit bundle representative in the K group which have a direct geometric meaning.

By dimension reason, only the $7th$ Chern class contributes nontrivially. Therefore, the net contribution is $FSW(c\cup c_7(\mathcal{V}),C-4E_1)$. By applying the family blowup formula seven times, one reduces the invariant to $FSW(c\cup c_7(\mathcal{V}),C-4E_1)$, where the fiber bundle is the pull back of $M_2\mapsto M$ by $M_8\mapsto M$. The graphical operation corresponds to the reduction of subredundant vertexes. Because the fiber bundle is pulled back from M, one can push forward along $M_8\mapsto M$, and the invariant is reduced to $FSW(f_*(c\cup c_7(\mathcal{V})),C-4E_1)$ on M. By using $c=PD(Y(\Gamma))$ and the fibration structure $g:Y(\Gamma)\mapsto M$, the invariant is equal to

$FSW(g_*(c_7(\mathcal{V})), C-4E_1)$. Suppose $g_*(c_7(\mathcal{V})) = r[pt] \in H^0(M, Z)$, then the final answer will simply be $r \cdot FSW(1, C-4E_1)$. One learns that $FSW(1, C-4E_1)$ calculates the number of singular curves with a fourth order point.

Using Vainsencher's notation, it is expressible as $r \cdot \sharp\Sigma((4); S)$. The integer r can be calculated by using the family index theorem. As we do not plan to enumerate the polynomial explicitly by the direct calculation, I skip the detail calculation here. The key point is that the integer r is derived from the topological information of the obstruction bundle, which is constructed from the curve $W = E_1 - E_2 - E_3 - \cdots - E_8$ and $\pi_1^* C$. The number r is independent of C as it is obtained by integrating $c_7(\mathcal{V})$ along $g : Y(\Gamma) \mapsto M$.

Summary 1. In the previous discussion, I only list the admissible decompositions which contribute to the invariants. There are many other admissible decompositions which do not contribute to the invariants. In this sense, Vainsencher's original argument was much more economical than ours. As my goal is to prove the existence of the universal formula rather than to enumerate the invariants directly, it should not be viewed as a defect.

9.6 A Simple comparison of my scheme with the one from the excess intersection theory

Finally, I want to make the link between my approach and the one from the excess intersection theory, e.g., [16]. I point out the relationship for the reader who has special algebraic-geometric interest in the topic. I do not plan to give a full length discussion here but may consider the details along the line in the sequel to this paper. I only discuss the concrete example when $n = 8$, given the fact that all the general cases follow similarly.

Consider the class $C - 2\sum_i E_i$ with $n = 8$. The admissible decomposition class discussed in the previous section was given by $C - 4E_1 + 2W$ with $W = E_1 - E_2 - E_3 \ldots - E_8$. Let Γ be the admissible graph attached to W, the graph starting at 1 with $2, \ldots 8$ being its direct descendents.

As was discussed already, the obstruction bundle is of complex 14 dimension. It can be decomposed into $\mathcal{V}_1 \oplus \mathcal{V}_2$; each is seven dimensional. The virtual bundle $\mathcal{V}_1$ is closely related to the obstruction bundle of the -8 curve. According to the general recipe derived from the family switching formula [39], the mixed invariant that contributes to the counting is given by $FSW(c_7(\mathcal{V}_1) \cdot \sum_i c_i(\mathcal{V}_2), C - 4E_1)$ with $B = M_8$. Let

$\pi : \mathcal{X} \mapsto M_8$ be the fiber bundle projection map, then the vector bundle $\mathcal{V}_1$ is virtually isomorphic to $[R\pi_* \mathcal{O}] - [R\pi_* \mathcal{O}(W)]$ over $Y(\Gamma)$. And it is easy to see that $c_7(\mathcal{V}_1)$ is Poincare dual to the compactification of the stratum, $Y(\Gamma)$. As $g : Y(\Gamma) \mapsto M_1 = M$ has a fibration structure, the mixed invariant can be recast into the $\int_{g^{-1}(p)} c_7(\mathcal{V}_2) \cdot FSW(1, C - 4E_1)$, where the pure invariant is evaluated on $M_1 = M$. This is the answer predicted by the family Seiberg-Witten theory in light of Gromov theory, as was worked out in the previous subsection. On the other hand, let us investigate the question in light of algebraic geometry [16].

