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q–ANGELESCU POLYNOMIALS

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1. Introduction: The Angelescu polynomial is defined by

$$\overline{\wedge}_n(x) = e^x D^n[e^{-x} A_n(x)], \tag{1.1}$$

where $\frac{A_n(x)}{n!}$ is an Appell set of polynomials. In a recent commnication I have studied the polynomial

$$\overline{\wedge}_n^{(a)}(x) = x^{-a} e^x D^n[x^a e^{-x} A_n(x)], \tag{1.2}$$

where a is a constant. In the present paper we shall make a stady of a q-Angelescu polynomial which is an extension of the Angelescu polynomial defined by (1.1), q-Appell sets were first defined and studeied by Sharma and Chak [2] who called these sets as q-harmonic. Later on, these sets were studied by Al-Salam [1].

We define q-Angelescu polynomials as

$$\overline{\wedge}_{n,q}(x) = e_q(xq^n) D_q{}^n[E_q(-x) P_n(x)], \tag{1.3}$$

where $P_n(x)$ is a q-Appell set satisfying the property

$$D_q\{P_n(x)\} = [n] P_{n-1}(x)$$

and

$$D_q\{f(x)\} = \frac{f(xq) - f(x)}{x(q-1)}.$$

2. Preliminaries: Let α be real or complex and let

$$[\alpha] = (1 - q^\alpha).$$

We shall use the notation

$$(a;k)=(1-q^{a})\dots(1-q^{a+k-1}),\quad (a)_0=1,$$
$$\begin{bmatrix}\alpha\\ k\end{bmatrix}=(-1)^k q^{k(2\alpha-k+1)/2}\frac{(-\alpha;k)}{(1;k)}$$

and

$${}_1\emptyset_1\left(a;b;x;\frac{1}{2}\right)=\sum_{n=0}^{\infty}\frac{q^{n(n-1)/1}(a;n)x^n}{(1;n)(b;n)}.$$

Let us recall the well-known formula

$$[a+b]_n=\sum_{k-0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}a^{n-k}b^k. \tag{2.1}$$

There are two q-analogues of exponential function e^x in common use. They are

$$e_q(x)=\prod_{n=0}^{\infty}(1-q^n x)^{-1}=\sum_{k=0}^{\infty}\frac{x^k}{[k]!} \tag{2.2}$$

and

$$E_q(x)=\prod_{n=0}^{\infty}(1+q^n x)=\sum_{k=0}^{\infty}\frac{q^{k(k-1)/2}x^k}{[k]!} \tag{2.3}$$

where $[k]!=[1]\dots[k]$.

3. A Generating function for $\overline{\wedge}_{n,q}(x)$.

By definition

$$\begin{aligned}\overline{\wedge}_{n,q}(x)&=e_q(xq^n){D_q}^n[E_q(-x)P_n(x)]\\ &=e_q(xq^n)\sum_{r=0}^{n}\begin{bmatrix}n\\ r\end{bmatrix}{D_q}^r\{E_q(-x)\}\Big|_{x=xq^{r-n}}D_q^{n-r}\{P_n(x)\}\\ &=e_q(xq^n)\sum_{r=0}^{n}\begin{bmatrix}n\\ r\end{bmatrix}(-1)^r q^{r(r-1)/2}E_q(-xq^n)\times\\ &\qquad\times P_r(x)[n][n-1]\dots[r+1].\end{aligned}$$

Hence, after suitable adjustement, we have the generating relation

$$\sum_{n=0}^{\infty}\frac{\overline{\wedge}_{n,q}(x)t^n}{(1;n)}=\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)/2}t^n P_n(x)}{(1;n)[1-t]_{n+1}}, \tag{3.1}$$

where ${}_1\emptyset_0(n+1;-;t)$ is written as $\{[1-t]_{n+1}\}^{-1}$.

It does not seem possible to express the right hand expression in simpler terms as in the case of Angelescy polynomials $\overline{\wedge}_n(x)$.

4. A Characterisation for $\overline{\wedge}_{n,q}(x)$.

We now proceed to give a characterization for the q-Appell polynomial $\overline{\wedge}_{n,q}(x)$.

