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A NOTE ON GENERALIZED LINE GRAPHS

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In this paper we will find all graphs G such that G and its complement (denoted by $\overline{G}$) are generalized line graphs. We consider only finite undirected graphs without loops or multiple lines. The basic terminology follows [**1**].

The theorem we are going to prove as a generalization of a result of L. W. Beineke [**2**], who found all graphs G such that G and $\overline{G}$ are line graphs. In a series of papers F. Harary et al. (see, for example, [**3**]) considered problems about graphs and their complements sharing a given property; our problem fits in their investigation.

We now give the basic definitions.

Definition 1. The cocktail-party graph on $2n$ points denoted by $CP(n)$ is the regular graph on $2n$ points of degree $2n-2$.

Definition 2. A generalized line graph, denoted by $L(H; a_1, \dots, a_n)$, is constructed from a graph H with n points $v_1, \dots, v_n$ and nonnegative integers $a_1, \dots, a_n$ in the following way; it consists of disjoint copies of $L(H)$ and $CP(a_i)$, $i = 1, 2, \dots, n$ with additional lines joining a point in $L(H)$ with a point in $CP(a_i)$ if the point in $L(H)$ corresponds to a line in H that has v_i as an endpoint.

Definition 3. A generalized cocktail-party graph (GCP) is a graph obtained by the deletion of independent lines from the complete graph K_n. Any point of degree $n-1$ is said to be of l-type, while the others are said to be of a-type.

In this paper we will refer to the following theorems from [**4**] or [**5**].

THEOREM A. *A graph G is a generalized line graph if and only if its lines can be partitioned into GCPs such that each point is in at most two GCPs, two GCPs have at most one common point and if two GCPs have a common point, then it is of l-type in both of them.*

THEOREM B. *A graph G is a generalized line graph if and only if it does not contain any of the* 31 *graphs in Fig.*1 *as an induced subgraph.*

An immediate consequence of Theorem B is:

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THEOREM C. *A graph G is a generalized line graph if and only if its complement $\overline{G}$ does not contain complements of any of the* 31 *graphs in Fig.* 1 *as an induced subgraph.*

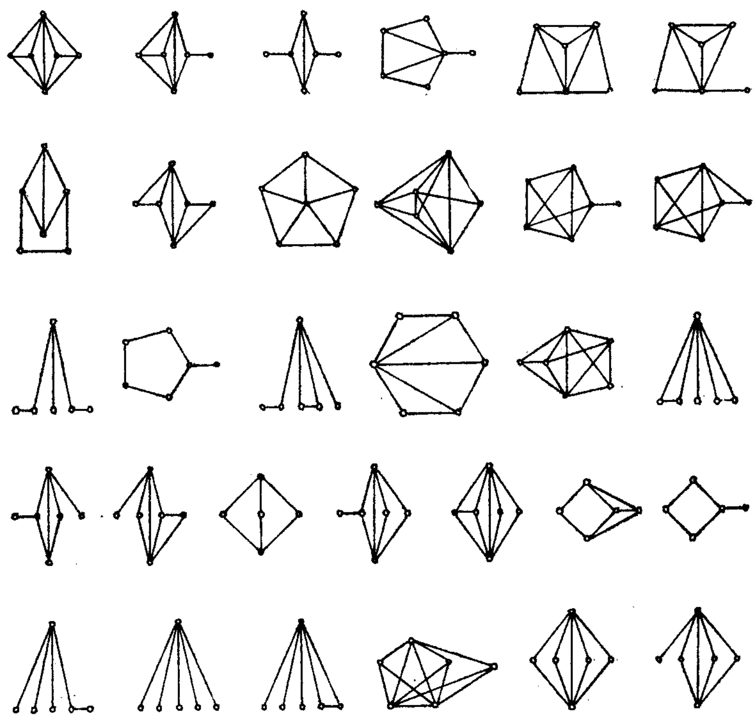

Fig. 1

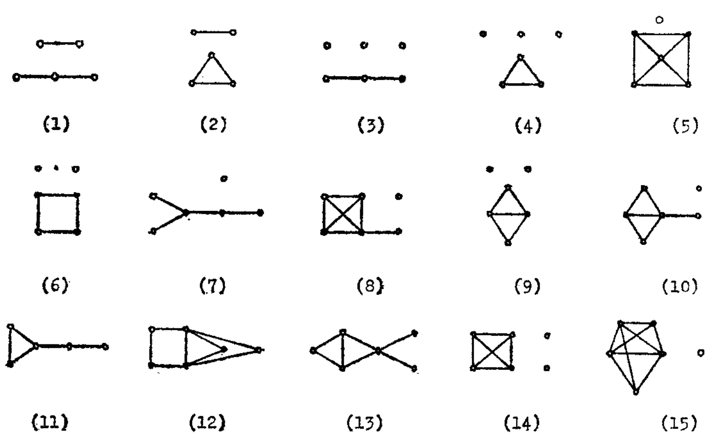

Fig. 2

In Fig. 2 we give only those of the complements of graphs in Fig. 1 which we will refer to in the further text.

The next theorem is a theorem of L. W. Beineke from [**2**]. We give it in a somewhat modified form.