Given a singular curve with fourth order singular point, the first blown up point is fixed without choice. The proper transformation of the singular curve is then $C - 4E_1$. Instead, the family invariant used to count the nodal curve is $C - 2E_1 - 2E_2 - 2E_3 \cdots - 2E_8$, then we find that $C - 2E_1$ splits into $C - 4E_1 + 2E_1$ with a nonreduced component. A nonreduced component is viewed as a curve with an infinite number of singularities, then the consecutive blow-ups can be applied on the curve $2E_1$, producing a high dimensional locus over which the curve $C - 4E_1$ exists. In fact, this is the main ill symptom of the counting scheme studied by Vainsencher [59]. In particular, the same problem also shows up when $n > 8$ in a more complicated way.

Let us study the particular example by the residual intersection theory. I will show that under the transversality assumption (which can be avoided by using the Seiberg-Witten theory approach), the calculation based on the residual intersection theory does match our calculation from the point of view of Gromov-Taubes theory.

It is easy to see that, given a singular curve with a fourth order point at $p \in M$, the locus corresponding to it can be identified with $g^{-1}(p) \subset Y(\Gamma) \subset M_8$. In other words, the Euler class of the appropriate obstruction vector bundle is not represented by a transversal section.

To handle this situation, let me recall the formula developed in Fulton's book [16]. All the terminologies in the following are understood in the algebraic context.

Theorem 9.2 (Residual Formula for Top Chern Classes). *Let s be a section of a rank e vector bundle E on a purely n dimensional scheme X. Assume that $Z(s)$ contains D, an effective Cartier divisor on X. Then there is a section s' of $E \otimes \mathcal{O}(-D)$ such that the canonical homomorphism from $E \otimes \mathcal{O}(-D)$ to E takes s' to s. (Locally the functions defining s are divisible by an equation for D, and the quotients define*

s'.) In addition, $Z(s')$ is the residual scheme to D in $Z(s)$ and

$$\mathcal{Z}(s) = \mathcal{Z}(s') + \sum_i (-1)^{i-1} c_{e-i}(E) \cap D^{i-1} \cdot [D]$$

in $A_{n-e}(Z(s))$. In particular, if s' is a regular section, then $\mathcal{Z}(s') = [Z(s')]$, which gives an explicit formula for $\mathcal{Z}(s)$.

The above theorem is a word by word reproduction of the Example 14.1.4 in Fulton's book [16]. Let us consider the case of M being algebraic.

In my set up, I first take $X = M_8$. In Fulton's formulation, the space D must be a Cartier Divisor. I assume that I have chosen a linear subsystem of $|C|$ such that there are a finite number of singular curves with fourth order singular points among them. Moreover, I assume for simplicity that the scheme of the singular curves with fourth order points is reduced and transversal. Notice that in algebraic geometry, one usually imposes extra conditions on very ampleness of the complete linear system to achieve this condition. Let $C_1, C_2 \cdots C_r$ be, respectively, these r different singular curves, and let $p_1, p_2, \cdots p_r$ be their singular points. Then the degenerated locus is given by $Z = \cup_{i \leq r} g^{-1}(p_i) \subset Y(\Gamma)$. This locus is understood as the projection of $Z(s) \subset |C| \times M_8$ into M_8. To calculate their contributions to the curve counting, one needs to blow up the loci to make them into Weil divisors. Therefore, I must consider X to be the blown up manifold of M_8 along the locus $\cup_{i \leq r} g^{-1}(p_i)$, and the Cartier divisor D is chosen to be the (union of) exceptional divisor. As the original locus $\cup_{i \leq r} g^{-1}(p_i)$ is smooth, D has a projective space bundle structure over its base. Since Z is complex 7 dimensional, the divisor D is of complex 15 dimension and the fiber projective spaces are of complex 8 dimensional.

As Z breaks up into different connected components, we may assume that $r = 1$, and the net result we get should be multiplied by the number of the singular curves to get the total result.

