

AN ORDERING OF THE SET OF SENTENCES OF PEANO ARITHMETIC

Aleksandar Ignjatović

Abstract. We consider a partial ordering of the set of sentences of Peano arithmetic P induced by a theory T extending P , which orders sentences according to the complexity of their “proofs”. Using some properties of the ordering induced by the theory $P + \neg\text{Con}_p$ we prove that P doesn't have the Joint Embedding Property. We also describe models for P which do not enrich the ordering induced by P , i.e. models satisfying $\langle_{\text{Th}(\mathfrak{M})} = \langle_P$, and we prove that for every consistent theory $T, T \supset P$, there is a theory $T' \supset P$ such that the ordering induced by the theory T' is a linear extension of the ordering induced by the theory T .

By L_P we denote the language of P and by $S(P)$ the set of sentences of P . Any consistent extension of P we denote by T , and N stands for the structure of natural numbers. By $\mathfrak{M}, \mathfrak{N}, \dots$ we denote nonstandard models for P , and by $|\mathfrak{M}|, |\mathfrak{N}|, \dots$ their domains respectively. If \mathfrak{M} and \mathfrak{N} are models for P , then $\mathfrak{M} \subset_{\Sigma_1} \mathfrak{N}$ means that for all Σ_1 -formulas φ and all $a_1, \dots, a_n \in |\mathfrak{M}|$, $\mathfrak{M} \models \varphi[a_1, \dots, a_n]$ implies $\mathfrak{N} \models \varphi[a_1, \dots, a_n]$; similarly, we write $\mathfrak{M} \prec_{\Sigma_1} \mathfrak{N}$ when $\mathfrak{M} \models \varphi[a_1, \dots, a_n]$ holds iff $\mathfrak{N} \models \varphi[a_1, \dots, a_n]$ holds.

We use the following model-theoretical consequence of Matijasevič's theorem.

LEMMA 0. *Let $\mathfrak{M}, \mathfrak{N}$ be models for P ; then $\mathfrak{M} \subset \mathfrak{N}$ implies $\mathfrak{M} \prec_{\Sigma_0} \mathfrak{N}$ and $\mathfrak{M} \subset_{\Sigma_1} \mathfrak{N}$. \square*

For any sentences φ and ψ from $S(P)$, by $\varphi < \psi$ we denote the Σ_1 -sentence

$$\exists x(\text{Prf}_p(x, \ulcorner \varphi \urcorner) \wedge (\forall y \leq x) \neg \text{Prf}_p(y, \ulcorner \psi \urcorner)).$$

The following lemma enables us to introduce the ordering in $S(P)$. msk

LEMMA 1. *Let T be a consistent extension of P ; then the relation \langle_T defined by $\varphi \langle_T \psi$ iff $T \vdash \varphi < \psi$ is transitive and irreflexive. \square*

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Since the sentence $\varphi < \psi$ is Σ_1 , it is obvious that $\varphi <_P \psi$ holds iff $\varphi < \text{Th}(N)\psi$ holds, i.e. $<_P = <_{\text{Th}(N)}$. The order type of $<_P$ is $\omega + \Lambda$; the set of theorems of P has order type ω , and Λ is the type of the empty ordering of the countable set of sentences from $S(P)$ not provable in P .

The ordering induced by the theory of a model $\mathfrak{M} \models P$ is linear iff $\mathfrak{M} \models \neg \text{Con}_P$. If $\mathfrak{M} \models \text{Con}_P$, then the ordering consists of a linearly ordered set of sentences having “proofs” in \mathfrak{M} , i.e. sentences satisfying $\mathfrak{M} \models \text{Th}_P(\ulcorner \varphi \urcorner)$, and the countable remainder of $S(P)$ with empty ordering; it is obvious that if φ belongs to the first set and ψ to the second, then $\varphi <_{\text{Th}(\mathfrak{M})} \psi$ holds.

Suppose that \mathfrak{M} and \mathfrak{N} are models for P and $\mathfrak{M} \subset \mathfrak{N}$. Since $\varphi < \psi$ is a Σ_1 -sentence, $\varphi <_{\text{Th}(\mathfrak{M})} \psi$ implies $\varphi <_{\text{Th}(\mathfrak{N})} \psi$, i.e. $<_{\text{Th}(\mathfrak{M})} \subset <_{\text{Th}(\mathfrak{N})}$. Also, $<_T \subset <_{\text{Th}(\mathfrak{M})}$ holds for any $T \supset P$ and $\mathfrak{M} \models T$. The next proposition describes models for P satisfying $<_P = <_{\text{Th}(\mathfrak{M})}$.

PROPOSITION 1. *Let \mathfrak{M} be a model for P ; then $<_P = <_{\text{Th}(\mathfrak{M})}$ iff $N \prec_{\Sigma_1} \mathfrak{M}$.*

Proof. Since $<_P = <_{\text{Th}(N)}$, from $N \prec_{\Sigma_1} \mathfrak{M}$, it immediately follows $\varphi <_P \psi$ iff $\varphi <_{\text{Th}(N)} \psi$ iff $\varphi <_{\text{Th}(\mathfrak{M})} \psi$ i.e. $<_P = <_{\text{Th}(\mathfrak{M})}$. Conversely, suppose that $<_P = <_{\text{Th}(\mathfrak{M})}$ holds, and let φ be a Σ_1 -sentence true in \mathfrak{M} . Since for Σ_1 -sentences φ , $P \vdash \varphi \rightarrow \text{Th}_P(\ulcorner \varphi \urcorner)$ holds (see 5.3.4 in [SM]), we get that $\mathfrak{M} \models \text{Th}_P(\ulcorner \varphi \urcorner)$. Then, for every other sentence $\psi \in S(P)$, either $\varphi <_{\text{Th}(\mathfrak{M})} \psi$ or $\psi <_{\text{Th}(\mathfrak{M})} \varphi$ holds. This and the assumption $<_P = <_{\text{Th}(\mathfrak{M})}$ imply that φ belongs to the linearly ordered part of $<_P$. Thus, φ is a theorem of P and consequently $\mathfrak{M} \models \varphi$. Since for all Σ_1 -sentences φ , $N \models \varphi$ implies $\mathfrak{M} \models \varphi$, we get $N \prec_{\Sigma_1} \mathfrak{M}$. square