THEOREM D. *Both G and $\overline{G}$ are line graphs if and only if either of them is equal to:* 1° *a clique or its complement;* 2° *an induced subragph of some of the graphs in Fig.* 3.

Now, let $\mathcal{S}$ denote the set of all graphs G such that G and $\overline{G}$ are generalized line graphs. Clearly, any GCP belongs to $\mathcal{S}$. The same applies to the complements of $GCPs$. Now we will show that all other members of $\mathcal{S}$, denote the corresponding set by $\mathcal{S}_0$ are small graphs with at most 9 points.

LEMMA 1. *If $G \in \mathcal{S}_0$ is not connected, it contains only one nontrivial component and at most two isolated points.*

Proof. Suppose G has two nontrivial components, one of which is not a line. Then G contains graphs (1) or (2) of Fig. 2 as induced subgraphs. If G has just one nontrivial component which is not a line and at least three isolated points, then it contains graphs (3) or (4) of Fig. 2 as induced subgraphs. □

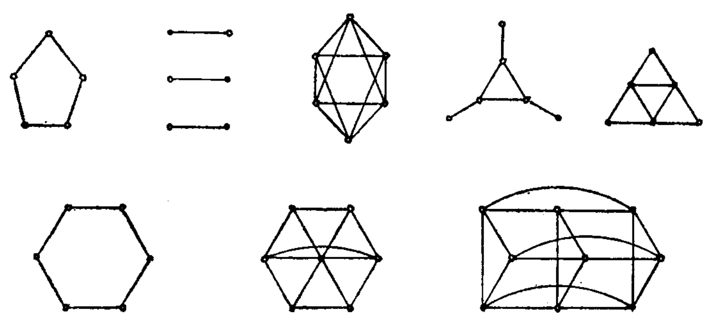

Fig. 3

LEMMA 2. *If $G \in \mathcal{S}_0$ contains at least one GCP*[1] *with at least two lines removed, then G is an induced subgraph of the graph of Fig.* 4.

Proof. Let G contain a GCP with at least two lines removed. Since G itself is not a GCP, it must contain at least one additional point. Now, because of graph (5) of Fig. 2, our GCP cannot have more than two lines removed. Since the additional point can be joined with at most two points of this GCP, (5) also implies that our GCP has at most six points. If this GCP is equal to C_4, then, because of graph (6) of Fig. 2, G contains at most one isolated point. Thus, G is contained in the graph of Fig. 4. Otherwise, if our GCP is equal to $K_5 - 2K_2$

[1] It is assumed that G is decomposed according to Theorem A. The same will be assumed later on.

or $K_6 - 2K_2$, any point not contained in it is adjacent to all l-type points of this GCP. Using (1) and (6) we easily complete the proof of the Lemma. □

LEMMA 3. *If* $G \in \mathcal{S}_0$ *contains two disjoint* $GCPs$*, each of them having one line removed, then* G *is an induced subgraph of some of the graphs displayed in Fig.* 5.

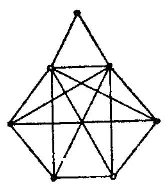

Fig. 4

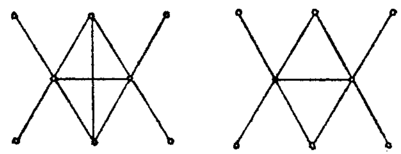

Fig. 5

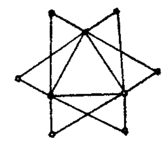

Fig 6

Proof. Suppose C_1 and C_2 are particular $GCPs$ as supposed in the lemma. Let x_1 and x_2 be any of the l-type points of C_1 and C_2, respectively. They are adjacent because of (1) (see Fig. 2). It follows from Theorem A that x_1 and x_2 are unique l-type points in C_1 and C_2. On the other hand, the line x_1x_2 belongs to some GCP, say C_3. Because of (7) there are no more $GCPs$ in G, not isolated points. By (8) and (9) C_3 cannot contain K_5 or $K_5 - x$ as an induced subgraph. Thus C_3 is one of the graphs $K_2, K_3, K_4 - x$ and K_4.□

LEMMA 4. *If* $G \in S_0$ *contains two* $GCPs$ *with a common point, each of them having one line removed, then* G *is an induced subgraph of the graph displayed in Fig.* 6.

Proof. Suppose C_1 and C_2 are particular $GCPs$ as supposed in the lemma. Because of (9), C_1 and C_2 can be equal only to $K_{1,2}$ or $K_4 - x$. If C_1 and C_2 are both equal to $K_{1,2}, G$ has no more lines by Theorem A, but it can have at most two isolated points. Furthermore, at least one the $GCPs$ is equal to $K_4 - x$. Because of (10), G has no isolated points and by Lemma 1 it is connected. Now, assume $C_1 = K_4 - x$ and $C_2 = K_{1,2}$. Let x_1 be the l-type point of C_1 which is not contained in C_2. If there are more than two $GCPs$ in G, one of them, say C_3, meets C_1 at x_1. If C_3 is not a clique, we get a case already discussed in Lemma 3. By (1), C_3 can have at most one line. Also, from the same reason, G has no more $GCPs$. Finally, assume that C_1 and C_2 are both equal to $K_4 - x$. Let x_1 and x_2 be the l-type points of C_1 and C_2, each of them belonging to just one GCP. Because of (1) x_1 and x_2 must be adjacent (therefore they constitute a new GCP, say C_3) and C_3 can be a clique with at most three points. By the same reason, if C_3 is equal to K_3, G has no more $GCPs$. Otherwise, if C_3 is not a clique, it must be equal to $K_4 - x$.□

From Lemmas 3 and 4, it follows that all other graphs G have at most one GCP with just one line removed.

