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DETERMINATION OF THE INVARIANT INTERVAL FOR POSITIVE LINEAR OPERATORS BY A NONSTATIONARY ITERATIVE PROCEDURE

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One of the essential conditions contained in the known fixed-point theorems [**2**] is the existence of a set which can be mapped into itself by means of the operators under consideration. Some possibilities for constructing such a set in R^n have been given [**4**] while in [**5**] the results were generalized for linear partially ordered spaces. A continuation of these investigations is work [**1**] in which an invariant interval for T'.

(1) $$T'x = Tx + f \quad (T \text{ —matrix with non-negative elements})$$

is obtained by the sequence

(2) $$x_n = Tx_{n-1} + f, \quad x_0 \in R^n \qquad (n = 1, 2, 3, \dots)$$

This work is concerned with the similar problems and is related to the nonstationary iterative procedures of the form:

(3) $$\begin{cases} z_n = T_n z_{n-1} + f, & z_0 \in B \quad (n = 1, 2, 3, \dots) \\ T_n x = Tx + \varrho_n, & \varrho \in B \end{cases}$$

in a partially ordered Banach space, B. However, for a wider application of the results [**1**] it is necessary to use a computer by which the sequence (2) is transformed into the sequence (3). When solving numerically the equations

(4) $$x = Tx + f$$

we often have to replace operator T by its approximative value (approximations, interpolations, quadrature formula) which in the course of forming the iterative procedure may vary from step to step

In determining the degree of technical tolerances and in mathematical formulations of problems in physics, chemistry and technology, has also to be replaced by an aproximative one, i.e. an iterative procedure of the form (3) is to be applied. It is natural to assume that such a sequence reflects some properties of the starting operator and, on the basis of the series it is possible to determine an invariant interval for the operator.

This work deals with the iterative procedures (3) with a positive linear operator T. It is supposed that the ϱ_n values can be majorized (minorized) with the elements generated by action of a positive linear operator on the particular element of the space B which is expressed through the sequence (3).

For the sake of simplicity we shall introduce the notations for functions which appear in the following theorems and which depend on the positive linear operators A_k and B_k, and which are, on the other hand, determined in each theorem separately.

$$Gz_k(s) = z_k + s\delta z_k \tag{5}$$

$$Iz_k(s, t, p, q) = [Gz_k(s) + t_k,\ Gz_k(s) + t_k(p) + q_k] \tag{6}$$

$$\begin{cases} R_k(A, B) = T_{k-1}B_k z_{k-2} + B_k z_{k-1} + (B_k A_k + B_k^2 + A_k)z_{k-2} \\ R'_k(A) = T_{k-1}A_k z_{k-2} + A_k(z_{k-1} + z_{k-2}) \end{cases} \tag{7}$$

$$\begin{cases} r_k(A, B) = 2B_k z_{k-1} + 2B_k A_k z_{k-2} B_k^2 z_{k-2} B_k f \\ r'_k(A) = 2A_k z_{k-1} + A_k^2 z_{k-2} - A_k f \end{cases} \tag{8}$$

$$\begin{cases} F_k(A, B) = T_{k-1}B_k z_{k-2} + (A_k + B_k + B_k A_k)z_{k-1} - \\ \qquad - (A_k + B_k A_k)(A_k + B_k)z_{k-2} + B_k^2 z_{k-2} \\ F'_k(A, B) = T_{k-1}A_k z_{k-2} + (2A_k + A_k^2)z_{k-1} - (A_k + A_k^2)\cdot \\ \qquad \cdot (A_k + B_k)z_{k-2} - A_k^2 z_{k-2} \end{cases} \tag{9}$$

$$\begin{cases} f_k(A, B) = 2B_k(E + A_k)z_{k-1} - 2B_k A_k(A_k + B_k)z_{k-2} + B_k^2 z_{k-2} - B_k f \\ f'_k(A, B) = 2A_k(E + A_k)z_{k-1} - 2A_k^2(A_k + B_k)z_{k-2} - A_k^2 z_{k-2} - A_k f, \end{cases} \tag{10}$$

where E denotes the identity operator.

