PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 30 (44), 1981, pp. 59–64

AN UPPER ESTIMATION FOR THE EIGENFREQUENCES OF VIBRATING LIAPUNOFF BODIES (FIRST BOUNDARY VALUE PROBLEM)

I. Joó, L.L. Stachó

1. For any bounded (open) domain $\Omega \subset R^n$ and for $j = 1, 2, \cdots$ we define the j-th cigenfrequency $\lambda_j(\Omega)$ of the homogeneous Ω shaped and at its boundary $\partial\Omega$ fixed vibrating body by

$$(1) \qquad \Lambda_j(\Omega) = \inf_{L \in M_j} \sup_{f \in L} \left(\int_\Omega \left\| \operatorname{grad} f(x) \right\|^2 dx \Big/ \int_\Omega |f(x)|^2 \, dx \right)^{1/2}$$

where M_j denotes the collection of the j-dimensional subspaces of the Soboleff space $W_0^{1,2}$.*

As it is well-known (*cf.* [**1**]), if $\partial\Omega$ is an $(n-1)$-dimensional C^2-submanifold of R^n, then the eigenvalues of the boundary value problem

$$\Delta f + \Lambda^2 \cdot f = 0, \ \ f \in C_0^\infty(\overline{\Omega})$$

are given by (1). On the other hand, it is also shown (e.g. [**1, 3**]) that all the mappings $\Omega \mapsto \Lambda_j(\Omega)$ are continuous with respect to the topology on the set of the bounded R^n-domains defined by the usual Hausdorff distance.

While for all dimensions it is clairified (cf. [**2**]) that

$$\Lambda_j(\Omega) \geq \Lambda_j \left(\left\{ x \in R^n : \|x\| < \left(\frac{\mathrm{vol}_n \Omega}{\omega_n} \right)^{1/n} \right\} \right) \qquad (j = 1, 2, \dots)$$

where vol_n denotes the n-dimensional Hausdorff measure and $\omega_n \equiv \mathrm{vol}_n\{x \in R^n : \|x\| < 1\}$, it is not at all known over two dimensions what kind of effective upper estimates can be given for the value of $\Lambda_j(\Omega)$ depending on some geometric parameters of Ω. However, for convex Ω-s it was proved (cf. [**3**]) that the analogous

*i.e. $f \in W_0^{1,2}$ if grad f exists in the weak sense and belongs to $L^2(\Omega)$ and supp f is contained in some copmact set which does not meet $\partial\Omega$.

of the best known two dimensional estimates (see [**4**]) hold in general (and can not be improved). The purpose of this paper is to extend a theorem of G. PÓLYA [**5**] concerning convex Ω-s to a larger class of geometrical figures (for generalized Liapunoff bodies, defined in the next sections).

2. A bounded domain $\Omega(\subset R^n)$ whose boundary is an $(n-1)$-dimensional C^2-submanifold in R^n is called a *Liapunoff body* if the Minkowskian curvature of $\partial\Omega$ with respect to the outward from Ω oriented normal vectors is non-negative at any point of $\partial\Omega$. (Remark here that the convexity of Ω is equivalent to the non-negativeness of the main curvatures of $\partial\Omega$ separately).

According to some recent results in geometric measure theory, it is possible to give a generalization of the concept of Minkowskian curvature which applies to the boundary of any open subset $\Omega \subset R^n$. This can be carried out as follows:

It is shown in [**6**, Theorem 5] that by setting

(2)
$$K \equiv \{(x,k) : x \in \partial\Omega,\ \|k\| = 1,\ \exists \varrho > 0\ x + \varrho k \in \Omega \text{ and dist } (x+\varrho k, \partial\Omega) = \varrho\}$$

