J. DIFFERENTIAL GEOMETRY
57 (2001) 23-46

VOLUMES OF TUBES IN HYPERBOLIC 3-MANIFOLDS

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Abstract

We give the first explicit lower bound for the length of a geodesic in a closed orientable hyperbolic 3-manifold M of lowest volume. We also give an upper bound for the tube radius of any shortest geodesic in M. We explain how these results might be the first steps towards a rigorous computer assisted effort to determine the least volume closed orientable hyperbolic 3-manifold(s).

1. Introduction

In this paper we prove the following Theorem[1] .

Theorem 1.1. *If W is a maximal tube of radius r about a geodesic γ in the complete orientable hyperbolic 3-manifold M and either* length(γ) ≤ 0.069 *or* $r \geq 1.483$*, then* volume$(W) \geq 0.95$.

The fact that the volume of the "Weeks" manifold—the hyperbolic 3-manifold obtained by performing $(5, 1), (5, 2)$ Dehn surgery on the two components of the Whitehead Link—is less than 0.943 provides us with the following Corollary.

Received July 7, 2000, and, in revised form, September 21, 2000. The first author partially supported by NSF grants DMS-9505253 and DMS-0071852, the second author by NSF grant DMS-9801736, and the third author by NSF grants DMS-9505253 and DMS-0071852.

[1] Using the ideas introduced in the present paper, the bounds of this theorem and/or its corollary have been improved in [13], [9], and [2]. For Corollary 1.2, the current best announced bound for length is 0.1036, and for radius is 1.032; see [2]. We note that our proof of Theorem 1.1 obtains slightly stronger numbers than those in the statement of the Theorem. We decided to use the weaker numbers (from an earlier version of our paper) in the statement of the Theorem because of the numerous references to them in the literature.

Corollary 1.2. *A shortest geodesic in a lowest volume closed orientable hyperbolic* 3*-manifold has length greater than* 0.069 *and tube radius less than* 1.483*.*

This theorem and corollary constitute the vital first step of a project whose goal is to determine the lowest volume closed hyperbolic 3-manifold(s). Specifically, our plan is to extend the computer-aided parameter space analysis of [8] from solid tubes of radius less than $\log(3)/2$ to solid tubes of sufficiently large radius to enable us to answer the low-volume question. The first step in our plan is to show that the relevant parameter space for this question is a compact set which is feasible to study.

The natural parameters which classify solid tubes around shortest geodesics in hyperbolic 3-manifolds are the complex length of the shortest geodesic, the "complex radius" of the solid tube, and the complex length of the "following transformation" (see [8] or Chapter 6; these are, respectively, L, D, and R). Reasonable bounds are straightforward to determine for all but the real length of the shortest geodesic.

The above theorem provides what we hope is a usable lower bound on the real length of the shortest geodesic. In §6, we discuss our evidence for believing that this lower bound enables us to determine a parameter space that will be reasonable to analyze.

Before focusing on the proof of Theorem 1.1, we will spend a few paragraphs discussing low-volume hyperbolic 3-manifolds. By Mostow Rigidity, the volume of a complete finite-volume hyperbolic 3-manifold is a topological invariant. Thurston [14], building on work of Gromov and Jørgensen, showed that the order type of the set of volumes of complete finite volume hyperbolic 3-manifolds is ω^ω and that at most finitely many manifolds can have the same volume. In particular, there is a closed hyperbolic 3-manifold realizing the minimum volume v_1.

The Weeks manifold (see above) has volume $0.9427\ldots$ and is the lowest volume closed manifold yet discovered. This manifold provides an upper bound of $v_1 \leq 0.9427\ldots$.

For several years, the best lower bound for v_1 was about 0.001 (see [11], [10], [6]). But recently, research on solid tubes around shortest geodesics in hyperbolic 3-manifolds by Gabai, Meyerhoff, and N. Thurston (see [8]), using the work of Gehring and Martin (see [7]), improved this lower bound to $0.16668\ldots$. Przeworski (see [12]) sharpened the Gehring-Martin approach and produced a lower bound of 0.276. This bound is actually a lower bound on the volume of a solid tube in the hyperbolic 3-manifold as is the $0.166\ldots$ bound.

A complete proof of Theorem 1.1 which relies on results from §2–§5 will be given at the end of this Introduction. The proof focuses on a solid tube of radius r in a closed hyperbolic 3-manifold. In particular, we study lifts of the tube to $\mathbb{H}^3$ and study how close these lifts are to each other. As such, before describing our proof, we will develop some definitions and facts related to tubes and the placement of lifts of tubes.

1.1 Orthoclasses and spectra

Let γ be a simple closed geodesic in the complete orientable hyperbolic 3-manifold $M = \mathbb{H}^3/G$ where G is identified with $\pi_1(M)$. Let γ_i denote the lifts of γ to the universal covering $\mathbb{H}^3$ with γ_0 denoting a fixed lift. We say that two lifts γ_i, γ_j are *conjugate* if there exists a $w \in \pi_1(M)$ such that $w(\gamma_i) = \gamma_0$ and $w(\gamma_0) = \gamma_j$. Let $H \subset \pi_1(M)$ be the subgroup generated by γ. We can assume that it is the subgroup of $\pi_1(M)$ which stabilizes γ_0. Partition $\{\gamma_i\} - \gamma_0$ into equivalence classes called *orthoclasses* by saying that γ_i is equivalent to γ_j if either γ_i is conjugate to γ_j or $\gamma_i = h(\gamma_j)$ where $h \in H$.

Lemma 1.3. *Each orthoclass contains exactly two H-orbits.*

Proof. Because H is the stabilizer of γ_0, an orthoclass contains at most two H-orbits. Conversely if γ_i and γ_j are conjugate via w, then γ_i and γ_j lie in distinct H-orbits (a fact first proven for horoballs by Adams [1]). To see this let α denote the oriented geodesic arc from γ_i to γ_0. Then $w(\alpha)$ is the oriented geodesic arc from γ_0 to γ_j. If $\gamma_j = h(\gamma_i)$ for $h \in H$, then $w^{-1}h(\alpha) = -\alpha$, where $-\alpha$ denotes α with the opposite orientation. Thus $w^{-1}h$ is a nontrivial covering transformation that has a fixed point, which is a contradiction. q.e.d.

Associated to an orthoclass is a positive real number which is the real distance from any element in that class to γ_0. Let $\mathcal{O}(1), \mathcal{O}(2), \ldots$ denote the orthoclasses ordered so that if $O(i)$ denotes the corresponding real distance, then $O(1) \leq O(2) \leq \cdots$. The solid tube V_0 of radius $r = O(1)/2$ projects to a tube W in M with the property that the interior of W is embedded and W is only immersed. Thus W is called a *maximal tube* and $r = O(1)/2$ is called the *tuberadius* of γ.

For each i, let V_i denote the tube of radius $O(1)/2$ about γ_i. The orthoclass equivalence relation of $\{\gamma_i\} - \gamma_0$ induces an equivalence relation on $\{V_i\} - V_0$. Call such an equivalence class an *orthotube class* and let $\mathcal{OT}(k)$ denote the class corresponding to $\mathcal{O}(k)$. Define $OT(k) = O(k) - O(1)$ which equals the distance from V_0 to any element in $\mathcal{OT}(k)$. The reason for studying the *orthotube spectrum* (which is the sequence $OT(1)$, $OT(2)$, $\ldots$), rather than the *ortholength spectrum* is that the concept of orthotube spectrum generalizes in the obvious way to the notion of *orthohoroball spectrum* for noncompact complete hyperbolic 3-manifolds. We note that it is useful to think of manifolds with very thick tubes as being geometrically close to cusped manifolds, and hence to treat them somewhat like cusped manifolds.