Sharma and Chak [2] gave the following characterization for the q-Appell polynomials: –

A polynomial set $\{P_n(x)\}$ is q-Appell, if and only if, there is a set of constants $\{a_k\}$ such that

$$P_n(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} a_{n-k} x^k, \qquad a_0 \neq 0. \tag{4.1}$$

Therefore, from (1.3) and (4.1), we have

$$\begin{aligned} \overline{\wedge}_{n,q}(x) &= e_q(xq^n) D_q{}^n \left[E_q(-x) \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} a_{n-k} x^k \right] \\ &= e_q(xq^n) \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} a_{n-k} D_q{}^n [E_q(-x) x^k] \\ &= e_q(xq^n)(-1)^n q^{n(n-1)/2} \sum_{k=0}^{n} ({}^{-}n;k) \begin{bmatrix} n \\ k \end{bmatrix} q^k \times \\ &\times a_{n-k} {}_1\emptyset_1 \left(n+1; n+1-k; -xq^{n-k}; \frac{1}{2} \right). \end{aligned} \tag{4.2}$$

Using a transformation due to Slater [4], namely,

$${}_1\emptyset_1(a;c;x) = \prod_{n=0}^{\infty} \{1/(1-xq^n)\} {}_1\emptyset_1 \left(c-a; c; -xq^a; \frac{1}{2} \right) \tag{4.3}$$

we have, from (4.2) that

$$\overline{\wedge}_{n,q}(x) = (-1)^n q^{n(n-1)/2} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-n;k) a_{n-k} q^k \times {}_1\emptyset_1(-k; n+1-k; xq^n), \tag{4.4}$$

where ${}_1\emptyset_1(-k; n+1-k; xq^n)$ is a q-Laguerre polynomial.

Thus, we arrive at the characterization that a polynomial $\overline{\wedge}_{n,q}(x)$ is q-Angelescu, if and only if, there is a set of constants $\{ak\}$ such that

$$\overline{\wedge}_{n,q}(x) = (-1)^n q^{n(n-1)/2} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-n;k) a_{n-k} q^k {}_1\emptyset_1(-k; n+1-k; xq^n)$$

with $a_0 \neq 0$.

The form (4.4) will be some times denoted as $\overline{\wedge}_{n,q}(x;a_n)$.

Particular Cases

(i) Taking $a_k = 1$ in (4.4), we get that

$$\overline{\wedge}_{n,q}(x;1) = (-1)^n q^{n(n-1)/2} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-n;k) q^k {}_1\emptyset_1(-k;n+1-k;xq^n) \tag{4.5}$$

is an q-Angelescu polynomial.

(ii) Next, if we take

$$a_k = (-1)^k q^{k\left(k+\frac{1}{2}\right)} \text{ in (4.4)}$$

we get the q-Angelescu polynomial

$$\overline{\wedge}_{n,q}\left(x;(-1)^k q^{k\left(k+\frac{1}{2}\right)}\right) = (-1)^n q^{3n^2/2} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-n;k) \times \\ \times q^{k\left(k+\frac{1}{2}\right)-2nk} {}_1\emptyset_1(-k;n-k+1;xq^n).$$

This can be alternatively written as

$$\overline{\wedge}_{n,q}\left(x;(-1)^k q^{k\left(k+\frac{1}{2}\right)}\right) = e_q(xq^n) q^{3n^2/2} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-n;k) \times \\ \times q^{k\left(k+\frac{1}{2}\right)-2nk} {}_1\emptyset_1\left(n+1;n+1-k;-xq^{n-k};\frac{1}{2}\right).$$

5. Another characterization of q-Angelescy polynomial

We shall now give a characterization of $\overline{\wedge}_{n,q}(x)$ in terms of moment constants.

Al-Salam [1] has proved the following characterization of q-Appell polynomial:

"A polynomial set $\{P_n(x)\}$ is a q-Appell set, if and only if, there is a function $\beta(x;q) \equiv \beta(x)$ of bounded variation on $(0,\infty)$ so that

$$b_n = \int_0^\infty x^n d\beta(x), \text{ exists for all } n = 0,1,2,\dots, b_0 \neq 0, \tag{5.1}$$

$$P_n(x) = \int_0^\infty [x+t]_n d\beta(t). \tag{5.2}$$

Using (5.2), we have

$$\begin{aligned}\overline{\wedge}_{n,q}(x) &= \exists_q(xq^n){D_q}^n\left[E_q(-x)\int_0^\infty [x+t]_n d\beta(t)\right]\\ &= e_q(xq^n)\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\int_0^\infty t^k d\beta(t){D_q}^n[E_q(-x)x^{n-k}]\\ &= e_q(xq^n)\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix} b_k {D_q}^n[E_q(-x)x^{n-k}]\\ &= (1;n)e_q(xq^n)\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}\frac{b_k q^{k(k-1)/2}}{(1;k)}\times\\ &\quad\times {}_1\emptyset_1\left(n+1;k+1;-xq^k;\frac{1}{2}\right).\end{aligned}$$