(3)
$$h(x,k) \equiv \sup\{\xi > 0 : \text{dist}(x+\varrho k,\ \partial\Omega) = \varrho,\ \forall \varrho \in [0,\xi]\} \quad \text{for } (x,k) \in K,$$

one always can find a σ-finite Borel measure μ on K and Borel measurable functions $a_j : K \to R\,(j = 0,\dots,n-1)$ such that for all $f \in L^1(\Omega)$ we have

$$\int\limits_\Omega f(y)\,dy = \int\limits_K \int\limits_0^{h(x,k)} f(x+\varrho k) \sum_{j=0}^{n-1} a_j(x,k)\varrho^j\,d\varrho\,d\mu(x,k). \tag{4}$$

Here $d\mu$ and $a_0,\dots,a_{n-1}$ are necessarily determined only up to the signed measures

$$d\alpha_j \equiv a_j\,d\mu \qquad (j = 0,\dots,n-1) \tag{5}$$

in the sense that if (4) is satisfied when $d\mu$ and $a_0,\dots a_{n-1}$ are replaced $dy\ d\tilde\mu$ and $\tilde a_0,\dots,\tilde a_{n-1}$, respectively, then we have

$$\int\limits_E a_j\,d\mu = \int\limits_E \tilde a_j\,d\tilde\mu \qquad (j = 0,\dots,n-1)$$

for all such $E \subset K$ that $\int_E a_j\,d\mu$ or $\int_E \tilde a_j\,d\tilde\mu$ makes sense. Thus, for $d\mu$ (and hence also $\delta\tilde\mu$)-almost every $(x,k) \in K$, the polynomials $\varrho \mapsto \sum_{j-0}^{n-1} a_j(x,k)\varrho^j$ and $\varrho \mapsto \sum_{j=0}^{n-1} \tilde a_j(x,k)\varrho^j$ differ only in a positive constant factor.

We shall call the measure $d\alpha_j$ defined by (5), which depends only on the geometric parametars of Ω, the *j-th curvature measure of the boundary of* Ω. This terminology is motivated by the relation (6) below. The formula (4) can be considered as a generalization of the main theorem in [**11**].

In the classical case, when $\partial\Omega$ is C^2-smooth, we have $(x,k)\in K$ if and only if k is the toward Ω oriented normal vector (of unit length) of the surface $\partial\Omega$ at the point $x(\in\partial\Omega)$. Now there is a natural choice for $d\mu$ and $a_0,\dots a_{n-1}$: We can define $d\mu$ by

$$\mu(E)=\mathrm{vol}_{n-1}\{x\in\partial\Omega:\exists k\ (x,k)\in E\}$$

(for the Borel measurable subsets E of K; vol_{n-1} denoting the $(n-1)$-dimensional Hausdorff measure). Then $a_0(x,k),\dots,a_{n-1}(x,k)$ are the coefficients of the polynomial

$$\varrho\mapsto\sum_{j=0}^{n-1}a_j(x,k)\varrho^j\equiv\prod_{i-1}^{n-1}(1-\varrho k_i(x)) \tag{6}$$

where $k_0(x),\dots,k_{n-1}(x)$ denote the main curvatures with respect to the outer normal of $\partial\Omega$ at the point x. Thus, in this special case, the curvature measures $d\alpha_j$ (defined by (5) and (6)) are all absolutely continuous with respect to $d\alpha_0$ and the Minkowskian curvature $k_1+\ldots+k_{n-1}$ of $\partial\Omega$ coincides with $-\frac{d\alpha_1}{d\alpha_0}$. Therefore, to save the most properties of the classiacal case, we define generalized Liapunoff bodies in the following way:

Definition. A bounded domain Ω in R^n is said to be a *generalized Liapunoff body* if all its cutvature measures $\alpha_j(j=0,\dots,n-1)$ introduced above are absolutely continuous with respect to α_0 and the function $-\frac{d\alpha_1}{d\alpha_0}$ (which we shall call now the Minkowskian curvature of $\partial\Omega$) is non-negative.

