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# Issues related to Rubio de Francia's Littlewood–Paley inequality

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Abstract. Let $\mathrm{S}_\omega f = \int_\omega \widehat{f}(\xi) e^{ix\xi}\, d\xi$ be the Fourier projection operator to an interval $\omega$ in the real line. Rubio de Francia's Littlewood–Paley inequality (Rubio de Francia, 1985) states that for any collection of disjoint intervals $\Omega$, we have

$$\Bigl\| \Bigl[ \sum_{\omega \in \Omega} |\mathrm{S}_\omega f|^2 \Bigr]^{1/2} \Bigr\|_p \lesssim \|f\|_p, \qquad 2 \le p < \infty.$$

We survey developments related to this inequality, including the higher dimensional case, and consequences for multipliers.

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# 1. Introduction

Our subject is a group of topics related to Rubio de Francia's extension [36] of the classical Littlewood–Paley inequality. We are especially interested in presenting a proof that highlights an approach in the language of time-frequency analysis, and addresses the known higher dimensional versions of this theorem. It is hoped that this approach will be helpful in conceiving of new versions of these inequalities. A first result in this direction is in the result of Karagulyan and the author [27]. These inequalities yield interesting consequence for multipliers, and these are reviewed as well.

Define the Fourier transform by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi}\, dx.$$

In one dimension, the projection onto the positive frequencies

$$\mathrm{P}_+ f(x) := \int_0^\infty \widehat{f}(\xi) e^{ix\xi}\, d\xi$$

is a bounded operator on all $L^p(\mathbb{R})$, $1 < p < \infty$. The typical proof of this fact first establishes the $L^p$ inequalities for the Hilbert transform, given by

$$\mathrm{H}\, f(x) := \lim_{\epsilon\to\infty} \int_{|y|>\epsilon} f(x-y) \frac{dy}{y}.$$

The Hilbert transform is given in frequency by a constant times

$$\mathrm{H}\, f(x) = c \int \widehat{f}(\xi)\, \mathrm{sign}(\xi)\, \mathrm{e}^{ix\xi} \; d\xi.$$

We see that $\mathrm{P}_+$ is linear combination of the identity and H. In particular $\mathrm{P}_+$ and H enjoy the same mapping properties.

In this paper, we will take the view that $L^p(\mathbb{R}^d)$ is the tensor product of $d$ copies of $L^p(\mathbb{R})$. A particular consequence is that the projection onto the positive quadrant

$$\mathrm{P}_+\, f(x) := \int_{[0,\infty]^d} f(\xi)\, \mathrm{e}^{ix\cdot\xi} \; d\xi$$

is a bounded operator on all $L^p(\mathbb{R}^d)$, as it is merely a tensor product of the one-dimensional projections.

A rectangle in $\mathbb{R}^d$ is denoted by $\omega$. Define the Fourier restriction operator to be

$$\mathrm{S}_\omega\, f(x) = \int_\omega \mathrm{e}^{ix\cdot\xi}\, \widehat{f}(\xi)\; d\xi.$$

This projection operator is bounded on all $L^p(\mathbb{R}^d)$, with constant bounded independently of $\omega$. To see this, define the modulation operators by

$$\mathrm{Mod}_\xi\, f(x) := \mathrm{e}^{ix\cdot\xi}\, f(x). \tag{1.1}$$

Observe that for $\xi = (\xi_1, \ldots, \xi_d)$, the interval $\omega = \prod_{j=1}^d [\xi_j, \infty)$, we have $\mathrm{S}_\omega = \mathrm{Mod}_{-\xi}\, \mathrm{P}_+\, \mathrm{Mod}_\xi$. Hence this projection is uniformly bounded. By taking linear combinations of projections of this type, we can obtain the $L^p$ boundedness of any projection operator $\mathrm{S}_\omega$, for rectangles $\omega$.

The theorem we wish to explain is:

**1.2. Theorem.** *Let $\Omega$ be any collection of disjoint rectangles with respect to a fixed choice of basis. Then the square function below maps $L^p(\mathbb{R}^d)$ into itself for $2 \le p < \infty$:*

$$\mathrm{S}^\Omega\, f(x) := \Bigl[ \sum_{\omega\in\Omega} |\mathrm{S}_\omega\, f(x)|^2 \Bigr]^{1/2}.$$

In one dimension this is Rubio de Francia's Theorem [36]. His proof pointed to the primacy of a BMO estimate in the proof of the theorem. The higher dimensional form was investigated by J.-L. Journé [26]. His original argument has been reshaped by F. Soria [40], S. Sato, [37], and Xue Zhu [43]. In this instance, the product BMO is essential, in the theory as developed by S.-Y. Chang and R. Fefferman [21, 14, 13].

We begin our discussion with the one-dimensional case, followed by the higher dimensional case. We adopt a 'time-frequency' approach to the theorem, inspired in part by the author's joint work with Christoph Thiele [29, 28]. The same pattern is adopted for the multiplier questions. The paper concludes with notes and comments.

We do not keep track of the value of generic absolute constants, instead using the notation $A \lesssim B$ iff $A \le KB$ for some constant $K$. Write $A \simeq B$ iff $A \lesssim B$ and $B \lesssim A$. For a rectangle $\omega$ and scalar $\lambda > 0$, $\lambda\omega$ denotes the rectangle with the same center as $\omega$ but each side length is $\lambda$ times the same side length of $\omega$. We use the notation $\mathbf{1}_A$ to denote the indicator function of the set $A$, that is, $1_A(x) = 1$ if $x \in A$ and is otherwise 0. Averages of integrals over a set are written as

$$\fint_A f\,dx := |A|^{-1} \int_A f\,dx.$$

For an operator $T$, $\|T\|_p$ denotes the norm of $T$ as an operator from $L^p(\mathbb{R}^d)$ to itself. In addition to the Modulation operator defined above, we will also use the translation operator

$$\mathrm{Tr}_y\, f(x) := f(x-y).$$

We shall assume the reader is familiar with the norm bounds for the one-dimensional maximal function

$$\mathrm{M}\, f(x) = \sup_t \fint_{[-t,t]} |f(x-y)|\,dt$$

The principal fact we need is that it maps $L^p$ into itself for $1 < p < \infty$. In $d$ dimensions, the strong maximal function refers to the maximal function

$$\mathrm{M}\, f(x) = \sup_{t_1,\dots,t_d>0} \fint_{[-t_1,t_1]\times\cdots[-t_d,t_d]} |f(x_1-y_1,\dots,x_d-y_d)|\,dy_1\cdots dy_d$$

Note that this maximal function is less than the one-dimensional maximal function applied in each coordinate in succession.

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## 2. The one-dimensional argument

In this setting, we give the proof in one dimension, as it is very much easier in this case. In addition, some of the ideas in this case will extend immediately to the higher dimensional case.

**2.1. Classical theory.** We should take some care to recall the classical theory of Littlewood and Paley. Let $\Delta$ denote the dyadic intervals

$$\Delta := \bigl\{\epsilon[2^k, 2^{k+1}) \,:\, \epsilon \in \{\pm 1\},\ k \in \mathbb{Z}\bigr\}.$$

The classical theorem is that:

**2.1. Theorem.** *For all $1 < p < \infty$, we have*

$$\bigl\|\mathrm{S}^{\Delta} f\bigr\|_p \simeq \|f\|_p. \tag{2.2}$$

We will not prove this here, but will make comments about the proof. If one knows that

$$\bigl\|\mathrm{S}^{\Delta} f\bigr\|_p \lesssim \|f\|_p, \qquad 1 < p < \infty, \tag{2.3}$$

then a duality argument permits one to deduce the reverse inequality for $L^{p'}$ norms, $p' = p/(p-1)$. Indeed, for $g \in L^{p'}$, choose $f \in L^p$ of norm one so that $\|g\|_{p'} = \langle f, g\rangle$ Then

$$\begin{aligned}
\|g\|_{p'} &= \langle f, g\rangle \\
&= \int \sum_{\omega\in\Delta} \mathrm{S}_\omega f \overline{\mathrm{S}_\omega g}\, dx \\
&\le \langle \mathrm{S}^{\Delta} f, \mathrm{S}^{\Delta} g\rangle \\
&\le \|\mathrm{S}^{\Delta} f\|_p \|\mathrm{S}^{\Delta} g\|_{p'} \\
&\lesssim \|\mathrm{S}^{\Delta} g\|_{p'}.
\end{aligned}$$

One only need prove the upper inequality for the full range of $1 < p < \infty$.

In so doing, we are faced with a common problem in the subject. Sharp frequency jumps produce kernels with slow decay at infinity, as is evidenced by the Hilbert transform, which has a single frequency jump and a nonintegrable kernel. The operator $\mathrm{S}^{\Delta}$ has infinitely many frequency jumps. It is far easier to to study a related operators with smoother frequency behavior, for then standard aspects of Calderón–Zygmund Theory are at one's disposal. Our purpose is then to introduce a class of operators which mimic the behavior of $\mathrm{S}^{\Delta}$, but have smoother frequency behavior.

Consider a smooth function $\psi_+$ which satisfies $\mathbf{1}_{[1,2]} \le \widehat{\psi_+} \le \mathbf{1}_{[\frac{1}{2},\frac{5}{2}]}$. Notice that $\psi * f$ is a smooth version of $S_{[1,2]} f$. Let $\psi_- = \overline{\psi_+}$. Define the dilation operators, of scale $\lambda$, by

$$\mathrm{Dil}^{(p)}_{\lambda} f(x) := \lambda^{-1/p} f(x/\lambda), \qquad 0 < p \le \infty,\ \lambda > 0. \tag{2.4}$$

The normalization chosen here normalizes the $L^p$ norm of $\mathrm{Dil}^{(p)}_{\lambda}$ to be one.

Consider distributions of the form

$$K=\sum_{k\in\mathbb{Z}}\sum_{\sigma\in\{\pm\}}\varepsilon_{k,\sigma}\,\mathrm{Dil}^{(1)}_{2^k}\,\psi_\sigma,\qquad \varepsilon_{k,\sigma}\in\{\pm1\},\tag{2.5}$$

and the operators $\mathrm{T}\,f=K*f$. This class of distributions satisfies the standard estimates of Calderón–Zygmund theory, with constants independent of the choices of signs above. In particular, these estimates would be

$$\begin{gathered}\sup_\xi|\widehat{K}(\xi)|<C\,,\\ |K(y)|<C|y|^{-1}\,,\\ |\tfrac{d}{dy}K(y)|<C|y|^{-2}\,,\end{gathered}$$

for a universal constant $C$. These inequalities imply that the operator norms of T on $L^p$ are bounded by constants that depend only on $p$.

The uniformity of the constants in the operator norms permits us to average over the choice of signs, and apply the Khintchine inequalities to conclude that

$$\Bigl\|\Bigl[\sum_{k\in\mathbb{Z}}\sum_{\sigma\in\{\pm\}}\Bigl|\mathrm{Dil}^{(1)}_{2^k}\,\psi_\sigma*f\Bigr|^2\Bigr]^{1/2}\Bigr\|_p\lesssim\|f\|_p,\qquad 1<p<\infty.\tag{2.6}$$

This is nearly the upper half of the inequalities in Theorem 1.2. For historical reasons, "smooth" square functions such as the one above, are referred to as "$G$ functions."