Given the fact that the situation starts from counting the nodal curves, the obstruction bundle E is given by the following expression:

Let $C(i) = C - 2\sum_{j \leq i} E_j$ be the cohomology classes intertwining C and $C(8) = C - 2\sum E_j$. Recall that I defined $f_l : M_{l+1} \mapsto M_l$ on page 400. One defines $h_{a,b} : M_a \mapsto M_b$ to be the composition of various f_l. In fact, one has $h_{a,b} = f_b \circ f_{b+1} \cdots f_{a-1}$. Let the relative tangent bundle of $f_l : M_{l+1} \mapsto M_l$ be denoted by $\mathcal{E}_l$. Then $\mathcal{E}_l$ is a complex rank two vector bundle over M_{l+1}. Based on these notations, the obstruction bundle E can be written as

$$\oplus_{0\leq i} C(i) \otimes (h^*_{n,i+1}\mathcal{E}^*_i \oplus \mathbf{C}).$$

It is clear that the vector bundle E is of rank 24.

To relate the two different approaches, one has the following crucial proposition whose proof was hidden in the family switching formula:

Proposition 9.2. *Let E be the obstruction bundle defined above; then the restriction of E to $Y(\Gamma)$ has an alternative decomposition in the $\mathbf{K}$ group.*

$$E|_{Y(\Gamma)} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3.$$

The virtual vector bundles $\mathcal{V}_1$ and $\mathcal{V}_2$ are defined as before and the vector bundle $\mathcal{V}_3$ is complex 10 dimensional.

$$\mathcal{V}_3 = g^*(\mathbf{S}^3(T^*M \oplus \mathbf{C}) \otimes C).$$

The decomposition of the obstruction bundle into factors constitutes the main ingredient of the hidden link.

Let us see how the proposition, along with the residual intersection theory in [16], lead to an alternative calculation of the nonisolated contribution of the counting parallel to ours.

From Fulton's theory, the contribution to the counting of the top chern classes is given by

$$\sum_i (-1)^{i-1} c_{e-i}(E) \cap D^{i-1} \cdot [D].$$

By using D as the exceptional divisor, which has a projective space bundle structure over Z, one reduces this explicit contribution to an evaluation over D, where D is also viewed as the tautological line bundle of the projective space bundle over Z. On the other hand, let us denote the normal bundle of $Z \subset M_8$ by N_Z. Then the scheme D is isomorphic to $P(N)$. According to the previous proposition, the equivalence class of the vector bundle E enjoys a special decomposition restricted to Z. In particular, the same decomposition still survives after one pulls back the vector bundle from Z to D. Thus, I can assume at this moment that the vector bundle E is simply the direct sum of these three complex vector bundles of ranks $7, 7$ and 10, respectively.

By using the defining property of the Segre classes, one finds that the previous expression can be reexpressed as $c(E) \cdot s(N)[P(N)]$, where

c and s represent the total Chern classes and Segre classes, respectively. It immediately leads to a dramatic simplification once one realizes that $\mathcal{V}_1 \oplus \mathbf{C}^2 = N$!

From here it follows that $c(E) \cdot s(N)[P(N)] = c(\mathcal{V}_2 \oplus \mathcal{V}_3)[Z]$. By using the fact that Z is the typical fiber of $g : Y(\Gamma) \mapsto M$, and the fact that $\mathcal{V}_3$ is pulled back from M by g^*, one further reduces the previous expression to $c(\mathcal{V}_2)[Z]$. Because Z is 7 dimensional, the expression finally simplifies to $c_7(\mathcal{V}_2)[Z]$. The answer here should be multiplied by the number of singular curves with fourth order points, which finally leads to the final expression, identical to the one given by family Seiberg-Witten theory. One notices that the the points $p_1, p_2, \ldots p_r$ can be interpreted as the projection of the zero locus of the canonical section of $\mathcal{V}_3$ on $|C| \times M$. According to the standard theory, it can be reduced further to $\int_M c_2(\mathcal{V}_3)$, as was discussed in [29]. The same expression essentially appears in the family blowup formula reducing $FSW_M(1, C - 4E_1)$ to a mixed invariant.

A much more detailed discussion along this line will be presented in Part II of the paper.

10. List of notations

A A connection of some positive $spin_c$ spinor bundle which satisfies the Seiberg-Witten equations. See page 393.

$A_i(\Gamma_0, \mathbf{M}_0)$ the four formal power series in q which appear in the factorization of the generating function of the normalized modified invariants with type $\llcorner \Gamma_0, \mathbf{M}_0 \lrcorner$. See Theorem 7.5 on page 522.

adm(l) the set of the admissible graphs marked by $\{1, 2, \cdots, l\}$. See page 415.

adm$(\mathbf{I})$ the set of the admissible graphs marked by a finite set $\mathbf{I} \subset \mathbf{N}$. See page 415.

adm$(n)'$ the equivalence classes of admissible graphs in adm(n) under the action of admissibility preserving elements in the symmetric group $\mathbf{S}_l$. See Proposition 4.10 on page 433.