The first two of the following useful formulas are well known and probably go back to Lobachevsky, see [5]. The third formula is due to Gehring-Martin [7] and is a consequence of the first two formulae.

Lemma 1.4. *Let W be a tube of radius r about a closed geodesic of length l. Then*

- $\text{area}(\partial W) = 2\pi l \sinh(r)\cosh(r)$
- $\text{volume}(W) = \pi l \sinh^2(r)$
- $\text{volume}(W) = \frac{1}{2}\tanh(r)\text{area}(\partial W)$

We call the orthogonal projection of V_i to ∂V_0 the *shadow* of V_i, and denote it $S(V_i)$ (see [7] and [12]).

We are now in a position to describe the key ideas of this paper.

Our method is similar in spirit to that of Gehring and Martin (see [7] and [12]). They analyze the shadow of V_1 and use it (twice, via a variant of Lemma 1.3) to get a lower bound on the area of the boundary of the maximal tube W in terms of the radius of W. This, of course, produces a lower bound on the volume of the tube and hence on the volume of the associated manifold. Their work only uses $\mathcal{OT}(1)$. Our main contribution is to use $\mathcal{OT}(2)$ to obtain a greater lower bound on tube volume.

A technical difference from [7] is that we work on the cylinder of radius $2r$ centered about γ_0 while Gehring and Martin work (via shadows) on the cylinder of radius r. For example, when studying the placement of centers of tubes in $\mathcal{OT}(1)$ we work on the cylinder of radius $2r$, which is where these centers live. Note that the *center* of a tube is the point on the tube's core geodesic which lies nearest to γ_0.

Now, we return to our main innovation. We analyze the additional contribution to area (hence volume) coming from the next closest tubes.

That is, we analyze $\mathcal{OT}(2)$. The general theme here is to split into two cases: first, if the $\mathcal{OT}(2)$ tubes are substantially farther from V_0 than the $\mathcal{OT}(1)$ tubes, then this implies that the centers of the $\mathcal{OT}(1)$ tubes are even farther apart than we had thought. This gives us more area, thereby enhancing our volume bound.

Second, if the $\mathcal{OT}(2)$ tubes are not much farther from V_0 than the $\mathcal{OT}(1)$ tubes, then this tells us that the $\mathcal{OT}(2)$ tubes contribute significant area (either on the cylinder of radius r or on the cylinder of radius $2r$), hence enhancing our volume bound.

We found the optimal trade-off between "not much farther" and "substantially farther" to occur at 0.298. In particular, we prove that if $OT(2) \geq 0.298$ and length$(\gamma) \leq 0.0717$ or tuberadius$(\gamma) \geq 1.464$, then volume$(M) \geq 0.943$, and if $OT(2) \leq 0.298$, and length$(\gamma) \leq .0717$ or tuberadius$(\gamma) \geq 1.464$, then volume$(M) \geq 0.943$.

The details are carried out in §4 and §5. In §2 certain geometric lemmas which are used in the rest of the paper are proved. In §3 an upper bound on the length of γ is given in terms of the tuberadius of γ, and this is shown to be equivalent to a lower bound on the tuberadius in terms of the length in certain cases. This leads to a preliminary lower bound on the volume of the tubular neighbourhood in terms of either the length or the tuberadius. The needed lower bounds are obtained in §4 and §5.

Proof of Theorem 1.1. First assume $OT(2) \geq 0.298$. If $r \geq 1.464$, then Proposition 4.1 applies, and hence $\pi l \sinh(r)^2 \geq 0.943$. On the other hand if $l \leq 0.0717$, then by Proposition 3.1, r is (considerably) larger than 0.149. Therefore Proposition 4.1 applies and $\pi l \sinh(r)^2 \geq 0.943$.

Now, assume $OT(2) \leq 0.298$. If $r \geq 1.464$, Proposition 5.1 applies and hence $\pi l \sinh(r)^2 \geq 0.943$. On the other hand if $l \leq 0.0717$, then, as in the previous paragraph, we use Proposition 3.1 to obtain the needed control over r (specifically, $r > 0.2014$ is needed). Hence Proposition 5.1 applies and $\pi l \sinh(r)^2 \geq 0.943$.

In either case, if $l \leq 0.0717$ or $r \geq 1.464$ then the volume of M is at least 0.943, proving the theorem. q.e.d.

2. Geometric lemmas

The remainder of this paper makes heavy use of certain facts regarding skew quadrilaterals in $\mathbb{H}^3$, as well as facts about the intersection of a hyperbolic sphere with a hyperbolic cylinder. The proofs of these facts in turn make use of the Klein hyperboloid model of hyperbolic space, which is not as commonly used as the upper half-space model and hence may not be as familiar to all readers. Hence this chapter will present a brief review of the Klein hyperboloid model as well as the lemmas which depend on it; for more details about the hyperboloid model, see for example Thurston's book [15].

In the Klein hyperboloid model, $\mathbb{H}^3$ is the hypersurface

$$\{(x,y,z,t) \in \mathbb{R}^4 | -x^2 - y^2 - z^2 + t^2 = 1, t > 0\}$$

and geodesics are just the intersection of $\mathbb{H}^3$ with planes through the origin in $\mathbb{R}^4$. The orientation-preserving isometries of this model are given by the connected component of the identity in the Lie group

$$O(3,1) = \{A \in \mathrm{GL}(4,\mathbb{R}) | A^t Q A = Q\}$$

where Q is the diagonal matrix with diagonal entries $(1,1,1,-1)$. Elements of this group act on points in $\mathbb{H}^3 \subset \mathbb{R}^4$ by matrix multiplication on the left (view points as column vectors). The metric d_H is given by the formula

$$\cosh^{-1}(-x_1x_2 - y_1y_2 - z_1z_2 + t_1t_2). \tag{2.1}$$

Using this model, without loss of generality assume that γ_0 is the intersection of $\mathbb{H}^3$ with the plane $\{x = y = 0\}$, and that γ_0 is oriented in the positive z-direction. Let λ be the geodesic which is the intersection of $\mathbb{H}^3$ with the plane $\{y = z = 0\}$, oriented in the positive x-direction. Then γ_0 and λ intersect at right angles. Define a set of *cylindrical coordinates* for $\mathbb{H}^3 - \gamma_0$ as follows: a point $p \in \mathbb{H}^3 - \gamma_0$ has cylindrical coordinates (d, ϕ, τ), with $\tau > 0$, if and only if p is the image of a point $q \in \lambda$ under a loxodromic motion along γ_0 with complex length $d + i\phi$ where q lies at a distance of τ from γ_0 in the direction of λ. Then the cylidrical coordinates of any point in $\mathbb{H}^3 - \gamma_0$ are defined uniquely except for ϕ, which is defined uniquely modulo 2π. This coordinate system can clearly be extended to all of $\mathbb{H}^3$ by letting $\tau = 0$, although ϕ is no longer well-defined when $\tau = 0$.