To conclude the theorem as stated, one method uses an extension of the boundedness of the Hilbert transform to a vector-valued setting. The particular form needed concerns the extension of the Hilbert transform to functions taking values in $\ell^q$ spaces. In particular, we have the inequalities

$$\|\|\mathrm{H}\,f_k\|_{\ell^q}\|_p\lesssim C_{p,q}\|\|f_k\|_{\ell^q}\|_p,\qquad 1<p,q<\infty.\tag{2.7}$$

Vector-valued inequalities are strongly linked to weighted inequalities, and one of the standard approaches to these inequalities depends upon the beautiful inequality of C. Fefferman and E.M. Stein [19]

$$\int|\mathrm{H}\,f|^q g\;dx\lesssim\int|f|^q(\mathrm{M}|g|^{1+\epsilon})^{1/(1+\epsilon)}\;dx,\quad 1<q<\infty,\ 0<\epsilon<1.\tag{2.8}$$

The implied constant depends only on $q$ and $\epsilon$. While we stated this for the Hilbert transform, it is important for our purposes to further note that this inequality continues to hold for a wide range of Calderón–Zygmund operators, including those that occur in (2.5). This is an observation that goes back to J. Schwartz [38], with many extensions, especially that of Benedek, Calderón and Panzone [2].

The proof that (2.8) implies (2.7) follows. Note that we need only prove the vector-valued estimates for $1 < q \leq p < \infty$, as the remaining estimates follow by duality, namely the dual estimate of $\mathrm{H} : L^p(\ell^q) \longrightarrow L^p(\ell^q)$ is $\mathrm{H} : L^{p'}(\ell^{q'}) \longrightarrow L^{p'}(\ell^{q'})$, in which the primes denote the conjugate index, $p' = p/(p-1)$. The cases of $q = p$ are trivial. For $1 < q < p < \infty$, and $\{f_k\} \in L^p(\ell^q)$ of norm one, it suffices to show that

$$\Big\| \sum_k |\mathrm{H}\, f_k|^q \Big\|_{p/q} \lesssim 1.$$

To do so, by duality, we can take $g \in L^{(p/q)'}$ of norm one, and estimate

$$\begin{aligned} \sum_k \int |\mathrm{H}\, f_k|^q g \, dx &\lesssim \sum_k \int |f_k|^q (\mathrm{M}|g|^{1+\epsilon})^{1/(1+\epsilon)} \, dx \\ &\lesssim \Big\| \sum_k |f_k|^q \Big\|_{p/q} \|(\mathrm{M}|g|^{1+\epsilon})^{1/(1+\epsilon)}\|_{(p/q)'} \\ &\lesssim 1 \end{aligned}$$

provided we take $1 + \epsilon < (p/q)'$.

Now, the Fourier projection onto an interval $\omega$ can be obtained as a linear combination of modulations of the Hilbert transform. Using this, one sees that the estimate (2.7) extends to the Fourier projections onto intervals. Namely, we have the estimate

$$\| \|\mathrm{S}_\omega\, f_\omega\|_{\ell^2(\Omega)} \|_p \lesssim \| \|f_\omega\|_{\ell^2(\Omega)} \|_p, \qquad 1 < p < \infty.$$

This is valid for all collections of intervals $\Omega$. Applying it to (2.6), with $\Omega = \Delta$, and using the fact that $S_{\sigma[2^k,2^{k+1})} f = \mathrm{S}_{\sigma[2^k,2^{k+1})} \operatorname{Dil}^{(1)}_{2^k} \psi_\sigma * f$ proves the upper half of the inequalities of Theorem 2.1, which what we wanted.

For our subsequent use, we note that the vector-valued extension of the Hilbert transform depends upon structural estimates that continue to hold for a wide variety of Calderón–Zygmund kernels. In particular, the Littlewood–Paley inequalities also admit a vector-valued extension,

$$\| \|\mathrm{S}^\Delta\, f_k\|_{\ell^q} \|_p \simeq \| \|f_k\|_{\ell^q} \|_p, \qquad 1 < p, q < \infty. \tag{2.9}$$

**2.2. Well-distributed collections.** We begin the main line of argument for Rubio de Francia's inequality in one dimension. The first step, found by Rubio de Francia [36], is a reduction of the general case to one in which one can square function by a smoother object.

Say that a collection of intervals $\Omega$ is *well-distributed* if

$$\Big\| \sum_{\omega\in\Omega} \mathbf{1}_{3\omega} \Big\|_\infty \le 100. \tag{2.10}$$

Thus, after dilating the intervals in the collection by a factor of (say) 3, at most 100 intervals can intersect.

The well-distributed collections allow one to smooth out $\mathrm{S}_\omega$, just as one does $\mathrm{S}_{[1,2]}$ in the proof of the classical Littlewood–Paley inequality. The main fact we should observe here is that

**2.11. Lemma.** *For each collection of intervals $\Omega$, we can define a well-distributed collection* $\operatorname{Well}(\Omega)$ *for which*

$$\big\|\mathrm{S}^\Omega f\big\|_p \simeq \big\|\mathrm{S}^{\operatorname{Well}(\Omega)} f\big\|_p, \qquad 1<p<\infty.$$

**Proof.** The argument here depends upon inequalities for vector-valued singular integral operators. We define the collection $\operatorname{Well}(\Omega)$ by first considering the interval $[-\frac12,\frac12]$. Set

$$\operatorname{Well}([-\tfrac12,\tfrac12]) = \big\{[-\tfrac1{18},\tfrac1{18}], \pm[\tfrac12-\tfrac49(\tfrac45)^k, \tfrac12-\tfrac49(\tfrac45)^{k+1}] \,:\, k\ge 0\big\}.$$

It is straightforward to check that all the intervals in this collection have a distance to the boundary of $[-\frac12,\frac12]$ that is four times their length. In particular, this collection is well-distributed. It has the additional property that for each $\omega\in\operatorname{Well}([-\frac12,\frac12])$ we have $2\omega\subset[-\frac12,\frac12]$.

It is an extension of the usual Littlewood–Paley inequality that

$$\big\|\mathrm{S}_{[-1/2,1/2]} f\big\|_p \simeq \big\|\mathrm{S}^{\operatorname{Well}([-1/2,1/2])} S_{[-1/2,1/2]} f\big\|_p, \qquad 1<p<\infty.$$

This inequality continues to hold in the vector-valued setting of (2.9).

We define $\operatorname{Well}(\omega)$ by affine invariance. For an interval $\omega$, select an affine function $\alpha : [-\frac12,\frac12] \longrightarrow \omega$, we set $\operatorname{Well}(\omega) := \alpha(\operatorname{Well}([-\frac12,\frac12]))$. For collections of intervals $\Omega$, we define $\operatorname{Well}(\Omega) := \bigcup_{\omega\in\Omega}\operatorname{Well}(\omega)$. It is clear that $\operatorname{Well}(\Omega)$ is well-distributed for collections of disjoint intervals $\Omega$. By a vector-valued Littlewood–Paley inequality, we have

$$\big\|\mathrm{S}^\Omega f\big\|_p \simeq \big\|\mathrm{S}^{\operatorname{Well}(\Omega)} f\big\|_p, \qquad 1<p<\infty.$$

This completes the proof of our lemma. □

In the proof of the lemma, we see that we are 'resolving the frequency jump' at both endpoints of the interval. In the sequel however, we don't need to rely upon this construction, using only the general definition of well-distributed.

For the remainder of the proof, we assume that $\Omega$ is well-distributed. We need only consider a smooth version of the square function $\mathrm{S}^\Omega$, with the well-distributed assumption being critical to boundedness of the smooth operator on $L^2$.

Let $\varphi$ be a Schwartz function so that

$$\mathbf{1}_{[-1/2,1/2]} \le \widehat{\varphi} \le \mathbf{1}_{[-1,1]}. \tag{2.12}$$

Set $\varphi^\omega = \operatorname{Mod}_{c(\omega)} \operatorname{Dil}^{(2)}_{|\omega|^{-1}} \varphi$, and

$$\mathrm{G}^\Omega f = \Bigl[\sum_{\omega\in\Omega} |\varphi^\omega * f|^2\Bigr]^{1/2}.$$

We need only show that

$$\|\mathrm{G}^\Omega f\|_p \lesssim \|f\|_p, \qquad 2 \le p < \infty, \tag{2.13}$$

for well-distributed collections $\Omega$. Note that that the well-distributed assumption and the assumptions about $\varphi$ make the $L^2$ inequality obvious.

**2.3. The tile operator.** We use the previous lemma to pass to an operator that is easier to control than the projections $\mathrm{S}_\omega$ or $\varphi^\omega * f$. This is done in the time frequency plane. Let $\mathbf{D}$ be the dyadic intervals in $\mathbb{R}$, that is

$$\mathbf{D} := \bigl\{[j2^k, (j+1)2^k \;:\; j,k \in \mathbb{Z}\bigr\}.$$

Say that $s = I_s \times \omega_s$ is a *tile* if $I_s \in \mathbf{D}$, $\omega_s$ is an interval, and $1 \le |s| = |I_s|\cdot|\omega_s| < 2$. Note that for any $\omega_s$, there is one choice of $|I_s|$ for which $I_s \times \omega_s$ will be a tile. We fix a Schwartz function $\varphi$, and define

$$\varphi_s := \operatorname{Mod}_{c(\omega_s)} \operatorname{Tr}_{c(I_s)} \operatorname{Dil}^{(2)}_{|I_s|} \varphi,$$

where $c(J)$ denotes the center of $J$. We take $\varphi$ as above, a Schwartz function satisfying $\mathbf{1}_{[-1,1]} \le \widehat{\varphi} \le \mathbf{1}_{[-2,2]}$.

Choosing tiles to have area approximately equal to one is suggested by the Fourier uncertainty principle. We sometimes refer to $I_s$ and $\omega_s$ as *dual* intervals. With this choice of definitions, the function $\varphi_s$ is approximately localized in the time frequency plane to the rectangle $I_s \times \omega_s$. This localization is precise in the frequency variable. The function $\widehat{\varphi_s}$ is supported in the interval $2\omega_s$. But, $\varphi_s$ is only approximately supported near the interval $I_s$. Since $\varphi$ is rapidly decreasing, we trivially have the estimate

$$|\varphi_s(x)| \lesssim |I_s|^{-1/2}(1 + |I_s|^{-1}\,|x - c(I_s)|)^{-N}, \qquad N \ge 1.$$

This is an adequate substitute for being compactly supported in the time variable.

For a collection of intervals $\Omega$, we set $\mathcal{T}(\Omega)$ to be the set of all possible tiles $s$ such that $\omega_s \in \Omega$. Note that for each $\omega \in \Omega$, the set of intervals

$\mathcal{T}(\{\omega\}) = \{I \,:\, I \times \omega \in \mathcal{T}(\Omega)\}$ is a a partition of $\mathbb{R}$ into intervals of equal length. Associated to $\mathcal{T}(\Omega)$ is a natural square function

$$\mathrm{T}^{\Omega} f = \Bigg[ \sum_{s \in \mathcal{T}(\Omega)} \frac{|\langle f, \varphi_s \rangle|^2}{|I_s|} \mathbf{1}_{I_s} \Bigg]^{1/2}.$$

Our main lemma is that:

**2.14. Lemma.** *For any collection of well-distributed intervals $\Omega$, we have*

$$\| \mathrm{T}^{\Omega} f \|_p \lesssim \| f \|_p, \qquad 2 \le p < \infty.$$

Let us argue that this lemma proves (2.13), for a slightly different square function, and so proves Rubio de Francia's Theorem in the one-dimensional case. One task is to pass from a sum of rank one operators to a convolution operator. This is in fact a general principle, that we can formulate this way.