LEMMA 1. *Let* (4) *be a given equation in* B *with a positive linear operator* T. *Let for* j *in the sequence* (2) *exist:*

$$p_i q_i \in B \quad (i = j-1, j) \quad \textit{such that:}$$

$$p_i \le x_i \le q_i \quad (i = j-1, j) \tag{11}$$

$$\begin{cases} \mu_j(q_{j-1} - p_j) \le p \\ \eta_j(p_{j-1} - q_j) \ge q_j, \quad \textit{where} \end{cases} \tag{12}$$

(13) $$\delta x_i = x_i - x_{i-1} \quad (i = 1, 2, \dots)$$

if the sequence of iterations is increasing, i.e.

(14) $$\delta x_i = x_{i-1} - x_i,$$

if the sequence of iterations is decreasing, and μ_j and η_j are real numbers for which holds:

(15) $$0 < \mu_j \leq \eta_j.$$

Then, for δx_i defined by (13) *the operator T' maps the interval $Ix_j(\mu, 0, \eta, 0)$ into itself, and for δx_i defined in* (14) *does the same for the corresponding interval*
$Ix_j(-\eta,\, 0,\, -\mu,\, 0)$.

Proof. a) Let $\delta x_i = x_i - x_{i-1}$.
According to (11) we have

$$\delta x_{j-1} - \delta x_i \leq q_{j-1} - p_j;$$

then, according to (12)

(16) $$\mu_j(\delta x_{j-1} - \delta x_j) \leq \delta x_j.$$

Since $T\delta x_{j-1} = \delta x_j$, by applying operator T to the right and left side of inequality (16) we get:

(17) $$T(x_j + \mu_j \delta x_j) + f \geq x_j + \mu_j \delta x_j.$$

In an anlogous way it can be shown that

(18) $$T(x_j + \eta_j \delta x_j) + f \leq x_j + \eta_j x_j,$$

On the basis of (17) and (18) we get the first part of the statement.

b) Let $\qquad \delta x_i = x_{i-1} - x_i.$

Here, similar to the previous case, one can obtain relation (16) and then after applying operator T get:

$$T(x_{j-1} - x_j) + \mu_j T\delta x_j \geq \mu_j \delta x_j$$
$$T(x_j - \mu_j \delta x_j) + f \leq x_j - \mu_j \delta x_j.$$

In a similar way one can get:

$$T(x_j - \eta_j \delta x_j) + f \geq x_j - \eta_j \delta x_j.$$

Note 1. In proving lemma 1. we used the proof of the statement 1.1 from [**1**].

COROLLARY 1. *The quantities p_i and q_i $(i = j-1,\, j)$ defined in lemma* 1. *are non-negative.*

THEOREM 1. *Let a linear positive operator T be defined in B and let for some $k \geq 2$ in the sequence* (3) *the following hold:*

1. *There exist linear positive operators A_k and B_k such that:*

$$(1.1)\qquad (T - A_k)z_{k-2} \leq T_n z_{k-2} \leq (T + B_k)z_{k-2} \qquad (n = k-1,\, k)$$

$$(1.2)\qquad \delta z_{k-1} \geq (A_k + B_k)z_{k-2},\ \delta z_k = z_k - z_{k-1} \qquad (k = 1,\, 2,\, \ldots)$$

(1.3) *There are real numbers s_k and S_k such that:*

a) $0 < s_k \leq S_k$

b) $\delta z_k - s_k(\delta z_{k-1} - \delta z_k) \geq (1 + 2s_k)A_k z_{k-2} + (1 + s_k)R_k(A,\, B)$

c) $S_k(\delta z_{k-1} - \delta z_k) - \delta z_k \geq (1 + 2S_k)B_k z_{k-2} + (1 + S_k)R'_k(A)$,

where R_k and R'_k are defined in (7).