THEOREM 1. *If $\Omega\in R^n$ is a generalized Liapunoff body then the function $\varrho\mapsto\mathrm{vol}_{n-1}\partial(\Omega_{-\varrho})$ (where $\Omega_{-\varrho}$ denotes the inner parallel domain of radius $\varrho>0$ of Ω, i.e. $\Omega_{-\varrho}\equiv\{x\in\Omega:\mathrm{dist}(x,\partial\Omega)>\varrho\}$) is non-increasing for $0<\varrho<\infty$.*

Proof. Let Ω denote a generalized Liapunof body and define K and h as in (2) and (3). Choose $d\mu,\ a_0,\dots,a_{n-1}$ so that (4) be satisfied. It is proved in [**6** Theorem 5, Corollary] that here we necessarily have

$$\sum_{j=0}^{n-1}a_j(x,k)\varrho^j>0\ \text{ whenever }\ 0<\varrho<h(x,k)\qquad((x,k)\in K). \tag{7}$$

Remark that (7) is not a simple corollary of (4) and the positiveness of the operation $f\mapsto\int_\Omega f(y)\,dy$ because these facts ensure only $\sum_{j=0}^{n-1}a_j(x,k)\varrho^j\geq 0$ for $0<\varrho\leq h(x,k)$. It is easy to see from (2) and (3) that $\Omega_{-\varrho}=\{x+\xi k:(x,k)\in K$ and $\varrho<\xi\leq h(x,k)\}$ and hence

$$1_{\Omega-\varrho}(x+\xi k=1_{[\varrho,h(x,k)]}(\xi)\ \text{ for }\ (x,k)\in K\ \text{ and }\ \xi\in[0,h(x,k)]. \tag{8}$$

From (4) and (8) we obtain

$$\text{vol}_n \Omega_{-\varrho} \int\limits_K \int\limits_{\varrho}^{\infty} \varphi_{x,k}(\xi)\, d\xi\, d\mu(x,k) \tag{9}$$

where $\varphi_{x,k}(\xi) = \sum_{j=0}^{n-1} a_j(x,k)\xi^j \cdot 1_{[0,k(x,k)]}(\xi)$.

Recall that for μ-almost every $(x,k) \in K$, the polynomial $P_{x,k} : \xi \mapsto \sum_{j=0}^{n-1} a_j(x,k)\xi^j$ has only real roots (cf. [**6**, Theorem 5]) and that from the definition of Liapunoff bodies and (7) we have $P_{x,k}(0) > 0$ and $P'_{x,k}(0) = a_1(x,k) = \frac{d\alpha_1}{d\alpha_0}\Big|_{(x,k)} \le 0$ (for μ-almost every $(x,k) \in K$).

Since, in general, a polynomial $P : R \mapsto R$ having only real roots and such that $P(0) > 0$ and $P'(0) \le 0$ is constant or has positive root and P desreases on $[0, \min\{\xi > 0 : P(\xi) = 0\}]$ (cf. [10, Lemma]) it follows from (7) and the definiton of $P_{x,k}$ that the functions $\varphi_{x,k}$ are monotone decreasing on the whole $[0,\infty)$ for μ-almost all $(x,k) \in K$. Therefore, from (9) we deduce that the function

$$\varrho \mapsto -\frac{1}{2}\left(\frac{d^+}{d\xi}\Big|_{\varrho} + \frac{d^-}{d\xi}\Big|_{\varrho}\right)\text{vol}_n \Omega_{-\xi}$$

is well-defined for all $\varrho > 0$ and it is decreasing.

However, it is shown in [**7**] that the $(n-1)$-dimensional Minkowski content of $\partial(\Omega_{-\varrho})$ equals to $-\frac{1}{2}\left(\frac{d^+}{d\xi}\Big|_{\varrho} + \frac{d^-}{d\xi}\Big|_{\varrho}\right)\text{vol}_n \Omega_{-\xi}$. Since the boundary of any bounded parallel set is easily an $(n-1)$-rectifiable subset of R^n (for definitions see [**8**]), a well-known theorem of M. Kneser (cf. [**8**]) implies that $\text{vol}_{n-1}\partial\Omega_{-\varrho} = (n-1)$-Minkovski content $(\partial(\Omega_{-\varrho})) = \frac{1}{2}\left(\frac{d^+}{d\xi}\Big|_{\varrho} + \frac{d^-}{d\xi}\Big|_{\varrho}\right)\text{vol}_n \Omega_{-\xi}$. This completes the proof.

