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A NIL-EXTENSION OF A COMPLETELY SIMPLE SEMIGROUP

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Abstract. We describe semigroups which are nil-extensions of completely simple semigroups and in particular nil-extension of left groups or rectangular bands.

In this paper we consider power regular semigroups in which indempotents are primitive. These semigroups are nil-extensions of a completely simple semigroups (Theorem 1).

Power regular semigroups are considered in [1]. A semigroup S is *power regular* if for every $a \in S$ there exists $m \in N$ such that $a^m \in a^m S a^m$. A semigroup S is *power completely regular* if for every $a \in S$ there exist $x \in S$ and $m \in N$ such that $a^m = a^m x a^m$, $a^m x = x a^m$.

If e, f are idempotents of a semigroup S, we shall write $e \leq f$ if $ef = fe = e$. An idempotent is called *primitive* if it is nonzero and is minimal in the set of nonzero idempotents relative to the partial order $\leq$. By *nil-extension* we mean an ideal extension by a nil-semigroup. A semigroup S with zero 0 is a *nil-semigroup* if for every $a \in S$ there exists $n \in N$ such that $a^n = 0$. By E denote the set of all idempotents of a semigroup.

For undefined notions and notations we refer to [2], [4] and [7].

LEMMA 1. *If S is power regular semigroup all of whose idempotents are primitive, then S is power completely regular with maximal subgroups given by $G_e = eSe$ $(e \in E)$.*

Proof. For $a \in S$ there exist $x \in S$ and $m \in N$ such that $a^m = a^m x a^m$. For $a^k \in S$, where $k > m$, there exist $y \in S$ and $n \in N$ such that $a^{kn} = a^{kn} y a^{kn}$. Assume that $e = a^m x$ and $f = a^{kn} y a^m x$. Then

$$\begin{aligned} f^2 &= a^{kn}ya^mxa^{kn}ya^mx = a^{kn}y(a^mxa^m)a^{kn-m}ya^mx = a^{kn}ya^ma^{kn-m}ya^mx \\ &= a^{kn}ya^{kn}ya^mx = a^{kn}ya^mx = f \\ ef &= a^mxa^{kn}ya^mx = a^mxa^ma^{kn-m}ya^mx = a^{kn}ya^mx = f \\ fe &= a^{kn}ya^mxa^mx = a^{kn}ya^mx = f. \end{aligned}$$

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Hence, $ef = fe = f$. so $e = f$. From this it follows that

$$a^m = a^m x a^m = e a^m = f a^m = a^{kn} y a^m x a^m \in a^{m+1} S a^m$$

i.e. S is power completely regular [1, Proposition 3.2].

Let $e \in E$ and $u \in G_e$, then $u = eue \in eSe$ and thus $G_e \subseteq eSe$. Conversely, let $u \in eSe$, i.e. u ebe for some $b \in S$. Then $u^p \in G_f$ for some $p \in N$ and $f \in E$, so

$$ef = eu^p(u^p)^{-1} = e(ebe)^p(u^p)^{-1} = f$$

and dually $fe = f$. Hence, $e = f$. Therefore, $u^p \in G_e$. From this and Lemma 1 of [6] we have that $u^{p+1} \in G_e$, so

$$e = u^{p+1}(u^{p+1})^{-1} = u \cdot u^p(u^{p+1})^{-1} = u^p(u^{p+1})^{-1} \cdot u$$

and since $eu = e(ebe) = ebe = u = ue$ we have that $u \in G_e$ and therefore $eSe \subseteq G_e$.

LEMMA 2. *The unity e of a minimal bi-ideal B of S is a primitive idempotent in S.*

Proof. For an arbitrary idempotent $f \in S$, if $f = ef = fe$, then $f = efe \in eSe \subseteq B$, so $e = f$ (since B is a subgroup of S [5, Lemma 2.6]).

LEMMA 3. *Let K be the union of all minimal bi-ideals of S. Then k is a completely simple kernel of S.*

Proof. By Lemma 2.5 [5] K is an ideal of S. By Lemma 2 we have that every idempotent from K is primitive and since K is a union of groups we have that K is completely simple [4, Corollary III 3.6.]