LEMMA 5. *If $G \in \mathcal{S}_0$ contains just one GCP with one line removed, then G is an induced subgraph of some of the graphs in Fig. 7.*

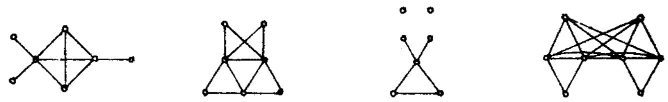

Fig. 7

Proof. Suppose C_1 is the GCP with just one line removed. Let C_2 be a clique having a common point x with C_1 and C_3 a clique disjoint with C_1. Then, by (1), x is adjacent to just one endpoint of every line in C_3. This is turn implies that C_3 is a line, i.e. K_2. On the other hand, by (11), C_1 is equal to $K_{1,2}$. furthermore, by (14), C_2 is a clique with at most four points. Any other GCP (if it exists), meets C_2 or even C_3 as well. But the latter is forbidden by (7) or (13). Since isolated points are not allowed by (7), G is an induced subgraph of the first graph of Fig.7. Thus in all other possibilities every clique meets C_1. For the same reason as before, any of these cliques has at most four points. Assume now that there is a clique C_2 with four points. In that case, by (1), C_1 must be equal to $K_{1,2}$. It follows from (8) that G has no isolated points and consequently it is an induced subgraph of the first graph of Fig. 7. Next, asume there is a clique C_2 with three points. If C_1 is not $K_{1,2}$ i.e. it has more than one l-type point, then, because of (1), each of these points is adjacent to just one point of C_2 which iz not in C_1. If C_1 has more than two points of l-type, then (12) or (1) appear in G. By (10) isolated points are forbidden and therefore G is an induced subgraph of the second graph of Fig. 7. Finally, assume all cliques are lines. From (15), it follows than C_1 has at most six points(G is not a GCP). If C_1 has five or six points, i.e. if it is equal to $K_5 - x$ or $K_6 - x$, then, by (15), G is an induced subgraph of the last graph of Fig. 7. If C_1 has four points, the same follows from (9) or (10). Thus, it remains that C_1has three points, i.e. it is equal to $K_{1,2}$. Now, it immediately follows that G is an induced subgraph of the third graph of Fig. 7. □

Now, we have to consider the graphs $G \in \mathcal{S}_0$ which contain only cliques. Of course, these graphs are line graphs. If their complements are line graphs as well, we can use the result of L. W. Beineke (Theorem D). Otherwise, if they are not line graphs, but generalized line graphs, it is clear that they were already encountered in some of the lemmas. Thus, collecting the former conclusions, we arrive to the following theorem.

THEOREM (Main result). *Both G and $\overline{G}$ are generalized line graphs if and only if either of them is equal to: 1° a generalized cocktail-party graph, or its complement; 2° an induced subgraph of some of the graphs of Fig. 8.*

It is known (see, for example, [6]) that, in comparison with line graphs, generalized line graphs comprise a larger class of graphs that have their spectra bounded from below by -2. Actually, this is the main reason for the current interest in generalized line graphs.

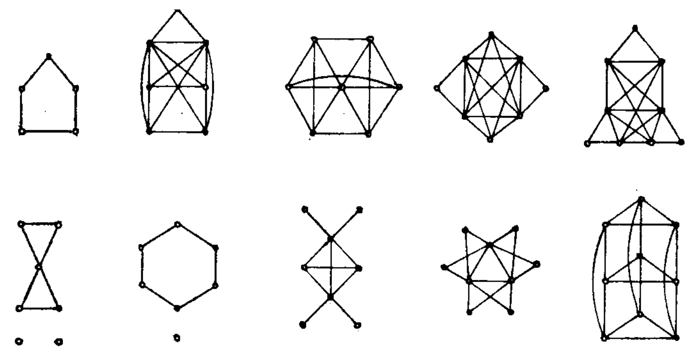

Fig. 8

It is pointed out in [**7**] that the complements of graphs having their spectra bounded from below by -2 are the graphs with second largest eigenvalue not exceeding 1. Thus, if some graphs G belongs to $\mathcal{S}$ then all its eigenvalues except possibly the largest one are located in the segment [-2, 1]. This is the main spectral implication of our result. The problem now arises to characterize all graphs with this particular property, i.e. to find all graphs whose all eigenvalues, except possibly the largest one, are located in the segment $[-2, 1]$.

Added in proof. The sixth and the seventh graph from Fig. 3 are not drawn correctly. They should be as the seventh and the third graph from Fig. 8.

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