Then the operator T' maps the interval $Ix_2(\mu,\, 0,\, \eta,\, 0)$ into itself. Here x_n is defined by (2) *for $x_0 = z_{k-2}$.*

Proof. We will show that the quantities

$$(19)\qquad \begin{cases} p_1 = \delta z_{k-1} - B_k z_{k-2} \\ q_1 = \delta z_{k-1} + A_k z_{k-2} \\ p_2 = \delta z_k - R_k(A,\, B) - A_k z_{k-2} \\ q_2 = \delta z_k + R'_k(A) + B_k z_{k-2} \end{cases}$$

satisfy inequality (11) for $j = 2$, whilst x_n is defined in (2) for $x_0 = z_{k-2}$ and δx_i in (14). Let us form the sequences:

$$(20)\qquad \begin{cases} y_n = (T + B_k)z_{n-1} + f, & y_0 = z_{k-2} \\ l_n = (T - A_k)l_{n-1} + f, & l_0 = z_{k-2} \\ \overline{z}_n = T_{k+n-2}\overline{z}_{n-1}, & \overline{z}_0 = z_{k-2} \\ x_n = Tx_{n-1} + f, & x_0 = z_{k-2} \end{cases}$$

and let us show that

$$(21)\qquad l_i \leq \overline{z_i} \leq y_i \quad (i = 0,\, 1,\, 2).$$

For $i = 0$ and $i = 1$ the statement is obtained from (20) and condition 1.1.

From 1.1. follows that $(A_k + B_k)z_{k-2} \geq 0$ and from 1.2. that

$$(22)\qquad \delta\overline{z}_1 \geq 0 \ \text{ and}$$

$$(23)\qquad (T - A_k)\overline{z}_0 + f + \varrho_{k-1} - B_k\overline{z}_0 \geq \overline{z}_0 \ \text{ respectively}$$

$$l_1 \geq \overline{z}_0$$

According to (20), (21) for $i = 1$ and (23) we have:

$$l_2 \leq T\overline{z}_1 - A_k\overline{z}_0 + f \leq \overline{z}_2 \leq T\overline{z}_1 + B_k\overline{z}_0 + f \leq (T + B_k)y_1 + f = y_2$$

Let us introduce the following notations:

$$(24)\qquad \begin{cases} a_i = y_i - \overline{z}_i & (i = 1,\, 2)\\ b_i = z_i - l_i & (i = 1,\, 2)\\ k_i = y_i - x_i & (i = 1,\, 2)\\ c_i = x_i - l_i & (i = 1,\, 2). \end{cases}$$

According to 1.1.

$$(25)\qquad \begin{cases} Tz_0 \le z_1 - f + A_k\overline{z}_0\\ THz_0 = T_{k-1}H\overline{z}_0 + \varrho_{k-1} \le T_{k-1}H\overline{z} + A_k\overline{z}_0 \end{cases}$$

By putting (25) into (24) we get:

$$(26)\qquad \begin{cases} k_2 \le R_k\ \ (A, B)\\ c_2 \le R'_k\ \ (A, R'_k(A)). \end{cases}$$

According to (21) we have $a_i \ge 0,\ \ b_i \ge 0\ \ (i = 1, 2)$, hence from (24) and (26) we get the relationship between x_i and $z_i\ \ (i = 1, 2)$

$$(27)\qquad \begin{cases} \overline{z}_1 - B_k\overline{z}_0 \le x_1 \le \overline{z}_1 + A_k\overline{z}_0\\ \overline{z}_2 - R_k(A,\, B) \le x_2 \le \overline{z}_2 + R_k(A). \end{cases}$$

On the basis of (27) we set the limitations for δx_i and we get that quantities (19) satisfy inequality (11). It comes out from 1.3. that (19) satisfies inequality (12) for $j = 2,\ \mu_2 = s_k,\ \eta_2 = S_k$; then a direct application of Lemma 1. leads to the statement of theorem.

Corollary 2. *Let conditions* 1.1. *be satisfied when* $A_k z_{k-2} \ge 0$ *and* $B_k z_k \ge 0$. *Then* $R'_k(A)$ *and* $R_k(A,\, B) \ge 0$.

