

## STRUCTURE CONNECTION IN AN ALMOST CONTACT METRIC MANIFOLD

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**Summary.** In 1970, semisymmetric connection were studied by Yano [1] in a Riemannian manifold and in 1972 Mishra [2] studied affine connection in an almost contact Riemannian manifold. In the present paper we have defined a structure connection in a Riemannian manifold and studied its properties in an almost contact metric manifold. It is seen that structure connection play fundamental role in an almost contact metric manifold.

### 1. Introduction

Let  $M_n$  be a  $n(= 2m + 1)$  dimensional  $C^\infty$ -manifold and let there exist a vector valued function  $f$ , a vector field  $t$  and a  $l$ -form  $A$  in  $M_n$  such that

$$(1.1) \quad \overline{X} + X = A(X)t, \quad \overline{X} \stackrel{\text{def}}{=} f(X)$$

for arbitrary vector field  $X$ , then  $M_n$  is called an almost contact manifold and the structure  $(f, t, A)$  is an almost contact structure. In an almost manifold the following hold [2]

$$(1.2) \quad \begin{aligned} \text{rank}(f) &= n - 1, \quad \bar{t} = 0, \quad A(\overline{X}) = 0 \\ A(t) &= 1. \end{aligned}$$

Let the almost contact manifold  $M_n$  be endowed with the nonsingular metric tensor  $g$  satisfying

$$(1.3) \quad g(\overline{X}, \overline{Y}) = g(X, Y) - A(X)A(Y)$$

Then  $M_n$  is called an almost contact metric manifold or Grayan manifold.

From (1.3) we obtain

$$(1.4) \quad g(Y, t) = A(Y)$$

Putting  $F(X, Y) = g(\bar{X}, Y)$ , we have

$$(1.5) \quad F(\bar{X}, \bar{Y}) = F(X, Y); \quad F(X, Y) = -F(Y, X).$$

If  $D$  be the Riemannian connection in an almost contact metric manifold, then

$$(1.6) \quad (D_x A)(Y) = g(D_x t, Y).$$

In an almost contact metric manifold Nijenhuis tensor  $N$  is given by

$$(1.7a) \quad N(X, Y) = N_D(X, Y) = (D_{\bar{X}}f)(Y) - (D_{\bar{Y}}f)(X) - \\ - \overline{(D_X f)(Y)} + \overline{(D_Y f)(X)}$$

$$(1.7b) \quad N(X, Y, Z) = N_D(X, Y, Z) = (D_{\bar{X}}F)(Y, Z) - (D_{\bar{Y}}F)(X, Z) \\ + (D_X F)(Y, \bar{Z}) - (D_Y F)(X, \bar{Z})$$

where

$${}^1N(X, Y, Z) = g((D_X F)(Y), Z).$$

An almost contact manifold  $M_n$  is said to be normal if the almost complex structure  $J$  on  $M_n \times R$  given by

$$(1.8) \quad J\left(X, h \frac{d}{dt}\right) = \left(f(X) - ht, A(X) \frac{d}{dt}\right)$$

where  $h$  is  $C^\infty$  - real valued function on  $M_n$ , is integrable. From this we have an almost contact manifold is normal if

$$(1.9) \quad N(X, Y) + dA(X, Y)t = 0.$$

An almost contact metric manifold  $M_n$  in which

$$(1.10) \quad F(X, Y) = (D_X A)(Y) - (D_Y A)(X) = (dA)(X, Y)$$

is called an almost Sasakian manifold (1.10) is equivalent to

$$(1.11) \quad (D_X F)(Y, Z) + (D_Y F)(Z, X) + (D_Z F)(X, Y) = 0.$$

An almost Sasakian manifold is said to be  $k$ -contact Riemannian manifold if  $A$  is a Killing vector i.e. if

$$(1.12) \quad (D_X A)(Y) + (D_Y A)(X) = 0.$$

Thus in a  $K$ -contact Riemannian manifold we have

$$(1.13) \quad F(X, Y) = 2(D_X A)(Y) = -2(D_Y A)(X).$$

An almost contact manifold with symmetric affine connection  $D$  is said to be affinely Sasakian if it is normal and

