PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 28 (42), 1980, pp. 43–50

# M-CONVEXITY AND BEST APPROXIMATION

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**Abstract**. The notion of M-convexity is introduced in Metric Spaces. The relations betwen M-convex, strictly M-convex and uniformly M-covex metric spaces are studied. The Best approximation properties for M-convex subsets of metric spaces are considered and many new results derived.

**Introduction.** The problem of best approximation has been extensively studied in normed linear spaces. The study of similar problems in metric linear spaces was initiated by G. Albinus [1, 2, 3] and I. Singer [12]. In metric linear spaces the proximinality of finite dimensional linear subspaces has been studied by K. Iseki [7] and in Frechet spaces, similar problems have been considered by V.N. Nikolski [10]. The consideration of best approximation problems in normed linear spaces was made by B.A. Hirschfeld [6] and A.M. Flomin [5]. In metric spaces many of the results of metric linear spaces were extended by I. Singer [12] and other. In this paper we have studied some such problems in a new kind of metric space which admits extensions of many results true in metric linear spaces and normed linear spaces.

In Section 1, we have defined M-convexity for metric spaces. The idea is essentially due to K. Menger who has survived in the prefix of the concept. Strict M-convexity and uniform M-convexity have been defined then and the relations among these spaces have been studied. In Sectton 2 M-convexity is defined for subsets and Chebyshev and proximinal properties are studied there.

DEFINITION 1.1. A metrix space $(X, d)$ is said to be M-convex if for every $x$, $y$ in $X$, $x \neq y$, there exists a $z$ in $X$ different from $x$ and $y$ such that

$$d(x,y) = d(x,z) + d(z,y).$$

One can immediately see that

PROPOSITION 1.2. Every normed linear linear space is an M-convex metric space.

We now give two exmples to illustrate the fact that not every metric space is M-convex and also the fact not every M-convex metrix space is a normed linear space.

EXAMPLE 1.3. Let $K$ be a non-convex closed subset of $R^n$ equipped with the relative topology. Then it is easy to see that $K$ is a metric space which is not M-convex.

EXAMPLE 1.4. Let $U$ denote the unit ball of $R^2$ i.e.

$$U = \{(x, y) \in R^2; x^2 + y^2 \leq 1\}$$

Then, if $U$ is equipped with the usual Euclidean metric, then $U$ becomes an M-convex metric space which is not a normed linear space.

DEFINITION 1.5. A metric space $(X, d)$ is said to be strictly M-convex if for every $x$, $y$, $t$ in $X$, all different and $r > 0$, there exists a $z$ in $X$ different from $x$, $y$ and $t$ such that

$$d(x, y) = d(x, z) + d(z, y) \tag{1}$$

$$d(x, t) \leq r, d(y, t) \leq r \text{ imply } d(z, t) < r \tag{2}$$

An example of an M-convex metric space which is not strictly M-convex is the following

EXAMPLE 1.6. Consider the metrix space $(X, d)$ where $d$ is defined as

$$d(x, y) = \max\{|x_1 - y_1|, \quad |x_2 - y_2|\}$$

$$\text{where } x = (x_1, x_2), \quad y = (y_1, y_2)$$

$$\text{and} \qquad X = \{x, y) \in R^2; \quad x > 0, \ y > 0\} \cup (0, 0).$$

Then it is easy to see th that $(X, d)$ is M-convex. To see that it is not strictly M-convex, consider

$$x = (1/3, 1/3) \qquad y = (2/3, 2/3)$$

$$t = (0, 0) \qquad r = 2/3.$$

and check that does not exist any $z$ in $X$ satisfying both the requirements of the definition.

