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ON A CONJECTURE OF A. IVIĆ AND W. SCHWARZ

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Introduction. In their paper A. Ivić and W. Schwarz [**1**] investigated the following system of arithmetical functional equations:

$$f^k = I * (f \circ q_r) \tag{1}$$

$$f \circ q_r = \mu^2 * f \tag{2}$$

where $k, r \geq 2$ are integers and $f(1) = 1$. Here $*$ denotes the Dirichlet convolution of arithmetical functions, defined by

$$(f * g)(n) = \sum_{d|n} f(d) g\Big(\frac{n}{d}\Big),$$

and $\circ$ denotes the ordinary composition of functions. I is the constant function with value 1, q_r is the r-th power function and μ^2 is the characteristic function of square-free numbers. An artihmetical function f is multiplicative if $f(mn) = f(m)f(n)$ when m and n are coprime, and f, is prime-independent if $f(p^m)$ depends only on m bat not on p, in other words there is a function t such that $f(p^m) = t(m)$ for all primes p and integers m. Multiplicative functions form a commutative group under the convolution and multiplicative prime-independent functions form a subgroup of this group. It is easy to check that

$$\tau^2 = I * (\tau \circ q_2), \qquad \tau \circ q_2 = \mu^2 * \tau$$

where τ denotes the divisor function, that is $\tau = I * I$.

This is a special case of the system (1–2) namely $k = r = 2$. A. Ivić and W. Schwarz [**1**] conjectured that this is the only nonnegative solution of the system (1–2) for $k, r \geq 2$. They could reach the following remarkable partial results:

THEOREM A: *If f is a multiplicative prime-independent solution of the system (1–2) with $k = r = 2$, then $f \equiv 0$ or $f = \tau$ or $f(p^m) = \chi(m+1)$ where $\chi = 1, -1,\ 0$ if $m \equiv 1, -1,\ 0$ mod 3 respectively.*

Theorem B*: If* $k \geq 3$, $r \geq 2$ *but* $r \neq 2^{k-1}$ *then there are no nonnegative solutions* f *of the system* (1–2) *with* $f(1) = 1$.

The aim of this paper is to prove the conjecture in the remaining cases. Our results runs as follows:

Theorem: *Let* $k, r \geq 2$ *are integers. The only nonnegative solution of the system* (1–2) *with* $f(1) = 1$ *is* $f = \tau$, $k = r = 2$.

We will use the idea of A. Ivić and W. Schwarz. Let $\omega(n)$ be the number of distinct prime factors of n. It is well-known that $I * \mu^2 = 2^\omega$, thus if f satisfies the system (1–2) then

$$f^k = 2^\omega * f, \tag{3}$$

and if f satisfies (3) and either (1) or (2) then f also satisfies the other one. A. Ivić and W. Schwarz investigated the equation (3) separately and got the following result:

Theorem C: *If* $k \geq 2$ *then there is exactly one nonnegative solution* f *of* (3) *with* $f(1) = 1$*, this is multiplicative and prime-independent.*

(Actually their result was somewhat stronger.) The proof of Theorem A, Theorem B and Theorem C is to be found in A. Ivić and W. Schwarz [**1**]. All the other statements mentioned in this section are well-known; see for example G. H. Hardy and E. M. Wright [**2**].

Proof. To prove the Theorem it suffices to investigate the system (2–3) which is equivalent to the system (1–2).

Theorem C says that for a given $k \geq 2$ we have a unique nonnegative function f satisfying (3) and $f(1) = 1$. In what follows f denotes this function, and the question is whether f satisfies (2) for a certain value of r. f is multiplicative and prime-independent thus we can abbreviate $f(p^m)$ by f_m and take $f_0 = f(1) = 1$. (2) means that $f_{mr} = f_{m-1} + f_m$ for $m \geq 1$, specially

$$f_r = 1 + f_1 \tag{4}$$

and (3) means that for $m \geq 1$

$$f_m^k = 2(f_0 + \cdots + f_{m+1}) + f_m. \tag{5}$$

Generally for $a > 0$ let x_a be the unique positive solution of the equation $x^k - x - a = 0$. Then x_a is monotonic in a, in other words $x^k - x - a \gtreqless 0$ if and only if $x \gtreqless x_a (x > 1)$. Take $a_m = 2(f_0 + \cdots + f_{m-1})$ for $m \geq 1$, then from (5) we have $f_m = x_{a_m}$; f_m is monotonic because a_m is trivially monotonic. Thus there is at most one value of r for which (4) is valid with a fixed k. This and Theorem A proves the Theorem in the case $k = 2$ and after Theorem B it remains to show that (4) is false with $k \geq 3$ and $r = 2^{k-1}$. The proof is based on giving a good lower bound for f_m.