$$(1.14) \quad fX = D_X t.$$

An almost contact manifold with a symmetric affine connection  $D$  is called an affinely almost cosymplectic manifold if

$$(1.15) \quad D_X f = 0, \quad D_X A = 0.$$

## 2. Structure Connection in a Riemannian Manifold

Let  $M_n$  be a  $C^\infty$  - Riemannian manifold of dimension  $n$ . Let  $f^*$ ,  $t^*$  and  $A^*$  be respectively (1.1) tensor field, a vector field and a 1-form in  $M_n$  and let  $D$  be a Riemannian connection in  $M_n$ . We define a connection  $B$  in  $M_n$  by

$$(2.1a) \quad B_X(Y) = D_X Y + A^*(Y)X - g(X, Y)G_*A^* + F^*(X, Y)t^*$$

and

$$(2.1b) \quad B_X g = 2A^*(X)g,$$

$$(2.1c) \quad g(f^*X, Y) + g(X, f^*Y) = 0$$

where  $g$  is a Riemannian metric and  $F^*(X, Y) = g(f^*(X), Y)$  and

$$g(G_*A^*, X) = A^*(X).$$

DEFINITION (2.1). A connection  $B$  in a Riemannian manifold  $M_n$  given by (2.1a), (2.1b) and (2.1c) is called a structure connection in  $M_n$ .

The torsion tensor  $S$  of structure connection  $B$  is given by

$$(2.2) \quad S(X, Y) = A^*(Y)X - A^*(X)Y + 2F^*(X, Y)t^*.$$

Let us put

$$(2.3) \quad B_X Y = D_X Y + P(X, Y)$$

(2.3) gives

$$(2.4) \quad S(X, Y) = P(X, Y) - P(Y, X).$$

The property (2.1a) of the definition can be written in the form

$$(2.5) \quad g(P(X, Y), Z) + g(P(X, Z), Y) = -2A^*(X)g(Y, Z).$$

Let

$$Q: T(M_n)^* \times T(M_n) \times T(M_n) \rightarrow \mathcal{Z}(M_n)$$

be the mapping defined by

$$(2.6) \quad Q(\omega, X, Y) = g(S(X, G_*\omega), Y) + g(S(Y, G_*\omega), X),$$

where  $T(M_n)$  is a set of all vector fields on  $M_n$  and  $T(M_n)^*$  is a set of all  $l$ -forms on  $M_n$  and  $\mathcal{Z}(M)$  is a set of all  $C^\infty$  - functions on  $M_n$ .

Taking account of (2.2) we have

$$(2.7) \quad \begin{aligned} Q(\omega, X, Y) &= 2A^*(G_*\omega)g(X, Y) - A^*(X)\omega(Y) \\ &\quad - A^*(Y)\omega(X) + 2\omega(f^*(X))g(t^*, Y) + 2\omega(f^*Y)g(t^*, X). \end{aligned}$$

According (2.4) and (2.5)

$$(2.8) \quad \begin{aligned} Q(\omega, X, Y) &= 2A^*(G_*\omega)g(X, Y) - 2A^*(X)\omega(Y) \\ &\quad - 2A^*(Y)\omega(X) - \omega(P(X, Y) + P(Y, X)). \end{aligned}$$

From (2.7) and (2.8) we deduce for every  $l$ -form

$$\begin{aligned} \omega(P(X, Y) + P(Y, X)) + A^*(X)\omega(Y) + A^*(Y)\omega(X) \\ + 2\omega(f^*X)g(t^*, Y) + 2\omega(f^*Y)g(t^*, X) = 0 \end{aligned}$$

and therefore

$$(2.9) \quad \begin{aligned} P(X, Y) + P(Y, X) + A^*(X)Y + A^*(Y)X + 2f^*(X)g(t^*, Y) \\ + 2f^*(Y)g(t^*, X) = 0 \end{aligned}$$

from (2.2) and (2.4) we obtain

$$(2.10) \quad P(X, Y) - P(Y, X) = A^*(Y)X - A^*(X)Y + 2F^*(X, Y)t^*$$

from (2.9) and (2.10) we have

$$(2.11) \quad \begin{aligned} P(X, Y) &= -A^*(X)Y - f^*(X)g(t^*, Y) - f^*(Y)g(t^*, X) \\ &\quad + F(X, Y)t^*. \end{aligned}$$