DEFINITION 1.7. A metrix space $(X, d)$ is said to be uniformly M-*convex* if for every pair of positive numbers $\varepsilon$ and $r$, there corresponds a positive number $\delta$ such that for every triplet $x$, $y$, $t$ in $X$ all different and satisfying $d(x, y) \geq \varepsilon$, $d(x, t) < r + \delta$, $d(y, t) < r + \delta$ there exists a $z$ in $X$ with the properties

$$d(x, y) = d(x, z) + d(z, y) \tag{1}$$

$$d(z, t) < r. \tag{2}$$

As an example of a uniformly M-convex metric space we site the following example.

EXAMPLE 1.8. Consider $(M, d)$ with metric $d$ defined as $d(x, y) = |x - y|$.

To see that it is uniformly M-convex, it is enough to check $d(x, y) \geq \varepsilon$, $d(x, 0) < r'$, $d(y, 0) < r'$, there exists a $z$ in $X$ defferent from $x$, $y$, $t$ such that

$$d(x, y) = d(x, z) + d(z, y) \tag{1}$$

$$d(z, 0) < r. \tag{2}$$

This can be verified easily.

One can readily see now.

PROPOSITION 1.9. Every uniformly M-convex metric space is strictly M-convex and not conversely.

DEFINITION 1.10. A metric space $(X, d)$ is said to be *totally complete* if every bounded closed subset of $X$ is compact.

We can now prove the following theorem.

THEOREM 1.11. *Every totaly complete strictly M-convex metrix space is unifomly M-convex.*

PROOF. Let $(X, d)$ be a totally complete M-convex metric space. Equip $X \times X$ with a metric $\rho$ defined as

$$\rho((x_1, y_1),\ (x_2, y_2)) = \{d_2(x_1, x_2) + d^2(y_1, y_2)\}^{\frac{1}{2}}$$

clearly, $(X \times X, \rho)$ is totally complete.

Then $S_t = \{(x, y) \in X \times X;\ d(x, t) \leq r\}$ is a closed and bounded subset of $X \times X$ and hence compact for every $t \in X$.

Define $\Phi_t : S_t \to R$ as

$$\Phi_t(x, y) = r - d(z, t) \text{ where } d(x, z) + d(z, y) = (x, y).$$

Then $\Phi_t$ is continuous and positive on $S_t$ and therefore there exists $\delta > 0$ such that

$$r - d(z, t) \geq \delta \text{ for all } t \text{ in } X$$
$$\text{i.e. } d(z, t) \leq r - \delta < r.$$

Hence the result.

2. We now define M-convex subsets of a metrix space.

DEFINITION 2.1. A subset $G$ of a metric space $(X, d)$ is said to be M-convex if for every $x, y \in G$, $x \neq y$, there exists a $z$ in $G$ such that $d(x, z) + d(z, y) = d(x, y)$.

We remark here that there are metric spaces no subset of which is M-convex. As an example of such a metrix space we refer to the following.

EXAMPLE 2.2. Consider $X = R$ with the metrix $d$ defined as

$$d(x,y) = \frac{|x-y|}{1+|x-y|}.$$

It is worthy to note at this stage that there is no relation as such betwen convexity and M-convexity in metric linear spaces. The example 2.2 provides an illustration to the fact that in metric linear spaces there are convex sets which are not M-convex while example 2.3 below proves the other direction.

EXAMPLE 2.3. Consider the metric linear space $(X, d)$ where $X = R^2$ and $d$ is defined as

$$d(x,y) = \max\{|x_1 - y_1|,\ \ |x_2 - y_2|\} \text{ where } x - (x_1, y_2)$$
$$y = (y_1, y_2).$$

Consider the set

$$G = \{(z_1, z_2);\quad 0 \leq z_1 \leq 2,\ \ z_2 = 0\} \cup \{(z_1, z_2);\quad z_1 = 2,\ \ 0 \leq z_2 \leq 1\}$$

Then it is easy to check that $G$ is M-convex but not convex.

We however remark that in normed linear spaces this is not true.