$\mathcal{ASW}(\eta, C_0)$ the mixed algebraic Seiberg-Witten invariant of the class C_0, where η is an element in the cycle class group $\mathcal{A}(B)$. See page 403.

$\mathcal{ADM}$ the set of admissible decompositions with $C_0 = C - \mathbf{M}(E)E$ fixed. See page 437.

$\mathcal{ADM}(s)$ the set of level s admissible decompositions in $\mathcal{ADM}$. See page 437.

B the base manifold of a fiber bundle $\mathcal{X} \mapsto B$. See page 395.

$Bl_X Y$ the blown up manifold by blowing up the almost complex submanifold $X \subset Y$. See page 468.

C in the symplectic setting, a cohomology class on $H^2(M, \mathbf{Z})$ or in the holomorphic geometry setting, a divisor class on M. See page 393.

C_0 a fiberwise class over $\mathcal{X} \mapsto B$. In the fiber bundle $M_{l+1} \mapsto M_l$, it is taken to be of the form $C - \mathbf{M}(E)E$. See page 405.

$\mathbf{C}_\Gamma$ the preexceptional cone attached to Y_Γ which is generated by a finite number of type I exceptional classes. See page 415.

$\mathcal{C}_j$ the j-Mori cone of an almost complex structure j. $\mathcal{C}_j^Q$ denotes the subset of $\mathcal{C}_j$ with energy bounded above by Q. Please consult Definition 8.1 on page 526.

codim(Γ) codimension of an admissible graph, defined by counting the number of 1-edges on Γ. See page 416.

comp(Γ) the number of connected components of a graph.

core$(\Gamma, \mathbf{M})$ the core of a topological type of singular curves $(\Gamma, C - \mathbf{M}(E)E)$. See Definition 5.3 on page 436.

$\chi(M)$ the Euler number of the manifold M.

$\mathrm{d}_{\mathbf{R}}(C_0)$ the real dimension of fiberwise Seiberg-Witten moduli space of the class C_0, which is equal to $C_0^2 - K \cdot C_0$. See page 394.

$\mathrm{d}_{\mathbf{C}}(C_0)$ the complex dimension of fiberwise Seiberg-Witten moduli space of the class C_0, which is equal to $\frac{C_0^2 - K\cdot C_0}{2}$. See page 396.

$\dim_{\mathbf{R}} B$ the real dimension of the manifold B.

$\mathbf{D}$ a decomposition class which consists of a collection of decompositions of the form $(C - \sum m_i e_i, \sum m_i e_i)$. See Definition 4.4 on page 408.

$\gg, \ll$ the subordinate partial ordering among different decomposition classes. See Definition 4.5 on page 409.

Δ_{ab} the (a,b) diagonal in M^l. See the proof of Proposition 3.1 on page 402.

Δ_l the relative diagonal map from M_l to $P_l = M_l \times_{M_{l-1}} M_l$. See the proof of Proposition 5.4 on page 451.

$\Delta(q)$ the weight 12 modular form of $SL_2(\mathbf{Z})$. See Theorem-Corollary 1.1 on page 389.

$\Delta_{\mathrm{II}}(q)$ the difference between $\mathcal{F}_{nor}(q)$ and $F_{RT}(q)$ which involves the contribution from the type II exceptional curves. See page 538.

$\mathbf{EC}_b(C_0)$ the exceptional cone of a class C_0 over $b \in B$. See Definition 4.2 on page 405.

Edge(Γ) the set of 1-edges of Γ. See Definition 4.9 on page 414.

$E_i(j), i \leq j, 1 \leq j \leq l$ the cohomology class in $H^*(M_l, \mathbf{Z})$ associated to the (i,j) blowing up. Occasionally, the same symbol also denotes the divisor class of the exceptional divisor. See page 402.

E_i the fiberwise exceptional class associated to the i-th blowing up. See page 416.

e_i the cohomology class of an exceptional curve. In this paper, it refers mostly to a type I exceptional class. See page 405, Definition 4.3 on page 408 and page 416.