**2.15. Lemma.** *Let $\varphi$ and $\phi$ be real-valued Schwartz functions on $\mathbb{R}$. Then*

$$\fint_{[0,1]} \sum_{m \in \mathbb{Z}} \langle f, \mathrm{Tr}_{y+m} \varphi \rangle \, \mathrm{Tr}_{y+m} \phi \; dy = f * \Phi,$$

$$\textit{where} \qquad \Phi(x) = \int \overline{\varphi(u)} \phi(x+u) \, du.$$

*In particular, $\widehat{\Phi} = \overline{\widehat{\varphi}} \widehat{\phi}$.*

The proof is immediate. The integral in question is

$$\iint_{\mathbb{R}} f(z) \overline{\varphi(z-y)} \phi(x-y) \; dy dz$$

and one changes variables, $u = z - y$.

**Proof of (2.13).** We need to pass from the discrete operator to a square function of convolution operators. Let

$$\chi(x) := (1+|x|)^{-10}, \qquad \chi_{(I)} = \mathrm{Dil}^{(1)}_{|I|} \mathrm{Tr}_{c(I)} \chi,$$

and set for $\omega \in \Omega$,

$$\mathrm{H}_{\omega} f = \sum_{\substack{s \in \mathcal{T}(\Omega) \\ \omega_s = \omega}} \langle f, \varphi_s \rangle \varphi_s \,.$$

By Cauchy–Schwarz, we may dominate

$$
\begin{aligned}
|\mathrm{H}_\omega f| &\le \sum_{\substack{s\in\mathcal{T}(\Omega)\\ \omega_s=\omega}} |\langle f,\varphi_s\rangle\varphi_s| \\
&\lesssim \sum_{\substack{s\in\mathcal{T}(\Omega)\\ \omega_s=\omega}} \frac{|\langle f,\varphi_s\rangle|}{\sqrt{|I_s|}} |\chi_{(I_s)} * \mathbf{1}_{I_s}|^2 \\
&\lesssim \left[\sum_{\substack{s\in\mathcal{T}(\Omega)\\ \omega_s=\omega}} \frac{|\langle f,\varphi_s\rangle|^2}{|I_s|} |\chi_{(I_s)} * \mathbf{1}_{I_s}|\right]^{1/2}.
\end{aligned}
$$

We took some care to include the convolution in this inequality, so that we could use the easily verified inequality $\int|\chi_{(I)} * f|^2 g\,dx \le \int|f|^2\chi_{(I)} * g\,dx$ in the following way: the square function $\|\mathrm{H}_\omega f\|_{\ell^2(\Omega)}$ is seen to map $L^p$ into itself, $2<p<\infty$ by duality. For functions $g\in L^{(p/2)'}$ of norm one, we can estimate

$$
\begin{aligned}
\sum_{s\in\mathcal{T}(\Omega)} \frac{|\langle f,\varphi_s\rangle|^2}{|I_s|}\int|\chi_{(I_s)} * \mathbf{1}_{I_s}|g\,dx &\le \sum_{s\in\mathcal{T}(\Omega)} \frac{|\langle f,\varphi_s\rangle|^2}{|I_s|}\int \mathbf{1}_{I_s}\chi_{(I_s)} * g\,dx \\
&\le \int|\mathrm{T}^\Omega f|^2 \sup_I \chi_{(I)} * g\,dx \\
&\lesssim \|\mathrm{T}^\Omega f\|_p^2\|\mathrm{M}\,g\|_{(p/2)'} \\
&\lesssim \|f\|_p^2.
\end{aligned}
$$

Here, $(p/2)'$ is the conjugate index to $p/2$, and M is the maximal function.

Thus, we have verified that

$$
\|\|\mathrm{H}_\omega f\|_{\ell^2(\Omega)}\|_p \lesssim \|f\|_p, \qquad 2<p<\infty.
$$

We now derive a convolution inequality. By Lemma 2.15,

$$
\lim_{T\to\infty} \fint_{[0,T]} \mathrm{Tr}_{-y}\,\mathrm{H}_\omega\,\mathrm{Tr}_y\,f\,dy = \psi^\omega * f
$$

for all $\omega$, where $\widehat{\psi^\omega} = |\widehat{\varphi^\omega}|^2$. □

Thus, we see that a square function inequality much like that of (2.13) holds; this completes the proof of Rubio de Francia's Theorem in the one-dimensional case, aside from the proof of Lemma 2.14.

**2.4. Proof of Lemma 2.14.** The proof of the boundedness of the tile operator $\mathrm{T}^{\Omega}$ on $L^2$ is straightforward, yet finer facts about this boundedness are very useful in extending the boundedness to $L^p$ for $p>2$. This is the subject of the next proposition.

**2.16. Proposition.** *Let $\psi$ be a smooth, rapidly decreasing function, satisfying in particular*

$$\big|\psi(x)\big| \lesssim (1+|x|)^{-20}. \tag{2.17}$$

*For any interval $\omega$, we have*

$$\sum_{s\in\mathcal{T}(\{\omega\})} |\langle f,\psi_s\rangle|^2 \lesssim \|f\|_2^2. \tag{2.18}$$

*Moreover, if $\mathbf{1}_{[-1,1]} \le \widehat{\psi} \le \mathbf{1}_{[-2,2]}$ we have the following more particular estimate. For all intervals $\omega, I$ satisfying $\rho := |I||\omega|^{-1} > 1$, and $t>0$,*

$$\sum_{\substack{s\in\mathcal{T}(\{\omega\})\\ I_s\subset I}} |\langle f,\psi_s\rangle|^2 \lesssim (t\rho)^{-5}\|\psi^{3\omega} * f\|_2^2 \qquad f \textit{ supported on } [tI]^c. \tag{2.19}$$

In the second inequality observe that we assume $|I||\omega|^{-1}>1$, so that the rectangle $I\times\omega$ is too big to be a tile. It is important that on the right-hand side we have both a condition on the spatial support of $f$, and in the norm we are making a convolution with a smooth analog of a Fourier projection.

**Proof.** The hypothesis (2.17) is too strong; we are not interested in the minimal hypotheses here, but it is useful for this proof to observe that we only need

$$\big|\psi(x)\big| \lesssim (1+|x|)^{-5} \tag{2.20}$$

to conclude the first inequality (2.18).

The inequality (2.18) can be seen as the assertion of the boundedness of the map $f\mapsto\{\langle f,\psi_s\rangle : s\in\mathcal{T}(\{\omega\})\}$ from $L^2$ to $\ell^2(\mathcal{T}(\{\omega\}))$. It is equivalent to show that the formal dual of this operator is bounded, and this inequality is

$$\Big\|\sum_{s\in\mathcal{T}(\{\omega\})} a_s\psi_s\Big\|_2 \lesssim \|a_s\|_{\ell^2(\mathcal{T}(\{\omega\})}\,. \tag{2.21}$$

Observe that

$$\langle\psi_s,\psi_{s'}\rangle \lesssim \Delta(s,s') := (1+|I_s|^{-1}|c(I_s)-c(R_{s'}|)^{-5}.$$

Estimate

$$\begin{aligned}(2.22)\qquad \Bigl\lVert \sum_{s\in\mathbb{Z}} a_s\psi_s \Bigr\rVert_2 &\le \sum_s |a_s| \sum_{s'} |a_{s'}|\Delta(s,s') \\ &\le \lVert a_s\rVert_{\ell^2} \Bigl[\sum_s \Bigl|\sum_{s'} |a_{s'}|\Delta(s,s')\Bigr|^2\Bigr]^{1/2} \\ &\lesssim \lVert a_s\rVert_{\ell^2} \Bigl[\sum_s \sum_{s'} |a_{s'}|^2\Delta(s,s')\Bigr]^{1/2} \\ &\le \lVert a_s\rVert_{\ell^2}^2 .\end{aligned}$$

Here, we use Cauchy–Schwarz, and the fact that the $L^2$ norm dominates the $L^1$ norm on probability spaces.

Turning to the proof of the more particular assertation (2.19), we first note a related inequality. Assume that $\psi$ satisfies (2.17).

$$(2.23)\qquad \sum_{\substack{s\in\mathcal{T}(\{\omega\})\\ I_s\subset I}} |\langle f,\psi_s\rangle|^2 \lesssim (t\rho)^{-5}\lVert f\rVert_2^2\,, \qquad f \text{ supported on } (tI)^c.$$

As in the statement of the lemma, $\rho=|I||\omega|^{-1}>1$. Here, we do not assume that $\psi$ has compact frequency support, just that it has rapid spatial decay. On the right-hand side, we do not impose the convolution with $\psi^{3\omega}$.

For an interval $I$ of length at least one, and $t>1$, write $\psi=\psi_0+\psi_\infty$ where $\psi_\infty(x)$ is supported on $|x|\ge \frac14 t\rho$, equals $\psi(x)$ on $|x|\ge\frac12 t\rho$, and satisfies the estimate

$$|\psi_\infty(x)| \lesssim (t\rho)^{-10}(1+|x|)^{-5}\,.$$

That is, $\psi_\infty$ satisfies the inequality (2.20) with constants that are smaller by an order of $(t\rho)^{-10}$.

Note that if $f$ is supported on the complement of $tI$, we have $\langle f,\psi_s\rangle = \langle f,\psi_\infty\rangle$ for $\lambda_s\in I$. Thus, (2.23) follows.

We now prove (2.17) as stated. We now assume that $\psi$ is a Schwartz function satisfying $\mathbf{1}_{[-1,1]}\le\widehat{\psi}\le\mathbf{1}_{[-2,2]}$. Then certainly, it satisfies (2.17), so that (2.23) holds. We also have that for all tiles $s\in\mathcal{T}(\{\omega\})$,

$$\langle f,\psi_s\rangle = \langle \psi^{3\omega}*f,\psi\rangle = \langle \psi^{3\omega}*\psi^{3\omega}*f,\psi\rangle\,.$$

Write $\psi^{3\omega}*\psi^{3\omega}*f = F_0+F_\infty$, where $F_0=[\psi^{3\omega}*f]\mathbf{1}_{\frac{t}{2}I}$.

Then, since $\psi$ is decreasing rapidly, we will have

$$\lVert F_0\rVert_2 \lesssim (t\rho)^{-10}\lVert\psi^{3\omega}*f\rVert_2.$$

Therefore, by the $L^2$ inequality (2.18)

$$\sum_{s\in\mathcal{T}(\{\omega\})} |\langle F_0,\psi_s\rangle|^2 \lesssim (t\rho)^{-10}\|\psi^{3\omega}*f\|_2^2\,.$$

On the other hand, the inequality (2.23) applies to $F_\infty$, so that

$$\sum_{\substack{s\in\mathcal{T}(\{\omega\})\\ I_s\subset I}} |\langle F_\infty,\psi_s\rangle|^2 \lesssim (t\rho)^{-5}\|F_\infty\|_2^2\,.$$

But certainly $\|F_\infty\|_2 \le \|\psi^{3\omega}*f\|_2 \lesssim \|f\|_2$. So our proof of the more particular assertation (2.19) is finished. □

Let us now argue that the tile operator $\mathrm{T}^\Omega$ maps $L^2$ into itself, under the assumption that $\Omega$ is well-distributed. For $\omega\in\Omega$, let $\mathcal{T}(\omega)$ be the tiles $s\in\mathcal{T}(\Omega)$ with $\omega_s=\omega$. It follows from Proposition 2.16 that we have the estimate

$$\sum_{s\in\mathcal{T}(\omega)} |\langle f,\varphi_s\rangle|^2 \lesssim \|f\|_2^2\,.$$

For a tile $s$, we have $\langle f,\varphi_s\rangle = \langle \mathrm{S}_{2\omega}\,f,\varphi_s\rangle$, where we impose the Fourier projection onto the interval $2\omega_s$ in the second inner product. Thus, on the right-hand side above, we can replace $\|f\|_2^2$ by $\|\mathrm{S}_{2\omega}\,f\|_2^2$.