DEFINITION 2.4. A subset $G$ of a metric space $(X, d)$ is said to be *promiminal* if for every $x$ in $X$ there exists at least one $\xi$ in $G$, called the best approximating element of $x$ in $G$ such that

$$d(x,\xi) = d(x,G) \equiv \inf_{z \in G} d(x,z)$$

The set $G$ is said to be *Chebyshev* if to every $x \in X$ there exists exactly one $\xi \in G$ such that

$$d(x,\xi) = d(x,G).$$

We shall denote by $\Pi_{G(x)}$ the set of best approximating elements of $x$ in $G$.

Thus we can define the setvalued map $\Pi_G$ from $X$ into subsets of $G$ as

$$\Pi_G(x) = \{z \in G;\quad d(x,z) = d(x,G)\}$$

We can define another real-valued function $e_G$ on $X$ as $e_G(x) = d(x,G)$.

Clearly $e_G$ is uniformly continuous. Regarding continuity of $\Pi_{G(x)}$, it is known that $\Pi_G$ is continuous at every point of $G$ if $G$ is Chebyshev.

We can prove the following theorems now.

THEOREM 2.5. *If $(X,d)$ is an M-convex metric space and $G$ is a Chebyshev subset of $X$ then if $z$ corresponds the M-convex element of $x$ and $\Pi_{G(x)}$*

$$\Pi_G(z) = \Pi_G(x).$$

PROOF. By the definition of $z$, we have

$$d(x,z) + d(z,\Pi_G(x)) = d(x,\Pi_G(x)).$$

Now, if $\xi \in G$, then

$$\begin{aligned} & d(z,\xi) \geq d(x,\xi) - d(x,z) \\ & d(x,\Pi_G(x)) - d(x,z) \\ & = d(z,\Pi_G(x)) \end{aligned}$$

This implies that $\Pi_G(x)$ is a best approximating element of $z$. Since $G$ is Chebyshev, $\Pi_G(x) = \Pi_G(z)$.

THEOREM 2.6. *If $(X,d)$ is a metric space, $G$ is a subset of $X$ and $y_0 \in G$, then $\Pi_G^{-1}(y_0)$ is closed.*

*Further, if $x_0 \in \Pi_G^1(y_0)$ and $d(x,z) + d(z,y_0) = d(x_0,y_0)$ for some $z \in X$, then $z \in \Pi_G^{-1}(y_0)$.*

PROOF. By definition

$$\begin{aligned} \Pi_G^{-1}(y_0) &= \{x \in X; \quad d(x,y_0) = d(x,G)\} \\ &= \bigcap_{y \in G} \{x \in X; \quad d(x,y_0) \leq d(x,y)\} \end{aligned}$$

By the continuity of the metric $d$, the first part of the result is then obvious. Now since $x_0 \in \Pi_G^{-1}(y_0)$, we have

$$d(x_0,y_0) \leq d(x_0,y) \text{ for all } y \in G.$$

Since $z$ satisfies $d(x_0,z) + d(z,y_0) = d(x_0,y_0)$, we write

$$\begin{aligned} d(x,y_0) &= d(x_0,y_0) - d(x_0,z) \\ &\leq d(x_0,y_0) \leq d(x_0,y) \text{ for all } y \in G \end{aligned}$$

$\therefore z \in \Pi_G^{-1}(y_0)$.

COROLLARY 2.7. If $G$ is Chebyshev in $X$, then $\Pi_G^{-1}(\Pi_G(x))$ is closed for every $x$ in $X$.

In general, proximinal sets or Chebyshev sets are neither convex nor M-convex. L.N.H. Bunt [4] and T.S. Motzkin have given conditions under which every Chebyshev set is convex. These conditions are however sufficient but not necessary. One such result is the following.

THEOREM. *In a finite dimensional smooth Banach Space, every Chebyshev set is convex and hence M-convex. But the problem whether the result is true for infinite dimensional Banach Spaces remains still open. Another interesting open problem is the following.*

Whether in a Hilbert Space, every Chebyshev set is convex? Under the present context, we can ask whether in a Hilbert space, every Chebyshev set is M-convex.