ϵ an edge of an admissible graph. See page 424.

$e(\epsilon)$ the ending vertex of an edge ϵ. See page 424.

$\mathcal{F}(\Gamma_0, \mathbf{M}_0; M)$ the generating fuction of the modified family invariant on M with duplicated singularities of type $\left\lfloor \Gamma_0, \mathbf{M}_0 \right\rfloor$. The symbol $\overline{\mathcal{F}}(\Gamma_0, \mathbf{M}_0; M)$ denotes the normalized generating function by the change of variable $t \mapsto |G_{\Gamma_0, \mathbf{M}_0}| t$. See page 519.

$FSW_B(1, C_0)$ the pure family Seiberg-Witten Invariant of the class C_0, associated to the $spin_c$ structure parameterized by $\mathcal{L} = 2C_0 - K_M$ over the base B. We use $FSW_B\langle 1, \mathcal{L}\rangle$ when we use $\mathcal{L}$ to parameterize the invariant. See page 395.

$FSW_B(c, C_0)$ the mixed family Seiberg-Witten Invariant of the class C_0, associated to the $spin_c$ structure $2C_0 - K_M$ over the base B. The class c denotes a cohomology class $\in H^{\cdot}(B, \mathbf{Z})$. See page 395.

$FSW(\mathbf{D})$ the short hand notation of the mixed family invariant determined by the decomposition class $\mathbf{D}$. See page 448.

$FSW^*(1, C_0), FSW^*(c, C_0)$ the pure and mixed modified family invariant. Please consult Subsection 5.1 on page 437.

$FSW^*(\mathbf{D})$ the short hand notation of the modified mixed family invariant determined by the decomposition class $\mathbf{D}$. See page 448.

$F_{RT}(q)$ the compound Ruan-Tian power series. See page 537.

$F_k(q)^{K3}$ the generating function enumerating the number of nodal curves on $K3$. See Theorem-Corollary 1.1 on page 389.

$F_k(q)^{T^4}$ the generating function enumerating the number of nodal curve on T^4. See Theorem-Corollary 1.2 on page 389.

$\mathcal{F}^{for}(q)$ the formal power series of normalized family invariant enumerating the nodal curves in $|L|$ on M. See Theorem 1.1 on page 390.

$\mathcal{F}^{for}_{\widetilde{M}}(q)$ the formal power series of normalized family invariant enumerating the nodal curves in $|L - E|$ on the blown up $\widetilde{M}$. See Theorem 1.2 on page 390.

$f_l : M_l \mapsto M_{l-1}$ The natural projection map from M_l to M_{l-1} which gives M_l a fiber bundle structure over M_{l-1}. Please see page 400.

$G_2(q)$ the weight two quasi-modular form, which is an Einsenstein series. Please see Theorem-Corollary 1.1 on page 389.

tg_n the number n nodes nodal curves in a linear system. Used by Vainsencher in [59]. See page 554.

γ the trivial admissible graph with no edges. γ_l denotes the trivial admissible graph with l free vertexes. See page 418.

Γ An admissible graph. Please consult Subsection 4.3 on page 412.

$\Gamma(-1)$ the admissible graph constructed from Γ by removing the l-th vertex along with all the 1-edges ending at l. Please consult Subsection 4.4.1 on page 418.

Γ^i the admissible graph derived from Γ by removing all the edges **Not** starting from the vertex marked i. Please see page 417.

Γ_{e_i} the admissible graph attached to e_i such that $Y(\Gamma_{e_i})$ can be identified with the existence locus of the type I exceptional curve dual to e_i. Please consult page 442.

$\Gamma_{\mathbf{D}}$ the admissible graph attached to a decomposition class in the concrete universal family. Please see page 445.

$(\Gamma, C - \mathbf{M}(E)E)$ the topological type of singular curves in the class C. It is the shorthand notation of $(\Gamma, C - \sum m_i E_i)$. Please see page 436.

G_Γ the subgroup of the symmetric group $\mathbf{S}_l$ preserving the admissible graph Γ. Please consult Definition 4.12 on page 428.

$G(\Gamma)$ the subgroup of the symmetric group $\mathbf{S}_l$ which maps the admissible $\Gamma \in \text{adm}(l)$ to an admissible graph in $\text{adm}(l)$. Please consult Definition 4.12 on page 428.

