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ON ALGEBRAS ALL OF WHOSE SUBALGEBRAS ARE SIMPLE; SOME SOLUTIONS OF PLONKA'S PROBLEM

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Abstract. For each cardinal number $\alpha \geq 1$, we construct two types of grupoids $\langle X_\alpha ; \circ\rangle$ and $\langle X_\alpha ; *\rangle$ which are hereditarily simple and have subgrupoids of all small orded. If $\alpha \geq \aleph_0$, we show that they both admit only discrete topology to become topological grupoids. An application of the grupoid $\langle X_\alpha ; *\rangle$ in the theory of non-associative rings is indicated.

1. Introduction. An algebra $\mathfrak{A}$ is simple if and only if its lattice of congruences is isomorphic to the two element chain. It is said to be hereditarily simple if every subalgebra is simple.

J. Plonka of the Polish Academy of Sciences (private communication) has asked whether there exists an infinite hereditarily simple universal algebra $\mathfrak{A} = \langle A, F\rangle$ such that for any cardinal number $1 \leq \alpha \leq \mid A \mid$ there exists a subalgebra $\mathfrak{B}$ of order $\alpha = \mid \mathfrak{B} \mid$.

The infinite chain, the left zero semigroups of the right zero semigroups are example of semigroups with arbitrary small order of subalgebras. Unifortunately, they are not hereditarily simple.

The construction of quasi-primal algebras which was given by Stone in [**3**, p. 404] provides hereditarily simple algebras with arbitrarily small order of subalgebras. However, all the subalgebras are finite.

The aim of this note is to present two types of grupoids, i. e. universal algebras of type $\langle 2\rangle$ which provide solution to Plonka's problem. We show that those grupoids of infitine cardinalities admit only discrete topology to become topological grupoids. Using one type of grupoids and division rings we can construct a large class of simple non-associative rings.

2. Hereditarily Simple Groupoids. For each cardinal number $\alpha \geq 1$, let X_α be a set with such a cardinal number.

We introduce here two different types of grupoids:

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(I) Fix an element, say e, in X_α. Define a binary operation $\circ$ on X_α as follows:

(1) $x \circ e = e \circ x = x$ for all x in X_α.

(2) $x \circ x = x$ for all x in X_α.

(3) $x \circ y = e$ if $x \neq y$ in $X_\alpha \backslash \{e\}$.

(II) Fix an element, say 0, in X_α. Define a binary operation $*$ on X_α as follows:

(1) $x * 0 = 0 * x = 0$ for all x in X_α.

(2) $x * x = 0$ for all x in X_α.

(3) $x * y = x$ if $x \neq y$ in $X_\alpha \backslash \{0\}$.

We call the elements 0 and e in $\langle X_\alpha; *\rangle$ and $\langle X_\alpha; \circ\rangle$ respectively *distinquished* elements.

We have the following result:

THEOREM 2.1. *The grupoid* $\langle X_\alpha; \circ\rangle$*(or*$\langle X_\alpha; *\rangle$*) has the following properties:* (1) *a subset A is a subrupoid if and only if it contains the distinguished element;* (2) *each subgrupoid is simple.*

Proof (1) is obvious.

(2) If A is a subgrupoid of $\langle X_\alpha; \circ\rangle$ then by (1) it is isomorphic to X_β for $\beta = \mid A \mid$. Therefore we need to show that each grupoid of the form $\langle X_\alpha; \circ\rangle$ is simple.

Let θ be a non-identity congurence of X_α and x, y are two distinct elements such that $x\theta y$. Consider the following cases:

Case 1. $x = e$. From $e\theta y$ we have for any $z \in X_\alpha \backslash \{e, y\}$, $z \circ e\theta z \circ y$, i. e. $z\theta e$. Hence $\theta = X_\alpha \times X_\alpha$.

Case 2. $x, y \in X_\alpha \backslash \{e\}$. Since $x \circ x\theta x \circ y$ we obtain $x\theta e$ which reduces to case 1. Hence $\theta = X_\alpha \times X_\alpha$.

All these cases show that θ is the universal conguence. Hence $\langle X_\alpha; \circ\rangle$ is simple.

The proof for $\langle X_\alpha; *\rangle$ is similar to the above proof and we omit it. □

COROLLARY 2.2. *For any* $\alpha \geq \aleph_0$ *the grupoids* $\langle X_\alpha; \circ\rangle$ *and* $\langle X_\alpha; *\rangle$ *are solutions of Plonka's problem.* □

Remark. Mc Nulty and Shallon [**8**] construced a grupoid $\mathcal{G}(G)$ from a graph G which they called graph algebra. The grupoid $\langle X_\alpha; *\rangle$ is in fact a graph algebra $\mathcal{G}(X_\alpha^*)$ of a complete graph on $X_\alpha \backslash \{0\} = X_\alpha^*$.

A neighborhood of a vertex in the graph is the set of all vertices adjacent to that vertex. McNulty and Shallon showed that the graph algebra $\mathcal{G}(G)$ is simple if and only if for any pair of distinct vertices in G it has distinct neigborhoods.

Since for each vertex x of the complete graph X_α^* the neighborhood $N(x)$ of x is $X_\alpha \backslash \{x\}$, by the result of McNulty and Shallon we can give another proof of the simplicity of $\langle X_\alpha; *\rangle$.

3. Special Feature of Grupoids $\langle X_\alpha;\circ\rangle$ and $\langle X_\alpha;*\rangle$. Recall that a triple $\langle A;F;T\rangle$, which consist of a universal algebra $\langle A;F\rangle$, is a topological algebra if each operation f of F is continuous under T. A universal algebra is called a DT-algebra ([**6**]) if the only topology if can be equipped with to become a topological algebra is the disrete topology.