Before we prove our next theorem we need the following definition.

DEFINITION 2.8. In a metric space $(X, d)$, a Menger set denoted as $M_{<x,y>}$ for a pair of distinct points $x$, $y$ is defined as the set of elements $z$ in $X$ such that

$$d(x,z) + d(z,y) = d(x,y)$$
$$\text{i.e, } M_{<x,y>} = \{z \in X; \quad d(x,z) + d(z,y) = d(x,y)\}.$$

One can immediately see that Menger sets can be emptysets, singleton sets or arbitrarily large sets. For example, we recall that $(R, d)$ with $d(x,y) = \frac{|x-y|}{1+|x-y|}$ has empty Menger sets for every pair of points of $x$, $y$ while $(R, d')$ with $d'(x,y) = |x-y|$ is such that the Menger set for every pair of distinct points is uncountable.

One can immediattely note

PROPOSITION 2.9. Every Menger set is closed.

DEFINITION 2.10. If a metric space has only singleton Menger sets for every pair of distinct elements, then it will be called Mengerian. We can prove the following theorem now.

THEOREM 2.11. *Every* M-*convex proximinal set in a strictly* M-*convex Mengerian metric space is Chebyshev.*

PROOF. Suppose $G$ is an M-convex promiminal set in the strictly M-convex Mengerian metric space $(X, d)$.

If possible, for some $x_0 \in X$, let $y_1, y_2 \in G$ be two best approximating elements i.e. $\Pi_G(x_0) = \{y_1, y_2\}$.

Then $d(x_0, y_1) = d(x_0, y) = \inf_{\xi \in G} d(x, \xi) = r$ say.

Since $X$ is strictly M-convex, there exists $z \in X$, $x \neq z \neq y$ such that $d(x,z) < r$

$$\text{and } d(y_1, z) + d(z, y_2) = d(y_1, y_2).$$

But since $G$ is M-convex and $X$ is Mengerian, $z \in G$.

This contradicts the definition of $r$. Hence the proof.

DEFINITION 2.12. A set $G$ in a metric space $(X, d)$ is said to be *approximatively compact* if for every sequence $y_n$ in $G$ with $\lim_n d(x, y_n) = d(x, G)$, there exists a subsequence $y_{n_k}$ coverging to an element of $G$.

THEOREM 2.13. *In an unifomly M-convex Mengrian metric space every complete M-convex set is approximatively compact.*

PROOF. Let $G$ be an M-convex complete set in uniformly M-convex Mengerian metric space $(X, d)$.

Let $y_n$ be a sequence in $G$ satisfying

$$\lim d(x, y_n) = d(x, G) = r \text{ (say)}$$

Let $\varepsilon > 0$ be arbitrary.

Since $X$ is uniformly M-convex, we can find a $\delta > 0$ satisflying some inequality relations.

Since $\lim_n d(x, y_n) = r$, we can choose a positive integer $N$ such that

$$d(x, y_n) > r + \delta \text{ whenever } n \geq N.$$

Let $n, m \geq N$. Then by the inequality relations, we get

$$d(x, y_n) < r + \delta$$
$$d(x, y_m) < r + \delta.$$

If possible let $d(y_n, y_m) \geq \varepsilon$.

Then these imply that there exists a $y \in X$ such that

$$d(y_n, y_0) + d(y_0, y_m) = d(y_n, y_m)$$

and

$$d(x, y_0) < r.$$

Since $X$ is M-convex and Mengerian, $y \in G$ and thus we arrive at a contradiction that $r = d(x, G)$.

Therefore $d(y_n, y_m) < \varepsilon$ for $m, n \geq N$.

i.e. $y_n$ is a Cauchy sequence.

By the completeness of $G$, the result follows then.