Given the preceding set-up, the following lemma is quite easy to prove in the Klein model:

Lemma 2.1. *The hyperbolic distance between two points p_0, $p_1 \in$*

$\mathbb{H}^3$ is given by the formula

$$\cosh(d_H(p_0,p_1)) = \cosh(d_1-d_0)\cosh(\tau_0)\cosh(\tau_1) - \cos(\phi_1-\phi_0)\sinh(\tau_0)\sinh(\tau_1) \tag{2.2}$$

where (d_0,ϕ_0,τ_0) and (d_1,ϕ_1,τ_1) are the cylindrical coordinates of p_0 and p_1 respectively.

Proof. Assume that $d_0=\phi_0=0$; the proof in the general case is similar. Then p_0 is the point $(\sinh(\tau_0),0,0,\cosh(\tau_0))$. Furthermore the loxodromic motion along γ_0 with complex length $d_1+i\phi_1$ is given by the matrix

$$\begin{pmatrix} \cos(\phi_1) & -\sin(\phi_1) & 0 & 0 \\ \sin(\phi_1) & \cos(\phi_1) & 0 & 0 \\ 0 & 0 & \cosh(d_1) & \sinh(d_1) \\ 0 & 0 & \sinh(d_1) & \cosh(d_1) \end{pmatrix}.$$

Thus p_1 must be the point

$$(\cos(\phi_1)\sinh(\tau_1),\sin(\phi_1)\sinh(\tau_1),\sinh(d_1)\cosh(\tau_1),\cosh(d_1)\cosh(\tau_1)).$$

Hence by the distance formula for the hyperboloid model,

$$\cosh(d_H(p_0,p_1)) = \cosh(d_1)\cosh(\tau_0)\cosh(\tau_1) - \cos(\phi_1)\sinh(\tau_0)\sinh(\tau_1)$$

as desired. q.e.d.

Lemma 2.1 (which can be thought of as a law of cosines for skew quadrilaterals in $\mathbb{H}^3$) can also be proved using more traditional hyperbolic trigonometry, but the above proof has a more direct feel to it.

The following submanifold of $\mathbb{H}^3$ will also be useful. Let $C\subset\mathbb{H}^3$ be the boundary of the cylinder of radius $2r$ about γ_0. Note that C is a Euclidean surface, isometric to the boundary of a right circular cylinder in $\mathbb{E}^3$ of radius $\sinh(2r)$, and every geodesic in $\mathcal{O}(1)$ is tangent to C. Denote the induced Euclidean metric on C by $d_E(p,q)$, to distinguish it from the hyperbolic metric $d_H(p,q)$, and let $u\in C$ be the point with cylindrical coordinates $(0,0,2r)$.

Lemma 2.2. *If $p\in C$ has cylindrical coordinates $(d,\phi,2r)$ with $-\pi<\phi\le\pi$, then*

$$d_E(p,u)^2 = (d\cosh(2r))^2 + (\phi\sinh(2r))^2.$$

Proof. Suppose first that $d = 0$. Then the shortest path in C from u to p is a circular arc with radius $2r$ and angle $|\phi|$. By elementary hyperbolic geometry (see for example [5]) such a curve has length $|\phi|\sinh(2r)$. On the other hand, if $\phi = 0$ then the shortest path in C from u to p is a curve lying in a plane containing γ_0, at a constant distance of $2r$ from γ_0; the length of such a curve is again a known result and equals $|d|\cosh(2r)$. Since C is a Euclidean manifold and the curves $\{d = 0, \tau = 2r\}$ and $\{\phi = 0, \tau = 2r\}$ intersect at right angles at u, the result follows from the Pythagorean theorem. q.e.d.

A consequence of the above lemma is that $x = d\cosh(2r)$ and $y = \phi\sinh(2r)$ are natural Euclidean coordinates for C.

Now for any $t > 0$ and $\rho \geq 0$, let $B(t, \rho)$ be the region consisting of all points $c \in C$ such that $d_H(c, q) \leq t$, where q is the unique point in $\mathbb{H}^3$ with cylindrical coordinates $(0, 0, 2r + \rho)$. The boundary of $B(t, \rho)$ is the intersection of C with a sphere of radius t centered at q; it is easy to see that for $\rho < t < 4r + \rho$, the boundary of $B(t, \rho)$ will be a single non-empty closed curve. The following lemma describes the size of $B(t, \rho)$ as a subset of the Euclidean manifold C:

Lemma 2.3. *If $0 \leq \rho < t < 4r + \rho$, then the region $B(t, \rho) \subset C$ contains a disk centered at u with radius*

$$R_{t,\rho} = \sqrt{\sinh(2r)\cosh(2r)\coth(2r+\rho)} \times \cosh^{-1}\left(\frac{\sinh(2r)\sinh(2r+\rho) + \cosh(t)}{\cosh(2r)\cosh(2r+\rho)}\right). \tag{2.3}$$

Proof. By Lemma 2.1, the boundary of $B(t, \rho)$ can be described by the equation

$$\cosh(d)\cosh(2r)\cosh(2r+\rho) - \cos(\phi)\sinh(2r)\sinh(2r+\rho) = \cosh(t)$$

where d, ϕ, and $\tau = 2r$ are cylindrical coordinates on $\mathbb{H}^3$. Since $|\cos(\phi)| \leq 1$,

$$|d| \leq \cosh^{-1}\left(\frac{\sinh(2r)\sinh(2r+\rho) + \cosh(t)}{\cosh(2r)\cosh(2r+\rho)}\right). \tag{2.4}$$

Let a be the quantity on the right-hand side of Equation (2.4).

The next step is to find a lower bound on ϕ in terms of d. The following derivation is based on a similar derivation in [7]. By elementary calculus, for all $|d| \leq a$

$$\begin{aligned} \cosh(a) - \cosh(d) &\geq \frac{a^2 - d^2}{2} \\ \Rightarrow \quad \frac{\sinh(2r)\sinh(2r+\rho) + \cosh(t)}{\cosh(2r)\cosh(2r+\rho)} - \cosh(d) &\geq \frac{a^2 - d^2}{2}. \end{aligned}$$

Rearranging the last inequality, for any point with cylindrical coordinates $(d, \phi, 2r)$ on the boundary of $B(t, \rho)$ we get

$$\begin{aligned} 1 - &\frac{\cosh(2r)\cosh(2r+\rho)}{\sinh(2r)\sinh(2r+\rho)} \left(\frac{a^2 - d^2}{2} \right) \\ &\geq \frac{\cosh(d)\cosh(2r)\cosh(2r+\rho) - \cosh(t)}{\sinh(2r)\sinh(2r+\rho)} = \cos(\phi). \end{aligned}$$

By another application of elementary calculus, the left-hand side of the above inequality is less than

$$\cos\left(\sqrt{\coth(2r)\coth(2r+\rho)(a^2 - d^2)} \right).$$

Hence

$$\cos\left(\sqrt{\coth(2r)\coth(2r+\rho)(a^2 - d^2)} \right) \geq \cos(\phi).$$

Assume that $-\pi < \phi \leq \pi$. Since $\phi = 0$ when $|d| = a$, and since the boundary of $B(t, \rho)$ is a single continuous curve, for all points $(d, \phi, 2r)$ on the boundary of $B(t, \rho)$

$$|\phi| \geq \sqrt{\coth(2r)\coth(2r+\rho)(a^2 - d^2)}.$$

Converting into Euclidean coordinates, $|x| \leq a\cosh(2r)$ and

$$|y| = |\phi \sinh(2r)| \geq \sqrt{\tanh(2r)\coth(2r+\rho)}\sqrt{(a\cosh(2r))^2 - x^2}$$

for any point on the boundary of $B(t, \rho)$ with Euclidean coordinates (x, y). Thus the region $B(t, \rho)$ contains an ellipse with axes parallel to the $\{x, y\}$-coordinate axes. Since tanh is an increasing function,

$$\tanh(2r)\coth(2r+\rho) \leq 1.$$

Hence the minor axis of this ellipse is in the y-direction. Thus the region $B(t, \rho) \subset C$ contains a circle centred at u with radius

$$a\sqrt{\sinh(2r)\cosh(2r)\coth(2r+\rho)}.$$

This proves the lemma. q.e.d.