κ the residual relative obstruction bundle of a switching process $C_0 \mapsto C_0 - \sum e_i$. Please consult page 443 in Subsection 5.2.

L an sufficiently very ample line bundle on M. See Main Theorem 1.1 on page 382.

$|L|$ the linear system of L, which is defined to be $\mathbf{P}(H^0(M, L))$. See Main Theorem 1.1 on page 382.

$\mathcal{L}$ a fiberwise $spin_c$ structure of $\mathcal{X} \mapsto B$. Please consult Section 2 on page 393.

M an algebraic surface or a symplectic four-manifold.

$M[n]$ The n-th Fulton-Mcpherson space of M.

M_l The l-th universal space of M. It can be constructed from M^l by $\frac{l(l+1)}{2}$ consecutive blowing ups. Please consult page 400 in Section 3.

$\mathbf{M}(E)$ the multiplicity function defined on $\text{Ver}(\Gamma)$, $\mathbf{M}(E) : \text{Ver}(\Gamma) \mapsto \mathbf{Z}$ which characterizes the resolution multiplicities. See page 436.

$\mathbf{M}(E)E$ the short hand notation of $\sum m_i E_i$, where m_i defines the multiplicity function $\mathbf{M}$. See page 437.

μ a smooth section of the fiberwise self dual two form of $\mathcal{X} \mapsto B$. Please see page 393.

$\mathcal{M}_{red}$ the reduced family moduli space. Please consult page 459.

$\mathcal{M}_S$ the reduced family moduli space over the space S. Please see Definition 6.2 on page 461.

$\mathcal{M}^{Kähler}$ the Kähler family Seiberg-Witten moduli space.

$\mathcal{M}^r_{Kähler}$ the $-r\omega$ deformed version of the Kähler family moduli space. Please consult Proposition 8.4 on page 540.

$\mathcal{M}_{red}(\omega, r, Z)$ the fiber product of the reduced $-r\omega$ deformed Kähler family moduli space with $Z \mapsto Y(\gamma)$. Please consult page 542.

$\mathcal{M}^{Kähler}_{\mathbf{D}}$ the reduced Kähler Seiberg-Witten family moduli space attached to the decomposition class $\mathbf{D}$.

$\mathcal{M}_{pert}$ the perturbed version of the reduced family Seiberg-Witten moduli space by the nested perturbation. See Proposition 6.3 on page 468.

$\mathcal{M}_{pert;k}$ the perturbed version of the reduced family Seiberg-Witten moduli space by k nested perturbations. Please consult page 469.

$\mathcal{M}^{res}_{pert}$ the residual part of the perturbed reduced family Seiberg-Witten moduli space. Please consult Proposition 6.3 on page 468.

$\mathcal{N}_S B$ the tubular neighborhood of a submanifold S in B. Similarly $\mathcal{N}_X Y$ represents a tubular neighborhood of an almost complex submanifold X in an almost complex manifold Y. Please see page 464 and page 467.

$N(\Gamma, \mathbf{M}(E))$ the contribution to the family invariant $FSW_{Y(\Gamma)}(1, C - \mathbf{M}(E)E)$ from the smooth curves which are expected to be propotional to the number of singular curves in $|C|$ with the prescribed topological type specified by $(\Gamma, \mathbf{M})$. Please see page 475.

$n_L(\delta)$ the virtual number of δ nodes curves in a generic δ dimensional sub-linear system of $|L|$. Please see Main Theorem 1.1 on page 382.

$n_L(\Gamma, L - \sum m_i E_i)$ the virtual number of singular curves of topological type $(\Gamma, L - \sum m_i E_i)$ in the linear system $|L|$. See Main Theorem 1.2 on page 383.

Obs the cokernel semibundle of the Seiberg-Witten deformation complex. Please see page 458 and page 461.

$\mathcal{OBS}$ the obstruction bundle of the Kuranishi model. Please consult page 467.

ω a family of fiberwise symplectic forms over the fiber bundle of four-manifolds $\mathcal{X} \mapsto B$. Please see page 393.

Perm(C) the set of permissible decomposition of the class C. See page 537.