3. The following geometric estimation is given in [**10**] for the eigenfrequences $\Lambda_j(\Omega)$:

Theorem 2. *Let Ω be such a bounded in R^n that $\sup_{\varrho>0} \text{vol}_{n-1}\partial(\Omega_{-\varrho}) < \infty$. Then, by setting $l(\Omega) \equiv \text{vol}_n\Omega / \sup_{\varrho>0} \text{vol}_{n-1}\partial(\Omega_{-\varrho})$, we have*

$$\Lambda_1(\Omega)^2 \le \frac{\pi}{2} \cdot l(\Omega)^{-1}$$

The ideas of the proof of Theorem 2 are essentially based upon those of the article [**5**].

Thus Theorem 1 directly yields our chief observation

THEOREM 3. *If Ω is a generalized Liapunoff body in R^n then*

$$\Lambda_1(\Omega)^2 \le \frac{\pi}{2} \cdot \frac{\lim_{\varrho \downarrow 0} \mathrm{vol}_{n-1} \partial(\Omega_{-\varrho})}{\mathrm{vol}_n \Omega} \tag{10}$$

In particular, if $\partial\Omega$ is a C^2-smooth hypersurface, then

$$\Lambda_1(\Omega)^2 \le \frac{\pi}{2} \cdot \frac{\mathrm{vol}_{n-1} \partial\Omega}{\mathrm{vol}_n \Omega}$$

Remark. One can prove that for any generalized Liapunoff body $\Omega \subset R^n$ we have on the right hand side of (10)

$$\lim_{\varrho \downarrow 0} \mathrm{vol}_{n-1} \partial(\Omega_{-\varrho}) = \int_{\partial\Omega} \text{cardinality } \{k : (x,k) \in K\} d\,\mathrm{vol}_{n-1}(x).$$

Proof. By [**6**, Lemma 9] we can fix disjoint Borel subsets $B_1, B_2, \dots$ of K and open sets $\Omega^{(1)}, \Omega^{(2)}, \dots \subset R^n$ with positive reach (for def. see [**6**] or [**11**]) such that by setting $\varrho_m \overset{\mathrm{def}}{=} \inf\left\{ \frac{1}{2}\mathrm{reach}\ \Omega^{(m)},\ h(x,k) : (x,k) \in B_m \right\}$ and $K_m \overset{\mathrm{def}}{=} \{(x,k) : x \in \partial\Omega^{(m)},\ \|k\| = 1,\ \exists \varrho > 0\ x + \varrho k \in \Omega^{(m)},\ \mathrm{dist}(x, \partial\Omega^{(m)}) = \varrho\}$ we have

$$K = \bigcup_{m=1}^{\infty} B_m, \ \ \varrho_m > 0 \quad \text{and} \quad B_m \subset K_m \quad (m = 1, 2, \dots).$$

Using [**6**, Theorem A, B] we can see that for each point $y \in \partial(\Omega^{(m)}_{\varrho_m})$ there exists a unique pair $(x_m(y),\ k_m(y))$ in K_m with the property $y = x_m(y) + \varrho_m k_m(y)$ and, by [**8**, 3.2.3], for any fixed $\xi \in R$, the mapping $T^m_\xi : y \mapsto y + \xi k(y)$ satisfies

$$\int_{T^m_\xi(S)} \mathrm{card}(T^m_\xi)^{-1}(z) \mathrm{vol}_{n-1} z = \int_S [1 + (\xi - \varrho_m) k^m_1(y)] \cdots [1 + (\xi - \varrho_m) k_{n-1}(y)] d\mathrm{vol}_{n-1}(y),$$

where $k^m_1, \dots, k^m_{n-1}$ are the main curvatuires of $\partial(\Omega^{(m)}_{-\varrho_m})$ (cf. [**6**, Theorem B]) defined vol_{n-1} almost everywhere on $\partial(\Omega^{(m)}_{-\varrho m})$. Hence in particular,

$$\int_{\{x : \exists k (x,k) \in B_m\}} \mathrm{card}\{k : (x,k) \in B_m\} d\mathrm{vol}_{k-1}(x) = \int_{\{x + \varrho_m k : (x,k) \in B_m\}} [1 - \varrho_m k^m_1(y)] \cdots [1 - \varrho_m k^m_{n-1}(y)] d\mathrm{vol}_{n-1}(y). \tag{11}$$