The above expression for $\overline{\wedge}_{n,q}(x)$ can also be written in the alternative form [using 4.3]

$$\overline{\wedge}_{n,q}(x) = (1;n)\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}\frac{b_k q^{k(k-1)/2}}{(1;k)}{}_1\emptyset_1(k-n;k+1;xq^n).$$

Thus, we have the characterization that a polynomial set $\overline{\wedge}_{n,q}(x)$ is a q-Angelescu set, if and only if, there exist constants b_k such that there is a function $\beta(x)$ of bounded variation on $(0,\infty)$ so that

$$b_n = \int_0^\infty x^n d\beta(x), \qquad n \geq 0, \quad b_0 \neq 0,$$

with

$$\overline{\wedge}_{n,q}(x) = (1;n)\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}\frac{b_k q^{k(k-1)/2}}{(1;k)}{}_1\emptyset_1(k-n;k+1;xq^n).$$

6. Transformation relations of certain particalar q-Angelesca polynomial

Consider the polynomials (4.4)

$$\overline{\wedge}_{n,q}(x;1) = (-1)^n q^{n(n-1)/2}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(-n;k){}_1\emptyset_1(k-n;n+1-k;xq^n) \qquad (6.1)$$

Using the following known transformation due to Slater [4], namely

$${}_1\emptyset_1(a;b+1;x) = \frac{(1-q^b)}{x}\{{}_1\emptyset_1(a;b;x) - {}_1\emptyset_1(a-1;b;xq^{1-a})\}$$

on the right hand side of (6.1), we get

$$\overline{\wedge}_{n,q}(x;1) = \frac{(1-q^n)(q^n-1)}{qx}\overline{\wedge}_{n-1,q}(xq;a'_{n-1}) - \frac{(-1)^n q^{n(n-1)/2}}{x}\times$$
$$\times \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-n;k)(1-q^{n-k})q^k {}_1\emptyset_1(-k-1;n-k;xq^{n+k+1}),$$

where $a'_{n-1} = \frac{q^n}{(1-q^n)}$.

Similarly from (6.1) and using Slater's [4; 2.1, 2.2, 2.6] transformations we get

$$\overline{\wedge}_{n,q}(x;1) = \frac{(-1)^n q^{n(n+1)/2}}{(q-q^{n+1})} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-n;k)q^k(1-q^{1-k})\times$$
$$\times {}_1\emptyset_1(2-k;n-k+1;xq^k)+$$
$$+\frac{(-1)^n q^{n(n+1)/2}(x-1)}{(q^n-1)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-n;k) {}_1\emptyset_1(1-k;n-k+1;xq^n)+$$
$$+\frac{(q^n+q^{n-1}-1)(-1)^n q^{n(n-1)/2}}{(q^n-1)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-n;k)q^k {}_1\emptyset_1(1-k;n-k+1;xq^n),$$

$$\overline{\wedge}_{n,q}(x;1) = (-1)^n q^{n(n-1)/2}(q^{n+3}-q) \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-n;k)\frac{q^k}{(q^{n+1-k}-1)}\times$$
$$\times {}_1\emptyset_1(-k-2;n-k+1;xq^n)+$$
$$+(-1)^n q^{(n-1)/2} q^{n-1}(q^2-x) \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-n;k)}{(q^{n+1-k}-1)} {}_1\emptyset_1(-k-1;n-k+1;xq^n)+$$
$$+(-1)^n q^{n(n-1)/2}(q-q^{n+2}-q^{n+3}) \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-n;k)q^k}{(q^{n+1-k}-1)} {}_1\emptyset_1(-k-1;$$
$$n-k+1;xq^n)$$

and

$$\overline{\wedge}_{n,q}(x;1) = \frac{(-1)^n q^{n(n-1)/2}}{(q^n-1)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-n;k)(q^k-1)(q^k-q^n)\times$$
$$\times {}_1\emptyset_1(-k+1;n-k;xq^n)+$$
$$+\frac{(-1)^n q^{n(n-1)/2}}{x(q^n-1)} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-n;k)[(q^k-q^n)+(x-1)(1-q^{n-k})]q^k\times$$
$$\times {}_1\emptyset_1(-k;n-k;xq^n)$$

respectively.