Finally, the well-distributed assumption implies that

$$\sum_{\omega\in\Omega}\|\mathrm{S}_{2\omega}\,f\|_2^2 \lesssim \|f\|_2^2\,.$$

The boundedness of the tile operator on $L^2$ follows.

To prove the remaining inequalities, we seek an appropriate endpoint estimate. That of BMO is very useful. Namely for $f\in L^\infty$, we show that

$$\|(\mathcal{T}^\Omega f)^2\|_{\mathrm{BMO}} \lesssim \|f\|_\infty^2\,. \tag{2.24}$$

Here, by BMO we mean dyadic BMO, which has this definition.

$$\|g\|_{\mathrm{BMO}} = \sup_{I\in\mathbf{D}} \fint_I \left| g - \fint_I g\right|\,dx. \tag{2.25}$$

The usual definition of BMO is formed by taking a supremum over all intervals, not just the dyadic ones. It is a useful simplification for us to restrict the supremum to dyadic intervals. The $L^p$ inequalities for $\mathrm{T}^\Omega$ are deduced by an interpolation argument, which we will summarize below.

There is a closely related notion, one that in the one-parameter setting coincides with the BMO norm. We distinguish it here, as it is a useful distinction for us in the higher parameter case. For a map $\alpha : \mathbf{D} \longrightarrow \mathbb{R}$, set

$$\|\alpha\|_{\mathrm{CM}} = \sup_{J\in\mathbf{D}} |J|^{-1} \sum_{I\subset J} |\alpha(I)|. \tag{2.26}$$

"CM" is for Carleson measure. The inequality (2.24) is, in this notation

$$\left\| \left\{ \sum_{\substack{s\in\mathcal{T}(\Omega)\\ I_s=J}} |\langle f,\varphi_s\rangle|^2 \,:\, J\in\mathbf{D} \right\} \right\|_{\mathrm{CM}} \lesssim \|f\|_\infty^2. \tag{2.27}$$

Or, equivalently, that we have the inequality

$$\sum_{\substack{s\in\mathcal{T}(\Omega)\\ I_s\subsetneq J}} |\langle f,\varphi_s\rangle|^2 \lesssim |J|\|f\|_\infty^2\,.$$

Notice that we can restrict the sum above to tiles $s$ with $I_s \subsetneq J$ as in the definition of BMO we are subtracting off the mean.

**Proof of (2.24).** Our proof follows a familiar pattern of argument. Fix a function $f$ of $L^\infty$ norm one. We fix a dyadic interval $J$ on which we check the BMO norm. We write $f = \sum_{k=1}^\infty g_k$, where $g_1 = f\mathbf{1}_{2J}$, and

$$g_k = f\mathbf{1}_{2^kJ-2^{k-1}J}, \qquad k>1.$$

The bound below follows from the $L^2$ bound on the tile operator.

$$\sigma(k) := \sum_{\substack{s\in\mathcal{T}(\Omega)\\ I_s\subsetneq J}} |\langle g_k,\varphi_s\rangle|^2 \lesssim \|g_k\|_2^2 \lesssim 2^k|J|.$$

For $k>5$, we will use the more particular estimate (2.19) to verify that

$$\sigma(k)^2 := \sum_{\substack{s\in\mathcal{T}(\Omega)\\ I_s\subsetneq J}} |\langle g_k,\varphi_s\rangle|^2 \lesssim 2^{-4k}\|g_k\|_2^2 \lesssim 2^{-4k}|J|. \tag{2.28}$$

Yet, to apply (2.19) we need to restrict attention to a single frequency interval $\omega$, which we do here.

$$\sum_{\substack{s\in\mathcal{T}(\Omega)\\ I_s\subsetneq J\,,\omega_s=\omega}} |\langle g_k,\varphi_s\rangle|^2 \lesssim 2^{-10k}\|\varphi^{3\omega} * g_k\|_2^2\,, \qquad \omega\in\Omega\,.$$

This is summed over $\omega \in \Omega$, using the estimate

$$\sum_{\omega\in\Omega} \|\varphi^{3\omega} * g_k\|_2^2 \lesssim \|g_k\|_2^2 \lesssim 2^k |J|$$

to prove (2.28).

The inequality (2.28) is summed over $k$ in the following way to finish the proof of the BMO estimate, (2.24).

$$\sum_{\substack{s\in\mathcal{T}(\Omega)\\ I_s\subsetneq J}} |\langle f, \varphi_s\rangle|^2 = \sum_{\substack{s\in\mathcal{T}(\Omega)\\ I_s\subsetneq J}} \left| \sum_{k=1}^{\infty} k^{-1} \cdot k^1 \cdot \langle g_k, \varphi_s\rangle \right|^2 \lesssim \sum_{k=1}^{\infty} k^2 \sigma(k)^2 \lesssim |J| \,. \tag{2.29}$$

□

We discuss how to derive the $L^p$ inequalities from the $L^2$ estimate and the $L^\infty \longrightarrow$ BMO estimate.

The method used by Rubio de Francia [36], to use our notation, was to prove the inequality $[(\mathrm{T}^\Omega f)^2]^\sharp \lesssim \mathrm{M}|f|^2$, where $g^\sharp$ is the (dyadic) sharp function defined by

$$g^\sharp(x) = \sup_{\substack{x\in I\\ I\in\mathbf{D}}} \fint_I \left| g(y) - \fint_I g(z)\, dz \right| dy.$$

One has the inequality $\|g^\sharp\|_p \lesssim \|g\|_p$ for $1 < p < \infty$. The proof we have given can be reorganized to prove this estimate.

We have not presented this argument since the sharp function does not permit a good extension to the case of higher parameters, which we discuss in the next section. On the other hand, a proof of the (standard) interpolation result between $L^p$ and BMO [3] is based upon the John–Nirenberg inequality, Lemma 2.30 below; a proof based upon this inequality does extend to higher parameters. We present this argument now.

One formulation of the inequality of F. John and L. Nirenberg is:

**2.30. Lemma.** *For each $1 < p < \infty$, we have the estimate below valid for all dyadic intervals $J$,*

$$\left\| \sum_{I\subset J} \frac{\alpha(I)}{|I|} \mathbf{1}_I \right\|_p \lesssim \|\alpha\|_{\mathrm{CM}} |J|^{1/p}.$$

*The implied constant depends only on $p$.*

**Proof.** It suffices to prove the inequality for $p$ an integer, as the remaining values of $p$ are available by Hölder's inequality. The case of $p = 1$ is the definition of the Carleson measure norm. Assuming the inequality for $p$, consider

$$\begin{aligned}\int_J \left[\sum_{I\subset J} \frac{\alpha(I)}{|I|}\mathbf{1}_I\right]^{p+1} dx &\le 2\sum_{J'\subset J}\frac{|\alpha(J')|}{|J'|}\int_{J'}\left[\sum_{I\subset J'}\frac{\alpha(I)}{|I|}\mathbf{1}_I\right]^p dx\\ &\lesssim \|\alpha\|_{CM}^p \sum_{J'\subset J}|\alpha(J')|\\ &\lesssim \|\alpha\|_{CM}^{p+1}|J|.\end{aligned}$$

Notice that we are strongly using the grid property of the dyadic intervals, namely that for $I, J \in \mathbf{D}$ we have $I\cap J \in \{\emptyset, I, J\}$.

For an alternate proof, see Lemma 3.11 below. □

We prove the following for the tile operator $\mathrm{T}^{\Omega}$:

$$\|\mathrm{T}^{\Omega}\,\mathbf{1}_F\|_p \lesssim |F|^{1/p}, \qquad 2<p<\infty, \tag{2.31}$$

for all sets $F\subset\mathbb{R}$ of finite measure. This is the restricted strong type inequality on $L^p$ for the tile operator—that is we only prove the $L^p$ estimate for indicator functions.

The $L^p$ inequality above is obtained by considering subsets of tiles, $\mathcal{T}\subset\mathcal{T}(\Omega)$, for which we will need the notation

$$\mathrm{T}^{\mathcal{T}}\,\mathbf{1}_F := \left[\sum_{s\in\mathcal{T}}\frac{|\langle \mathbf{1}_F,\varphi_s\rangle|^2}{|I_s|}\mathbf{1}_{I_s}\right]^{1/2}.$$

As well, take $\operatorname{sh}(\mathcal{T}) := \bigcup_{s\in\mathcal{T}} I_s$ to be the *shadow of $\mathcal{T}$*.

The critical step is to decompose $\mathcal{T}(\Omega)$ into subsets $\mathcal{T}_k$ for which

$$\|(\mathrm{T}^{\mathcal{T}_k}\,\mathbf{1}_F)^2\|_{\mathrm{BMO}} \lesssim 2^{-2k}, \qquad |\operatorname{sh}(\mathcal{T}_k)|\lesssim 2^{2k}|F|, \qquad k\ge 1. \tag{2.32}$$

We have already seen that the BMO norm is bounded, so we need only consider $k\ge 1$ above. Then, by the John–Nirenberg inequality,

$$\|\mathrm{T}^{\mathcal{T}_k}\,\mathbf{1}_F\|_p \lesssim 2^{-k(1-2/p)}|F|^{1/p}.$$

This is summable in $k$ for $p>2$.

The decomposition (2.32) follows from this claim. Suppose that $\mathcal{T}\subset\mathcal{T}(\Omega)$ satisfies

$$\left\|(\mathrm{T}^{\mathcal{T}}\,\mathbf{1}_F)^2\right\|_{\mathrm{BMO}} \lesssim \beta.$$

We show how to write it as a union of $\mathcal{T}_{\mathrm{big}}$ and $\mathcal{T}_{\mathrm{small}}$ where

$$\left\|(\mathrm{T}^{\mathcal{T}_{\mathrm{small}}}\,\mathbf{1}_F)^2\right\|_{\mathrm{BMO}} \lesssim \tfrac{\beta}{4}, \qquad |\operatorname{sh}(\mathcal{T}_{\mathrm{big}})| \lesssim \beta^{-1}|F|\,.$$

The decomposition is achieved in a recursive fashion. Initialize

$$\mathbf{J} := \emptyset, \quad \mathcal{T}_{\text{big}} := \emptyset, \quad \mathcal{T}_{\text{small}} := \emptyset, \quad \mathcal{T}_{\text{stock}} := \mathcal{T}.$$

While $\big\|(\mathrm{T}^{\mathcal{T}_{\text{stock}}}\mathbf{1}_F)^2\big\|_{\text{BMO}} \geq \frac{\beta}{4}$, there is a maximal dyadic interval $J \in \mathbf{D}$ for which

$$\sum_{\substack{s\in\mathcal{T}_{\text{stock}}\\ I_s\subset J}} |\langle \mathbf{1}_F, \varphi_s\rangle|^2 \geq \tfrac{\beta}{4}|J|.$$

Update

$$\mathbf{J} := \mathbf{J}\cup\{J\}, \quad \mathcal{T}_{\text{big}} := \mathcal{T}_{\text{big}} \cup \{s\in\mathcal{T}_{\text{stock}} \,:\, I_s\subset J\},$$
$$\mathcal{T}_{\text{stock}} := \mathcal{T}_{\text{stock}} - \{s\in\mathcal{T}_{\text{stock}} \,:\, I_s\subset J\}.$$

Upon completion of the While loop, update $\mathcal{T}_{\text{small}} := \mathcal{T}_{\text{stock}}$ and return the values of $\mathcal{T}_{\text{big}}$ and $\mathcal{T}_{\text{small}}$.