The following theorem is a generalization of a result of Munn [6, Theorem 2].

THEOREM 1. *The following conditions are equivalent on a semigroup S:*

(i) *S is power regular and all idempotents of S are primitive;*

(ii) *S is a nil-extension of a completely simple semigroup;*

(iii) $(\forall a, b \in S)$ $(\exists m \in N)$ $(a^m \in a^m b S a^m)$.

Proof. (i) $\Rightarrow$ (ii). By Lemma 1 we have that S is power completely regular and maximal subgroups of S are of the form $G_e = eSe$ $(e \in E)$. Since $G_e (e \in E)$ is a minimal bi-ideal [5, Lemma 2.6], then by Lemma 3 we have that S has a completely simple kernel K. It is clear that for every $a \in S$ there exists $m \in N$ such that $a^m \in K$.

(ii) $\Rightarrow$ (i). This implication follows immediately.

(ii) $\Rightarrow$ (iii). If S is nil-extension of a completely simple semigroup, then for $a, b \in S$, a^m, $a^m b a^m \in G_e$ for some $m \in N$ (Lemma 1), so $a^m = a^m b a^m x$ for some $x \in G_e$. From this it follows that $a^m = a^m b a^m x (a^m)^{-1} a^m \in a^m b S a^m$.

(iii) $\Rightarrow$ (ii). For $a = b$ we have that $a^m \in a^{m+1} S a^m$, so by [1, Proposition 3.2] S is power completely regular. Let S have a proper ideal I. For $e \in E$ and

$b \in I$ we have $e \in ebSe \subseteq 1$. Hence, the intersection of all ideals of S is nonempty, i.e. S has a minimal ideal K. Since K is power completely regular we have that K is completely simple (Theorem 2. [6]). For $a \in S$ and $b \in K$ we have that $a^m \in a^m bSa^m \subseteq K$ for some $m \in N$.

THEOREM 2. *The following conditions on a semigroup S are equivalent:*

(i) *S is a nil-extension of a left group;*

(ii) *S is power regular and E is a left zero band;*

(iii) $(\forall a, b \in S)$ $(\exists m \in N)$ $(a^m \in a^m S a^m b)$.

Proof. (i) $\Rightarrow$ (ii). This implication follows immediately.

(ii) $\Rightarrow$ (iii). By Theorem 1 we have that S contains a completely simple kernel K which is, in fact, a left group. For $a, b \in S$ there exist $m, n \in N$ such that a^m, $b^n \in K$, so $a^m = xb^{n+1}$, $b^n = ya^m$ for some $x, y \in K$. Since $a^m \in G_e$ for some $e \in E$ we have $a^m = a^m(a^m)^{-1}xb^n b = a^m(a^m)^{-1}xya^m b \in a^m S a^m b$.

(iii) $\Rightarrow$ (i). If the condition (iii) holds, then for $a \in S$ we have that $a^m \in a^m S a^m a = a^m S a^{m+1}$ for some $m \in N$ and therefore by Proposition 3.2. [1] S is power completely regular. For $e, f \in E$ we have that $f = fxfe$ for some $x \in S$, so $fe = (fxfe)e = f$, i.e. E is a left zero band. Hence, $K \cup_{e \in E} G_e$ is a left group (see [2, Ex. e. § 1.11.]

COROLLARY 1. *S is a left group iff* $(\forall a, b \in S)$ $(a \in aSab)$.

THEOREM 3. *Let S be a semigroup. If*

$$(\forall a \in S)(\exists_1 x \in S)(\exists m \in N)(a^m = xa^{m+1}) \tag{1}$$

then S is a nil-extension of a left group.