π_i the composition of the projection map $M_l \mapsto M^l$ and $M^l \mapsto M$ to the $i-$copy of M. Please consult page 400.

p_a the projection morphism from the total space of the $\mathbf{P}^1$ bundle $\tilde{\tilde{\Xi}}_a$ to its base. Please consult page 444.

pc the partial compactification of $Y_{\Gamma_{\mathbf{D}}}$. See page 496.

Ψ A Dirac spinor in $\Gamma(\mathcal{S}^+)$. Please see page 393.

ψ a spinor in $\Gamma(\mathcal{S}^+)$. Please consult page 458.

ST_a^b the stablization map from adm(a) to adm$(b), b > a$ by joining $b-a$ free vertexes. See page 556.

$\mathcal{S}_r$ a strata parameterized by r over which the preexceptional cone is a constant cone. Please consult page 407.

$\mathcal{S}_{\mathbf{D}}$ the strata associated to $\mathbf{D}$. It is called the support of $\mathbf{D}$. Please consult Definition 4.5 on page 409.

$\mathcal{S}^{\pm}$ the positive and negative spinor bundles associated to a $spin_c$ structure. See page 458.

$s(\epsilon)$ the starting vertex of an edge ϵ. See page 424.

σ_i the section of the $\mathbf{P}^1$ bundle $\tilde{\tilde{\Xi}}'_j$ induced from the intersection with the type I exceptional curves dual to $e_i; e_i \cdot e_j = 1$. Please consult page 447.

$\sharp\Sigma((2^{[n]};S)$ the expected number of singular curves on an algebraic surface S with singularities of type $2^{[n]}$ in [59]. Please consult page 554.

$\triangle_i; i = 1,2$ the two classes of vertexes in $\mathrm{Ver}(\Gamma_i)$. Please see page 430.

$\mathrm{Ver}(\Gamma)$ the set of vertexes of Γ. Please see Definition 4.9 on page 414.

$\mathcal{X}$ a fiber bundle of smooth four-manifolds over certain base manifold B. Please see page 395.

$X_{j;k}$ a collection of almost complex submanifolds of Y which are parameterized by two indices $(j,k) \in \mathbf{J} \times \mathbf{N} \cup \{0\}$. Please consult page 467.

Ξ_a the total space of the universal type I exceptional curve associated to e_a, which has a $\mathbf{P}^1$ fibration structure. Please consult page 442.

$\widetilde{\Xi}_a$ the relative minimal model of Ξ_a which has a $\mathbf{P}^1$ fiber bundle structure. Please consult Proposition 5.1 on page 442.

Y_Γ the locally closed strata in M_l associated to Γ. Please consult the remark right after Proposition 4.2 on page 415.

$Y(\Gamma)$ the closure of Y_Γ in M_l. Please consult Proposition 4.3 on page 416.

$Y_{\mathbf{D}} = Y_{\Gamma_{\mathbf{D}}}$ the support of a type I decomposition class.

$Y_{j;k}$ the difference $X_{j;k} - \cup_{p;q} X_{p;q}$. Please consult Definition 6.3 on page 468.

$\Gamma > \Gamma'$ the partial order $>$ among admissible graphs by the degeneration relationship. Please consult Definition 4.8 on page 413.

level a nonnegative integer attached to a decomposition class to measure its order among the set of the decomposition classes. Please consult Axiom 4.1 on page 410.

$\mathcal{V}_a$ the relative obstruction bundle associated to the switching process $C - \mathbf{M}(E)E - \sum_{b\in J} e_b \mapsto C - \mathbf{M}(E) - \sum_{b\in J} e_b - e_a$. Please see page 443 and the proof of Proposition 5.2 on page 443.

$\overline{\mathcal{V}}_a$ the relative obstruction sheaf corresponding to $\mathcal{V}_a$.

Υ_a a divisor in $\widetilde{\Xi}_a$ which is constructed by restricting $-C + \mathbf{M}(E)E + \sum_{b\in J} e_b + e_a$. Its degree is equal to $e_a \cdot (\mathbf{M}(E)E + \sum_{b\in J} e_b + e_a)$. Please consult page 445.

$\Upsilon_{a,1}$ **or** $\Upsilon_{a,2}$ the different versions of Υ_a in two different switching processes where the orders of e_{a_1} and e_{a_2} are reversed. Please see page 447.

$\zeta_{\mathbf{D}}$ the sum of all the type I exceptional class $\sum_i e_i$ which are involved in a decomposition class $\mathbf{D}$. See page 498.

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