Observe that by the $L^2$ bound for the tile operator we have

$$\begin{aligned}\beta|\mathrm{sh}(\mathcal{T}_{\text{big}})| &\lesssim \beta\sum_{J\in\mathbf{J}}|J| \\ &\lesssim \sum_{s\in\mathcal{T}_{\text{big}}}|\langle \mathbf{1}_F,\varphi_s\rangle|^2 \\ &\lesssim |F|.\end{aligned}$$

This completes the proof of (2.32). Our discussion of the restricted strong type inequality is complete.

## 3. The case of higher dimensions

We give the proof of Theorem 1.2 in higher dimensions. The tensor product structure permits us to adapt many of the arguments of the one-dimensional case. (Some arguments are far less trivial to adapt however.) For instance, one can apply the classical Littlewood–Paley inequality in each variable separately. This would yield a particular instance of a Littlewood–Paley inequality in higher dimensions. Namely, for all dimensions $d$,

$$\big\|\mathrm{S}^{\Delta^d} f\big\|_p \simeq \|f\|_p, \qquad 1<p<\infty, \tag{3.1}$$

where $\Delta^d = \bigotimes_1^d \Delta$ is the $d$–fold tensor product of the lacunary intervals $\Delta$, as in Theorem 1.2.

Considerations of this type apply to many of the arguments made in the one-dimensional case of Theorem 1.2. In particular the definition of well-distributed, and the Lemma 2.11 continues to hold in the higher dimensional setting.

As before, the well-distributed assumption permits defining a "smooth" square function that is clearly bounded on $L^2$. We again choose to replace a convolution square function with an appropriate tile operator.

The definition of the smooth square function—and of tiles—requires a little more care. For positive quantities $t = (t_1, \dots, t_d)$, dilation operators are given by

$$\operatorname{Dil}_t^{(p)} f(x_1, \dots, x_d) = \left[\prod_{j=1}^{d} t_j^{-1/p}\right] f(x_1/t_1, \dots, x_j/t_d), \qquad 0 \le p \le \infty\,,$$

with the normalization chosen to preserve the $L^p$ norm of $f$.

A rectangle is a product of intervals in the standard basis. Writing a rectangle as $R = R_{(1)} \times \cdots \times R_{(d)}$, we extend the definition of the dilation operators in the following way:

$$\operatorname{Dil}_R^{(p)} := \operatorname{Tr}_{c(R)} \operatorname{Dil}^{(p)}_{(|R_{(1)}|,\dots,|R_{(d)}|)}\,.$$

For a Schwartz function $\varphi$ on $\mathbb{R}^d$, satisfying

$$\mathbf{1}_{[-1/2,1/2]^d} \le \widehat{\varphi} \le \mathbf{1}_{[-1,1]^d}$$

we set

$$\varphi^\omega = \operatorname{Mod}_{c(\omega)} \operatorname{Dil}^{(1)}_{(|\omega_{(1)}|^{-1},\dots,|\omega_{(d)}|^{-1})} \varphi. \tag{3.2}$$

For a collection of well-distributed rectangles $\Omega$, we should show that the inequality (2.13) holds.

We substitute the smooth convolution square function for a sum over tiles. Say that $R \times \omega$ is a *tile* if both $\omega$ and $R$ are rectangles and for all $1 \le j \le d$, $1 \le |\omega_{(j)}| \cdot |R_{(j)}| < 2$, and $R_{(j)}$ is a dyadic interval. Thus, we are requiring that $\omega$ and $R$ be dual in each coordinate separately. In this instance, we refer to $\omega$ and $R$ as dual rectangles.

Write $s = R_s \times \omega_s$. As before, let $\mathcal{T}(\Omega)$ be the set of all tiles $s$ such that $\omega_s \in \Omega$. Define functions adapted to tiles and a tile operator by

$$\varphi_s = \operatorname{Mod}_{c(\omega_s)} \operatorname{Dil}^{(2)}_{R_s} \varphi$$

$$\mathrm{T}^\Omega f = \left[\sum_{s \in \mathcal{T}(\Omega)} \frac{|\langle f, \varphi_s\rangle|^2}{|R_s|} \mathbf{1}_{R_s}\right]^{1/2}.$$

The main point is to establish the boundedness of this operator on $L^2$ and an appropriate endpoint estimate. The analog of Proposition 2.16 is in this setting is:

**3.3. Proposition.** *Assume only that the function $\varphi$ satisfies*

$$|\varphi(x)| \lesssim (1+|x|)^{-20d}\,. \tag{3.4}$$

*Let $\Omega = \{\omega\}$. Then, we have the estimate*

$$\sum_{s\in\mathcal{T}(\{\omega\})} |\langle f, \varphi_s\rangle|^2 \lesssim \|f\|_2^2\,. \tag{3.5}$$

*Now let $\varphi$ be a smooth Schwartz function satisfying $\mathbf{1}_{[-1,1]^d} \le \widehat{\varphi} \le \mathbf{1}_{[-2,2]^d}$. For a subset $U \subset \mathbb{R}^d$ of finite measure, $0<a<1$, and a function $f$ supported on the complement of $\{\mathrm{M}\,\mathbf{1}_U > a\}$, we have the estimate*

$$\sum_{\substack{R_s\subset U\\ \omega_s=\omega}} |\langle f, \varphi_s\rangle|^2 \lesssim a^{15d} \|\varphi^{3\omega} * f\|_2^2\,. \tag{3.6}$$

The more particular assertion (3.6) has a far more complicated form than in the one-dimensional setting. That is because when we turn to the endpoint estimate, it is a Carleson measure condition; this condition is far more subtle, in that it requires testing the measure against arbitrary sets, instead of just intervals, or rectangles.

**Proof.** The hypothesis (3.4) is more than enough to conclude (3.5). We need only assume

$$|\varphi(x)| \lesssim (1+|x|)^{-5d}\,. \tag{3.7}$$

After taking an appropriate dilation and modulation, we can assume that $\omega = [-\frac{1}{2}, \frac{1}{2}]^d$. We view the inequality (3.5) as the boundedness of the linear map $f \mapsto \{\langle f, \varphi_s\rangle \,:\, s \in \mathcal{T}(\{\omega\})\}$ from $L^2(\mathbb{R}^d)$ into $\ell^2(\mathbb{Z}^d)$. We then prove that the dual to this operator is bounded, that is we verify the inequality

$$\Bigl\| \sum_{s\in\mathcal{T}(\{\omega\})} a_s \varphi_s \Bigr\|_2 \lesssim \|a_s\|_{\ell^2(\mathcal{T}(\{\omega\}))}\,.$$

Observe that

$$|\langle \varphi_s, \varphi_{s'}\rangle| \lesssim (1+\mathrm{dist}(R_s, R_{s'}))^{-5d}\,, \qquad s, s' \in \mathcal{T}(\{\omega\})\,.$$

The remaining steps of the proof are a modification of (2.22).

As in the one-dimensional setting, the more particular assertion is proved in two stages. First we assume only that the function $\varphi$ satisfy (3.4), and prove

$$\sum_{\substack{R_s\subset U\\ \omega_s=\omega}} |\langle f, \varphi_s\rangle|^2 \lesssim a^{15d} \|f\|_2^2 \tag{3.8}$$

for functions $f$ supported on the complement of $\{\mathrm{M}\,\mathbf{1}_U > a\}$.

Take $\widetilde{\varphi}(x)$ to be a function which equals $a^{-15d}\varphi(x)$ provided $|x| \geq \frac{2}{a}$. With this, $\varphi$ satisfies (3.7) with a constant independent of $a$.

For a subset $U \subset \mathbb{R}^d$ of finite measure, and function $f$ supported on the complement of $\{\mathrm{M}\,\mathbf{1}_U > a^d\}$, and tile $s$ with $R_s \subset U$, we have $a^{-15d}\langle f, \varphi_s\rangle = \langle f, \widetilde{\varphi}_s\rangle$. Thus, (3.8) follows from the $L^2$ estimate we have already proved.

We can then prove the assertation of the lemma. Take $\varphi$, $f \in L^2$, $0 < a < 1$, and $U \subset \mathbb{R}^d$ as in (3.6). Then, for all tiles $s = R_s \times \omega$, we have

$$\langle f, \varphi_s\rangle = \langle \varphi^{3\omega} * f, \varphi_s\rangle.$$

We write $\varphi^{3\omega} * f = F_0 + F_\infty$, where

$$F_0 = [\varphi^{3\omega} * f]\mathbf{1}_{\{\mathrm{M}\,\mathbf{1}_U > 2a\}}.$$

The rapid decay of $\varphi$, with the fact about the support of $f$, show that $\|F_0\|_2 \lesssim a^{15d}\|\varphi^{3\omega} * f\|_2$. Thus, the estimate below follows from the $L^2$ inequality (3.4):

$$\sum_{\substack{R_s \subset U \\ \omega_s = \omega}} |\langle F_0, \varphi_s\rangle|^2 \lesssim a^{15d}\|\varphi^{3\omega} * f\|_2^2\,.$$

As for the term $F_\infty$, we use the estimate (3.8) to see that

$$\sum_{\substack{R_s \subset U \\ \omega_s = \omega}} |\langle F_\infty, \varphi_s\rangle|^2 \lesssim a^{15d}\|\varphi^{3\omega} * f\|_2^2\,.$$

This completes our proof of (3.6). □

We can now prove the $L^2$ boundedness of the square function. Using the well-distributed assumption and (3.5), we can estimate

$$\begin{aligned} \sum_{s \in \mathcal{T}(\Omega)} |\langle f, \varphi_s\rangle|^2 &= \sum_{\omega \in \Omega} \sum_{s \in \mathcal{T}(\{\omega\})} |\langle S_{2\omega} f, \varphi_s\rangle|^2 \\ &\lesssim \sum_{\omega \in \Omega} \|S_{2\omega} f\|_2^2 \\ &\lesssim \|f\|_2^2\,. \end{aligned}$$

The endpoint estimate we seek is phrased this way. For all subsets $U \subset \mathbb{R}^d$ of finite measure, and functions $f$ of $L^\infty$ norm one,

$$|U|^{-1} \sum_{\substack{s \in \mathcal{T}(\Omega) \\ R_s \subset U}} |\langle f, \varphi_s\rangle|^2 \lesssim 1. \tag{3.9}$$