Proof. Let (1) be satisfied in a semigroup S. Then $a^m = xa^{m+1} = x^2 a a^{m+1}$. From this and from (1) it follows that

$$x = x^2 a. \tag{2}$$

Furthermore, for x there exist $y \in S$ and $n \in N$ such that $x^n = yx^{n+1}$ and

$$y^2 = yx. \tag{3}$$

From (2) and (3) it follows that

$$y^2 = yy^2 x = y^3 x^2 a = y^3 x x^2 a^2 = y^2 x^2 a^2 = y^2 xa = ya = yx^m a^{m+1}.$$

For $k = \max(m, n)$ we have

$$y^2 = yx^{m+1} = yx^{k+1}a^{k+2} = yx^{n+1}x^{k-n}a^{k+2} = x^n a^{k-n} a^{k+2} = x^k a^{k+2} = xa^3,$$

so $y = y^2 x = xa^3 x$. Further,

$$\begin{aligned} y^{m+2} = y^m y^2 = y^m ya = y^{m-1} y^2 a \\ = y^{m-1} ya^2 = \ldots = ya^{m+1} = xa^3 xa^{m+1} = xa^3 a^m = a^{m+2}. \end{aligned}$$

From this it follows that $y^{m+2}x^{m+1} = a^{m+2}x^{m+1}$ and by (3) we have $y = a^{m+2}x^{m+1}$. Hence

$$a^m = xa^{m+1} = x^n a^{m+n} = yx^{n+1}a^{m+n} = a^{m+2}x^{m+1}x^{n+1}a^{m+n}$$

so $a^m \in a^{m+1}Sa^m$, i.e. S is power completely regular.

Let $e, f \in E$. Then $(ef)^m = x(ef)^{m+1} = xe(ef)^{m+1}$ for some $x \in S$ and $m \in N$. By uniqueness we have that $x = x^2ef$ and $x = xe$. From this it follows that $x = xe = xf$, so $x = (ef)^m$. Furthermore, $(ef)^m = (ef)^m = e = (efe)^m$ and

$$(ef)^m = (efe)^m f = (ef)^{m+1} = (ef)(ef)^{m+1} = e(ef)^{m+1}.$$

Therefore, $ef = e$. So by Theorem 2 S is a nil-extension of a left group.

Definition 1. S is a power group if S is a power regular with exactly one idempotent.

Theorem 4. *The following conditions are equivalent on a semigroup S:*

(i) *S is a power group;*

(ii) *S is a nil-extension of a group;*

(iii) $(\forall a, b \in S)\ (\exists m \in N)\ (a^m \in ba^m Sa^m b)$

Proof. (i) $\Rightarrow$ (ii) This implication follows immediately.

(ii) $\Rightarrow$ (iii) Let S be nil-extension of a group G. For $a, b \in S$ we have that $a^m, a^m b, ba^m sa^m b \in G$ for some $m \in N$ and for each $s \in S$, and then $a^m = ba^m sa^m bx$ for some $x \in G$, i.e. $a^m = ba^m sa^m bx(a^m b)^{-1}a^m b \in ba^m Sa^m b$.

(iii) $\Rightarrow$ (i) It is clear that S is power regular. We shall prove that S has only one idempotent. If e and f are idempotents from S, then $e = xf$, $f = ey$ for some $x, y \in S$, so $ef = xff = xf = e$, $ef = eey = ey = f$ thus $e = f$.

Corollary 2. *The following conditions are equivalent on a semigroup S:*

(i) *S is a regular semigroup with only one idempotent;*

(ii) *S is a group;*

(iii) $(\forall a, b \in S)\ (a \in baSab)$.

Remark. (i) $\Rightarrow$ (ii) is Corollary IV.3.6. of [4].

Lemma 4. *Let S be a semigroup. If*

$$(\forall a \in S)(\exists_1 x \in S)(\exists m \in N)(a^m = a^m xa^m) \tag{4}$$

then S is a power group.

Proof. Assume that (4) holds. Then for $e, f \in E$ we have

$$(ef)^m = (ef)^m g(ef)^m \tag{5}$$

for some $g \in S$ and $m \in N$ and by uniqueness we have that

$$g = g(ef)^m g \tag{6}$$

It follows from $(ef)^m fg(ef)^m = (ef)^m$ that

$$fg = g \tag{7}$$

Similarly,

$$ge = g. \tag{8}$$

If $m = 1$, then by (6), (7) and (8) we have that $g = g^2$.