Using Corollary 2.9.1 from [MI], asserting that $N \prec_{\Sigma_1} \mathfrak{M}$, iff $\bigcap \{ \mathfrak{N} \mid \mathfrak{N} \subset_e \mathfrak{M}, \mathfrak{N} \cong \mathfrak{M} \} = N$, we get the following algebraic characterization of models satisfying $<_P = <_{\text{Th}(\mathfrak{M})}$.

COROLLARY 1. $<_P = <_{\text{Th}(\mathfrak{M})}$ iff $\bigcap \{ \mathfrak{N} \mid \mathfrak{N} \subset_e \mathfrak{M}, \mathfrak{N} \cong \mathfrak{M} \} = N$. □

We use the following lemma to prove that, although the ordering $<_{\text{Th}(\mathfrak{M})}$ is linear for any model \mathfrak{M} of the theory $P + \neg \text{Con}_P$, the ordering $<_{P + \neg \text{Con}_P}$ itself is not linear.

LEMMA 2. *There is a sentence varphi , independent of the theory $P + \neg \text{Con}_P$, such that $P \vdash \varphi \leftrightarrow ((\varphi \rightarrow \text{Con}_P) < (\neg \text{Con}_P \rightarrow \varphi))$.*

Proof. Gödel’s diagonalization technique and usual arguments for sentences of the Rosser type.

PROPOSITION 2. *There are two sentences $\sigma, \psi \in S(P)$, such that neither $\sigma <_{P + \neg \text{Con}_P} \psi$, nor $\psi <_{P + \neg \text{Con}_P} \sigma$ holds.*

Proof. Let σ be the sentence $\varphi \rightarrow \text{Con}_P$ and ψ the sentence $\neg \text{Con}_P \rightarrow \varphi$, where φ is the sentence from Lemma 2. From the same lemma we get that there are two models \mathfrak{M}_1 and \mathfrak{M}_2 of the theory $P + \neg \text{Con}_P$ such that $\mathfrak{M}_1 \models \varphi$ and $\mathfrak{M}_1 \models \neg \varphi$. It is obvious that $\sigma <_{\text{Th}(\mathfrak{M}_1)} \psi$ and $\neg(\sigma <_{\text{Th}(\mathfrak{M}_2)} \psi)$. But $\mathfrak{M}_2 \models \neg \text{Con}_P$ implies that the ordering $<_{\text{Th}(\mathfrak{M}_2)}$ is linear; so we get $\psi <_{\text{Th}(\mathfrak{M}_2)} \sigma$. Thus, neither $\sigma <_{P + \neg \text{Con}_P} \psi$ nor $\psi <_{P + \neg \text{Con}_P} \sigma$ holds.

Models \mathfrak{M}_1 and \mathfrak{M}_2 described in the proof of Proposition 2 don't have a common extension which is a model for P : $\mathfrak{M}_1 \subset \mathfrak{N}$, $\mathfrak{M}_2 \subset \mathfrak{N}$ and $\mathfrak{N} \models P$ would imply $\sigma <_{\text{Th}(\mathfrak{N})} \psi$ and $\psi <_{\text{Th}(\mathfrak{N})} \sigma$, which is a contradiction. Thus, we get the following corollary.

COROLLARY 2. *Peano arithmetic does not have the Joint Embedding Property, i.e. there are two models of P which cannot be embedded in any common extension which is a model for P .* \square

Let T be a consistent extension of P ; if the theory $T' = T + \neg\text{Con}_P$ is consistent and $\mathfrak{M} \models T'$, then the ordering $<_{\text{Th}(\mathfrak{M})}$ is a linear extension of the ordering $<_T$. Using the following lemma [KR] we prove that the ordering $<_T$ induced by any consistent extension T of P can be extended in a similar way.

LEMMA 3 (Kreisel). *The theory $P + \neg\text{Con}_P$ is a Π_1 -conservative extension of P , i.e. for any Π_1 -sentence φ , $P + \neg\text{Con}_P \vdash \varphi$ iff $P \vdash \varphi$.*

PROPOSITION 3. *For any consistent theory $T \supset P$ there is a theory $T' \supset P$ such that the ordering $<_{T'}$ is a linear extension of $<_T$.*

Proof. Let $S = \{\varphi < \psi \mid \varphi <_T \psi; \varphi, \psi \in S(P)\}$. The theory $P + \neg\text{Con}_P + S$ is consistent: for any finite $S_0 \subset S$, Lemma 3 and $P + \neg\text{Con}_P \vdash \neg\bigwedge_{\varphi \in S_0} \varphi$ would imply $P \vdash \neg\bigwedge_{\varphi \in S_0} \varphi$; this is a contradiction, since $T \supset P$ is consistent and $T \vdash \bigwedge_{\varphi \in S_0} \varphi$. Let \mathfrak{M} be a model for the theory $P + \neg\text{Con}_P + S$; the ordering $<_{\text{Th}(\mathfrak{M})}$ is a linear extension of $<_T$ induced by the consistent extension $T' = \text{Th}(\mathfrak{M})$ of P .

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Prirodnomatematički fakultet
34000 Kragujevac, Jugoslavija

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