When operators A_k and B_k are commutative with operator T it is possible to omit the explicit appearance of operator T_{k-1} in condition 1.3. of Theorem 1. Namely, operator T_{k-1} is defined through its action on element z_{k-2} and often we cannot determine $T_{k-1}x$ for $x \ne z_{k-2}$.

Theorem 2. *Let conditions* 1.1. *and* 1.2. *of theorem* 1. *be satisfied. Let operator* T *be commutative with operators* A_k *and* B_k *and condition* 3.1 *hold, where* r_k *and* r'_k *are substituted for* R_k *and* R'_k*, respectively* (r_k *and* r'_k *are defined in* (8)). *Then, the statement of theorem* 1. *holds.*

Proof. It can be carried out in a similar way to that of the previous theorem. We put $c_2 \le r'_k(A),\ k_2 \le r_k\ (A,\, B)$.

Condition 1.2. plays a significant role in forming the relation:

$$(28)\qquad l \le l_1 \le l_2,$$

since nothing can be stated on the monotinity of operator $T - A_k$. When a sequence of iterations is monotonously decreasing then the conditions for (28) are somewhat different.

THEOREM 3. *Let T be a linear positive operator defined in B and let for some $k \geq 2$ in the sequence* (3) *the following relations hold:*

There are linear positive operators A_k and B_k such that

3.1. $(T - A_k)x \leq T_n x \leq (T + B_k)x \quad (x = z_{k-2'}\ z_{k-1} - (A_k + B_k)z_{k-2})$
$(n = k - 1,\ k)$

3.2. *There are real numbers s_k and S_k such that:*

a) $0 < s_k \leq S_k$

b) $s_k(\delta z_{k-1} - \delta z_k) - \delta z_k \leq -(1 + 2s_k)B_k W_k - (1 + s_k)F'_k(A, B)$

c) $S_k(\delta z_{k-1} - \delta z_k) - \delta z_k \geq (1 + 2S_k)A_k W_k - (1 + S_k)F_k(A,\ B)$

where

$$\delta z_k = z_{k-1} - z_n \quad (k = 1,\ 2,\ \ldots)$$

$$W_k = z_{k-1} - (A_k + B_k)z_{k-2},\ \ F_k \text{ and } F'_k \text{ are defined by } (9)$$

Then, the interval $Ix_2(-\eta,\ 0,\ -\mu,\ 0)$, $\mu_2 = s_k$, $\eta_2 = S_k$ is mapped by T' into itself x_n is defined in (2) *for $x_0 = z_{k-2}$.*

Proof. We will consider the sequence (10) and show that (21) holds,

According to 3.1. $\overline{z}_1 \geq W_k$ and $l_1 \geq W_k$

$$l_2 \leq T\overline{z} - A_k W_k + f \leq T\overline{z}_1 + \varrho_k + f = \overline{z}_2 \leq T\overline{z}_1 + B_k W_k + f \leq (T + B_k)\overline{z}_1 + f \leq y_2.$$

From 3.1. we get

$$\begin{cases} T\overline{z}_0 \leq \overline{z}_1 - f + A_k W_k \\ TH\overline{z}_0 \leq T_{k-1}H\overline{z}_0 + A_k W_k \quad (H = A_k, B_k). \end{cases} \tag{29}$$

After substituting (29) into (24) we get:

$$\begin{cases} c_2 \leq F'_k(A,\ B) \\ k_2 \leq F_k(A,\ B). \end{cases} \tag{30}$$

Like in theorem 1, we get that the quantities

$$\begin{cases} p_1 = d\overline{z}_1 - A_k W_k \\ q_1 = \delta\overline{z}_1 + B_k W_k \\ p_2 = \delta\overline{z}_2 - F'_k(A,\ B) - B_k W_k \\ q_2 = \delta\overline{z}_2 + F(A,\ B) + A_k W_k \end{cases} \tag{31}$$

satisfy inequalities (11) and (12) when $j = 2$, $\mu_2 = s_k$, $\eta_2 = S_k$.

After applying lemma 1. the statement of the theorem comes out.