3. The first order tuberadius formula

In this section, we use the technology developed in §1 and §2 to show how the radius of a maximal tube and the length of its core (closed) geodesic control each other. This is the content of Proposition 3.1.

The tube-volume estimates that follow from Proposition 3.1 are not strong enough to prove Theorem 1.1. In fact, the analysis in the present section is to some extent a warm-up for the more intricate analyses of §4 and §5. We also note that the first part of Proposition 3.1 was originally proved by Gehring and Martin in [7], although this only becomes apparent after some trigonometric manipulation.

Proposition 3.1. *If γ is a geodesic of length l and tuberadius r in a complete orientable hyperbolic 3-manifold, then*

$$l \geq \frac{\sqrt{3}\cosh(2r)}{2\pi\sinh(2r)}\left(\cosh^{-1}\left(\frac{\sinh^2(2r)+\cosh(2r)}{\cosh^2(2r)}\right)\right)^2. \tag{3.1}$$

Furthermore if $l \leq 0.110629$, then the right side of (3.1) is invertible and we obtain an implicit lower bound on r as a function of l.

Proof. Recall that for any tube $V_j \neq V_0$, the *center* v_j of V_j is the point on γ_j which is closest to γ_0. Thus the center of a tube in $\mathcal{OT}(n)$ must lie at a distance of $O(n)$ from γ_0. Let C and u be defined as in the previous chapter; in particular, C is a tube in $\mathbb{H}^3$ of radius $2r$ (*not* radius r). Without loss of generality there exists a tube $U \in \mathcal{OT}(1)$ such that u is the center of U.

The images of u under H (the stabilizer of γ_0; see §1) are the vertices of a tiling of C by quadrilaterals. The area A of a fundamental region for the action of H on C is the area of one of these quadrilaterals. However, A is also the area of the boundary of the manifold $\overline{C}/H$, where $\overline{C}$ is the solid cylinder bounded by C. Hence by Lemma 1.4,

$$A = 2\pi l\cosh(2r)\sinh(2r).$$

If we can produce a lower bound for A in terms of r alone then we can exploit it and the above area formula to get a lower bound for l in terms of r. We do this in the next three paragraphs.

According to Lemma 2.3, the distance $d_E(p,q)$ between the centers of any two tubes in $\mathcal{OT}(1)$ is bounded below by $R_{2r,0}$. Then each such center is contained in a circle of radius $R_{2r,0}/2$ (in C) containing no other such centers, and any two such circles have disjoint interiors. Note however that any fundamental region for the action of H contains the centers of at least two tubes in $\mathcal{OT}(1)$, one from each H-orbit. Thus,

$$A \geq 2\pi \left(\frac{R_{2r,0}}{2} \right)^2 .$$

As in [1] and [10], improve the above estimate by a constant factor as follows: a packing of C by circles of radius $R_{2r,0}/2$ lifts to a circle packing of the Euclidean plane, which can be no denser than the hexagonal packing. The hexagonal packing of the plane has density $\pi/(2\sqrt{3})$, hence

$$\begin{aligned} A &\geq \frac{\pi {R_{2r,0}}^2}{2} \frac{2\sqrt{3}}{\pi} \\ &= {R_{2r,0}}^2 \sqrt{3}. \end{aligned}$$

Comparing this area formula and the above area formula (involving l and r), we see that

$$l \geq \frac{{R_{2r,0}}^2 \sqrt{3}}{2\pi \cosh(2r)\sinh(2r)}. \tag{3.2}$$

Substituting the value of $R_{2r,0}$ from Equation (2.3) into Equation (3.2) above proves the first part of Proposition 3.1.

We now prove the second part of Proposition 3.1. That is, we show how to bound r in terms of l. A computer examination of the function on the right-hand side of Equation (3.1) shows that it has the asymptotic properties one would expect. Namely, as r goes to infinity the lower bound on l goes to 0 and the corresponding lower bound on the volume of W (which is just $\pi l \sinh^2(r)$ by Lemma 1.4) approaches $\sqrt{3}/2$. However the behavior of the function near 0 is problematic: the function goes to 0 as $r \to 0^+$ and seems to have a local maximum near $r = 0.5$. Thus, Equation (3.1) alone does not give a lower bound for r in terms of l.

To get around this problem, we use an earlier theorem by Meyerhoff and Zagier [11] as sharpened by Cao-Gehring-Martin [3].

Theorem 3.2 (Meyerhoff-Zagier; Cao-Gehring-Martin). *Let γ be a geodesic in a complete orientable hyperbolic 3-manifold. If the real length l of γ is less than*

$$\frac{\sqrt{3}}{2\pi}(\sqrt{2}-1) \approx 0.11418$$

then there exists an embedded solid tube around γ whose radius r satisfies

$$\sinh^2(r) = \frac{\sqrt{1-(4\pi l/\sqrt{3})}}{(4\pi l/\sqrt{3})} - \frac{1}{2}.$$

This theorem gives a lower bound for r in terms of l, namely

$$r \geq \sinh^{-1}\left(\sqrt{\frac{\sqrt{1-(4\pi l/\sqrt{3})}}{(4\pi l/\sqrt{3})} - \frac{1}{2}}\right)$$

and given that $l > 0$, the above expression can be written as

$$l \geq \frac{\sqrt{3}}{2\pi}\left(\frac{-1+\sqrt{2+4\sinh^2(r)+4\sinh^4(r)}}{\left(1+2\sinh^2(r)\right)^2}\right). \tag{3.3}$$

Comparing the expression on the right hand side of (3.3) with that on the right hand side of (3.1), one sees that the (3.3) estimate of l is greater than the one from (3.1) when $r < 0.22926$, at which point both estimates say that $l \geq 0.110629$. Furthermore, an examination of Equation (3.1) by computer indicates that the function on the right hand side of (3.1) is increasing for r less than 0.52396, and decreasing for r greater than 0.52397, and the function equals 0.110629 when r equals 0.22926... or 0.98296....

Hence for $r > 0.98296$ and $l < 0.110629$ the function on the right hand side of (3.1) is invertible. Hence Equation (3.1) can be expressed as a lower bound on r in terms of l which is valid when $l \leq 0.110629$. This completes the proof of Proposition 3.1. q.e.d.

The asymptotic value of $\pi l \sinh^2(r)$ for the volume of the maximal tube W given by this estimate when $l \to 0^+$ (or equivalently when $r \to \infty$) is $\sqrt{3}/2$. Note that the corresponding asymptotic value obtained from the Meyerhoff-Zagier estimate alone is $\sqrt{3}/4$, so the combined result is an improvement by a factor of two. Unfortunately, $\sqrt{3}/2$ is approximately 0.866 while the volume of the Weeks manifolds is approximately 0.943. So it is necessary to find ways to improve the result of Proposition 3.1 in particular cases.