DEFINITION 2.14. A metric space $(X, d)$ is said to have *P-property* if for a fixed $p$ in $X$, every sequence $y_n$ in a M-convex set $G$ of $X$ satisfying $\lim_n d(p, y_n) = d(p, G)$ has a Cauchy subsequence.

The following theorem can now be proved.

THEOREM 2.15. *A complete M-convex subset G of a metric space $(X, d)$ having P-property is Chebyshev.*

PROOF. Let $p \in X$ and $r = d(x, G)$.

So there exists a sequence $y_n$ in $G$ such that

$$\lim_n d(p, y_n) = r$$

By P-property, $y_n$ has a Chauchy subsequence $y_{n_k}$ in $G$. Since $G$ is complete, $y_n$ converges to some $y \in G$.

By continuity of the metric, we then get

$$d(p, y) = r.$$

If possible now let $y_1, y_2 \in G$ be such that

$$d(p, y_1) = d(p, y_2) = r.$$

Define a sequence $z_n$ as follows

$$\begin{aligned} z_n &= y_1 \text{ if } n \text{ is even} \\ &= y_2 \text{ if } n \text{ is odd.} \end{aligned}$$

Then $\lim_n d(p, z) = d(p, z_1) = r = d(p, z_2)$.

By P-proporety $z_n$ has a Cauchy subsequence $z_{n_k}$ i.e., for given $\xi > 0$, there exists a positive integer $N$ such that $n_k, m_k \geq N$ implies $d(z_{n_k}, z_{m_k}) < \xi$.

Since $\varepsilon$ is arbitrary, $y_1 = y_2$.

This proves that $G$ is Chebyshev.

THEOREM 2.16. *Every uniformly M-convex Mengerian metric space has P-property.*

PROOF. Embodied in Theorem 2.14.

*Acknowledgement:* The author wishes to thank Dr. M. K. Das for constant encouragement and helpful criticism in the preparation of this paper.

REFERENCES

[1] G. Albinus; *Einge Beitrage zur Approximations Theorie in metrishen Vektorraumen*, Wiss Z. d. Tech. Univ. Dresden, 15, 1–4 (1966).

[2] G. Albinus; *Uber Best approximationen in metrischen Vektorraumen*, Dissertation, Dresden, 1966.

[3] G. Albinus; *Normartige Metriken auf metrisierbaren lokal konvexen topologischen Vektorraumen*, Math. Nach, 37, 177–196 (1968).

[4] L.N.M. Bunt; *Bijdrage to t de theorie der konvekse puntverzamelingen*, Thesis, University of Groningen, publ. Amsterdam, 1934.

[5] A.M. Flomin; *Some problems of Approximation Theory in locally convex topological vector space* (Russian), Thesis, Phy. Math. Sc., Moscow, 1966.

[6] R.A. Hirschfeld; *On best approximations in normed vector spaces I and II*, Nieuw. Arch. voor Wisk, 6, 41–51, 99–107 (1958).

[7] K. Iseki; *An approximation problem in quasi-normed spaces*, proc. Japan Acad., 35, 465–466 (1959).

[8] C. Kuratowski; *Use condition metrique pour la retraction des ensembles*, Comptes rendus Soc. Sci lettres Varsovie, Cl, III, 28, 156–158 (1936).

[9] K. Menger; *Untersuchungen über allgemeine Metrik*, Math. Ann. 100, 75–103 (1928).

[10] V.N. Nikolski; *Best Approximation by elements of convex sets in normed linear spaces*, (Russian) Kalimin – Gos. Pred. Inst Usen. Pa. p 29 (1963).

[11] W. Nitka; *Une generalisation du theoreme de Kuratovski sur la caracterisation metrique de la retraction*, Colloq. Math. 8, 35–37 (1961).

[12] I. Singer; *Best Approximation in Normed Linear spaces by elements of Linear Subspaces*, Springer-Verlag, 1970.

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