Using the notation of (3.10), this inequality is equivalent to

$$\left\| \left\{ \sum_{\substack{s\in\mathcal{T}\\ R_s=R}} \frac{|\langle f,\varphi_s\rangle|^2}{|R|}\mathbf{1}_R \,:\, R\in\mathbf{D}^d \right\} \right\|_{\mathrm{CM}} \lesssim \|f\|_\infty\,.$$

Write $f=\sum_{k=1}^\infty g_k$ where

$$\begin{aligned} g_1 &= f\mathbf{1}_{\{\mathrm{M}\,\mathbf{1}_U\ge\frac12\}},\\ g_k &= f\mathbf{1}_{\{2^{-k}\le\mathrm{M}\,\mathbf{1}_U\le 2^{-(k-1)}\}}, \qquad k>1. \end{aligned}$$

Using the boundedness of the maximal function on, e.g., $L^2$, and the $L^2$ boundedness of the tile operator, we have

$$|U|^{-1}\sum_{\substack{s\in\mathcal{T}(\Omega)\\ R_s\subset U}} |\langle g_k,\varphi_s\rangle|^2 \lesssim |U|^{-1}\|f_k\|_2^2 \lesssim 2^{2k}.$$

For the terms arising from $g_k$, with $k\ge 5$, we can use (3.6) with $a=2^{-k/d}$ to see that

$$\begin{aligned} \sum_{\substack{s\in\mathcal{T}(\Omega)\\ R_s\subset U}} |\langle g_k,\varphi_s\rangle|^2 &= \sum_{\omega\in\Omega} \sum_{\substack{s\in\mathcal{T}(\Omega)\\ \omega=\omega_s\,,\ R_s\subset U}} |\langle g_k,\varphi_s\rangle|^2\\ &\lesssim 2^{-10d}\sum_{\omega\in\Omega}\|\varphi^{3\omega}*g_k\|_2^2\\ &\lesssim 2^{-10k}\|g_k\|_2^2\\ &\lesssim 2^{-8k}|U|. \end{aligned}$$

Here, we have used the fact that the strong maximal function is bounded on $L^2$. The conclusion of the proof of (3.9) then follows the lines of (2.29).

To deduce the $L^p$ inequalities, one can again appeal to interpolation. Alternatively, the restricted strong type inequality can be proved directly using the John–Nirenberg inequality for the product Carleson measure. This inequality is recalled in the next section, and argument is formally quite simliar to the one we gave for one dimension. Details are omitted.

**Carleson measures in the product setting.** The subject of Carleson measures are central to the subject of product BMO, as discovered by S.-Y. Chang and R. Fefferman [13, 14].

A definition can be phrased in terms of maps $\alpha$ from the dyadic rectangles $\mathbf{D}^d$ of $\mathbb{R}^d$. This norm is

$$\|\alpha\|_{\mathrm{CM}} = \sup_{U\subset\mathbb{R}^d} |U|^{-1} \sum_{R\subset U} \alpha(R). \tag{3.10}$$

What is most important is that the supremum is taken over all sets $U \subset \mathbb{R}^d$ of finite measure. It would of course be most natural to restrict the supremum to rectangles, and while this is not an adequate definition, it nevertheless plays an important role in the theory. See the lemma of Journé [25], as well as the survey of Journé's Lemma of Cabrelli, Lacey, Molter, and Pipher [11].

Of importance here is the analog of the John–Nirenberg inequality in this setting.

**3.11. Lemma.** *We have the inequality below, valid for all sets $U$ of finite measure.*

$$\Big\| \sum_{R\subset U} \frac{\alpha(R)}{|R|} \mathbf{1}_R \Big\|_p \lesssim \|\alpha\|_{\mathrm{CM}} |U|^{1/p}, \qquad 1<p<\infty.$$

**Proof.** We use the duality argument of Chang and Fefferman [14]. Let $\|\alpha\|_{\mathrm{CM}} = 1$. Define

$$F_V := \sum_{R\subset V} \frac{\alpha(R)}{|R|} \mathbf{1}_R.$$

We shall show that for all $U$, there is a set $V$ satisfying $|V| < \frac{1}{2}|U|$ for which

$$\|F_U\|_p \lesssim |U|^{1/p} + \|F_V\|_p. \tag{3.12}$$

Clearly, inductive application of this inequality will prove our lemma.

The argument for (3.12) is by duality. Thus, for a given $1<p<\infty$, and conjugate index $p'$, take $g \in L^{p'}$ of norm one so that $\|F_U\|_p = \langle F_U, g\rangle$. Set

$$V = \{\mathrm{M}\, g > K|U|^{-1/p'}\}$$

where $M$ is the strong maximal function and $K$ is sufficiently large so that $|V| < \frac{1}{2}|U|$. Then

$$\langle F_U, g\rangle = \sum_{\substack{R\subset U\\ R\not\subset V}} \alpha(R) \fint_R g\, dx + \langle F_V, g\rangle.$$

The second term is at most $\|F_V\|_p$ by Hölder's inequality. For the first term, note that the average of $g$ over $R$ can be at most $K|U|^{-1/p'}$. So by the definition of Carleson measure norm, it is at most

$$\sum_{\substack{R\subset U\\ R\not\subset V}} \alpha(R) \fint_R g\,dx \lesssim |U|^{-1/p'} \sum_{R\subset U} \alpha(R) \lesssim |U|^{1/p},$$

as required by (3.12). □

# 4. Implications for multipliers

Let us consider a bounded function $m$, and define

$$\mathrm{A}_m f(x) := \int m(\xi)\widehat{f}(\xi)\,d\xi.$$

This is the multiplier operator given by $m$, and the Plancherel equality implies that the operator norm of $A$ on $L^2$ is given by $\|m\|_\infty$. It is of significant interest to have a description of the the norm of $A$ as an operator on $L^p$ only in terms of properties of the function $m$.

Littlewood–Paley inequalities have implications here, as is recognized through the proof of the classical Marcinciewcz Theorem. Coifman, Rubio de Francia and Semmes [16] found a beautiful extension of this classical theorem with a proof that is a pleasing application of Rubio de Francia's inequality. We work first in one dimension. To state it, for an interval $[a,b]$, and index $0<q<\infty$, we set the $q$ variation norm of $m$ on the interval $[a,b]$ to be

$$\|m\|_{\mathrm{Var}_q([a,b])} := \sup\left\{\left[\sum_{k=0}^{K} |m(\xi_{k+1}) - m(\xi_k)|^q\right]^{1/q}\right\} \tag{4.1}$$

where the supremum is over all finite sequences $a=\xi_0<\xi_1<\xi_2<\cdots<\xi_{K+1}=b$. Set $\|v\|_{V_q([a,b])} := \|m\|_{L^\infty([a,b])} + \|m\|_{\mathrm{Var}_q([a,b])}$ Note that if $q=1$, this norm coincides with the classical bounded variation norm.

**4.2. Theorem.** *Suppose that $1<p,q<\infty$, satisfying $|\frac12-\frac1p|<\frac1q$. Then for all functions $m\in L^\infty(\mathbb{R})$, we have*

$$\|\mathrm{A}_m\|_p \lesssim \sup_{I\in\mathbf{D}} \|m\|_{V_q(I)}.$$

Note that the right-hand side is a supremum over the Littlewood–Paley intervals $I\in\mathbf{D}$. The theorem above is as in the Marcinciewcz Theorem, provided one takes $q=1$. But the theorem of Coifman, Rubio de Francia and Semmes states that even for the much rougher case of $q=2$, the right-hand side is an upper bound for all $L^p$ operator norms of the multiplier norm $\mathrm{A}_m$. In addition, as $q$ increases to infinity, the $V_q$ norms approach that of $L^\infty$, which is the correct estimate for the multiplier norm at $p=2$.

**4.1. Proof of Theorem 4.2.** The first lemma in the proof is a transparent display of the usefulness of the Littlewood–Paley inequalities in decoupling scales.

**4.3. Lemma.** *Suppose the multiplier $m$ is of the form $m = \sum_{\omega \in \mathbf{D}} a_\omega \mathbf{1}_\omega$, for a sequence of reals $a_\omega$. Then,*

$$\|\mathrm{A}_m\|_p \lesssim \|a_\omega\|_{\ell^\infty(\mathbf{D})}, \qquad 1 < p < \infty.$$

*Suppose that for an integer $n$, that $\mathbf{D}_n$ is a partition of $\mathbb{R}$ that refines the partition $\mathbf{D}$, and partitions each $\omega \in \mathbf{D}$ into at most $n$ subintervals. Consider a multiplier of the form*

$$m = \sum_{\omega \in \mathbf{D}_n} a_\omega \mathbf{1}_\omega.$$

*For $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{q}$, we have*

$$\|\mathrm{A}_m\|_p \lesssim n^{1/q} \|a_\omega\|_{\ell^\infty(\mathbf{D}_n)}. \tag{4.4}$$

**Proof.** In the first claim, for each $\omega \in \mathbf{D}$, we have $\mathrm{S}_\omega \mathrm{A}_m = a_\omega \mathrm{S}_\omega$, so that for any $f \in L^p$, we have by the Littlewood–Paley inequalities

$$\begin{aligned}
\|\mathrm{A}_m f\|_p &\simeq \|S^{\mathbf{D}} \mathrm{A}_m f\|_p \\
&\simeq \Bigg\| \Bigg[ \sum_{\omega \in \mathbf{D}} |a_\omega|^2 |\mathrm{S}_\omega f|^2 \Bigg]^{1/2} \Bigg\|_p \\
&\lesssim \|a_\omega\|_{\ell^\infty(\mathbf{D})} \|S^{\mathbf{D}} f\|_p.
\end{aligned}$$

The proof of (4.4) is by interpolation. Let us presume that $\|a_\omega\|_{\ell^\infty(\mathbf{D}_n)} = 1$. We certainly have $\|\mathrm{A}_m\|_2 = 1$. On the other hand, with an eye towards applying the classical Littlewood–Paley inequality and Rubio de Francia's extension of it, for each $\omega \in \mathbf{D}$, we have

$$|\mathrm{S}_\omega \mathrm{A}_m f| \le n^{1/2} \Bigg[ \sum_{\substack{\omega' \in \mathbf{D}_n \\ \omega' \subset \omega}} |\mathrm{S}_{\omega'} f|^2 \Bigg]^{1/2}.$$

Therefore, we may estimate for any $2 < r < \infty$,

$$\begin{aligned}
\|\mathrm{A}_m f\|_r &\lesssim \|\mathrm{S}^{\mathbf{D}} \mathrm{A}_m f\|_r \\
&\lesssim n^{1/2} \|\mathrm{S}^{\mathbf{D}_n} f\|_r \\
&\lesssim n^{1/2} \|f\|_r.
\end{aligned} \tag{4.5}$$

To conclude (4.4), let us first note the useful principle that $\|\mathrm{A}_m\|_p = \|\mathrm{A}_m\|_{p'}$, where $p'$ is the conjugate index. So we can take $p > 2$. For the choice of $\frac{1}{2} - \frac{1}{p} < \frac{1}{q}$, take a value of $r$ that is very large, in fact

$$\frac{1}{2} - \frac{1}{p} < \frac{1}{r} < \frac{1}{q}$$

and interpolate (4.5) with the $L^2$ bound. □

Since our last inequality is so close in form to the theorem we wish to prove, the most expedient thing to do is to note a slightly technical lemma about functions in the $V_q$ class.