For example, if $OT(2) = 0$ then Lemma 1.3 together with the proof of Proposition 3.1 yield

Proposition 3.3. *If γ is a geodesic of length l and tuberadius r in a complete orientable hyperbolic 3-manifold, and if $OT(2) = 0$, then*

$$l \geq \frac{\sqrt{3}\cosh(2r)}{\pi \sinh(2r)} \left(\cosh^{-1}\left(\frac{\sinh^2(2r)+\cosh(2r)}{\cosh^2(2r)}\right)\right)^2. \tag{3.4}$$

Furthermore if $l \leq 0.1134$, then the right side of formula (3.4) is invertible and provides an implicit lower bound on r as a function of l.

Note that the asymptotic volume of W obtained from Proposition 3.3 when $l \to 0^+$ is $\sqrt{3}$. (Proposition 3.3 is not used in the rest of this paper; instead, Proposition 5.1 is used.)

4. The second order tuberadius formula

In this section the result of Proposition 3.1 is improved when $OT(2)$ is sufficiently large. That is, we now take into account information from the next closest orthoclass of solid tubes. Specifically, we find that the fact that the next nearest orthoclass (to V_0) is relatively far away means that the members of the $\mathcal{OT}(1)$ orthoclass are farther apart than we accounted for in Proposition 3.1. This produces a better area bound on C (the cylinder of radius $2r$) and hence better length-radius control and volume bounds. In §5 we deal with the case where the next nearest orthoclass $\mathcal{OT}(2)$ is relatively close to V_0.

Proposition 4.1. *If γ is a geodesic of length l and tuberadius r in a complete orientable hyperbolic 3-manifold and if $OT(2) \geq \rho$, where $0 < \rho < 2r$, then*

$$l \geq \frac{\sqrt{3}\cosh(2r)}{2\pi \sinh(2r)} \left(\cosh^{-1}\left(\frac{\sinh^2(2r)+\cosh(2r+\rho)}{\cosh^2(2r)}\right)\right)^2. \tag{4.1}$$

Furthermore if $\rho = 0.298$ then the right-hand side of formula (4.1) is strictly decreasing in r and hence invertible, and provides an implicit lower bound on r as a function of l; these bounds imply that if either $r \geq 1.464$, or $l \leq 0.0717$ and $r > 0.149$, then $\pi l \sinh(r)^2 \geq 0.943$ (provided that $OT(2) \geq \rho = 0.298$).

Proof. Let A and B be distinct tubes in $\mathcal{OT}(1)$. Since $OT(2) \geq \rho > 0$, by Lemma 4.2 at the end of this section $d(A, B) \geq \rho$.

Hence the center of each tube in $\mathcal{OT}(1)$ is contained in an open ball of hyperbolic radius $2r + \rho$ which contains the center of no other tube in $\mathcal{OT}(1)$. By Lemma 2.3, the intersection of this ball with the cylinder C contains a disk of radius $R_{2r+\rho,0}$. Replacing $R_{2r,0}$ with $R_{2r+\rho,0}$ in formula (3.2) proves the first part of Proposition 4.1. The proof that the right hand side of (4.1) is decreasing when $\rho = 0.298$ is a matter of computer analysis.

In particular, the estimate of Proposition 4.1 when $\rho = 0.298$ says $\pi l \sinh^2(r) \geq 0.943$ when $r \geq 1.464$, and that $r \geq 1.464$ when $l \leq 0.0717$ (assuming $r > 0.149$).

It remains to prove the following:

Lemma 4.2. *Suppose $A, B \in \mathcal{OT}(1)$, $A \neq B$, and $OT(2) > 0$. Then $d(A, B) \geq OT(2)$.*

Proof. Suppose instead that $d(A, B) < OT(2)$. Let $j \in \pi_1(M)$ be such that $j(A) = V_0$. Then $d(V_0, j(B)) < OT(2)$, which implies $j(B) \in \mathcal{OT}(1)$. In other words $j(B)$ and V_0 are tangent and consequently B and $j^{-1}(V_0) = A$ are tangent. We now show that this implies the existence of an elliptic element of order 3 in $\pi_1(M)$, a contradiction.

There are two cases to consider. Either A and B lie in the same H-orbit, or they lie in conjugate H-orbits. These cases will be handled in turn in the next two subsections, which will complete the proof.

4.1 A and B lie in the same H-orbit

Suppose that A and B lie in the same H-orbit, that is $B = \sigma(A)$ where $\sigma \in H$. Since $j(B) \in \mathcal{OT}(1)$, $j(B)$ is a translate of either A or $j(V_0)$ under H by Lemma 1.3.

Suppose that $j(B) = \alpha(A)$, $\alpha \in H$. Then $j\sigma j^{-1}(V_0) = \alpha j^{-1}(V_0)$, and hence $j\sigma j^{-1} = \alpha j^{-1}\beta$ for some $\beta \in H$. Let $f = j^{-1}\alpha$, $g = \sigma$, and $h = \alpha^{-1}\beta^{-1}\alpha$. Then $g, h \in H$, $f \notin H$, and $gfh = f^2$. At this point, resort to direct computation in the upper half-space model of hyperbolic three-space. Then g, h, and f are all elements of $\mathrm{PSL}(2, \mathbb{C})$. In addition we assume without loss of generality that the line γ_0 is the line from 0 to ∞, and hence all elements of H have this line as their axis. Furthermore by multiplying h by $-I$ if necessary assume that $gfh = f^2$ as matrices in $\mathrm{SL}(2, \mathbb{C})$.

Then g and h will be diagonal matrices with diagonal entries (a, a^{-1}) and (b, b^{-1}) respectively, while

$$f = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

for some x, y, z, w satisfying $xw - yz = 1$. Making the appropriate substitutions, the matrix equation $gfh = f^2$ becomes

$$\begin{pmatrix} abx & ab^{-1}y \\ a^{-1}bz & a^{-1}b^{-1}w \end{pmatrix} = \begin{pmatrix} x^2+yz & y(x+w) \\ z(x+w) & w^2+yz \end{pmatrix}.$$

Taking the products of the (1,2) and (2,1) entries on each side,

$$yz = yz(x+w)^2.$$

Note that if $z = 0$ then f fixes ∞, which would imply that $f \in H$, a contradiction. Similarly if $y = 0$ then f fixes 0, also a contradiction. So $yz \neq 0$, and hence $(x+w)^2 = 1$. Hence f is elliptic of order 3, a contradiction.

Now suppose instead that $j(B) = \alpha j(V_0)$ for some $\alpha \in H$. Then $j\sigma j^{-1} = \alpha j \beta$ for some $\beta \in H$. This time let $f = \beta j$, $g = \beta\alpha^{-1}\beta^{-1}$, and $h = \sigma$. Then again $gfh = f^2$, and so again f is elliptic of order 3, a contradiction.