From the monotonicity of x_a we get

(6) $$x_a > (a+1)^{1/k}.$$

The definition of a_m and the trivial bound $f_m \geq f_o = 1$ give us that $a_m \geq 2m$ and (6) leads to $f_m \geq (2m+1)^{1/k}$, which is also true for $m = 0$. Combining this whith the definition of a_m we get

$$a_m \geq 2 \sum_{i=0}^{m-1} (2i+1)^{1/k}.$$

Using the convexity of the function $x^{1/k}$ we get

$$(2i+1)^{1/k} > \frac{1}{2} \int_{2i}^{2i+2} x^{1/k}\, dx$$

which leads to

$$a_m \geq \int_0^{2m} x^{1/k}\, dx.$$

Finally (6) gives us that

(7) $$f_m > \Big(\frac{k}{k+1} (2m)^{\frac{k+1}{k}} \Big)^{1/k},$$

and with $r = 2^{k-1}$ we obtain

(8) $$f_r > 2 \Big(\frac{2k}{k+1} \Big)^{1/k}.$$

Numerical calculations show that for $3 \leq k \leq 9$

$$1 + f_1 = 1 + x_2 > 2 \Big(\frac{2k}{k+1} \Big)^{1/k}$$

so this approach is too rough to prove these cases. For $k \geq 10$ we get from the monotonicity of x_a that

$$x_2 < \Big(\frac{25}{8} \Big)^{1/k}$$

and trivially from (8)

$$f_r > 2 \Big(\frac{20}{11} \Big)^{1/k}.$$

It is easy to check that for $k \geq 10$

$$2 \Big(\frac{20}{11} \Big)^{1/k} > 1 + \Big(\frac{25}{8} \Big)^{1/k},$$

which means that f cant't satisfy equation (4) and therefore equation (2). This proves the Theorem if $k \geq 10$. For $3 \leq k \leq 9$ we have the numerical data:

k	$1+f_1$	f_r
3	$2.5213\ldots$	$2.5109\ldots$
4	$2.3532\ldots$	$2.3492\ldots$
5	$2.6771\ldots$	$2.2702\ldots$
6	$2.2148\ldots$	$2.2231\ldots$
7	$2.1796\ldots$	$2.1911\ldots$
8	$2.1544\ldots$	$2.1677\ldots$
9	$2.1353\ldots$	$2.1497\ldots$

This completes the proof of our theorem and gives the affirmative answer for the conjecture.

Remarks. From the monotonicity of x_a and f_m it is easy to prove that

$$f_m \leq (2m+1)^{\frac{1}{k-1}}$$

and an argument similar to the one above gives the upper bound

$$f_m \leq \left(\frac{k-1}{k}(2m)^{\frac{k}{k-1}} + \frac{k+1}{k}\right)^{\frac{1}{k-1}} \tag{9}$$

and an easy special case of this is

$$f_r < 2\left(\frac{2k+3/2}{k+1}\right)^{1/k}$$

with $r = 2^{k-1}$. Comparing this with (8) we get

$$f_r = 2 + \frac{\log 4}{k} + O(k^{-2})$$

but

$$1 + f_1 = 2 + \frac{\log 3}{k} + O(k^{-2}).$$

If m is close to $3^{1/2}2^{k-2}$ then (7) and (9) gives us that

$$f_m = 2 + \frac{\log 3}{k} + O(k^{-2}).$$

This shows that in our proof $3^{1/2}2^{k-2}$ is a more critical value than 2^{k-1} which is the critical value of the proof of A. Ivić and W. Schwarz.

Using (7) we can improve our lower bound for a_m and we can get

$$f_r > 2\left(\frac{2k+1/2}{k+1}\right)^{1/k}$$

where $r = 2^{k-1}$. This proves the Theorem for $k \geq 8$. Some other improvements are possible.

REFERENCES

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