**4.6. Lemma.** *If $m \in V_q(I)$ is of norm one, we can choose partitions $\Pi_j$, $j \in \mathbb{N}$, of $I$ into at most $2^j$ subintervals and functions $m_j$ that are measurable with respect to $\Pi_j$, so that*

$$m = \sum_j m_j, \qquad \|m_j\|_\infty \lesssim 2^{-j}.$$

**Proof.** Let $\mathbf{P}_j = \{(k2^{-j}, (k+1)2^{-j}] \,:\, 0 \le k < 2^j\}$ be the standard partition of $[0,1]$ into intervals of length $2^{-j}$. Consider the function $\mu : I = [a,b] \longrightarrow [0,1]$ given by

$$\mu(x) := \|m\|^q_{\mathrm{Var}_q([a,x])}.$$

This function is monotone, nondecreasing, hence has a well-defined inverse function. Define $\Pi_j = \mu^{-1}(\mathbf{P}_j)$. We define the functions $m_j$ so that

$$\sum_{k=1}^{j} m_j = \sum_{\omega \in \Pi_j} \mathbf{1}_\omega \fint_\omega m \, d\xi.$$

That is, the $m_j$ are taken to be a martingale difference sequence with respect to the increasing sigma fields $\Pi_j$. Thus, it is clear that $m = \sum m_j$. The bound on the $L^\infty$ norm of the $m_j$ is easy to deduce from the definitions. □

We can prove the Theorem 4.2 as follows. For $\frac{1}{2} - \frac{1}{p} < \frac{1}{r} < \frac{1}{q}$, and $m$ such that

$$\sup_{\omega \in \mathbf{D}} \|m\mathbf{1}_\omega\|_{V_q(\omega)} \le 1,$$

we apply Lemma 4.6 and (4.4) to each $m\mathbf{1}_\omega$ to conclude that we can write $m = \sum_j m_j$, so that $m_j$ is a multiplier satisfying $\|A_{m_j}\|_p \lesssim 2^{j/r - j/q}$. But this estimate is summable in $j$, and so completes the proof of the theorem.

**4.2. The higher dimensional form.** The extension of the theorem above to higher dimensions was made by Q. Xu [42]. His point of view was to take an inductive and vector-valued approach. Some of his ideas were motivated by prior work of G. Pisier and Q. Xu [33, 34] in which interesting applications of $q$-variation spaces are made.

The definition of the $q$-variation in higher dimensions is done inductively. For a function $m : \mathbb{R}^d \longrightarrow \mathbb{C}$, define difference operators by

$$\mathrm{Diff}(m, k, h, x) = m(x + he_k) - m(x), \qquad 1 \leq k \leq d,$$

where $e_k$ is the $k$th coordinate vector. For a rectangle $R = \prod_{k=1}^{d} [x_k, x_k + h_k]$, set

$$\mathrm{Diff}_R(m) = \mathrm{Diff}(m, 1, h_1, x) \cdots \mathrm{Diff}(m, d, h_d, x), \qquad x = (x_1, \ldots, x_d).$$

Define

$$\|m\|_{\mathrm{Var}_q(Q)} = \sup_{\mathcal{P}} \Bigl[ \sum_{R \in \mathcal{P}} |\mathrm{Diff}_R(M)|^q \Bigr]^{1/q}, \quad 0 < q < \infty.$$

The supremum is formed over all partitions $\mathcal{P}$ of the rectangle $Q$ into subrectangles.

Given $1 \leq k < d$, and $y = (y_1, \ldots, y_k) \in \mathbb{R}^k$, and a map $\alpha : \{1, \ldots, k\} \to \{1, \ldots, d\}$, let $m_{y,\alpha}$ be the function from $\mathbb{R}^{d-k}$ to $\mathbb{C}$ obtained from $m$ by restricting the $\alpha(j)$th coordinate to be $y_j$, $1 \leq j \leq k$. Then, the $V_q(Q)$ norm is

$$\|m\|_{V_q(Q)} = \|m\|_\infty + \|m\|_{\mathrm{Var}_q(\mathbb{R}^d)} + \sup_{k,\alpha} \sup_{y \in \mathbb{R}^k} \|m_{k,\alpha}\|_{\mathrm{Var}_q(Q_{y,\alpha})}.$$

Here, we let $Q_{y,\alpha}$ be the cube obtained from $Q$ by restricting the $\alpha(j)$th coordinate to be $y_j$, $1 \leq j \leq k$.

Recall the notation $\Delta^d$ for the lacunary intervals in $d$ dimensions, and in particular (3.1).

**4.7. Theorem.** *Suppose that* $|\frac{1}{2} - \frac{1}{p}| < \frac{1}{q}$. *For functions* $m : \mathbb{R}^d \longrightarrow \mathbb{C}$, *we have the estimate on the multiplier norm of* $m$

$$\|\mathrm{A}_m\|_p \lesssim \sup_{R \in \Delta^d} \|m\mathbf{1}_R\|_{V_q(\mathbb{R}^d)}.$$

**4.3. Proof of Theorem 4.7.** The argument is a reprise of that for the one-dimensional case. We begin with definitions in one dimension. Let $B$ be a linear space with norm $\| \, \|_B$. For an interval $I$ let $\mathcal{E}(I, B)$ be the linear space of step functions $m : I \longrightarrow B$ with finite range. Thus,

$$m = \sum_{j=1}^{J} b_j \mathbf{1}_{I_j}$$

for a finite partition $\{I_1, \dots, I_j\}$ of $I$ into intervals, and a sequence of values $b_j \in B$. If $B = \mathbb{C}$, we write simply $\mathcal{E}(I)$. There is a family of norms that we impose on $\mathcal{E}(I, B)$.

$$\langle\!\langle m \rangle\!\rangle_{\mathcal{E}(I,B),q} = \left[\sum_{j=1}^{J} \|b_j\|_B^q\right]^{1/q}, \qquad 1 \le q \le \infty.$$

For a rectangle $R = R_1 \times \cdots R_d$, set

$$\mathcal{E}(R) := \mathcal{E}(R_1, \mathcal{E}(R_2, \cdots, \mathcal{E}(R_d, \mathbb{C}) \cdots)).$$

The following lemma is a variant of Lemma 4.3.

**4.8. Lemma.** *Let $m : \mathbb{R}^d \longrightarrow \mathbb{C}$ be a function such that $m\mathbf{1}_R \in \mathcal{E}(R)$ for all $R \in \Delta^d$. Then, we have these two estimates for the multiplier $\mathrm{A}_m$.*

$$\|\mathrm{A}_m\|_p \lesssim \sup_{R \in \Delta^d} \langle\!\langle m \rangle\!\rangle_{\mathcal{E}(R),2}, \qquad 1 < p < \infty, \tag{4.9}$$

$$\|\mathrm{A}_m\|_p \lesssim \sup_{R \in \Delta^d} \langle\!\langle m \rangle\!\rangle_{\mathcal{E}(I,B),q}, \qquad 1 < p < \infty,\ |\tfrac{1}{2} - \tfrac{1}{p}| < \tfrac{1}{q}. \tag{4.10}$$

**Proof.** The first claim, the obvious bound at $L^2$, and complex interpolation prove the second claim.

As for the first claim, take a multiplier $m$ for which the right-hand side in (4.9) is 1. To each $R \in \Delta^d$, there is a partition $\Omega_R$ of $R$ into a finite number of rectangles so that

$$m\mathbf{1}_R = \sum_{\omega \in \Omega_R} a_\omega \mathbf{1}_\omega,$$
$$\sum_{\omega \in \Omega_R} |a_\omega|^2 \le 1.$$

This conclusion is obvious for $d = 1$, and induction on dimension will prove it in full generality.

Then observe that by Cauchy–Schwarz,

$$|\mathrm{S}_R \mathrm{A}_m| \le \mathrm{S}^{\Omega_R}.$$

Set $\Omega = \bigcup_{R \in \Delta^d} \Omega_R$. Using the Littlewood–Paley inequality (3.1), and Rubio de Francia's inequality in $d$ dimensions, we may estimate

$$\begin{aligned} \|\mathrm{A}_m f\|_p &\simeq \|\mathrm{S}^{\Delta^d} \mathrm{A}_m f\|_p \\ &\lesssim \|\mathrm{S}^{\Omega} f\|_p \\ &\lesssim \|f\|_p. \end{aligned}$$

The last step requires that $2 \le p < \infty$, but the operator norm $\|\mathrm{A}_m\|_p$ is invariant under conjugation of $p$, so that we need only consider this range of $p$'s. □

We extend the notion of $\mathcal{E}(I,B)$. Let $B$ be a Banach space, and set $\mathcal{U}_q(I,B)$ to be the Banach space of functions $m : I \longrightarrow B$ for which the norm below is finite.

$$\|m\|_{\mathcal{U}_q} := \inf \Bigl\{ \sum_j \langle\!\langle m_j \rangle\!\rangle_{\mathcal{E}(I,B),q} \,:\, m = \sum_j m_j,\ m_j \in \mathcal{E}(I,B) \Bigr\}.$$

For a rectangle $R = R_1 \times \cdots R_d$, set

$$\mathcal{U}_q(R) := \mathcal{U}_q(R_1, \mathcal{U}_q(R_2, \cdots, \mathcal{U}_q(R_d, \mathbb{C}) \cdots)).$$

By convexity, we clearly have the inequalities

$$\begin{aligned} \|\mathrm{A}_m\|_p &\lesssim \sup_{R \in \Delta^d} \|m \mathbf{1}_R\|_{\mathcal{U}_2(R)}, && 1 < p < \infty, \\ \|\mathrm{A}_m\|_p &\lesssim \sup_{R \in \Delta^d} \|m \mathbf{1}_R\|_{\mathcal{U}_q(R)}, && 1 < p < \infty,\ |\tfrac{1}{2} - \tfrac{1}{p}| < \tfrac{1}{q}. \end{aligned}$$

As well, we have the inclusion $\mathcal{U}_q(R) \subset \mathrm{Var}_q(R)$, for $1 \le q < \infty$. The reverse inclusion is not true in general, nevertheless the inclusion is true with a small perturbation of indicies.

Let us note that the definition of the $q$-variation space on an interval, given in (4.1), has an immediate extension to a setting in which the functions $m$ take values in a Banach space $B$. Let us denote this space as $\mathrm{V}_q(I,B)$.

**4.11. Lemma.** *For all $1 \le p < q < \infty$, all intervals $I$, and Banach spaces $B$, we have the inclusion*

$$\mathrm{V}_p(I,B) \subset \mathcal{U}_q(I,B).$$

*For all pairs of intervals $I, J$, we have*

$$\mathrm{V}_q(I \times J, B) = \mathrm{V}_q(I, \mathrm{V}_q(J,B)).$$

*In addition, for all rectangles $R$, we have*

$$\mathrm{V}_p(R) \subset \mathcal{U}_q(R).$$

*In each instance, the inclusion map is bounded.*

The first claim of the lemma is proved by a trivial modification of the proof of Lemma 4.6. (The martingale convergence theorem holds for all Banach space valued martingales.) The second claim is easy to verify, and the last claim is a corollary to the first two.