4.2 A and B lie in conjugate H-orbits

Suppose now that $B = \sigma j(V_0)$ for some $\sigma \in H$. Then $\sigma j(A) = V_0$, and $\sigma j(V_0) = B$. Consider $(\sigma j)^{-1}(A)$. This tube is tangent to $(\sigma j)^{-1}(B) = V_0$ and hence is an element of $\mathcal{OT}(1)$. (Recall the assumption that $OT(2) > 0$.) Hence $(\sigma j)^{-1}(A)$ must be a translate of either A or $j(V_0)$ under H. If $(\sigma j)^{-1}(A) = \alpha(A)$ for some $\alpha \in H$, then the result of the previous section applies since $(\sigma j)^{-1}(A)$ is also tangent to $(\sigma j)^{-1}(V_0) = A$.

Suppose then that $(\sigma j)^{-1}(A) = \alpha j(V_0)$ for some $\alpha \in H$. Then $j^{-1}\sigma^{-1}j^{-1}(V_0) = \alpha j(V_0)$, and hence $j^{-1}\sigma^{-1}j^{-1} = \alpha j \beta$ for some $\beta \in H$. Let $f = j\sigma$, $g = \sigma^{-1}\alpha$, and $h = \sigma^{-1}\beta$. Then $g, h \in H$ while $f \notin H$, and $gfh = f^{-2}$. Proceeding as in the previous section, lift everything to $\mathrm{SL}(2, \mathbb{C})$ and assume that g and h are diagonal matrices with diagonal entries (a, a^{-1}) and (b, b^{-1}) respectively, and that

$$f = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

for some x, y, z, w satisfying $xw - yz = 1$. Then the matrix product $gfh = f^{-2}$ expands to

$$\begin{pmatrix} abx & ab^{-1}y \\ a^{-1}bz & a^{-1}b^{-1}w \end{pmatrix} = \begin{pmatrix} w^2+yz & -y(x+w) \\ -z(x+w) & x^2+yz \end{pmatrix}.$$

Multiplying the (1,2) and (2,1) entries on each side,

$$yz = yz(x+w)^2.$$

And as in the previous case, $yz \neq 0$. So $(x+w)^2 = 1$. Hence f is elliptic of order 3, a contradiction. This completes the proof of Lemma 4.2.

q.e.d.

5. Bounding the tuberadius when $OT(2)$ is small

In this section the result of Proposition 3.1 is improved when $OT(2)$ is sufficiently small. The significance of this condition is that the orthotube class $\mathcal{OT}(2)$ is relatively close to V_0 and makes a significant contribution to the area of the cylinder C.

Proposition 5.1. *If γ is a geodesic of length l and tuberadius r in a complete orientable hyperbolic 3-manifold, and if $r > 0.2014$ and $OT(2) = \rho \leq 0.298$, then*

$$l \geq \frac{1}{\cosh(2r)\sinh(2r)}\left(\frac{{R_{2r,0}}^2}{4} + \left(R_{2r,0.298} - \frac{R_{2r,0}}{2}\right)^2\right). \tag{5.1}$$

Furthermore if $l \leq 0.11014$ then the right-hand side of (5.1) is invertible and provides an implicit lower bound on r as a function of l; these bounds imply that if either $r \geq 1.464$, or $l \leq 0.0717$ and $r > 0.2014$, then $\pi l \sinh(r)^2 \geq 0.943$ (provided that $OT(2) \leq 0.298$).

Proof. Choose $V \in \mathcal{OT}(2)$, let v be the center of V, and let v' be the projection of v to the cylinder C. The interior of the hyperbolic ball centered at v of radius $2r$ will not contain the center of any tube in $\mathcal{OT}(1)$. Hence by Lemma 2.3 there is a disk in C centered at v' with radius $R_{2r,\rho}$ whose interior does not contain the center of any tube in $\mathcal{OT}(1)$. We use the following lemma to obtain a radius which is independent of ρ:

Lemma 5.2. *For fixed r and $0 < \rho < 2r$, $R_{2r,\rho}$ is a decreasing function of ρ.*

Proof: From Equation (2.3) one can see that because coth is decreasing for positive values and $\cosh^{-1}$ is increasing, it suffices to show that the function

$$\frac{\sinh(2r)\sinh(2r+\rho) + \cosh(2r)}{\cosh(2r)\cosh(2r+\rho)}$$

is decreasing in ρ. But by direct calculation,

$$\begin{aligned}\frac{\partial}{\partial\rho}&\frac{\sinh(2r)\sinh(2r+\rho)+\cosh(2r)}{\cosh(2r)\cosh(2r+\rho)}\\&=\frac{\sinh(2r)-\cosh(2r)\sinh(2r+\rho)}{\cosh(2r)\cosh^2(2r+\rho)}\leq 0\end{aligned}$$

because $\cosh(2r)\sinh(2r+\rho)\geq\sinh(2r)$. This proves the lemma.

q.e.d.

Hence the interior of the disk in C centered at v' with radius $R_{2r,0.298}$ does not contain the center of any tube in $\mathcal{OT}(1)$, provided that $r>0.149$. (The reason why we require that $r>0.2014$ and not $r>0.149$ is explained in the next paragraph.)

Now pack the cylinder C with disks as follows. Around the center of each tube in $\mathcal{OT}(1)$ place a disk of radius $R_{2r,0}/2$. Next, place a disk of radius $R_{2r,0.298}-R_{2r,0}/2$ around the projection to C of the center of each tube in $\mathcal{OT}(2)$ (for example, one such disk is centered at v'). We claim that none of these disks overlap (that is, their interiors are disjoint). First, none of the primary disks (that is, those of radius $R_{2r,0}/2$) overlap, by the same argument as given in the proof of Proposition 3.1. Second, by computer analysis, we see that $R_{2r,0.298}>R_{2r,0}/2$ for $r>0.2014$, hence the "secondary" disks do not overlap the "primary" disks. Finally, we use the following lemma to show that no two secondary disks overlap.

Lemma 5.3. *If V_i, $V_j\in\mathcal{OT}(2)$ are tubes with centers v_i, v_j respectively, and if the projections of v_i and v_j to C are the points ${v_i}'$ and ${v_j}'$ respectively, then*

$$d_E({v_i}',{v_j}')\geq 2R_{2r,0.298}-R_{2r,0}. \tag{5.2}$$

Proof. Without loss of generality, we can assume that $v_i'=u$, or in other words that v_i has cylindrical coordinates $(0,0,2r+\rho)$. Let the cylindrical coordinates of v_j be $(d,\phi,2r+\rho)$. Because $d_H(v_i,v_j)\geq 2r$, Lemma 2.1 says that

$$\cosh(d)\cosh^2(2r+\rho)-\cos(\phi)\sinh^2(2r+\rho)\geq\cosh(2r).$$

Then by the same arguments as in the proof of Lemma 2.3, the distance $d_E({v_i}',{v_j}')$ along C must be at least δ_ρ, where

$$\begin{aligned}\delta_\rho=&\sqrt{\sinh(2r)\cosh(2r)\coth(2r+\rho)}\\&\times\cosh^{-1}\left(\frac{\sinh^2(2r+\rho)+\cosh(2r)}{\cosh^2(2r+\rho)}\right).\end{aligned}\tag{5.3}$$

So it suffices to show that $\delta_\rho \geq 2R_{2r,0.298} - R_{2r,0}$. Now note that coth is decreasing for positive values, $\cosh^{-1}$ is increasing, and

$$\begin{aligned}\frac{\partial}{\partial\rho}\frac{\sinh^2(2r+\rho)+\cosh(2r)}{\cosh^2(2r+\rho)} &= (\cosh^4(2r+\rho))^{-1}\\ &\quad\times\Big(2\sinh(2r+\rho)\cosh(2r+\rho)\\ &\qquad - 2\cosh(2r)\sinh(2r+\rho)\cosh(2r+\rho)\Big)\\ &\leq 0.\end{aligned}$$

Hence δ_ρ is decreasing in ρ, so it suffices to show that $\delta_{0.298} \geq 2R_{2r,0.298} - R_{2r,0}$. But this inequality is a special case of the following lemma:

Lemma 5.4. *If $0 < \rho < 2r$, then $R_{2r,0} - R_{2r,\rho} \geq R_{2r,\rho} - \delta_\rho$.*

Proof. Note that if $R_{2r,\rho} - \delta_\rho < 0$, then the Lemma is true since $R_{2r,0} - R_{2r,\rho} \geq 0$ by Lemma 5.2. So assume that $R_{2r,\rho} - \delta_\rho \geq 0$.