7. An integral involving $\overline{\wedge}_{n,q}(x)$.

We now proceed to derive the value of a general integral involving $\overline{\wedge}_{n,q}(x)$. In particular, depending on the orthogonality of the Appell polynomial, assume that q-Appell polynomial $P_n(x)$ is orthogonal in the interval (α,β) with respect to a normalized weight function $w(x)$. Consider then the integral

$$\int_\alpha^\beta w(x)\overline{\wedge}_{n,q}(x)\overline{\wedge}_{m,q}(x)dx = \int_\alpha^\beta w(x)\sum_{r=0}^{r}\sum_{s=0}^{m}\begin{bmatrix}n\\r\end{bmatrix}\begin{bmatrix}m\\s\end{bmatrix}(-1)^{r+s}q^{r(r-1)/2}\times \times q^{s(s-1)/2}P_r(x)P_s(x)(r+1;n)(s+1;m)dx \tag{7.1}$$

Using the assumed orthogonality property for $P_n(x)$, we get

$$\int_\alpha^\beta w(x)\overline{\wedge}_{n,q}(x)\overline{\wedge}_{m,q}(x)dx = \sum_{r=s=0}^{\min(m,n)}\begin{bmatrix}n\\r\end{bmatrix}\begin{bmatrix}m\\s\end{bmatrix}(-1)^{r+s}\times$$
$$\times q^{r(r-1)/2+s(s-1)/2}(r+1;n)(s+1;m)$$
$$=0,\ \text{otherwise.}$$

As for example, consider the q-Appell polynomial $H_n(x)$, defined as

$$H_n(x)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}x^k.$$

Since the set $H_n\left(-xq^{-\frac{1}{2}}\right)$ is orthogonal over the unit circle with respect to the weight function

$$f(\emptyset)=\sum_{n=-\infty}^{\infty}q^{n^2/2}e^{in\emptyset},\quad |q|<1,$$

we have from (7.1), using orthogonality condition for $H_n\left(-zq^{-\frac{1}{2}}\right)$

$$\int_{|x|=1} f(x)\overline{\wedge}_{n,q}\left(-xq^{-\frac{1}{2}};1\right)\overline{\wedge}_{m,q}\left(-xq^{-\frac{1}{2}};1\right)dx$$
$$=K\sum_{r=s=0}^{\min(n,m)}\begin{bmatrix}n\\r\end{bmatrix}\begin{bmatrix}m\\s\end{bmatrix}(-1)^{r+s}q^{r(r-1)/2+s(s-1)/2}(r+1;n)(s+1;m)$$

$=0$, otherwise.

where $K=1$ if $f(x)$ is a normalized weight function, otherwise K is a suitable constant.

8. Still another integral involving an orthogonal q-Appell polynomial and $\overline{\wedge}_{n,q}(x)$.

Again considering a set of q-Appell polynomials $P_n(x)$ orthogonal with respect to a normalized weight function $w(x)$ ower the interval (α, β) we have

$$\int_\alpha^\beta w(x)P_m(x)\overline{\wedge}_{n,q}(x)dx$$

$$= \int_\alpha^\beta w(x)P_m(x)\sum_{r=0}^{n}\begin{bmatrix} n \\ r \end{bmatrix}(-1)^r q^{r(r-1)/2}(r+1;n)P_r(x)dx.$$

Again using the orthogonality property for $P_n(x)$, we have

$$\begin{aligned}\int_\alpha^\beta w(x)P_m(x)\overline{\wedge}_{n,q}(x)dx &= 0; \text{ if } m > n \\ &= (-1)^n q^{n(n-1)/2}(n+1;n);\ m = n \\ &= (-1)^m q^{m(m-1)/2}(m+1;n)\ m < n.\end{aligned}$$

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