## 5. Notes and remarks

**5.1. Remark.** L. Carleson [12] first noted the possible extension of the Littlewood–Paley inequality, proving in 1967 that Theorem 1.2 holds in the special case that $\Omega = \{[n, n+1) \,:\, n \in \mathbb{Z}\}$. He also noted that the inequality does not extend to $1 < p < 2$. A corresponding extension to homethetic parallelepipeds was given by A. Córdoba [17], who also pointed out the connection to multipliers.

**5.2. Remark.** Rubio de Francia's paper [36] adopted an approach that we could outline this way. The reduction to the well-distributed case is made, and we have borrowed that line of reasoning from him. This permits one to define a smooth operator $\mathrm{G}^\Omega$ in (2.13). That $\mathrm{G}^\Omega$ is bounded on $L^p$, for $2 < p < \infty$, is a consequence of a bound on the sharp function. In our notation, that sharp function estimate would be

$$(\mathrm{G}^\Omega f)^\sharp \lesssim (\mathrm{M}|f|^2)^{1/2}. \tag{5.3}$$

The sharp denotes the function

$$g^\sharp = \sup_J \mathbf{1}_J \left[ \fint_J \left| g - \fint_J g \right|^2 \, dx \right]^{1/2},$$

the supremum being formed over all intervals $J$. It is known that $\|g\|_p \simeq \|g^\sharp\|_p$ for $1 < p < \infty$. Notice that our proof can be adapted to prove a dyadic version of (5.3) for the tile operator $\mathrm{T}^\Omega$ if desired. The sharp function estimate has the advantage of quickly giving weighted inequalities. It has the disadvantage of not easily generalizing to higher dimensions. On this point, see R. Fefferman [20].

**5.4. Remark.** The weighted version of Rubio de Francia's inequality states that for all weights $w \in A^1$, one has

$$\int |\mathrm{S}^\Omega f|^2 \, w \, dx \lesssim \int |f|^2 \, w \, dx.$$

There is a similar conclusion for multipliers.

$$\int |\mathrm{A}_m f|^2 \, w \, dx \lesssim \sup_{R \in \Delta^d} \|m \mathbf{1}_R\|^2_{\mathrm{V}_2(R)} \int |f|^2 \, w \, dx.$$

See Coifman, Rubio de Francia, and Semmes [16], for one dimension and Q. Xu [42] for dimensions greater than 1.

**5.5. Remark.** Among those authors who made a contribution to Rubio de Francia's one-dimensional inequality, P. Sjölin [39] provided an alternate derivation of Rubio de Francia's sharp function estimate (5.3). In another direction, observe that (2.1), Rubio de Francia's inequality has the dual formulation $\|f\|_p \lesssim \|\mathrm{S}^\Omega f\|_p$, for $1 < p < 2$. J. Bourgain [10] established a dual endpoint estimate for the unit circle:

$$\|f\|_{H^1} \lesssim \|\mathrm{S}^\Omega f\|_1.$$

This inequality in higher dimensions seems to be open.

**5.6. Remark.** Rubio de Francia's inequality does not extend below $L^2$. While this is suggested by the duality estimates, an explicit example is given in one dimension by $\widehat{f} = \mathbf{1}_{[0,N]}$, for a large integer $N$, and $\Omega = \{(n, n+1) : n \in \mathbb{Z}\}$. Then, it is easy to see that

$$N^{1/2}\mathbf{1}_{[0,1]} \lesssim \mathrm{S}^\Omega f, \qquad \|f\|_p \simeq N^{p/(p-1)}, \quad 1 < p < \infty.$$

It follows that $1 < p < 2$ is not permitted in Rubio de Francia's inequality.

**5.7. Remark.** In considering an estimate below $L^2$, in any dimension, we have the following interpolation argument available to us for all well-distributed collections $\Omega$. We have the estimate

$$\sup_{s \in \mathbf{T}^\Omega} \mathbf{1}_{R_s} \frac{|\langle f, \varphi_s \rangle|}{\sqrt{|R_s|}} \lesssim \mathrm{M} f \tag{5.8}$$

where M denotes the strong maximal function. Thus, the right-hand side is a bounded operator on all $L^p$. By taking a value of $p$ very close to one, and interpolating with the $L^2$ bound for $\mathrm{S}^\Omega$, we see that

$$\Bigg\| \Bigg\| \mathbf{1}_{R_s} \frac{|\langle f, \varphi_s \rangle|}{\sqrt{|R_s|}} \Bigg\|_{\ell^q(\Omega)} \Bigg\|_p \lesssim \|f\|_p, \qquad 1 < p < 2, \ \frac{p}{p-1} < q < \infty. \tag{5.9}$$

By (2.9), this inequality continues to hold without the well-distributed assumption. Namely, using (2.7), for all disjoint collections of rectangles $\Omega$,

$$\| \|\mathrm{S}_\omega f\|_{\ell^q(\Omega)} \|_p \lesssim \|f\|_p \qquad 1 < p < 2, \ \frac{p}{p-1} < q < \infty.$$

**5.10. Remark.** Cowling and Tao [18], for $1 < p < 2$, construct $f \in L^p$ for which

$$\big\| \|\mathrm{S}_\omega f\|_{\ell^{p'}(\Omega)} \big\|_{L^p} = \infty\,,$$

forbidding one possible extension of the interpolation above to a natural endpoint estimate.

**5.11. Remark.** Quek [35], on the other hand, finds a sharp endpoint estimate

$$\big\| \| \mathrm{S}_\omega f \|_{\ell^{p'}(\Omega)} \big\|_{L^{p,p'}} \lesssim \|f\|_p, \qquad 1 < p < 2,\ p' = \tfrac{p}{p-1}.$$

In this last inequality, the space $L^{p,p'}$ denotes a Lorenz space. Indeed, he uses the weak $L^1$ estimate (5.8), together with a complex interpolation method.

**5.12. Remark.** The higher dimensional formulation of Rubio de Francia's inequality did not admit an immediately clear formulation. J.-L. Journé [26] established the theorem in the higher dimensional case, but used a very sophisticated proof. Simpler arguments, very close in spirit to what we have presented, were given by F. Soria [40] in two dimensions, and in higher dimensions by S. Sato [37] and X. Zhu [43]. These authors continued to focus on the $G$ function (2.13), instead of the time frequency approach we have used.

**5.13. Remark.** We should mention that if one is considering the higher dimensional version of Theorem 1.2, with the simplification that the collection of rectangles consists only of cubes, then the method of proof need not invoke the difficulties of the BMO theory of Chang and Fefferman. The usual one-parameter BMO theory will suffice. The same comment holds if all the rectangles in $\Omega$ are homeothetic under translations and application of a power of a fixed expanding matrix.

**5.14. Remark.** It would be of interest to establish variants of Rubio de Francia's inequality for other collections of sets in the plane. A. Cordoba has established a preliminary result in this direction for finite numbers of sectors in the plane. G. Karagulyan and the author [27] provide a result for more general sets of directions, presuming *a priori* bounds on the maximal function associated with this set of directions.

**5.15. Remark.** The inequality (2.7) is now typically seen as a consequence of the general theory of weighted inequalities. In particular, if $h \in L^1(\mathbb{R})$, and $\epsilon > 0$, it is the case that $(\mathrm{M}\,h)^{1-\epsilon}$ is a weight in the Muckenhoupt class $A^1$. In particular, this observation implies (2.7). See the material on weighted inequalities in E.M. Stein [41].

**5.16. Remark.** Critical to the proof of Rubio de Francia's inequality is the $L^2$ boundedness of the tile operator $\mathrm{T}^\Omega$. This is of course an immediate consequence of the well-distributed assumption. It would be of some interest to establish a reasonable geometric condition on the tiles which would be sufficient for the $L^2$ boundedness of the operator $\mathrm{T}^\Omega$. In this regard, one should consult the inequality of J. Barrionuevo and the author [1]. This inequality is of a weak type, but is sharp.

**5.17. Remark.** V. Olevskii [31] independently established a version of Theorem 4.2 on the integers.

**5.18. Remark.** Observe that, in a certain sense, the multiplier result Theorem 4.2 is optimal. In one dimension, let $\psi$ denote a smooth bump function $\psi$ with frequency support in $[-1/2, 1/2]$, and for random choices of signs $\varepsilon_k$, $k \geq 1$, and integer $N$, consider the multiplier

$$m = \sum_{k=1}^{N} \varepsilon_k \operatorname{Tr}_k \widehat{\psi}.$$

Apply this multiplier to the function $\widehat{f} = \mathbf{1}_{[0,N]}$. By an application of Khintchine's inequality,

$$\mathbb{E}\|\mathrm{A}_m f\|_p \simeq \sqrt{N}, \qquad 1 < p < \infty.$$

On the other hand, it is straightforward to verify that $\|f\|_p \simeq N^{p/(p-1)}$. We conclude that

$$\|\mathrm{A}_m\|_p \gtrsim N^{|1/2-1/p|}.$$

Clearly $\mathbb{E}\|m\|_{V_q} \simeq N^{1/q}$. That is, up to arbitrarily small constant, the values of $q$ permitted in Theorem 4.2 are optimal.

**5.19. Remark.** V. Olevskii [30] refines the notion in which Theorem 4.2 is optimal. The argument is phrased in terms of multipliers for $\ell^p(\mathbb{Z})$. It is evident that the $q$-variation norm is preserved under homeomorphims. That is let $\phi : \mathbb{T} \longrightarrow \mathbb{T}$ be a homeomorphism. Then $\|m\|_{V_q(\mathbb{T})} = \|m \circ \phi\|_{V_q(\mathbb{T})}$. For a multiplier $m : \mathbb{T} \longrightarrow \mathbb{R}$, let

$$\|m\|_{M_p^0} = \sup_{\phi} \sup_{\|f\|_{\ell^p(\mathbb{Z})}=1} \left\| \int_{\mathbb{T}} \widehat{f}(\tau) m \circ \phi(\tau) e^{in\tau} \, d\tau \right\|_p$$

Thus, $M_p^0$ is the supremum over multiplier norms of $m \circ \phi$, for all homeomorphims $\phi$. Then, Olevskii shows that if $\|m\|_{M_p^0} < \infty$, then $m$ has finite $q$-variation for all $|\frac{1}{2} - \frac{1}{p}| < \frac{1}{q}$.

**5.20. Remark.** E. Berkson and T. Gillespie [7, 8, 9] have extended the Coifman, Rubio, Semmes result to a setting in which one has an operator with an appropriate spectral representation.

**5.21. Remark.** The Rubio de Francia inequalities are in only one direction. K.E. Hare and I. Klemes [22, 23, 24] have undertaken a somewhat general study of necessary and sufficient conditions on a class of intervals to satisfy a the inequality that is reverse to that of Rubio. A theorem from [24] concerns a collection of intervals $\Omega = \{\omega_j : j \in \mathbb{Z}\}$ which are assumed to partition $\mathbb{R}$, and satisfy $|\omega_{j+1}|/|\omega_j| \to \infty$. Then one has

$$\|f\|_p \lesssim \|\mathrm{S}^{\Omega} f\|_p, \qquad 2 < p < \infty.$$

What is important here is that the locations of the $\omega_j$ are not specified. The authors conjecture that it is enough to have $|\omega_{j+1}|/|\omega_j| > \lambda > 1$.

**5.22. Remark.** Certain operator theoretic variants of issues related to Rubio de Francia's inequality are discussed in the papers of Berkson and Gillespie [4, 5, 6].

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