If $m > 1$, then by (6), (7) and (8) we obtain $g = g(ef)^m g = g(fe)^{m-1} g$ and by uniqueness we have that

$$(ef)^m = (fe)^{m-1} \tag{9}$$

It follows from (5) and (9) that $(fe)^{m-1} = (fe)^{m-1} g(fe)^{m-1} = (fe)^{m-1} eg(fe)^{m-1}$, so

$$eg = g. \tag{10}$$

Similarly,

$$gf = g. \tag{11}$$

By (7), (8), (9) and (10) we have that $g = g(ef)^m g = g^2$. Since g is an idempotent, then by uniqueness from (6) we obtain $g = (ef)^m$. Hence,

$$(ef)^{2m} = (ef)^m e(ef)^m = (ef)^m = (ef)^m f(ef)^m$$

and therefore $e = f$. Thus S is a power group.

Remark. The converse of Lemma 4 is not true. For example, the semigroup S given by table 1

1	a	b	c
a	a	b	a
b	b	a	b
c	a	b	a

2	a	b	c
a	a	b	b
b	b	a	a
c	b	a	a

is a power group. But, for c we have that $c^2 = a \in G = \{a, b\}$ and there exist $x = a$ and $x = c$ such that $c^2 = c^2 x c^2$.

It is easy to see that in the semigroup given by table 2 the condition (1) from Lemma 4 is satisfied.

Theorem 5. *The condition* (4) *from Lemma 4 holds iff there is only one idempotent e in S and for every $a \in S$ there exists $m \in N$ such that $a^m = a^m x a^m$, $xe = x$.*

Proof. If (4) holds, then by Lemma 4 S contains only one idempotent e. By uniqueness we have that $x = xa^m x$ and $a^m x = e$ implies $xe = x$

Conversely, assume that for $a \in S$ there exist $x, y \in S$ and $m \in N$ such that

$$a^m = a^m x a^m = a^m y a^m. \tag{12}$$

By uniqueness of the idempotent we have that $a^m x = xa^m$. Hence, a^m is in a subgroup G_e of S. By Lemma 1 [6] we have that $xe = ex$, $ye = ey$ and $xe, ye \in G_e$.

So by (12) we have that $a^m exa^m = a^m eya^m$ and thus $ex = ey$ by cancellation in G_e. Hence, $x = y$.

COROLLARY 3. [3] *S is a group iff* $(\forall a \in S)$ $(\exists_1 x \in S)$ $(a = axa)$.

THEOREM 6. *S is a nil-extension of a rectangular band iff*

$$(\forall a, b \in S)(\exists m \in N)(a^m = a^m ba^m).$$

Proof. Let S be a nil-extension of a rectangular band E. Then for $a, b \in S$ there exists $m \in N$ such that $a^m = e \in E$ and by Lemma 1 we have that $a^m ba^m = e$. Thus $a^m = a^m ba^m$.

Conversely, it is clear that $E \neq \emptyset$. For $e, f \in E$ we have $e = efe$ and $f = fef$ and if $ef = fe$, then $e = ef = f$. Thus E is a rectangular band. For $e \in E$ and $x \in S$ we have that $e = exe$, so $ex, xe \in E$, i.e. E is an ideal of S and clearly for every $a \in S$ there exists $m \in N$ such that $a^m \in E$. Therefore, S is a nil-extension of a rectangular band.

COROLLARY 4. [4] *S is a rectangular band iff* $(\forall a, b \in S)$ $(a = aba)$.

COROLLARY 5. *S is a nil-extension of a left zero band iff*

$$(\forall a, b \in S)(\exists m \in N)(a^m = a^m b).$$

COROLLARY 6. *S is a nil-semigroup iff* $(\forall a, b \in S)$ $(\exists m \in N)$ $(a^m ba^m = a^m b)$.

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