Let $g(r,v,w) = \cosh^{-1}\left(\frac{\sinh(2r+v)\sinh(2r+w)+\cosh(2r)}{\cosh(2r+v)\cosh(2r+w)}\right)$. Then, after dividing through by $\sqrt{\sinh(2r)\cosh(2r)}$, it must be shown that

$$\begin{aligned}&\sqrt{\coth(2r)}\ g(r,0,0) - \sqrt{\coth(2r+\rho)}\ g(r,\rho,0)\\ &\qquad \geq \sqrt{\coth(2r+\rho)}\ g(r,\rho,0) - \sqrt{\coth(2r+\rho)}\ g(r,\rho,\rho).\end{aligned}$$

We show this by proving that

$$\begin{aligned}&\sqrt{\coth(2r)}\ g(r,0,0) - \sqrt{\coth(2r+\rho)}\ g(r,\rho,0)\\ &\qquad \geq \sqrt{\coth(2r+\rho/2)}\ g(r,0,0) - \sqrt{\coth(2r+\rho/2)}\ g(r,\rho,0)\\ &\qquad \geq \sqrt{\coth(2r+\rho/2)}\ g(r,\rho,0) - \sqrt{\coth(2r+\rho/2)}\ g(r,\rho,\rho)\\ &\qquad \geq \sqrt{\coth(2r+\rho)}\ g(r,\rho,0) - \sqrt{\coth(2r+\rho)}\ g(r,\rho,\rho).\end{aligned}$$

The first and last inequalities are simple consequences of the fact that the coth is a decreasing function. (Note that our previous assumption implies that $g(r,\rho,0) - g(r,\rho,\rho) > 0$.) The middle inequality is harder to prove. After dividing through by $\sqrt{\coth(2r+\rho/2)}$ we need only show that $g(r,0,0) - g(r,\rho,0) \geq g(r,\rho,0) - g(r,\rho,\rho)$. This will follow from hyperbolic trigonometry applied to certain quadrilaterals with two adjacent right angles (these are sometimes called Saccheri quadrilaterals). The side of such a quadrilateral ending in the two right angles will be called the *base* of the quadrilateral. For convenience, let the four sides in clockwise order be called the west, north, east, and south sides, where the base is the west side.

Using hyperbolic trigonometry (see [5] pg.88), we see that $X = g(r,0,0)$ is the base of the Saccheri quadrilateral with three other sides of length $2r$; $Y = g(r,\rho,0)$ is the base of the Saccheri quadrilateral with three other sides of length $2r+\rho$, $2r$, $2r$ in clockwise order; and $Z = g(r,\rho,\rho)$ is the base of the Saccheri quadrilateral with three other sides of length $2r+\rho$, $2r$, $2r+\rho$ in clockwise order. Note that the two non-base vertices can be thought of as the centers of circles of radius r in all three quadrilaterals, and that in each quadrilateral these circles abut.

Overlap the first and third quadrilaterals in such a way that the base of the third is a subset of the base of the first, and the mid-points of the two bases coincide. Then, the first quadrilateral has excess base length $\frac{X-Z}{2}$ on both ends.

Now, overlap the first and second quadrilaterals, so that the base of the second is a subset of the base of the first, and the west-south vertices coincide. Then, the first quadrilateral has excess base length $X-Y$ on the west-north end. Further, the circle associated to the north-east vertex of the second quadrilateral must dip below the perpendicular bisector to the base of the first quadrilateral. But the north-east-vertex circle for the third quadrilateral does not dip below this line. Hence, $X - Y > \frac{X-Z}{2}$.

This shows that $X - Y \geq Y - Z$, proving the lemma. q.e.d.

By the above two lemmas, C can be packed with disks of two different radii. Specifically, C can be packed with disks of radius $R_{2r,0}/2$ around the centers of tubes in $\mathcal{OT}(1)$, and disks of radius $R_{2r,0.298} - R_{2r,0}/2$ around those points v' which are the projections to C of centers of tubes in $\mathcal{OT}(2)$. Furthermore, both $\mathcal{OT}(1)$ and $\mathcal{OT}(2)$ consist of two H-orbits by Lemma 1.3. Hence, if A is the area of a fundamental region on C as before, then

$$A \geq 2\pi \left(\frac{R_{2r,0}}{2}\right)^2 + 2\pi \left(R_{2r,0.298} - \frac{R_{2r,0}}{2}\right)^2.$$

And as before $A = 2\pi l \cosh(2r)\sinh(2r)$; this and the above equation prove the first part of Proposition 5.1.

5.1 Bounding r in terms of l

As with Equation (3.1), the right-hand side of Equation (5.1) is not an invertible function. However, comparing the estimate of Equation (5.1) with that of Equation (3.3), we see that the trade-off point is at $0.2442\ldots$. At this point, both estimates give $l \geq 0.11014$. The function on the right-hand side of (5.1) has a single local maximum at approximately $r = 0.591$, and it equals 0.11014 when r equals one of $0.24419\ldots$ and $1.2042\ldots$. Thus Equation (5.1) and Equation (3.3) together determine a lower bound for r in terms of l when $l \leq 0.11014$, or equivalently $r \geq 1.2042$. This proves the second part of Proposition 5.1.

Equation (5.1) also implies that $\pi l \sinh^2(r) \geq 0.943$ when $r \geq 1.464$. Furthermore, the implicit lower bound on r in terms of l implies that $r \geq 1.464$ when $l \leq 0.0717$. This completes the proof. q.e.d.

6. Good parameter space

In this section, we will present evidence for believing that Theorem 1.1 produces a reasonable solid tube parameter space for the low-volume question. We begin by giving a brief introduction to the solid tube parameter space as in [8].

Associated to a solid tube around an oriented shortest closed geodesic γ in a hyperbolic 3-manifold is a natural 2-generator (torsion-free) Kleinian group. The first generator f is a (primitive) covering transformation fixing a normalized lift γ_0 of γ. The second generator w is a covering transformation taking a lift γ_1 (nearest to γ_0) to γ_0. Such 2-generator groups can be parametrized by three complex numbers. The first parameter is the complex length of the core geodesic, and determines the transformation f. The second and third parameters determine w. The second parameter is the complex length of the transformation α taking γ_1 to γ_0 along their unique common perpendicular. It might seem as if α is w, but this is not (necessarily) correct, one may have to "follow" α by a transformation β whose fixed axis is γ_0. The complex length of β is the third parameter.