Thus from (2.3) and (2.11) structure connection  $B$  in a Riemannian manifold  $(M_n, g)$  is given by

$$(2.12) \quad \begin{aligned} B_X(Y) &= D_X(Y) - A^*(X)Y - f^*(X)g(t^*, Y) - f^*(Y)g(t^*, X) \\ &\quad + F(X, Y)t^*. \end{aligned}$$

### 3. Structure connection in an almost contact metric manifold

Let  $M_n$  be a almost contact metric manifold and  $(f, t, A)$  be an almost contact structure on  $M_n$  then a contact connection  $B$  in an almost contact metric manifold is given by

$$(3.1) \quad B_X Y = D_X Y - A(X)Y - \bar{X}A(Y) - \bar{Y}A(X) + g(\bar{X}, Y)t$$

the structure connection may also be written as

$$(3.2) \quad (B_X A)Y = (D_X A)Y + A(X)A(Y) - F(X, Y).$$

Thus we have the following theorem:

**THEOREM 3.1.** *Every almost contact metric manifold admits a structure connection  $B$  defined by (3.2). The structure connection  $B$  is uniquely determined by the contact form  $A$  and tensor field  $f$ .*

From (2.2) we have in an almost contact manifold the following results.

$$(3.3) \quad \begin{aligned} \text{a) } & S(\bar{X}, \bar{Y}) = 2F(X, Y)t \\ \text{b) } & S(X, t) = -\bar{X} \\ \text{c) } & S(\bar{X}, t) = \bar{X} \\ \text{d) } & S(\bar{\bar{X}}, t) + S(X, t) = 0 \\ \text{e) } & S(\bar{\bar{X}}, Y) = A(X)A(Y)t - A(Y)X - 2F(X, Y)t \\ \text{f) } & S(\bar{X}, Y) + S(\bar{X}, Y) = A(\bar{Y})X - A(X)Y \\ \text{g) } & A(S(X, Y)) = 2F(X, Y) \\ \text{h) } & \overline{\overline{S(\bar{X}, \bar{Y})}} + S(X, Y) - S(\bar{X}, \bar{Y}) \end{aligned}$$

Let us define

$$'S(X, Y, Z) = g(S(X, Y), Z)$$

$$'P(X, Y, Z) = g(P(X, Y), Z).$$

In an almost contact metric manifold with structure connection  $B$  we have

$$(3.4a) \quad B_X \bar{Y} = D_X \bar{Y} + g(X, Y)t$$

$$(3.4b) \quad \bar{B}_X \bar{Y} = \bar{D}_X \bar{Y}$$

$$(3.4c) \quad (B_X A)\bar{Y} = (D_X A)\bar{Y} - g(\bar{X}, \bar{Y})$$

$$(3.4d) \quad (B_X F)(Y, Z) = 2A(X)g(\bar{Y}, Z) + g((B_X f)(Y), Z)$$

$$(3.4e) \quad (B_X f)(Y) = (D_X f)(Y) - g(X, Y)t - A(Y)X$$

**THEOREM (3.2).** *In an almost contact metric manifold with structure connection  $B$ , we have*

$$(3.5a) \quad \begin{aligned} 'P(X, Y, Z) &= A(Z)F(X, Y) - A(X)F(Y, Z) \\ &\quad - A(Y)F(X, Z) - A(X)g(Y, Z) \end{aligned}$$

$$(3.5b) \quad 'P(\bar{X}, \bar{Y}, \bar{Z}) = 0 = 'S(\bar{X}, \bar{Y}, \bar{Z}).$$

PROOF. The proof follows from (1.4) and (2.11).

**THEOREM 3.3.** *If  $B$  a structure connection in an almost contact metric manifold  $M_n$  with a Riemannian connection  $D$  then*

$$(3.6a) \quad (B_X F)(Y, Z) = (D_X F)(Y, Z) - 2A(X)F(Y, Z) \\ + A(Y)g(X, Z) - A(Z)g(X, Y)$$

$$(3.6b) \quad (B_{\overline{X}} F)(\overline{Y}, \overline{Z}) = (D_{\overline{X}} F)(\overline{Y}, \overline{Z}).$$

PROOF. We know,

$$X(F(Y, Z)) = (B_X F)(Y, Z) + F(B_X Y, Z) + F(Y, B_X Z)$$

and

$$X(F(Y, Z)) = (D_X F)(Y, Z) + F(D_X Y, Z) + F(Y, D_X Z).$$

From the above equations, we get

$$(3.7) \quad (B_X F)(Y, Z) = (D_X F)(Y, Z) - {}'P(X, Y, \overline{Z}) + {}'P(X, Z, \overline{Y}).$$

Using (3.7), (1.4) and (2.11) we have the theorem.