The proof of the main Theorem in [**6**] shows (cf. [**6**, (5'), (5")]) that the measures $a_j d\mu$ in formula (5) are given by

$$\sum_{j=0}^{n-1} \varrho^j \int_B a_j d\mu = \int_{\{x + \varrho_m k (x,k) \in B\}} [1 + (\varrho - \varrho_m) k^m_1(y)] \cdots [1 + (\varrho - \varrho_m) k^m_{n-1}(y)] d\mathrm{vol}_{n-1}(y)$$

for $B \subset B_m$ and $\varrho \in R(m = 1, 2, \dots)$. Thus (11) yields

$$\int_{\partial\Omega} \mathrm{card}\{k : (x,k) \in K\} d\mathrm{vol}_{n-1}(x) = \int_K a_0(x,k) d\mu(x,k). \tag{12}$$

On the other hand, applying the functions $\varphi_{x,k}((x,k) \in K)$ introduced in formula (9), we see

$$\int_K a_0 d\mu = \int_K \lim_{\varrho \downarrow 0} \varphi_{x,k}(\varrho) d\mu(x,k) = \lim_{\varrho \downarrow 0} \int_K \varphi_{x,k}(\varrho) d\mu(x,k)$$

since the functions $\varphi_{x,k}$ are monotone decreasing for all fixed $(x,k) \in K$. Now, to complete the proof, we need only to remark that, by (9) and by [**8**, 3.2.34], we have

$$\mathrm{vol}_{n-1} \partial\Omega_{-\varrho} = -\frac{d}{d\varrho} \mathrm{vol}_n \Omega_{-\varrho} = -\frac{d}{d\varrho} \int_k \int_\varrho^\infty \varphi_{x,k}(\xi) d\xi d\mu(x,k) = \int_K \varphi_{x,k}(\varrho) d\mu(x,k)$$

for almost every $\varrho \in (0, \infty)$.

REFERENCES

[1] R. Courant, D. Hilbert, *Methods of Mathematical Physis*, Interscience, New York, 1953.

[2] R. Courant, *Beweis des Satzes dass von allen homogenen Membranen gegebenes Umfanges und gegebener Spannung die kreisförmige den tiefsten Grundton gibt*, Math. Zeitschr. **1**, 1918, 321–328.

[3] E. Makay, *The fundamental frequencies of two and three dimensional membranes*, MTA Mat. Kut. Int. Közl. **8**, 1963, A, 109–125.

[4] L. E. Payne, M. F. Weinberger, *Some Isometric Inequalities for Membrane Frequences and Torsional Rigidity*, J. Math. Analysis and Appl, **20**, 1961, 210–216.

[5] G. Pòlya, *Two more inequalities between physical and geometrical quantities*, J. Indian Math. Soc. **24**, 1960, 413–419.

[6] L. L. Stachó, *On curvature measures*, Acta Sci. Math. **41**, 1979, 191–207.

[7] L. L. Stachó, *On the volume function of parallel sets*, Acta Sci. Math. **38**, 1976, 365–374

[8] H. Federer, *Geometric Measure Theory*, Springer, New York, 1975.

[9] S. H. Gould, *Variational Inequalities for Eigenvalue Problems*, University of Toronto Press, Toronto, 1957.

[10] I. Joó, L. L. Stachó, *Generalization of an inequality of G. Pólya concerning the eigenfrequences of vibrating bodies*, to appear

[11] H. Federer, *Cyrvature measures*, Trans. Amer. Math. Soc. **93**, 1959, 418–491.

Bolyai Institute
University Szeged
Aradi Vertanuk Tere 1
6720 Szeged, Hungary