CORROLARY 3. *Let condition* 3.1 *be satisfied when $A_k W_k \geq 0$, $B_k W_k \geq 0$. Then, $F_k(A,\ B) \geq 0$ and $F'_k(A,\ B) \geq 0$.*

The further theorems we shall give without proof. The proofs are quite similar to the preceding ones and can be found elsewhere [**7**].

THEOREM 4. *Let us assume that the conditions of theorem* 3. *are satisfied when* F_k *and* F'_k *are replaced by* f_k *and* f'_k *respectively* (f_k *and* f'_k *are defined by* (10)) *and let operators* A_k *and* B_k *be commutative with operator* T. *Then the statement of the theorem is valid.*

THEOREM 5. *Let the linear operator* T *be defined in* B *and let for some* $k \geq 2$ *in the sequence* (3) *the following hold:*

There are linear positive operators A_k *and* B_k *such that:*

5.1. $(T + A_k)x \leq T_n x \leq (T - B_k)x \quad (n = k-1,\, k)$

$x = z_{k-2},\ W_k$

5.2. *There are real numbers* s_k *and* S_k *such that*

a) $0 < s_k \leq S_k$

b) $s_k(\delta_{k-1} - \delta z_k) - \delta z_k \leq (1 + s_k)F'_k(B, A) + (1 + 2s_k)A_k W_k$

c) $S_k(\delta z_{k-1} - \delta z_k) - \delta z_k \geq -(1 + S_k)F_k(B, A) - (1 + 2S_k)B_k W_k$, *where*

$$\delta z_k = z_k - z_{k-1} \quad (k = 1, 2, \ldots)$$

Then, the operator T' *maps the interval* $Ix_2(\mu, 0, \eta, 0)$, $\mu_1 = s_k$, $\eta_2 = S_k$ *into itself. Here,* x_n *is defined by* (2) *for* $x_0 = z_{k-2}$.

COROLLARY 4. *Let* 5.1. *be valid when* $A_k W_k \leq 0$ *and* $B_k W_k \leq 0$.
Then: $F_k(B, A) \leq 0, \quad F'_k(B, A) \leq 0$.

THEOREM 6. *Let us assume that the suppositions of theorem* 5. *are valid and let* F_k *and* F'_k *be replaced by* f_k *and* f'_k, *respectively. Let operators* A_k *nad* B_k *be commutative with operator* T. *Then the statement of theorem* 5. *is also valid.*

THEOREM 7. *Let the operator* T *be defined in* B *and for some* $k \geq 2$ *in the sequence* (3), *let us have:*

7.1. $(T + A_k)z_{k-2} \leq T_n z_{k-2} \leq (T - B_k)z_{k-2} \quad (n = k-1,\ k)$

7.2. $\delta z_{k-1} \geq -(A_k + B_k)z_{k-2}$

7.3. *There exist real numbers* s_k *and* S_k *such that:*

a) $0 < s_k \leq S_k$

b) $s_k(\delta z_{k-1} - \delta z_k) - \delta z_k \leq (1 + 2s_k)B_k z_{k-2} + (1 + s_k)R_k(B, A)$

c) $S_k(\delta z_{k-1} - \delta z_k) - \delta z_k \geq -(1 + 2S_k)A_k z_{k-2} - (1 + S_k)R'_k(B)$,

where

$$\delta z_k = z_{k-1} - z_k$$

Then, by means of the operator T' *the interval* $Ix_2(-\eta, 0, -\mu, 0)$, $\mu_2 = s_k\ \eta_2 = S_k$ *is mapped into itself. If this case* x_n *is defined by* (2) *for* $x_0 = z_{k-2}$.

Corollary 5. *Let* 7.1. *and* 7.2. *hold and let* $A_k\overline{z}_0 \le 0$ *and* $B_k\overline{z}_0 \le 0$. *Then,* $R'_k(B) \le 0$ *and* $R_k(B, A) \le 0$.