**COROLLARY 1.** *If  $M_n$  is an almost Sasakian then*

$$(3.8) \quad (B_X F)(Y, Z) + (B_Y F)(Z, X) + (B_Z F)(X, Y) = -2A(X)F(Y, Z) \\ - 2A(Y)F(Z, X) - 2A(Z)F(X, Y).$$

**COROLLARY 2.** *In an almost contact metric manifold with structure connection  $B$  we have*

$$(3.9a) \quad (B_{\overline{X}} F)(\overline{Y}, Z) = (D_{\overline{X}} F)(\overline{Y}, Z) \\ - A(Z)g(X, Y) + A(X)A(Y)A(Z)$$

$$(3.9b) \quad (B_X F)(\overline{\overline{Y}}, \overline{\overline{Z}}) + (B_X F)(\overline{Y}, \overline{Z}) = -4A(X)F(Y, Z)$$

**THEOREM 3.4.** *In an almost contact metric manifold with a structure connection  $B$  we have*

$$(3.10a) \quad N_B(X, Y) = N(X, Y) + S(\overline{X}, \overline{Y})$$

$$(3.10b) \quad {}'N(X, Y, Z) = {}'N_B(X, Y, Z) + 2A(Y)g(X, Z) - 2A(X)g(Y, Z)$$

where

$$N_B(X, Y) = (B_{\overline{X}} f)(Y) - (B_{\overline{Y}} f)(X) - (\overline{B_X f})(Y) + (\overline{B_Y f})(X).$$

PROOF. We have in an almost contact metric manifold Nijenhuis tensor  $N$  is given by

$$\begin{aligned}
N(X, Y) &= D_{\overline{X}}\overline{Y} - \overline{D_{\overline{X}}Y} - D_{\overline{Y}}\overline{X} + \overline{D_{\overline{Y}}X} - \overline{D_X\overline{Y}} + \overline{D_X\overline{Y}} + \overline{D_Y\overline{X}} - \overline{D_YX} \\
&= B_{\overline{X}}\overline{Y} - F(X, Y)t - B_{\overline{Y}}\overline{X} + F(Y, X)t - \overline{B_{\overline{X}}Y} \\
&= \overline{\overline{X}}A(Y) + \overline{B_{\overline{Y}}X} + \overline{\overline{Y}}A(X) - \overline{B_{\overline{X}}\overline{Y}} - A(X)\overline{\overline{Y}} \\
&\quad - \overline{\overline{Y}}A(X) + \overline{B_Y\overline{X}} + A(Y)\overline{\overline{X}} + \overline{\overline{X}}A(Y) + \overline{B_XY} \\
&\quad + A(X)\overline{\overline{Y}} + \overline{\overline{X}}A(Y) + \overline{\overline{Y}}A(X) - \overline{B_YX} - A(Y)\overline{\overline{X}} \\
&\quad - \overline{\overline{Y}}A(X) - \overline{\overline{X}}A(Y).
\end{aligned}$$

From this have the theorem (3.4).

THEOREM 3.5. *In a  $k$ -contact Riemannian manifold with structure connection  $B$  we have*

$$(3.11a) \quad (B_X A)(Y) + (B_Y A)(X) = 2A(X)A(Y)$$

$$(3.11b) \quad A(N_B(X, Y)) = F(X, Y).$$

PROOF. Using (1.12) and (3.2) we get first part of the theorem.