[8] studies the parameter space of such (marked) 2-generator groups and sees, in certain cases, which ones could possibly correspond to maximal solid tubes around *shortest* geodesics in a hyperbolic 3-manifold. [8] focused on maximal solid tubes of radius less than $\log(3)/2$ because this was the relevant tuberadius for the applications therein. The parameter space was reduced to a compact parameter space by exploiting the

method of [11] to show that if the shortest geodesic has length less than 0.0978, then it has a solid tube of radius $\log(3)/2$ (the other parameter bounds are easier to develop).

[8] showed that, with six families of exceptions (corresponding to six sub-boxes in the parameter space), closed hyperbolic 3-manifolds must have maximal tubes of radius greater than $\log(3)/2$ around their shortest geodesic(s). A large amount of the computation time in [8] was spent in the parameter space near the parameter points associated with these six exceptional families. The regions in the parameter space away from the exceptional points were relatively easy to eliminate (that is, to show that they could not have a maximal tube of the radius in question; maximality would not allow so small a tube).

We now make some comments about the solid tube parameter space required for the low-volume question. We have only begun analyzing this parameter space; as such, our comments are rather speculative, although based on experience with the [8] parameter space, and also based on some solid-tube analysis via a modified version of J. Weeks's program SnapPea (see [16]). The present paper is concerned with the most difficult parameter bound to obtain: the real length of the shortest geodesic. The question at hand is whether the bound of $l \geq 0.1036$ (see [2]; which improves the bounds of Theorem 1.1) is an effective bound.

Our belief is that $l \geq 0.1036$ will be a reasonable bound to work with. First, we note that in [8] a bound of $l \geq 0.0978$ was successfully used (though we recognize that in the present computer analysis, significantly larger tube radii will need to be analyzed). Second, we provide some (very preliminary) experimental data. We used a modified version of J. Weeks's program SnapPea [16] to analyze solid tubes around shortest geodesics as follows. We considered surgeries of the form $(p, q), 0 \leq |p|, |q| \leq 5$ on each of the cusps on the orientable manifolds in the Hodgson-Weeks census [16] of 5-tetrahedral cusped manifolds. (In this notation, *meridian* and *longitude* correspond to the two shortest curves in the associated Euclidean structure.) To each surgered manifold yielding a hyperbolic manifold we computed both the length and tube volume of a shortest geodesic. This nonrigorous experimental work has produced 12 manifolds with tube volumes < 0.943. A scatter plot of length versus tube volume is given in Figure 1. The solid line depicts tube volume 0.943. Despite the limited number of manifolds studied and the fact that the census manifolds are special (relatively combinatorially uncomplicated), it is striking that not many small volume tubes have appeared, that the lengths of the shortest geodesics in the exceptional

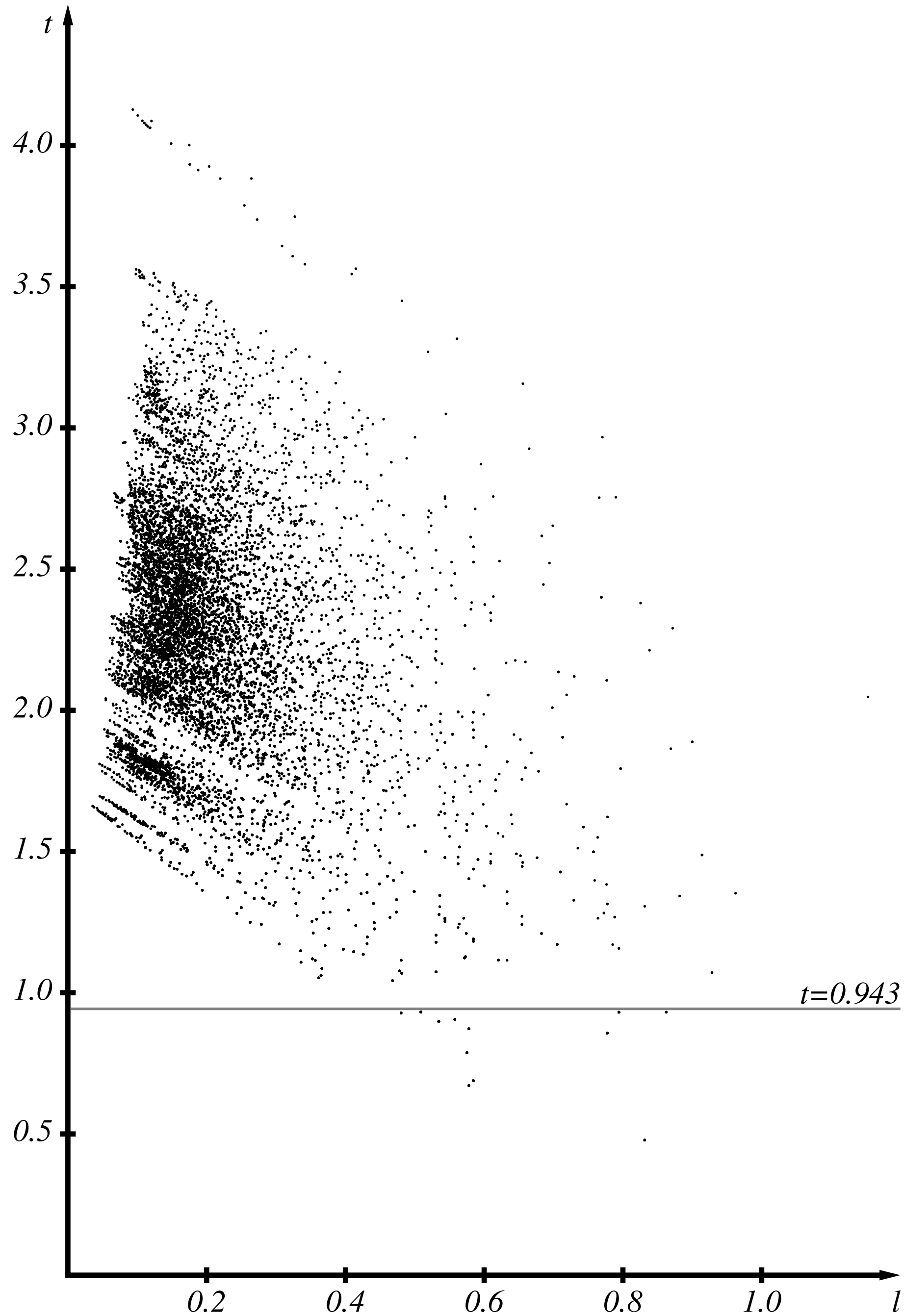

Figure 1: Tube volume versus length, for the shortest geodesics in a selection of closed manifolds.

manifolds are greater than 0.48, and when l is short the cloud of data points lies comfortably above the 0.943 tube volume line. Note that the closed manifolds considered here include all of the orientable manifolds in the closed census [16] up to volume 2.36 and most of the census manifolds up to volume 3.8. The exceptional manifolds are m003(2,1), m003(-1,3), m007(3,1), m003(1,3), m010(-3,1), m009(4,1), m007(1,2), m007(4,1), m007(3,2), m010(1,2), m006(-3,2), and m036(-3,2). Thus far all (resp. 11, 3) of the exceptional manifolds are obtained as surgeries on manifolds obtained by gluing 4 (resp. 3, 2) tetrahedra or less.

We are cautiously optimistic that there will not be too many exceptional parameter points and manifolds with which we must ultimately contend.

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