Theorem 8. *Let the suppositions in theorem* 7. *be satisfied and* R_k *and* R'_k *are replaced by* r_k *and* r'_k *respectively. Let operators* A_k *and* B_k *be commutative with operator* T. *Then the statement of theorem* 7. *is also valid.*

In the all above theorems the existence of the interval $Ix_2(u,\, 0,\, v,\, 0)$ has been established and it remains invariant under action of operator T' However, this interval cannot be determined since the sequence x_n is not known. Using the sequence z_k it is posible to determine the interval $Iz_k(U,\, m,\, V,\, n)$ so that:

$$\begin{cases} Ix_2(u,\, 0,\, v,\, 0) \le Iz_k(U,\, m,\, V,\, n) \\ U_k = r_2,\;\; V_k = v_2 \end{cases} \tag{29}$$

Let us introduce the notation

$$\begin{cases} p'_k = \delta z_{k-1} - p_1 \\ q'_k = q_1 - \delta z_{k-1}, \;\; \text{where } p_1, q_1 \text{ and } \delta z_{k-1} \text{ are defined in each the-} \end{cases} \tag{30}$$

orem separately,

$$\begin{cases} c_2 \le C_{\max} \\ k_2 \le K_{\max} \end{cases} \tag{31}$$

$C_{\max}$ and $K_{\max}$ are the majorizing functions for c_2 and k_2 and they have also been determined separately.

Now we will show that for the theorems concerned with the increasing sequence of iterations, m_k and n_k can be defined as:

$$\begin{cases} m_k = K_{\max}(1+U_k) - U_k q'_k \\ n_k = C_{\max}(1+V_k) + V_k p'_k. \end{cases} \tag{32}$$

For each of the described theorems it holds:

$$\begin{aligned} &\overline{z}_2 - K_{\max} \le x_2 \le \overline{z}_2 + C_{\max} \\ &\delta\overline{z}_2 - K_{\max} - q'_k \le \delta x_2 \le \delta\overline{z}_2 + C_{\max} + p'_k; \end{aligned}$$

hence

$$\begin{aligned} &x_2 + U_k\delta x_2 \ge \overline{z}_2 + U_k\delta\overline{z}_2 + m_k \\ &x_2 + V_k\delta x_2 \le \overline{z}_2 + V_k\delta\overline{z}_2 + n_k, \end{aligned}$$

which implies (29). In a similar way it is possible to get the theorems describing s monotonously decreasing sequence of iterations:

$$\begin{cases} n_k = C_{\max} - V_k(C_{\max} + q'_k) \\ m_k = -K_{\max} + U_k(K_{max} + p'_k). \end{cases} \tag{33}$$

THEOREM 9.1. *Let us suppose that by means of some of theorem* 1, 3, 5, *or* 7, *the intervals* Iz_k *and* Iz_{k+1} *have been obtained.*

9.2. Let $s_k \leq s_{k+1}, \quad S_k \geq S_{k+1}$

9.3. $p_i' \geq 0, \; q_i' \geq 0_{(i=1,k+1)} p_i'$ *and* q_i' *are defined by* (30).

Then, $Iz_{k+1} \subseteq Iz_k$.

Proof. We shall prove only a part of the statement concerning theorem 1. In an analogous way, it can be sown that it holds for the other theorems, too.

For theorem 1. (according to corollary 2) it holds:

$$\text{(34)} \qquad \begin{cases} R_k(A,\, B) = K_{\max} \geq 0 \\ R_k'(A) = C_{\max} \geq 0 \end{cases}$$

$$\text{(35)} \qquad \begin{cases} p_k' = B_k z_{k-2} \geq 0, \quad B_{k+1} z_{k-1} \geq 0 \\ q_k' = A_k z_{k-2} \geq 0, \quad A_{k+1} z_{k-1} \geq 0. \end{cases}$$

According to corollary 1, we have $p_2 \geq 0$ and $q_2 \geq 0$ (p_2 and q_2 are defined by (19)); hence, by substituting s_{k+1} and S_{k+1} for s_k and S_k, respectively, the interval Iz_{k+1} becomes expanded. Let consider the difference of the lower bounds of intervals Iz_{k+1} and Iz_k.