For the proof of 2nd, using the equations (1.13) and (3.2) we have in a  $k$ -contact Riemannian manifold.

$$(3.12) \quad (B_{\overline{X}}A)(\overline{Y}) - (B_{\overline{Y}}A)(\overline{X}) = -F(X, Y)$$

also

$$(3.13) \quad (B_{\overline{X}}A)(\overline{Y}) = -A((B_{\overline{X}}f)(Y)).$$

Again from (1.17) a, we have

$$(3.14) \quad A(N_B(\overline{X}, \overline{Y})) = A((B_{\overline{X}}f)(Y)) - A((B_{\overline{Y}}f)(X))$$

from (3.12), (3.13) and (3.14) we have the 2nd part of the theorem.

COROLLARY. *In a  $k$ -contact Riemannian manifold with structure connection  $B$ , we have*

$$(3.15a) \quad A(N_B(\overline{X}, \overline{Y})) = A(N_B(X, Y))$$

$$(3.15b) \quad A(N_B(\overline{\overline{X}}, \overline{\overline{Y}})) + A(N_B(\overline{X}, \overline{Y})) = 0.$$

**THEOREM 3.6** *In a normal contact metric manifold with structure connection  $B$*

$$(3.6) \quad N_B(X, Y) = F(X, Y)t = \frac{1}{2}S(\bar{X}, \bar{Y}).$$

**PROOF.** Using (1.9) (3.3a) and (3.10a), we have the theorem.

**THEOREM 3.7.** *In a  $k$ -contact Riemannian manifold with structure connection  $B$ , we have*

$$(3.17a) \quad B_X t = -A(X)t$$

$$(3.17b) \quad (B_X A)(Y) = A(X)A(Y).$$

**PROOF.** In a  $k$ -contact manifold we can easily obtain

$$(3.18) \quad D_X t = \bar{X}$$

(3.18) together with (3.1) gives the first part of the theorem. Using (3.2) and (1.13) we have 2nd part of the theorem.

**COROLLARY.** *An almost contact metric manifold  $M_n$  with structure connection  $B$  is a affinely Sasakian manifold if*

$$(3.19a) \quad A(N_B(X, Y)) = F(X, Y)$$

and

$$(3.19b) \quad B_X t = -A(X)t.$$

#### 4. Curvature of structure connection in an almost contact metric manifold

Let  $\tilde{R}$  be the curvature tensor of structure connection  $B$  and  $K$  be the curvature tensor of Riemannian connection  $D$  in an almost metric manifold, then

$$(4.1) \quad \tilde{R}(X, Y, Z) = B_X B_Y Z - B_Y B_X Z - B_{[X, Y]} Z.$$

We have following theorem:

**THEOREM (4.1)** *In an almost contact metric manifold the curvature tensor  $\tilde{R}$  is given by*

$$(4.2) \quad \tilde{R}(X, Y, Z) = K(X, Y, Z) + {}_X\theta_Y - {}_Y\theta_X,$$



where  ${}_X\theta_Y$  is given by

$$(4.3) \quad \begin{aligned} {}_X\theta_Y &= (D_Y A)(X)(Z + \bar{Z}) + (D_X F)(Y, Z)t \\ &\quad + F(Y, Z)D_X t + A(Y)A(Z)X + (D_Y A)(Z)\bar{X} \\ &\quad + F(X, Z)\bar{Y} + A(X)g(Y, Z)t + A(Z)(D_Y f)(X) \\ &\quad + A(X)(D_Y f)(Z). \end{aligned}$$

THEOREM (4.2). *In an affinely almost cosymplectic manifold we have*

$$(4.4a) \quad \begin{aligned} \tilde{R}(X, Y, Z) &= K(X, Y, Z) + A(Y)A(Z)X - A(X)A(Z)Y \\ &\quad - F(Y, Z)\bar{X} + F(X, Z)\bar{Y} + A(X)g(Y, Z)t \\ &\quad - A(Y)g(X, Z)t \end{aligned}$$

$$(4.4b) \quad (C_1^1 \tilde{R})(Y, Z) = \text{Ric}(Y, Z) + (n-1)A(Y)A(Z)$$

$$(4.4c) \quad C_1^1 E = r + \text{rank}(f)$$

where  $r = C_1^1 R$  is scalar tensor and

$$(4.5) \quad \begin{aligned} \text{Ric}(Y, Z) &= g(R(Y), Z) \\ (C_1^1 \tilde{R})(Y, Z) &= g(E(Y), Z). \end{aligned}$$

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