Hansen [**4**] provided the first example of a DT-grupoid. In [**6, 7**] we showed that any n-grupoid, i. e. algebra of type $\langle n\rangle$, is a subalgebra of a $DT-n$- grupoid. Other types of DT-algebras such as groups, rings, quasi-groups and loops have been investigated in [**1, 9, 10**] and [**11**].

We now prove:

THEOREM 3.3. *For any cardinal number $\alpha \geq \aleph_0$, the grupoid $\langle X_\alpha;*\rangle$ is a DT-grupoid.*

Proof, Let T be a Hausdorff topology such that $\langle X_\alpha;*;T\rangle$ is topological grupoid. We want to show that any one-element set $\{x\}$ is open in $\langle X_\alpha;T\rangle$, from which we will deduce that T is discrete. Consider the following cases:

Case 1. $x \neq 0$. Since $x*x=0$ we have that for each open neighborhood V of 0 there exists an open neighborhood U of x such that $U*U \subseteq V$. As $\langle X_\alpha;T\rangle$ is a Hausdorff space we can find an open neighborhood V of 0 which does not contain x. Then, by definition of our operation $*$, U must be either $\{x\}$ or $\{x,0\}$. If $U=\{x,0\}$ then, as $\{0\}$ is closed, we conclude that $\{x\}=U-\{0\}$ is open.

Case 2. $x=0$. Since $y*0=0$, if W is an open neighborhood of 0 that contains no x, then by continuity of $*$ we can find two open neighborhoods U,V of y, and 0, respectively, such that $U*V\subseteq W$. Then V must be equal to $\{0\}$ or $\{0,x\}$, for otherwise we would have $x\in W$, which contradicts the hypothesis.

By the argument of case 1 we conclude that $\{0\}$ is open. Thus T is a discrete topology. □

Using a similar argument, we also have the following theorem:

THEOREM 3.4. *For any cardinal number $\alpha \geq \aleph_0$, the grupoid $\langle X_\alpha;\circ\rangle$ is a DT-grupoid.*

4. Application of the Grupoid $\langle X_\alpha;*\rangle$ in the Theory of Non-Associative Rings. In this section we shall use the grupoids of Section 2 to construct some simple non-associative rings.

Let $\langle G;*\rangle$ be a grupoid. Let F be a division ring. Denote by $F[G]$ the set of all functuions from G to F with finite support, i. e. $f(a)=0$ for almost every $a\in G$. Let $H: G\times G \to F$ be a non-zero function.

Define $+$ and $\times$ on $F[G]$ as follows:

(1) $(f+g)(a)=f(a)+g(a)$ for any $f,g\in F[G]$ and $a\in G$.

(2) $(f\times g)(a)=\sum\limits_{b*c=a} H(b,c)f(b)\cdot g(c)$.

In general, $F[G]$ is a non-associative ring. We will denote f by $\sum ra$ where $f(a) = r$. If the grupoid G has the zero z we shall identify the element rz where $r \in F$ with the zero 0 of the ring. The ring is called the truncated grupoid ring over F associated with the grupoid G and the factor set $\{H(i,j)\}$ and will be denoted by $F[G;H]$.

If $H : G \times G \to F$ is the constant map $H(i,j) = 1$ for all $i,j \in G$, then $F[G;H]$ is the usual grupoid ring.

This construction of ring was orginally introduced by Bruck [**2**] for the loop G and he showed that for a suitable choice of the factor set $\{H(i,j)\}$, the truncated loop algebra is simple. Jenner [**5**] showed that if the factor set has the property that $H(i,j) \neq 0$ for $x_i * x_j \neq 0$, then the truncated loop algebra is simple.

We observe that the following holds:

$\circ$	e	a	b
e	e	a	b
a	a	e	e
b	b	e	e

Example 4.5. Let $\langle X_2, \circ \rangle = \{e, a, b\}$ with the following multiplication table and $F = GF(2)$ be the Galois field of order 2.

Then the grupoid ring $F[X_2]$ is not simple. In fact, $I = \{0, e+a, e+b, a+b\}$ is a proper ideal of $F[X_2]$.

However, for the type 2 grupoids we have:

THEOREM 4.6. *For any $\alpha \geq 1$ and any division ring F, if X_α is the type* 2 *grupoid and $H : X_\alpha \times X_\alpha \to F$, with property $H(i,j) \neq 0$ for any $i \neq j$, then the truncated grupoid ring $F[X_\alpha;H]$ is simple.*

Proof. It sufficies to show that any two-sided ideal (u) generated by a non-zero element u in $F[X_\alpha;H]$ is the whole ring.

If u has length greater than two, then by the property of the multiplication of the grupoid X_α we can find an element in (u) with length one.

Without loss of generality, we may assume $u = rx_j$. For any $x_i \in X_\alpha$ and $s \in F$ we have $H(i,j)^{-1}(s,r)^{-1}x_i \circ u = sx_i \in (u)$. Thus (u) contains all the generators of $F[X_\alpha;H]$ and hence $(u) = F[X_\alpha;H]$. $\square$

The above result will provide a large class of new simple non-associative rings. We deduce from Theorem 4.6 the following theorem:

THEOREM 4.7. *For any $\alpha \geq 1$, there exists a simple ring R with α generators such that R contains a simple subring which is generated by β generators for all $1 \leq \beta \leq \alpha$.*

We make the following conjecture:

CONJECTURE 4.8. *For any $\alpha \geq N_0$, and any finite Galois field F, the simple ring $F[X;H]$ in Theorem* 4.6 *is a DT-ring.*

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