$$\begin{aligned} &Gz_{k+1}(s) + m_{k+1} - Gz_k(s) - m_k \geq \delta z_{k+1} + s_k(\delta z_{k+1} - \delta z_k) - \\ &- (1 + s_k) R_{k+1}(A,\, B) - s_k A_{k+1} z_{k-1} + (1 + s_k) R_k(A,\, B) + A_k z_{k-2} \geq 0, \end{aligned}$$

since by condition 1.3. (theorem 1.)

$$\delta z_{k+1} - s_k(\delta z_k - \delta z_{k+1}) - (1 + s_k) R_{k+1}(A,\, B) - (1 + 2 s_k) A_{k+1} z_{k+1} \geq 0,$$

then, by (35)

$$\delta z_{k+1} - s_k(\delta z_{k+1} - \delta z_k) - (1+s_k) R_{k+1}(A,\, B) - s_k A_{k+1} z_{k-1} \geq (1+s_k) A_{k+1} z_{k+1} \geq 0.$$

In a similar way it can be shown that

$$Gz_k(S) + n_k - Gz_{k+1}(S) - n_{k+1} \geq 0$$

from which follows the statement of the theorem.

In the case when the operators A_k and B_k are commutative with T it is posible by applying theorems 2., 4., 6. and 8. to get an invariant interval for T', but we cannot state that the majorizing functions for c_2 and k_2 are positive. If that could be stated on the basic of some other conditions (for theorem 2. if $f \geq$ the iteration sequence is also increasing) then theorem 9. could be applied.

Condition 9.2. results from the properties of the corresponding stationary iterative sequence. Namely, if

$$s_n(\delta x_{n-1} - \delta x_n) \leq \delta x_n \leq S_n(\delta x_{n-1} - \delta x_n)$$

then, because of the monotonity and linearity of the operator T, one gets:

$$s_n(\delta x_n - \delta x_{n+1}) \leq \delta x_{n+1} \leq S_n(\delta x_n - \delta x_{n+1}). \tag{36}$$

In our case the sequence x_n is formed in dependence of thesequ ence z_n, It is assumed that $x_0 = z_{k-2}$. When the indices are changed from k to $k+1$ a new sequence is formed which has as the starting element z_{k-1}, so that (36) does not hold ($T\delta x_{k-1} \neq \delta x_k$). Condition 9.2. requires that the lower bound for ϱ_n should be non-positive and the upper one non-negative, which is achieved by a suitable choice of operators A_k and B_k.

By applying the above results we get the following:

1. The construction of an invariant interval for operator T' allows one to apply the fixed point theorems and to get the solution of equation (1) without the explicit contracting requirement.

2. From each interval Iz_k it is possible to get an a posteriori error estimation. The error changes from step to step and its estimation is getting more and more accurate as long as we can state that $Iz_{k+1} \subseteq Iz_k$.

3. In the case of a monotonously increasing iteration sequence

$s_k p_2 \geq K_{maz}$ where p_2 and k_{max} are defined by the corresponding theorems, by taking the lower bound of the interval as the starting element of the next iterative step, an acceleration of the iterative procedure is achieved ($Gz_k(s) + m_k \geq z_k$).

For a monotonously decreasing sequence, the acceleration is accomplished by taking an upper interval limit as the starting element in the following iterative step while

$$s_k p_2 \geq C_{max}.$$

4. Each determination of a new interval is an improvement with respect to the former one if $Iz_{k+1} \subseteq Iz_k$.

Then

$$u_{0k} \leq u_{0,k+1} \leq \cdots \leq v_{0,k+1} \leq v_{0,k},\ Iz_k = [u_{0,k},\, v_{0,k}]$$

which enables forming a two-sided non-stationary iterative procedure for solving equation (1).

5. In the special case when $A_k \equiv B_k \equiv 0$ the nonstationary procedure becomes a stationary one, and the theorem statements of this kind lead to the statements dealt with in [**1**] and [**3**].

In our next article we shall present the application of these result to an approximative solution of the systems of linear integral equations.

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